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Classification of topological defects in Abelian topological states

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We propose the most general classification of pointlike and linelike extrinsic topological defects in $(2+1)$-dimensional Abelian topological states. We first map generic extrinsic defects to boundary defects, and then provide a classification of the latter. Based on this classification, the most generic point defects can be understood as domain walls between topologically distinct boundary regions. We show that topologically distinct boundaries can themselves be classified by certain maximal subgroups of mutually bosonic quasiparticles, called Lagrangian subgroups. We study the topological properties of the point defects, including their quantum dimension, localized zero modes, and projective braiding statistics.

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A fundamental discovery in condensed matter physics has been the understanding of topologically ordered states of matter.[1,2] Topologically ordered states possess quasiparticle excitations with fractional statistics, topology-dependent ground-state degeneracies, and long-range entanglement, all of which are robust even without symmetry. The most common topological orders seen experimentally are the fractional quantum Hall (FQH) states, while there is increasing evidence that they may be observed in frustrated magnets.[3]

Recently, an entirely new window into the physics of topologically ordered states, called twist defects or extrinsic defects, has been discovered and attracted increasing interest.[4–21] An extrinsic defect is a pointlike or linelike defect either in a topological state or on the interface between two topologically distinct states, which leads to novel topological properties that are absent without the defect. For example, while it is well known that topological states can host robust gapless edge modes, it is only recently becoming apparent that the structure of gapped edges is also host to a rich set of universal physics. Topologically ordered states can have topologically distinct types of gapped boundaries, which are separated from each other by a quantum phase transition on the edge. Domain walls between topologically distinct boundaries are pointlike extrinsic defects that lead to a wide class of localized "parafermion" zero modes. This vastly generalizes the Majorana zero modes that are currently the subject of intense research[22] and represents a new direction for realizing non-Abelian statistics and universal topological quantum computation.

A simple example is a "genon":[4–6] Consider a branch-cut line in a bilayer topological state, across which the two layers are exchanged (Fig. 1). A genon is defined as an end point of the branch cut. It was observed that the bilayer system with genons is topologically equivalent to a single-layer system on a high-genus surface, yielding a topological degeneracy that grows exponentially with the number of genons, and a notion of (projective) braiding statistics that can be studied systematically.[5] Even when the topological state in each layer is Abelian, the genons have non-Abelian statistics. This has led to a recent experimental proposal for realizing a wide class of topological qubits in conventional bilayer FQH states,[8] and an understanding of how to realize universal topological quantum computation using nonuniversal, non-Abelian states.[5] Extrinsic defects with the same type of non-Abelian statistics and parafermion zero modes have also been proposed in other physical systems, such as lattice defects in certain exactly solvable Z_N rotor models,[9,10] and FQH states in proximity with superconductivity (SC) and ferromagnetism (FM).[11–13,23]

Given the panoply of such exotic physics discovered recently, there is a fundamental question about what is the general, unifying conceptual framework to understand these results. A deeper understanding, aside from being of intrinsic interest, is expected to aid in the challenge of identifying topological order in experimental or computational settings, and to provide a guide to further development of practical experimental proposals to probe these phenomena. In this Rapid Communication, we report the development of such a general understanding of extrinsic defects in Abelian topological states. For two-dimensional topological states, there are two general forms of extrinsic defects: *Line defects*, which separate two different or identical topological states, and *point defects*, which may exist in a single topological state, such as twist defects,[5] or at junctions between different line defects (Fig. 2). We demonstrate that all extrinsic defects can be mapped to *boundary defects*, i.e., boundary lines of topological states with point defects separating different boundary regions. Based on the understanding of boundary defects, we develop a classification of gapped line defects between Abelian topological states, extending previous results.[16–21,24] We prove that gapped line defects are *classified* by "Lagrangian subgroups," which consist of subgroups of topological quasiparticles that have trivial self and mutual statistics, and that are condensed on the boundary. This proves a recent conjecture[18] about the classification of "topological boundary conditions" in Abelian Chern-Simons theory, and extends a recent result[21]

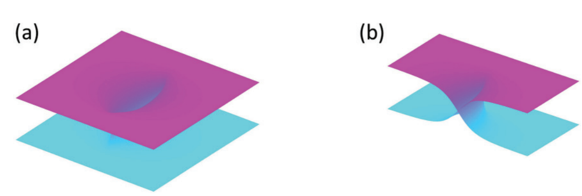

FIG. 1. (Color online) (a) Schematic picture of bilayer system with a pair of genons (Refs. 4,5). (b) Same as (a) with part of the system removed, to see the branch-cut line clearly.

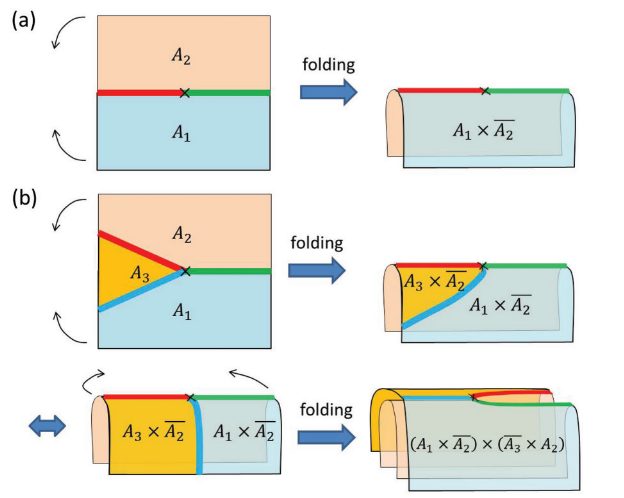

FIG. 2. (Color online) (a) Domain wall between two different gapped edges separating topological phases A_1 and A_2. By folding A_2 over, this can be mapped to a domain wall on the boundary separating $A_1 \times \bar{A}_2$ and the vacuum. (b) A junction where multiple gapped edges meet is also a possible type of point defect. On an infinite plane, by applying the folding trick multiple times, this can also be mapped to a domain wall on the boundary separating a topological phase and the vacuum.

that the existence of a Lagrangian subgroup is a necessary and sufficient condition for the existence of a gapped edge. It also explains how to understand on a deeper level the role of SC, FM, or twisted boundary conditions in recent works.[4,5,11–13]

The nontrivial point defects on the boundary are then classified by domain walls between topologically distinct line defects. We obtain the quantum dimension of general point defects, demonstrating that they are generally non-Abelian and can be understood in terms of the fractional statistics of bulk quasiparticle excitations. We show that the point defects localize a set of topologically protected zero modes, which can be understood as a localized, robust nonzero density of states at zero energy for a certain subgroup of the topological quasiparticles.

Abelian topological states and line defects. Abelian topological states in $2+1$ dimensions are generically described by Abelian Chern-Simons (CS) theories.[1,25] With N compact $U(1)$ gauge fields a^I, the most general Chern-Simons term has the form of $\mathcal{L}_{CS} = \frac{1}{4\pi}\sum_{I,J} K_{IJ}\epsilon^{\mu\nu\lambda}a^I_\mu\partial_\nu a^J_\lambda$. The coefficient matrix K is a nonsingular symmetric matrix, which is integer valued as required by gauge invariance. The K matrix, up to integer-valued congruent transformations, classifies Abelian topologically ordered states. A topological quasiparticle carries point charges $l_I \in \mathbb{Z}$ of a^I. The statistics of a quasiparticle labeled by the integer vector l is given by $\theta_l = \pi l^T K^{-1} l$, and the mutual statistics of two quasiparticles l, l' is $\theta_{ll'} = 2\pi l^T K^{-1} l'$. A quasiparticle with $l_I = K_{IJ}v_J, v_J \in \mathbb{Z}$ is considered as a local "electron" in the theory, which may be bosonic or fermionic depending on K. Therefore the topologically nontrivial quasiparticles are labeled by integer vectors l mod Kv, with the number of topologically distinct quasiparticles given by $|\text{Det}\, K|$.

Different K matrices can specify equivalent topological states if they have the same quasiparticle content. For example, $K' = W^T K W$, for W an integer matrix with $|\text{Det}\, W| = 1$, describes the same topological order. Another example is

$$K' = K \oplus P \tag{1}$$

with P an integer matrix with $|\text{Det} P| = 1$. Adding P does not introduce any new topological quasiparticles, so that K and K' describe the same topological order.

A general line defect in a topological state is a one-dimensional boundary between two topological states, A_1 and A_2 [see Fig. 2(a)]. Some line defects, such as the edge of chiral topological states, are robustly gapless.[1,21,26] In this work we will explore gapped line defects.

In order to understand the properties of general boundaries, it is helpful to apply a folding process[18,19] [see Fig. 2(a)]. By folding the upper half plane using a parity transformation relative to the line defect, A_2 is mapped to its parity conjugate $\bar{A}_2$, so that the line defect becomes a boundary between the topological state $A_1 \times \bar{A}_2$ and a topologically trivial gapped state. Therefore, to study gapped line defects, it suffices to consider all possible gapped boundaries between general topological phases and the trivial state.

Classification of gapped boundaries. The key feature of a gapped boundary of a topological phase is that some subgroup of the topological quasiparticles are *condensed* on the boundary, and can be created/annihilated on the boundary by *local* operators.[21] Physically this describes superselection sectors for how topological quasiparticles can be reflected/transmitted at line defects.[16,19] We first consider the genon[4,5] as an example.

Consider a simple bilayer topological state, the $(mm0)$ Halperin state,[27] with the K matrix $K = m\mathbb{I}_{2\times2}$, which describes two independent $1/m$-Laughlin FQH states. Here $\mathbb{I}_{2\times2}$ is the 2-dimensional identity matrix. Folding the state along a line [see Fig. 3(a)] we obtain a 4-layer system with the K matrix $K = \begin{pmatrix} m\mathbb{I}_{2\times2} & 0 \\ 0 & -m\mathbb{I}_{2\times2}\end{pmatrix}$. The boundary of such a state can be gapped by introducing either interlayer or intralayer backscattering. The genon is defined as the domain wall between these two types of boundaries.

These two boundary conditions can be distinguished by the behavior of quasiparticles at the boundary. Across the boundary gapped by intralayer backscattering, quasiparticles move between layers $1,\bar{1}$ and $2,\bar{2}$, so that quasiparticles of the type $l = (q_1,q_2,-q_1,-q_2)^T$ can be annihilated or created at the boundary. Such quasiparticles have bosonic self-statistics and mutual statistics, and thus can be considered to be "condensed" on the boundary. Similarly, across the boundary defined by interlayer backscattering, a different set of quasiparticles with $l = (q_1,q_2,-q_2,-q_1)^T$ are condensed.

We see that different gapped boundaries condense different subgroups of quasiparticles. In general, it has been proven that every gapped boundary must condense a subgroup of quasiparticles M, called a "Lagrangian subgroup," which has the following properties:[21]

(1) $e^{i\theta_{mm'}} = 1$ for all $m, m' \in M$;

(2) for all $l \notin M$, $e^{i\theta_{lm}} \neq 1$ for at least one $m \in M$.

For bosonic states (when all diagonals of K are even), we also have $e^{i\theta_m} = 1$ for all $m \in M$. This set of quasiparticles forms an Abelian group with group multiplication defined by particle fusion. The first condition defines the bosonic mutual statistics and bosonic or fermionic self-statistics, allowing $m \in M$ to be condensed on the boundary. The second condition guarantees that the boundary is completely gapped, since all other quasiparticles $l \notin M$ have nontrivial mutual statistics with particles in M, and thus are confined when quasiparticles in M are condensed.

In the following we will strengthen this result by proving that *every* Lagrangian subgroup M corresponds to a gapped boundary where M is condensed.

To explicitly write down the boundary condition corresponding to a Lagrangian subgroup, we introduce the edge theory of the CS theory $\mathcal{L}_{CS}$ defined above, which is given by the chiral Luttinger liquid theory[1,25] $\mathcal{L}_{\text{edge}} = \frac{1}{4\pi} K_{IJ}\partial_x\phi_I\partial_t\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J$. V_{IJ} is a real symmetric positive definite matrix, and ϕ_I are real compact scalar fields: $\phi_I \sim \phi_I + 2\pi$. If K has an equal number of positive and negative eigenvalues, then there are an equal number of left- and right-moving modes, which is a necessary but not sufficient condition for the edge to be gapped.

The electron annihilation operators Ψ_I and quasiparticle annihilation operators χ_l on the boundary are given by $\Psi_I = e^{iK_{IJ}\phi_J}$, $\chi_l = e^{il_I\phi_I}$, where l is an integer vector describing the quasiparticles. Naively, the condensation of a quasiparticle $m \in M$ can be described by adding a term $\frac{g}{2}(\chi_m + \chi_m^\dagger) = g\cos(m_I\phi_I)$ to $\mathcal{L}_{\text{edge}}$. However, such a term has two problems. First, it is not a local term, written in terms of local "electron" operators Ψ_I. Second, the condition $m^T K^{-1} m = 0$ must be satisfied in order for the phase $m_I\phi_I$ to obtain a classical value. With this condition, it is possible to perform a change of basis $\phi \to W\phi$ so that the theory is mapped to a standard nonchiral Luttinger liquid, with $\cos(m_I\phi_I)$ mapped to a conventional backscattering term. The first problem can be solved by multiplying an integer coefficient c_i, such that $c_i K^{-1} m_i \equiv \Lambda_i \in \mathbb{Z}$, and thus $\cos(c_i m_{iI}\phi^I) = \cos(\Lambda_i^T K\phi)$ is a local electron tunneling operator.

The second can be solved if we can find a set of generators $\{m_i\}$ of M satisfying $m_i^T K m_j = 0, \forall i,j$. Then, the term $g\sum_i \cos(c_i m_{iI}\phi_I)$ can be added to the Lagrangian and will condense the particles in M: $\langle e^{im^T\phi}\rangle \neq 0$ if $m \in M$. It is known that if one can find N such null vectors m_i for a $2N \times 2N$ K matrix, the edge can be completely gapped.[26]

However, it is not always possible to find such a null vector basis $\{m_i\}$ which fully generates M. For example, consider $K = \left(\begin{smallmatrix} 0 & 4 \\ 4 & 0 \end{smallmatrix}\right)$, which describes Z_4 topological order. This system has a Lagrangian subgroup generated by $m_1^T = (2,0)$, $m_2^T = (0,2)$. It is not possible to find a single null vector which generates this Lagrangian subgroup. Consequently, it is not clear what cosine term on the boundary leads to condensation of this Lagrangian subgroup.

This problem can be resolved by introducing a topologically equivalent K matrix with higher dimension, as is shown in Eq. (1). In the edge theory, adding additional trivial blocks P such as $P = \tau_x$ or τ_z, where τ_i are 2×2 Pauli matrices, corresponds to adding purely one-dimensional edge channels to the boundary, such as Heisenberg spin 1/2 chains. Thus we find the following:

Lemma: For each Lagrangian subgroup M of the topological state described by K, there exists a K' which is topologically equivalent to K and has $\text{rank}(K') = 2N'$, such that the same Lagrangian subgroup M of K' can be generated by N' null vectors $m'_i, i = 1,2,\ldots,N'$.

The proof of this conclusion will be presented in the Supplemental Material.[35] As a simple example that illustrates the main idea of the proof, consider the previous example, with $K = \left(\begin{smallmatrix} 0 & 4 \\ 4 & 0 \end{smallmatrix}\right)$, and $m_1^T = (2,0)$, $m_2^T = (0,2)$. We define $K' = \left(\begin{smallmatrix} K & 0 \\ 0 & \tau_x \end{smallmatrix}\right)$, and $m_1'^T = (2,0,0,1)$, $m_2'^T = (0,2,-1,0)$. Here, K' is topologically equivalent to K, and $m_i'^T K'^{-1} m'_j = 0$. Physically, this result implies that one can always condense the particles in a Lagrangian subgroup on the edge, as long as using additional trivial edge states is allowed.

We conclude that every Lagrangian subgroup M corresponds to a gapped boundary where M is condensed, providing a classification, in the absence of any symmetries, of topologically distinct gapped boundaries.

Classification and characterization of point defects. In general, a point defect is a junction where multiple different line defects meet. Under the folding process, which may be applied multiple times, the point defects can always be mapped to domain walls between two gapped edges (Fig. 2). Therefore it is sufficient to study the point defect at the domain wall between two gapped boundaries. Based on the above classification of gapped boundaries, the domain walls are classified by a pair of Lagrangian subgroups (M,M'), corresponding to the gapped boundaries surrounding the domain wall.

Consider a point defect labeled by (M,M'). In the genon example reviewed above, the simplest topological property of the point defect is its nontrivial quantum dimension. This can be understood from the fact that the bilayer system (on the sphere) with $2n$ genons has genus $n-1$, which leads to a topological ground-state degeneracy that grows exponentially in n. For Abelian states, the topological degeneracy can be obtained from the algebra of the Wilson loop operators, which measure the topological charge through non-contractible loops. For example, a sphere with 4 genons is equivalent to a torus, which has two non-contractible loops a,b [see Fig. 3(a)]. When each layer is a $1/m$ Laughlin state, the Wilson loop operator $W(c)$ is defined by creating a pair of charge $1/m$, $-1/m$ particles, taking one of them around the loop c, and then annihilating them. $W(a)$ and $W(b)$ satisfy the commutation relation $W(a)W(b) = W(b)W(a)e^{i2\pi/m}$, and each leave the system in its ground-state subspace, requiring the ground-state degeneracy to be an integer multiple of m.[5]

These Wilson loop operators can be generalized to the generic point defects. By folding the bilayer system with genons along a line containing the genons, the Wilson loops become Wilson lines of the particles $(1/m,0,-1/m,0)$ or $(1/m,0,0,-1/m)$, which terminate at the boundary since the corresponding particles are condensed at the boundary. For general defects, between two gapped boundaries A,B with Lagrangian subgroups M and M', respectively, the Wilson lines can be defined [Fig. 3(b)] by creating a boson $m \in M$ at the A boundary using a local operator, moving it along a path a connecting two A regions, and finally annihilating it using a local operator. We denote such an operator as $W_m(a)$ and similarly $W_{m'}(b)$ for moving particle $m' \in M'$ between two B regions. The commutation relation between $W_m(a)$ and $W_{m'}(b)$

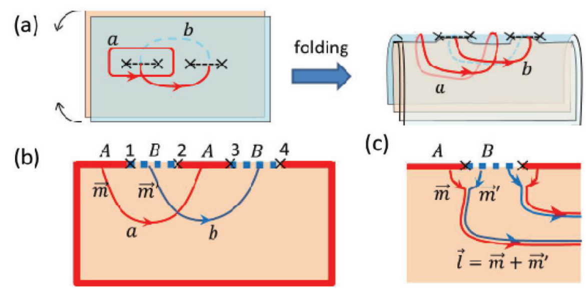

FIG. 3. (Color online) (a) Non-contractible loops in the bilayer system with 4 genons. Loop a is in the upper (blue) layer and loop b runs from upper layer to the lower (orange) layer across the branch cuts. After folding, the genons become domain walls between different gapped edges, and the non-contractible loops become Wilson lines that terminate on the boundaries. (b) The general Wilson lines in a system with boundary defects and point defects. Line a (b) defines the unitary operator $W_{\mathbf{m}}(a)$ [$W_{\mathbf{m}'}(b)$] which corresponds to adiabatic motion of bosonic quasiparticle $\mathbf{m}$ ($\mathbf{m}'$) along the paths a (b).

is determined by the mutual statistics of particles m and m', which is nontrivial when M and M' are different Lagrangian subgroups:

$$W_m(a)W_{m'}(b) = W_{m'}(b)W_m(a)e^{2\pi i m^T K^{-1} m'}. \quad (2)$$

Since these operators leave the system in the ground-state subspace, the ground states must form a representation of this algebra.

The degeneracy D required by one pair of non-contractible intersecting lines a,b is the dimension of the minimal representation of the algebra (2), which can be obtained by acting one set of operators, such as $\{W_m(a)\}$, on the eigenstates of the other set $\{W_{m'}(b)\}$. On a boundary with $2n$ defects between n pairs of alternating A and B regions, there will be $n-1$ pairs of noncommuting line operators satisfying the same algebra as above, leading to a degeneracy D^{n-1}. Therefore each point defect has a quantum dimension of $d = \sqrt{D}$.

If fermions exist microscopically, there may be an additional $\sqrt{2}$ factor in the quantum dimension, originating from the Majorana zero modes of purely one-dimensional physics,[28] which is independent of the above analysis.

Localized zero modes. A key feature of the point defects is that they localize a nonzero density of states at zero energy for a certain subgroup of quasiparticles. Such zero modes have been studied for specific types of defects,[5,11,12] and here we show that they exist in general point defects. Consider a point defect at $x = 0$ between two boundary regions A at $x < 0$ and B at $x > 0$, which are labeled by Lagrangian subgroups M and M'. For quasiparticles $m \in M, m' \in M'$, the boson creation operators $\chi_m(-\epsilon) = e^{i m_I \phi_I(-\epsilon)}$ and $\chi_{m'}(\epsilon) = e^{i m'_I \phi_I(\epsilon)}$ for $\epsilon > 0$ create condensed quasiparticles in A and B regions, respectively. Therefore $\chi_m(-\epsilon)\chi_{m'}(\epsilon)$ preserves the ground-state manifold. Taking the limit $\epsilon \to 0$ we obtain a local operator at the point defect $\gamma_l \equiv \lim_{\epsilon\to 0^+} \chi_m(-\epsilon)\chi_{m'}(\epsilon)$ with $l = m + m'$. By construction, bilinear combinations of γ_l on different defects preserve the ground-state manifold, which means γ_l is a zero-mode operator. The zero-mode creation process can be understood as the emission of a quasiparticle l which has fractional statistics, as is illustrated in Fig. 3(c). The zero modes γ_l are generalizations of the parafermion zero modes[29–31] studied in Refs. 5,8,11–13.

These zero modes can be braided by tuning their interactions, as discussed for specific examples in Refs. 5,11–13,32, 33. This leads to a notion of projective non-Abelian statistics for these defects; a general analysis for generic defects will be presented in a later work.[34]

Recently, we learned that some of the results on the classification of line defects have been independently found by Levin and included in Appendix A 3 of Ref. 21.

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[35]See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.88.241103 for proof of the Lemma presented in the main text.