



Field theory of nematicity in the spontaneous quantum anomalous Hall effect

Yizhi You and Eduardo Fradkin

Department of Physics and Institute for Condensed Matter Theory, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, Illinois, 61801-3080, USA

(Received 21 October 2013; published 23 December 2013)

We derive from a microscopic model the effective theory of nematic order in a system with a spontaneous quantum anomalous Hall effect in two dimensions. Starting with a model of two-component fermions (a spinor field) with a quadratic band crossing and short-range four-fermion marginally relevant interactions we use $1/N$ expansion and bosonization methods to derive the effective field theory for the hydrodynamic modes associated with the conserved currents and with the local fluctuations of the nematic order parameter. We focus on the vicinity of the quantum phase transition from the isotropic Mott Chern insulating phase to a phase in which time-reversal symmetry breaking coexists with nematic order, the nematic Chern insulator. The topological sector of the effective field theory is a background field (BF)/Chern-Simons gauge theory. We show that the nematic order parameter field couples with the Maxwell-type terms of the gauge fields as the space components of a locally fluctuating metric tensor. The nematic field has $z = 2$ dynamic scaling exponent. The low-energy dynamics of the nematic order parameter is found to be governed by a Berry phase term. By means of a detailed analysis of the coupling of the spinor field of the fermions to the changes of their local frames originating from long-wavelength lattice deformations, we calculate the Hall viscosity of this system and show that in this system it is not the same as the Berry phase term in the effective action of the nematic field, but both are related to the concept of torque Hall viscosity, which we introduce here.

DOI: [10.1103/PhysRevB.88.235124](https://doi.org/10.1103/PhysRevB.88.235124)

PACS number(s): 11.15.Yc, 71.10.Fd, 73.43.Lp, 73.43.Nq

I. INTRODUCTION AND MOTIVATION

The theory of topological phases of matter has been a central problem in condensed matter physics since the discovery of the quantum Hall effects^{1,2} in two-dimensional electron gases (2DEG) in large magnetic fields. The precisely observed (quantized or fractional) values of the Hall conductance is a manifestation of the fact that it is a topological invariant of the incompressible fluid.^{3–5} The fractional quantum Hall fluids, on the other hand, are explained by the universal properties encoded in the structure of their wave functions⁶ whose excitations (vortices) carry fractional charge and fractional statistics.^{6–8} The robustness of these properties is a consequence of their topological character. In addition to having fractionalized excitations, these topological fluids have a ground-state degeneracy that depends on the topology of the surface on which they reside, which is not a consequence of the spontaneous breaking of any global symmetry.⁹ The universal behavior of these topological fluids is encoded in an effective low-energy, the Chern-Simons, gauge theory.^{10–14}

There is now a growing body of (mostly theoretical) evidence that such topological phases of matter exist in several models of frustrated quantum antiferromagnets¹⁵ and in quantum dimer models.^{16,17} The recent discovery of topological insulators^{18–22} has opened a new arena in which these ideas play out. Interacting versions of simple models of topological Chern insulators, such as the Haldane model,²³ have topological phases with fractionalized excitations.^{24–27}

An interesting question is the interplay and possible coexistence of topological order and spontaneous symmetry breaking. For some filling fraction, the 2DEG is known to have a ferromagnetic quantum Hall ground state,^{28,29} in which spin rotational symmetry is spontaneously broken. Also, a state with a nematic “valley” order has also been seen in quantum Hall fluids on misoriented samples.^{30,31} On the other hand,

experiments in the 2DEG in the second Landau level found a nematic state in a regime in which the fractional (and integer) quantum Hall effect is absent.^{32–34} In this phase, the 2DEG is a uniform gapless electron fluid with a spontaneously broken spatial rotational symmetry.^{35,36}

Recent experiments by Xia and coworkers found that the 2DEG in the first Landau level in tilted magnetic fields has a strong tendency to break rotational invariance inside an incompressible fractional quantum Hall Laughlin state.^{37,38} Although in the experiments rotational invariance is broken explicitly by the tilted magnetic field, the temperature dependence of the transport anisotropy suggest that this state has a large nematic susceptibility and may be close to a phase transition to a nematic state. These experiments motivated Mulligan, Kachru and Nayak to develop a theory in which nematic order coexists with a fractional quantum Hall fluid.^{39,40} The possible existence of such states was anticipated by two early proposals of wave functions for anisotropic quantum Hall fluids.^{41,42}

The experiments of Xia and coworkers have also motivated the inquiry of the role of more microscopic, “geometrical,” degrees of freedom in the physics of these topological fluids.^{43–45} Recently, Maciejko and coworkers proposed an effective field theory of the anisotropic fractional quantum Hall state.⁴⁶ Using mainly symmetry arguments, they found that the nematic order parameter couples to the fractional quantum Hall fluid in the same way as the space components of a metric tensor. A similar effect was found earlier in a theory of a nematic charge $4e$ superconductor⁴⁷ involving, instead, the order parameter field of the superconductor. A key result of Ref. 46 is that the dynamics of the nematic degrees of freedom is governed by a Berry phase term in the effective action whose coefficient is the Hall viscosity of the topological fluid.^{48–52}

There are many aspects of this problem that remain unclear. In the case of the 2DEG, the existence of a compressible

nematic phase (in the second Landau level) suggests that it must be related to the anisotropy seen in the first Landau level, albeit in the incompressible phase. The theory of Ref. 39 suggests a possible mechanism (and an identification of the nematic degrees of freedom) solely in terms of the low-energy degrees of freedom of the quantum Hall fluid, but runs into difficulties in systems with Galilean invariance. In addition, that theory should also apply to the case of the integer quantum Hall effect. Although it is possible to write down a wave function for an anisotropic quantum Hall state by breaking rotational invariance explicitly at the microscopic level,⁴⁴ such an approach does not explain how it may come about from an isotropic incompressible state.

In this paper, we will investigate these problems by deriving an effective field theory for a Mott Chern insulator in a nematic phase in a simple microscopic lattice model recently proposed in by Sun and coworkers.⁵³ We will discuss in detail the case of the 2DEG in magnetic fields in a separate publication.⁵⁴ The model of Ref. 53 describes a correlated two-dimensional system of spinless fermions on a checkerboard square lattice in which two bands have a quadratic crossing at the corners of the (square) Brillouin zone. In the noninteracting system, the quadratic band crossing is protected by the C_4 point group symmetry of square lattice and by time-reversal invariance.

Due to the quadratic band crossing, this electronic system has a dynamical scaling exponent $z = 2$ (i.e., the energy scales with the square of the momentum). As a direct consequence of the $z = 2$ scaling, four fermion operators are naively marginal operators. This free-fermion system, which can be regarded as a fermionic version of a quantum Lifshitz model,⁵⁵ is at an infrared unstable fixed point of the renormalization group (RG). This semimetal fixed point is unstable to infinitesimal repulsive interactions to (a) a gapped phase with a spontaneously broken time-reversal invariance, i.e., a topological Mott Chern insulator with a spontaneous quantum anomalous Hall state,⁵⁶ (b) to a gapless semimetal nematic phase in which the point group symmetry breaks spontaneously from C_4 down to C_2 , and (c) to a gapped phase in which both time reversal symmetry breaking and the point group symmetry breaking coexist.⁵³

Models with quadratic band crossings describe the low-energy description of graphene bilayers,^{57–59} where there are two such crossings, and in the topologically protected surface states of 3D topological crystalline insulators.^{60,61} We discuss below some caveats on the relevance of this model to such systems. In particular, a (Mott) Chern insulating state has been conjectured to exist in bilayer graphene.⁵⁷

Due to the marginal relevance of local interactions, the behavior of the system in these phases can be investigated using controlled approximations, such as $1/N$ expansions and perturbative RG calculations. In contrast, in the case of the massless Dirac fermion, local interactions are irrelevant and a finite (and typically large) critical value of the coupling constant is needed to drive the system into a Mott Chern insulating phase.⁵⁶ Here, we will use the $1/N$ expansion and bosonization methods to derive an effective field theory of the Mott Chern insulator and of its quantum phase transition to a nematic Chern insulator in the context of the model of Ref. 53. The effective field theory includes the hydrodynamic degrees

of freedom of the conserved currents of the fermions, in the form of a background field (BF)/Chern-Simons gauge theory, and the local fluctuations of the nematic order parameter. In particular, in this theory, the nematic fluctuations are present at low energies, which is required to describe a continuous quantum phase transition to a nematic Mott Chern insulator.

We will also show that the effective low-energy dynamics of the nematic order parameter is indeed a Berry phase term, with a structure similar to that proposed by Maciejko and collaborators. We also find that the nematic fields can be regarded as providing a local fluctuating spatial metric for the hydrodynamic gauge fields of the Mott Chern insulator. However, we will also show that the nematic degrees of freedom do not couple to the fermionic degrees of freedom as a local frame field and hence, they cannot be identified with a local geometry. We show that the Hall viscosity, which in a system of spinors is the response of the system to a change of the local frames⁵⁰ (i.e., a long-wavelength distortion of the lattice), is not equal to the Berry phase of the nematic modes. Instead, the Berry phase is related to the concept of torque Hall viscosity which we introduce here. In addition, we find that in this system the Hall viscosity is not given by the coefficient of the q^2 term in the Hall conductance. Recently, Hoyos and Son showed that in Galilean-invariant one-component quantum Hall fluids systems these two coefficients should be equal to each other.⁵² These assumptions do not apply to multicomponent fermionic (spinor) systems as in the present case. We also find that the Hall viscosity and the Berry phase coefficient are related to the Hall torque viscosity.

This paper is organized as follows. In Sec. II, we present the model of interacting fermions in two dimensions with a quadratic band crossing and we discuss its phase diagram. In Sec. III, we develop an effective field theory of the interplay of nematic order and of the hydrodynamic gauge theory. In Sec. IV, we use the $1/N$ expansion to derive the effective action in the vicinity of the nematic transition inside the spontaneous quantum anomalous Hall phase, and use it to discuss briefly the nature of the two phases and the quantum and thermal critical behavior. In Sec. V, we present the effective field theory of the nematic fields in the presence of broken time-reversal invariance. Here, we discuss in detail the role played by the Hall viscosity in the effective field theory. In Sec. VI, we introduce the concept of Hall torque viscosity and discuss its relation with the Hall viscosity and with the Berry phase. Our conclusions are presented in Sec. VII. Details of the calculations, including the proofs of gauge invariance, are given in several Appendices.

II. THE QUADRATIC BAND-CROSSING MODEL AND ITS PHASES

In this paper, we will use the following simple model for a quadratic band crossing (QBC), introduced by Sun and collaborators.⁵³ We begin with a summary of the results of their work that will be useful for our analysis. One of the cases discussed by Sun *et al.* is a system of spinless fermions on a checkerboard lattice. This lattice has two sublattices, and the single-particle states are two-component spinors. The band structure of this system is described by the tight-binding

one-particle Hamiltonian

$$h_0(\mathbf{k}) = t(\cos k_1 - \cos k_2)\sigma_3 + 4t' \cos\left(\frac{k_1}{2}\right) \cos\left(\frac{k_2}{2}\right) \sigma_1, \quad (2.1)$$

where $\mathbf{k} = (k_1, k_2)$ are vectors of the first Brillouin zone (BZ), $|k_i| \leq \pi$ (with $i = 1, 2$), and σ_1 and σ_3 are two (real symmetric, 2×2) Pauli matrices. The lattice model also has a contribution proportional to the 2×2 identity matrix which, for a range of parameters, can be ignored.⁶² Tsai and coworkers⁶³ discussed a similar problem on the Lieb lattice.

In this system, the two bands cross at the Fermi energy at the corners of the BZ, (π, π) (and its symmetry related points). For a half-filled system, the Fermi energy is exactly at the band crossing points, and the ground state of the noninteracting system describes a semimetal with a quadratic band dispersion. Similar problems have been discussed in the context of bilayer graphene.^{57,58,64}

The band structure of this semimetal has a nontrivial Berry phase

$$i \oint_{\Gamma} d\mathbf{k} \cdot \langle \mathbf{k} | \nabla_{\mathbf{k}} | \mathbf{k} \rangle = n\pi, \quad (2.2)$$

where $|\mathbf{k}\rangle$ is a Bloch state at momentum \mathbf{k} of the BZ, and Γ is a closed curve on the BZ that encloses the quadratic band crossing point, (π, π) . For a two-band system with a QBC, the integer $n = 2$ ($n = \pm 1$ for Dirac fermions). In this case, the changes of the Chern number of the two bands are carried entirely by the (single) quadratic crossing. At the noninteracting level, the Berry phase here is protected by both discrete lattice symmetries and by time reversal invariance.

For momenta $\mathbf{k} = (\pi, \pi) - \mathbf{q}$ close to the crossing points (the corners of the BZ) we can approximate the one-particle Hamiltonian by expanding Eq. (2.1) about the crossing point. Let us denote by $\psi_{\alpha}(\mathbf{q})$ (with $\alpha = 1, 2$) a two-component Fermi field with wave vectors \mathbf{q} [measured from the (π, π) point]. The effective free fermion Hamiltonian, in momentum and in position space, is

$$\begin{aligned} H_0 &= \int \frac{d^2 q}{(2\pi)^2} \psi_{\alpha}^{\dagger}(\mathbf{q}) [(q_1^2 - q_2^2)\sigma_3 + 2q_1 q_2 \sigma_1]_{\alpha\beta} \psi_{\beta}(\mathbf{q}) \\ &= - \int d^2 x \psi_{\alpha}^{\dagger}(\mathbf{x}) [\sigma_3 (\partial_1^2 - \partial_2^2) + \sigma_1 2\partial_1 \partial_2]_{\alpha\beta} \psi_{\beta}(\mathbf{x}). \end{aligned} \quad (2.3)$$

Here, and from now on, we have set $t = t'$ for simplicity (and rescaled the energy scale so that $t = 1$). This is a special point of high (rotational) symmetry, which does not qualitatively change the results. In the case of bilayer graphene, one has two “valleys” (or species) of fermions whose free-fermion Hamiltonians are given by Eq. (2.3), with a different sign of t' , which plays the role of a chirality that distinguishes one valley from the other. Thus, for bilayer graphene, one has $|t| = |t'|$.

For a system of (spinless) fermions with a QBC with short-range repulsive microscopic interactions, the effective low-energy Hamiltonian is the sum of the free-fermion Hamiltonian H_0 of Eq. (2.3) and an interaction term H_{int} , which can be

succinctly written in the form

$$H_{\text{int}} = - \int d^2 x \frac{1}{2} [g_0 \Phi_0^2(\mathbf{x}) + g \Phi^2(\mathbf{x})], \quad (2.4)$$

where g_0 and g are two (positive) coupling constants. The operators $\Phi_0(\mathbf{x})$ and $\Phi(\mathbf{x})$ in Eq. (2.4) are, respectively, given by the (Hermitian) bilinears of fermion operators,

$$\Phi_0(\mathbf{x}) = \psi^{\dagger}(\mathbf{x}) \sigma_2 \psi(\mathbf{x}), \quad (2.5)$$

$$\Phi(\mathbf{x}) = \psi^{\dagger}(\mathbf{x}) \sigma \psi(\mathbf{x}). \quad (2.6)$$

Here, $\sigma = (\sigma_1, \sigma_3)$, and, for clarity, we have suppressed the spinor indices. For $t = t'$, the full Hamiltonian, $H = H_0 + H_{\text{int}}$ is invariant under time reversal and under arbitrary rotations. However, for $t \neq t'$, it is only invariant under the (discrete) point group C_4 .

The operator $\Phi_0(\mathbf{x})$ of Eq. (2.5) breaks time-reversal invariance and is the order parameter for time-reversal symmetry breaking. If $\langle \Phi \rangle \neq 0$, the system would have a gap and exhibit a zero-field quantum Hall effect with $\sigma_{xy} = e^2/h$ (i.e., an anomalous quantum Hall effect). The operator $\Phi(\mathbf{x})$ of Eq. (2.6) breaks rotational invariance and it is the nematic order parameter. In fact, Φ is invariant under a rotation by π and hence it is not a vector but a director, as it should be. Moreover, if we were to add terms proportional to the operators Φ_0 and Φ to the free-fermion Hamiltonian of Eq. (2.3), the QBC either gets gapped (if $\langle \Phi_0 \rangle \neq 0$) or splits into two massless Dirac fermions which are separated either along the x (or y) axis [is $\langle \Phi_1 \rangle \neq 0$] or along a diagonal (if $\langle \Phi_2 \rangle \neq 0$). Hence this state breaks rotational invariance (or C_4 or C_6 down to C_2). Hence a state with $\langle \Phi_0 \rangle \neq 0$ is a topological Chern insulator, while a state with $\langle \Phi \rangle \neq 0$ is a nematic semimetal. If spin and other degrees of freedom are also considered, other operators (and hence possible phases) that transform nontrivially under other symmetries must be considered, leading, for instance, to a state with a spin Hall effect, a ferromagnet, triplet nematic order, and others.^{53,57–59,64}

In the case of the theory of massless Dirac fermions (e.g., graphene), short-range interactions are irrelevant operators, rendering the semimetallic phase stable, and can only trigger a (quantum) phase transition if the coupling constants are larger than a critical value.⁶⁵ However, in the case of a theory of fermions with a QBC, short-range interactions of the form of Eq. (2.4) are marginally relevant and destabilize the QBC semimetal even for arbitrarily weak interactions⁵³ (see also the prescient work of Abrikosov and coworkers).⁶⁶

The kinematic differences between the two systems, Dirac and the QBC, lead to a change in the scaling behavior of the operators.⁵³ In particular, the Hamiltonian H_0 of Eq. (2.3) describes a quantum critical system of free fermions with dynamical exponent $z = 2$ and, hence, in this system, time scales as the square of a length, L^2 . For this reason, it has some similarities with systems in the quantum Lifshitz universality class.⁵⁵ Consequently, in a system with $z = 2$ dynamic scaling, in two space dimensions the fermion operator has scaling dimension $\Delta_{\psi} = 1$, $[\psi] = L^{-1}$, and all four-fermion operators have scaling dimension 4.

In *two* (space) dimensions, this means that all four fermion operators are *marginal* [in the renormalization group (RG)

sense] since here $d + z = 4$. Therefore the stability (or instability) of the free-fermion QBC semimetal, such as the surface states of the three-dimensional crystalline topological insulators,^{60,67,68} such as $\text{Pb}_{1-x}\text{Sn}_x\text{Te}$, is determined by quantum corrections. In contrast, systems with a QBC in *three* dimensions, such as the pyrochlore iridates $A_2\text{Ir}_2\text{O}_7$ (where A is a lanthanide or yttrium),^{69–71} short-range interactions are perturbatively irrelevant and the QBC semimetal is stable (up to a critical value of the coupling constants) (see, once again, Ref. 66).

One-loop renormalization group calculations show that, in two dimensions, in a system with microscopic repulsive interactions, and hence $g_0 > 0$ and $g > 0$, four-fermion operators of the form of Eq. (2.4) are *marginally relevant*,^{53,58,64} and, hence, weak repulsive interactions render the semimetal free-fermion ground state unstable. Several phases can occur depending on the details of the microscopic interactions. In Ref. 53, it was shown that in the case of the QBC of the checkerboard lattice a weak (infinitesimal) repulsive interaction drives the system into a state with a spontaneous anomalous quantum Hall effect (i.e., a Chern insulator with a spontaneously-broken time-reversal symmetry), with a subsequent phase transition to a nematic semimetal state. Sun and coworkers⁵³ also found a regime in which the nematic state and the Chern insulating state coexist. Thus, in this phase, the system has a spontaneously broken time-reversal invariance and also a spontaneously broken rotational invariance, and is a nematic Chern insulator. Such topological Mott insulators were proposed earlier on by Raghu, Qi, Honerkamp, and Zhang in the context of Dirac-type systems where they can only occur at relatively large values of the interactions.⁵⁶

III. EFFECTIVE GAUGE THEORY FOR THE ANISOTROPIC QAH STATE

Our goal is to derive an effective action for the spontaneous QAH phase and to describe the transition to a nematic QAH phase. To this end, we will generalize our system to one in which there are N “flavors” of fermions and to drive the effective field theory using a large- N expansion. Sun and coworkers have shown that, unlike the familiar case of the Luttinger liquids in one space dimensions, the renormalization group β function(s) for the $N = 1$ case has the same structure as the $N > 1$ case.⁵³ The resulting effective Lagrangian density for the spinor fermionic field $\psi_a(x)$ [with $a = 1, \dots, N$, $x = (x_0, \vec{x})$, and x_0 is the time coordinate] (here we are omitting the spinor indices):

$$\begin{aligned} \mathcal{L}_F[\bar{\psi}, \psi, a_\mu] = & \bar{\psi}_a(x) [i\gamma_0 D_0 - \gamma_1 (D_1^2 - D_2^2) \\ & - \gamma_2 (D_1 D_2 + D_2 D_1)] \psi_a(x) \\ & + \frac{g_0}{2N} \Phi_0(x)^2 + \frac{g}{2N} \Phi^2(x), \end{aligned} \quad (3.1)$$

where $\Phi_0(x)$ and $\Phi(x)$ are the fermion bilinears defined in Eqs. (2.5) and (2.6), respectively, suitably generalized for a system with N flavors of fermions. Minimal coupling of the fermions to the gauge field requires that we change of the Hamiltonian of the system to ensure its Hermiticity and gauge invariance.

In Eq. (3.1), we have used the standard 2×2 Dirac gamma matrices, given in terms of the three Pauli matrices

$$\gamma_0 = \sigma_2, \quad \gamma_1 = i\sigma_1, \quad \gamma_2 = -i\sigma_3 \quad (3.2)$$

and satisfy the Dirac (Clifford) algebra (with $\mu = 0, 1, 2$)

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} I, \quad (3.3)$$

where I is the 2×2 identity matrix and $\eta_{\mu\nu} = \text{diag}(1, -1, -1)$ is the Minkowski metric in $2 + 1$ space-time dimensions.

In the Lagrangian of Eq. (3.1), we introduced the coupling to a gauge field a_μ through the covariant derivatives

$$D_\mu = \partial_\mu - ia_\mu. \quad (3.4)$$

The coupling to a gauge field is needed both to describe the interactions with an external electromagnetic field A_μ and also to express the charge currents of the fermions in terms of a dual gauge field. This latter procedure leads to a hydrodynamic theory of the Chern insulating phase.⁷²

The hydrodynamic theory is derived using the procedure of functional bosonization of Ref. 73 and expanded in Ref. 74 (see also Ref. 75). Following the work of Chan *et al.*,⁷² we will derive the effective hydrodynamic theory by considering the partition function of the fermionic theory with the Lagrangian of Eq. (3.1) coupled to a dynamical gauge field a_μ whose field strength $\mathcal{F}_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ vanishes everywhere (in space and time), and hence is a gauge transformation. For a system with periodic boundary conditions, integrating the partition function over all gauge transformations (including large gauge transformations) amounts to averaging the partition function (and hence all its observables) over the torus of boundary conditions.

The averaged partition function is

$$\begin{aligned} Z[A_\mu] = & \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}a_\mu \prod_{x,\mu,\nu} \delta(\mathcal{F}_{\mu\nu}) \\ & \times \exp\left(i \int d^3x \mathcal{L}_F[\bar{\psi}, \psi, A_\mu + a_\mu]\right), \end{aligned} \quad (3.5)$$

where A_μ is a weak external electromagnetic field (used as a source), \mathcal{L}_F is the Lagrangian of Eq. (3.1). Using the representation of the δ function

$$\prod_{x,\mu,\nu} \delta(\mathcal{F}_{\mu\nu}) = \int \mathcal{D}b_\mu \exp\left(i \int d^3x b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda\right) \quad (3.6)$$

and the invariance of the measure under shifts $a_\mu \rightarrow a_\mu - A_\mu$, we find that the averaged partition function can be written in the equivalent form

$$\begin{aligned} Z[A_\mu] = & \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}a_\mu \mathcal{D}b_\mu \\ & \times \exp\left(i \int d^3x \mathcal{L}[\bar{\psi}, \psi, A_\mu, a_\mu, b_\mu]\right). \end{aligned} \quad (3.7)$$

The Lagrangian in the exponent of Eq. (3.7) is given by

$$\mathcal{L}[\bar{\psi}, \psi, A_\mu, a_\mu, b_\mu] = b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A_\lambda) + \mathcal{L}_F[\bar{\psi}, \psi, a_\mu], \quad (3.8)$$

where the Lagrangian \mathcal{L}_F on the right-hand side of Eq. (3.8) is given in Eq. (3.1). In the Chern insulating phase, this

expression leads to the BF topological field theory form of the hydrodynamic theory.^{72,76}

It is now straightforward to show⁷²⁻⁷⁴ that the fermionic currents j_μ can be expressed in terms of the dual hydrodynamic field $j_\mu \equiv \epsilon_{\mu\nu\lambda} \partial^\nu b^\lambda$ as an operator identity. This hydrodynamic identity is the starting point of the effective field theory of the fractional quantum Hall fluids.^{77,13,14}

On the other hand, the conserved and gauge-invariant fermionic currents j_μ have the explicit form

$$j_0 = \frac{\delta \mathcal{L}_F}{\delta a_0} = \bar{\psi}_a \gamma_0 \psi_a = \psi_a^\dagger \psi_a, \quad (3.9)$$

$$j_1 = \frac{\delta \mathcal{L}_F}{\delta a_1} = i \bar{\psi}_a (\gamma_1 D_1 + \gamma_2 D_2) \psi_a + \text{H.c.}, \quad (3.10)$$

$$j_2 = \frac{\delta \mathcal{L}_F}{\delta a_2} = i \bar{\psi}_a (-\gamma_1 D_2 + \gamma_2 D_1) \psi_a + \text{H.c.}, \quad (3.11)$$

where D_1 and D_2 denote the spatial components of the covariant derivative and where the summation over the index a has been assumed. Notice that unlike the relativistic Dirac theory but in close resemblance to the nonrelativistic case, the spatial components of the fermionic current depend explicitly on the gauge field a_μ , as expected for a theory with dynamical exponent $z = 2$.

We will now proceed to derive an effective action which is accurate in the large N limit (but which is qualitatively correct for all finite N). To this end, we will decouple the four-fermion interactions in the Lagrangian \mathcal{L}_F by means of a Hubbard-Stratonovich transformation. In terms of three real Hubbard-Stratonovich fields $M_0(x)$, which couples to the time-reversal-symmetry-breaking order parameter Φ_0 [of Eq. (2.5)], and $M_1(x)$ and $M_2(x)$, which couple to the components of the nematic order parameter Φ [of Eq. (2.6)], the Lagrangian \mathcal{L}_F of Eq. (3.1) takes the form

$$\begin{aligned} \mathcal{L}_F[\bar{\psi}, \psi, a_\mu, M_0, \mathbf{M}] &= \bar{\psi}_a(x) [i \gamma_0 D_0 - \gamma_1 (D_1^2 - D_2^2) - \gamma_2 (D_1 D_2 + D_2 D_1) \\ &\quad + M_0(x) + \mathbf{M}(x) \cdot \boldsymbol{\gamma}] \psi_a(x) \\ &\quad - \frac{N}{2g_0} M_0(x)^2 - \frac{N}{2g} \mathbf{M}^2(x). \end{aligned} \quad (3.12)$$

Upon integrating-out the fermionic fields, we obtain the following expression for the averaged partition function

$$\begin{aligned} Z[A_\mu] &= \int \mathcal{D}a_\mu \mathcal{D}b_\mu Z[a_\mu] \\ &\quad \times \exp \left[i \int d^3x N b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A_\lambda) \right], \end{aligned} \quad (3.13)$$

where we scaled the b_μ field by a factor of N for future convenience. The partition function $Z[a_\mu]$ is given by

$$Z[a_\mu] = \int \mathcal{D}M_0 \mathcal{D}\mathbf{M} \exp(i N S[a_\mu, M_0, \mathbf{M}]), \quad (3.14)$$

where

$$\begin{aligned} S[a_\mu, M_0, \mathbf{M}] &= - \int d^3x \left[\frac{1}{2g_0} M_0^2(x) + \frac{1}{2g} \mathbf{M}(x)^2 \right] \\ &\quad - i \text{Tr} \ln \mathcal{M}[a_\mu, M_0, \mathbf{M}] \end{aligned} \quad (3.15)$$

and \mathcal{M} is the differential operator

$$\begin{aligned} \mathcal{M}[a_\mu, M_0, \mathbf{M}] &= i \gamma_0 D_0 - \gamma_1 (D_1^2 - D_2^2) \\ &\quad - \gamma_2 (D_1 D_2 + D_2 D_1) + M_0(x) + \mathbf{M}(x) \cdot \boldsymbol{\gamma} \end{aligned} \quad (3.16)$$

is the action used in Eq. (3.14). Notice that the Hubbard-Stratonovich fields M_0 and \mathbf{M} have units of (momentum)² \equiv energy (which is consistent since $z = 2$.)

Putting it all together we find that the partition function of the full problem is

$$Z[A_\mu] = \int \mathcal{D}b_\mu \mathcal{D}a_\mu \mathcal{D}M_0 \mathcal{D}\mathbf{M} e^{i N S_{\text{eff}}[a_\mu, M_0, \mathbf{M}, A_\mu]}, \quad (3.17)$$

where the effective action is

$$S_{\text{eff}} = S[a_\mu, M_0, \mathbf{M}] + \int d^3x b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A_\lambda). \quad (3.18)$$

Here, $S[a_\mu, M_0, \mathbf{M}]$ is given by Eq. (3.15). Notice that from Eq. (3.12) the following identities hold:

$$\frac{\Phi_0}{N} = \frac{M_0}{g_0}, \quad \frac{\Phi}{N} = \frac{\mathbf{M}}{g}. \quad (3.19)$$

As usual, the correlation functions of the Hubbard-Stratonovich fields are (essentially) the same as those of the order parameters.

We can now proceed to solve this theory in the large N limit. The effective action we are seeking will be obtained in the leading order of the $1/N$ expansion which is equivalent to a one-loop approximation. [For a general discussion of large N (“vector”) field theories see, e.g., the extensive review of Ref. 78.]

In the large N limit, the partition function $Z[a_\mu]$ [of Eq. (3.14)] is well approximated by an expansion about the saddle points of the effective action S_{eff} of Eq. (3.15). Here, we will seek translationally invariant states, such as the phases with spontaneously broken time-reversal invariance, with $\langle \Phi_0 \rangle \neq 0$, and/or spontaneously broken rotational invariance, with $\langle \Phi \rangle \neq 0$. In what follows, the gauge field a_μ can be taken to be a weak perturbation (and hence it will not affect the saddle-point equations). Hence we will set $a_\mu = 0$ in the saddle-point equations. The effects of quantum fluctuations of the gauge field a_μ will appear in the $1/N$ corrections.

The saddle-point equations (the “gap equations”) are

$$\frac{\delta S_{\text{eff}}}{\delta M_0(x)} = 0 \Rightarrow \frac{m}{g_0} = -i \text{tr} G(x, x; m, \mathbf{M}), \quad (3.20)$$

$$\frac{\delta S_{\text{eff}}}{\delta \mathbf{M}(x)} = 0 \Rightarrow \frac{\mathbf{M}}{g} = -i \text{tr} [G(x, x; m, \mathbf{M}) \boldsymbol{\gamma}], \quad (3.21)$$

where a sum over repeated indices is assumed and the trace runs over the spinor indices. $S_{\alpha\beta}(x, x'; m, \mathbf{M})$ (with $\alpha, \beta = 1, 2$ being the spinor indices) is the Feynman (time-ordered) propagator of a fermionic field with $z = 2$ with constant values of the fields $M_0 \equiv m$ and \mathbf{M} ,

$$\begin{aligned} G_{\alpha\beta}(x, x'; m, \mathbf{M}) &= -i \langle T(\psi_\alpha(x) \bar{\psi}_\beta(x')) \rangle \\ &= \langle x, \alpha | [i \gamma_0 \partial_0 - \gamma_1 (\partial_1^2 - \partial_2^2) - \gamma_2 2\partial_1 \partial_2 \\ &\quad + m + \mathbf{M} \cdot \boldsymbol{\gamma}]^{-1} | x', \beta \rangle. \end{aligned} \quad (3.22)$$

In frequency and momentum space $q_\mu = (q_0, \mathbf{q})$, the Feynman propagator is (dropping the spinor indices)

$$G(p; m, \mathbf{M}) = \frac{1}{p_0 \gamma_0 - (p_1^2 - p_2^2) \gamma_1 - 2p_1 p_2 \gamma_2 - m - \mathbf{M} \cdot \boldsymbol{\gamma} - i\epsilon} \quad (3.23)$$

from where we read off the spectrum of (one-particle) fermionic excitations

$$E_\pm(\mathbf{q}; m, \mathbf{M}) = \pm E(\mathbf{q}; m, \mathbf{M}) \quad (3.24)$$

and

$$E(\mathbf{q}; m, \mathbf{M}) = \sqrt{(q_1^2 - q_2^2 + M_1)^2 + (2q_1 q_2 + M_2)^2 + m^2}. \quad (3.25)$$

Clearly, $M_0 = m$ is a (time-reversal symmetry breaking) mass gap, and \mathbf{M} breaks rotational invariance, by splitting the QBC into two Dirac cones, along a direction and by an amount set by \mathbf{M} .

Upon computing the traces over the spinor indices and after an integration over frequencies, the ‘‘gap’’ equations (3.21) can be put in the form

$$\frac{m}{g_0} = \int \frac{d^2 q}{(2\pi)^2} \frac{m}{E(\mathbf{q}; m, \mathbf{M})}, \quad (3.26)$$

$$\frac{M_1}{g} = \int \frac{d^2 q}{(2\pi)^2} \frac{q_1^2 - q_2^2 + M_1}{E(\mathbf{q}; m, \mathbf{M})}, \quad (3.27)$$

$$\frac{M_2}{g} = \int \frac{d^2 q}{(2\pi)^2} \frac{2q_1 q_2 + M_2}{E(\mathbf{q}; m, \mathbf{M})}, \quad (3.28)$$

where $E(\mathbf{q}; m, \mathbf{M})$ is given in Eq. (3.25). The integrals in Eqs. (3.26)–(3.28) are logarithmically divergent at large momenta \mathbf{q} and require a UV momentum cutoff $\Lambda \sim \pi/a$, where a is the lattice spacing. This logarithmic divergence is a consequence of the marginally relevant nature of the interactions.

In the $N \rightarrow \infty$ limit, the ground-state energy density of the system $\mathcal{E}(m, \mathbf{M})$ is

$$\mathcal{E}(m, \mathbf{M}) = \frac{N}{2g_0} m^2 + \frac{N}{2g} \mathbf{M}^2 - N \int \frac{d^2 q}{(2\pi)^2} E(\mathbf{q}; m, \mathbf{M}), \quad (3.29)$$

where we have filled up the negative energy states. This ground-state energy density has extrema at the values of m and \mathbf{M} which are the simultaneous solutions of Eqs. (3.26)–(3.28).

The saddle-point equations, Eqs. (3.26)–(3.28), have three types of uniform solutions: (a) an isotropic (or C_4 invariant) phase with $m \neq 0$ and $\mathbf{M} = 0$ in which time reversal invariance is spontaneously broken which is an insulating (Mott) phase with a spontaneous QAH effect, (b) a phase with $m = 0$ but with $\mathbf{M} \neq 0$ with a spontaneously broken rotational (or C_4) invariance, which is a nematic semimetal with a spectrum of two massless Dirac fermions, and (c) a coexistence phase with $m \neq 0$ and $\mathbf{M} \neq 0$, in which both time-reversal and rotational invariance are spontaneously broken, i.e., this is an insulating nematic QAH phase.

In Ref. 53, it was found that, for certain range of parameters the quantum phase transition from the QAH phase to the nematic QAH phase is continuous while the subsequent transition to a the nematic semimetal is first order. The details of the phase diagram depend also on the parameters t and t' , defined in the free fermion Hamiltonian of Eq. (2.1), which break the continuous symmetry under rotations down to the C_4 point-group symmetry (for the case of the checkerboard lattice).

In this paper, we will focus on the (isotropic or C_4 -symmetric) QAH phase and its continuous quantum phase transition to the nematic QAH phase in which both orders are present. In the $N \rightarrow \infty$ limit, the ground-state energy density of the QAH phase is

$$\mathcal{E}(m, \mathbf{M}) = \mathcal{E}_0 + \frac{m^2}{2g_0} - \frac{m^2}{8\pi} \ln \left(\frac{2\Lambda^2}{|m|} \right), \quad (3.30)$$

where $\mathcal{E}_0 = -\Lambda^2/(8\pi)$ (here and below Λ is a momentum cutoff, $\Lambda \sim \pi/a$) is the ground-state energy density of free fermions with a QBC and where we have kept the leading (divergent) terms in $\Lambda^2/|m| \rightarrow \infty$. In Eq. (3.30), we have omitted an overall factor of N .

The ground-state energy of Eq. (3.30) is minimized if the saddle-point equation Eq. (3.26) is satisfied, which now becomes

$$\frac{1}{g_0} = \frac{1}{4\pi} \ln \left(\frac{2\Lambda^2}{|m|} \right). \quad (3.31)$$

The solution of this equation is

$$|m| = 2\Lambda^2 \exp \left(-\frac{4\pi}{g_0} \right), \quad (3.32)$$

which has the characteristic form of a marginally relevant perturbation. From now on, we will assume that the leading instability of the system is to the QAH phase, which opens the finite gap m is the fermion spectrum and breaks spontaneously time-reversal invariance.

We will consider the case in which the onset of nematic order takes place inside the QAH phase. In this situation, the nematic order will be weak and its onset will not affect appreciably, to lowest order, the time-reversal-symmetry breaking mass gap m . With these assumptions, we can expand the ground-state energy of Eq. (3.29) in powers of the nematic order parameter \mathbf{M} up to quartic order, which has the form

$$\mathcal{E}(m, \mathbf{M}) = \mathcal{E}_{\text{QAH}} + r(m) \mathbf{M}^2 + u(m) \mathbf{M}^4 + O(\mathbf{M}^6), \quad (3.33)$$

where \mathcal{E}_{QAH} is the ground-state energy of the nematic phase, and the parameters $r(m)$ and $u(m)$ are

$$r(m) = \frac{1}{2g} - \frac{1}{8\pi} \ln \left(\frac{2\Lambda^2}{|m|} \right), \quad u = \frac{21}{256\pi} \frac{1}{m^2}. \quad (3.34)$$

From here we find that there is a (quantum) phase transition to a nematic QAH phase at a critical value g_c ,

$$\frac{1}{g_c} = \frac{1}{4\pi} \ln \left(\frac{2\Lambda^2}{|m|} \right). \quad (3.35)$$

Within these approximations, the transition takes place at $g_c = g_0$. For $g > g_c$, nematic order parameter \mathbf{M} has a nonvanishing

expectation value,

$$|\bar{\mathbf{M}}| = \left(\frac{-r(m)}{2u(m)} \right)^{1/2} = A |m| \left(\frac{1}{g_c} - \frac{1}{g} \right)^{1/2}, \quad (3.36)$$

where $A^2 = 64\pi/21$. Further, inside the nematic QAH phase, the QAH order parameter, m , becomes progressively suppressed until a first-order quantum phase transition to a nematic semimetal phase is reached.⁵³

IV. EFFECTIVE ACTION AND $1/N$ EXPANSION

We will now derive the effective field theory for the quantum fluctuations in the QAH phase close to the nematic quantum phase transition. To this end, we will compute the effects of quantum fluctuations to the lowest order in the $1/N$ expansion. In the QAH phase, the only field with a nonvanishing expectation value is the field M_0 , whereas the nematic field \mathbf{M} has a vanishing expectation value in the QAH phase (but not in the nematic phase). By gauge invariance the gauge fields a_μ and b_μ cannot have a nonvanishing expectation value (although their fluxes could).

The fluctuations of the time-reversal symmetry-breaking field M_0 are massive in the QAH phase (and in the nematic QAH phase). Since we are interested in the effective field theory close to the transition to the nematic QAH phase, we will not be interested in the fluctuations of this massive field, whose main effect is a renormalization of the effective parameters. Thus in what follows, we will ignore the fluctuations of the field M_0 about the $N = \infty$ expectation value $M_0 = m$.

We will now expand the effective action of Eq. (3.15) to lowest orders in the $1/N$ expansion. Let us denote by $G_0(x, x'; m)$

$$G_0(x, x'; m) \equiv \langle x | \mathcal{M}_0^{-1} | x' \rangle \quad (4.1)$$

the Feynman propagator of the fermions in the QAH phase given by Eq. (3.22). Here, we implicit the spinor indices and set the expectation value of the nematic field \mathbf{M} to zero and $M_0 = m$. In Eq. (4.1), \mathcal{M}_0 is the differential operator of Eq. (3.16) in the symmetric phase with broken time-reversal symmetry.

In momentum space, the propagator of Eq. (4.2) becomes

$$G_0(p) = \frac{p_0\gamma_0 - (p_1^2 - p_2^2)\gamma_1 - 2p_1p_2\gamma_2 + m}{p_0^2 - (p_1^2 + p_2^2) - m^2 - i\epsilon}. \quad (4.2)$$

The expansion in powers of $1/N$ can now be determined by using the expansion of the logarithm

$$\begin{aligned} \text{tr} \ln \mathcal{M} &= \text{tr} \ln (\mathcal{M}_0 + \delta\mathcal{M}) \\ &= \text{tr} \ln \mathcal{M}_0 + \text{tr} \ln (I + \mathcal{M}_0^{-1} \delta\mathcal{M}), \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} \text{tr} \ln (I + \mathcal{M}_0^{-1} \delta\mathcal{M}) &= \text{tr} (\mathcal{M}_0^{-1} \delta\mathcal{M}) - \frac{1}{2} \text{tr} (\mathcal{M}_0^{-1} \delta\mathcal{M})^2 \\ &\quad + \frac{1}{3} \text{tr} (\mathcal{M}_0^{-1} \delta\mathcal{M})^3 + \dots, \end{aligned} \quad (4.4)$$

where \mathcal{M}_0 and $\delta\mathcal{M}$ are the operators

$$\begin{aligned} \mathcal{M}_0 &= i\gamma_0\partial_0 - \gamma_1(\partial_1^2 - \partial_2^2) - \gamma_2 2\partial_1\partial_2 + m, \\ \delta\mathcal{M} &= \mathbf{M}(x) \cdot \boldsymbol{\gamma} + a_\mu \mathcal{J}^\mu - \mathbf{T}_{ij} a_i a_j, \end{aligned} \quad (4.5)$$

with the vertices \mathcal{J}_μ and given by

$$\mathcal{J}_0 = \gamma_0, \quad (4.6)$$

$$\mathcal{J}_1 = i\gamma_1\partial_1 + i\gamma_2\partial_2 + \text{H.c.}, \quad (4.7)$$

$$\mathcal{J}_2 = -i\gamma_1\partial_2 + i\gamma_2\partial_1 + \text{H.c.}, \quad (4.8)$$

where $i, j = 1, 2$ label the two spatial components of the gauge field a_μ , and the matrix \mathbf{T} is

$$\mathbf{T} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}, \quad (4.9)$$

where γ_1 and γ_2 are the two spatial Dirac gamma matrices.

The terms in the expansion of Eq. (4.4) that are quadratic in the nematic fields \mathbf{M} and on the hydrodynamic gauge field a_μ represent the leading quantum fluctuations about the $N = \infty$ limit. The effective action for the quantum fluctuations of the hydrodynamic gauge field a_μ and the nematic fields \mathbf{M} have the form

$$S_{\text{eff}}[a_\mu, \mathbf{M}] = S_{\text{eff}}[a_\mu] + S_{\text{eff}}[\mathbf{M}] + S_{\text{eff}}[a_\mu, \mathbf{M}]. \quad (4.10)$$

Here, $S[\mathbf{M}]$ describes the dynamics of the nematic field, and will be studied in detail in the next section. In this section, we focus on the effective action of the hydrodynamic gauge fields and on their coupling to the nematic fields, $S_{\text{eff}}[a_\mu] + S_{\text{eff}}[a_\mu, \mathbf{M}]$. The details of the Feynman diagrams and of the calculations included in this section can be found in Appendix A. The resulting effective Lagrangian is

$$\begin{aligned} &\mathcal{L}_{\text{eff}}[a_\mu] + \mathcal{L}_{\text{eff}}[a_\mu, \mathbf{M}] \\ &= \frac{N}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + N b_\mu \epsilon^{\mu\nu\rho} \partial_\nu (a_\rho - A_\rho) \\ &\quad + \frac{N}{8\pi} \left(\frac{1}{m} + \frac{M_1}{2m^2} \right) (\partial_0 a_1 - \partial_1 a_0)^2 \\ &\quad + \frac{N}{8\pi} \left(\frac{1}{m} - \frac{M_1}{2m^2} \right) (\partial_0 a_2 - \partial_2 a_0)^2 \\ &\quad + \frac{N}{8\pi} \frac{M_2}{m^2} (\partial_0 a_1 - \partial_1 a_0) (\partial_0 a_2 - \partial_2 a_0) \\ &\quad - \frac{N}{4\pi} (\partial_2 a_1 - \partial_1 a_2)^2. \end{aligned} \quad (4.11)$$

The effective gauge theory is a Maxwell-Chern-Simons theory. The first term is the Chern-Simons term from the nontrivial fermion band, the second term is the BF term obtained from the functional bosonization technique we used. It is straightforward to see that this effective action predicts that the QAH phase has a quantized Hall conductivity $\sigma_{xy} = Ne^2/h$, as expected for the quadratic band crossing case.⁵³

The rest of the terms in the effective action of Eq. (4.11) are the parity-even Maxwell terms and the local coupling of the fluctuation of the nematic fields to the hydrodynamic gauge field. The latter has the form of an effective spatial anisotropy. Hence, it is apparent from Eq. (4.11) that the nematic order parameters couple to the gauge fields as an effective spatial metric. To make this more clear, let us rewrite the Maxwell terms $\mathcal{L}_{\text{Maxwell}}$ in the form (for comparison, see

Ref. 47)

$$\mathcal{L}_{\text{Maxwell}} = -\frac{N}{8\pi\sqrt{2m}} f_{\mu\nu} g^{\mu\alpha} f_{\alpha\beta} g^{\beta\nu}, \quad (4.12)$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \quad (4.13)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2m} Q_{\mu\nu}, \quad (4.14)$$

where we have rescaled the time coordinate and temporal component of the gauge field $x_0 \rightarrow \frac{1}{\sqrt{2m}} x_0, a_0 \rightarrow \sqrt{2m} a'_0$ so as to renormalize the dielectric constant and make the ‘‘speed of light’’ be 1. The modified metric in the Maxwell term are composed of a regular flat metric of 2 + 1-dimensional Minkowski space-time, $\eta_{\mu\nu} = \text{diag}(1, -1, -1)$, locally modified by a traceless metric $Q_{\mu\nu}$ induced by the local spatial anisotropy. The traceless symmetric tensor $Q_{\mu\nu}$ only has nonvanishing spatial components,

$$Q_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_1 & M_2 \\ 0 & M_2 & -M_1 \end{pmatrix}, \quad (4.15)$$

From the expression of $Q_{\mu\nu}$, it is clear that this is the hydrodynamic theory of a gauge field on a manifold with a fluctuating nontrivial (purely spatial) metric due to the coupling to the nematic field. As the fluctuation of the nematic field modifies the local metric, in the anisotropic phase, where the tensor Q_{ij} (or, equivalently, \mathbf{M}) acquires a nonzero expectation value, the Maxwell term becomes anisotropic. This leads to anisotropic transport (at finite wave vector \mathbf{q}) in the nematic QAH. This phenomenon is equivalent to having an anisotropic dielectric tensor that plays the role of the metric tensor we introduced here.

V. EFFECTIVE THEORY OF THE NEMATICITY

Let us now derive the effective theory of the nematic field \mathbf{M} . The effective action $S_{\text{eff}}(\mathbf{M})$, obtained for the integration of the fermions and from the Hubbard-Stratonovich fields, has the form

$$S_{\text{eff}}(\mathbf{M}) = N \ln \det(G_0^{-1} - \mathbf{M} \cdot \boldsymbol{\gamma}) - \int d^3x \frac{N}{2g} \mathbf{M}^2. \quad (5.1)$$

By expanding the effective action to the quadratic order, we get

$$\begin{aligned} S_{\text{eff}} &= -\frac{N}{2} \text{tr}(G_0 \boldsymbol{\gamma} \cdot \mathbf{M} G_0 \boldsymbol{\gamma} \cdot \mathbf{M}) - \int d^3x \frac{N}{2g} \mathbf{M}^2 \\ &= -\frac{N}{2} \int \frac{d^3p}{(2\pi)^3} M_i(-p) \Gamma_{ij}(p) M_j(p) \\ &\quad - \frac{N}{2g} \int \frac{d^3p}{(2\pi)^3} |\mathbf{M}(p)|^2, \end{aligned} \quad (5.2)$$

where $\Gamma_{ij}(p)$ is the one-loop kernel

$$\Gamma_{ij}(p) = \int \frac{d^3k}{(2\pi)^3} \text{tr}[\gamma_i G_0(p+k) \gamma_j G_0(k)], \quad (5.3)$$

which is given by the self-energy diagram discussed in Appendix B.

Let us now define a 2×2 traceless symmetric tensor field \mathbf{Q} , which is natural to describe a nematic phase^{79,80}

$$\mathbf{Q} = \begin{pmatrix} M_1 & M_2 \\ M_2 & -M_1 \end{pmatrix}. \quad (5.4)$$

At long wavelengths and low frequencies, the effective Lagrangian of the nematic order parameter $\mathcal{L}[\mathbf{Q}]$ is

$$\begin{aligned} \frac{1}{N} \mathcal{L}_{\text{eff}}[\mathbf{Q}] &= -\chi(m) \epsilon^{bc} Q_{ab} \partial_0 Q_{ac} - r(m) \text{tr}(\mathbf{Q} \mathbf{Q}) \\ &\quad + \kappa_1 \text{tr}(\mathbf{Q} \mathbf{K} \mathbf{Q}) + \kappa_2 \text{tr}(\sigma_1 \mathbf{Q} \mathbf{K}' \mathbf{Q}) \\ &\quad - u(m) \text{tr}(\mathbf{Q} \mathbf{Q} \mathbf{Q}), \end{aligned} \quad (5.5)$$

where \mathbf{K} and \mathbf{K}' are the 2×2 symmetric matrix differential operators

$$\mathbf{K} = \begin{pmatrix} \partial_1^2 & \partial_1 \partial_2 \\ \partial_2 \partial_1 & \partial_2^2 \end{pmatrix}, \quad \mathbf{K}' = \begin{pmatrix} \partial_2 \partial_1 & \partial_2^2 \\ \partial_1^2 & \partial_2 \partial_1 \end{pmatrix}, \quad (5.6)$$

and σ_1 is the (symmetric and real) Pauli matrix.

The coefficients $r(m)$ and $u(m)$ in Eq. (5.5) were given already in Eq. (3.34). The coefficient $\chi(m)$ shown in Eq. (5.5), is given by

$$\chi(m) = \frac{1}{64\pi} \frac{1}{m}. \quad (5.7)$$

The coefficient $\chi(m)$ depends on *both* the *magnitude* and the *sign* of the parameter m , i.e., on the expectation value of the order parameter that measures the spontaneous breaking of time-reversal invariance in the Mott Chern insulator. This behavior is reminiscent of the parity anomaly of a Dirac fermion in 2 + 1 dimensions.^{81,82} In the next section, we will see shortly that $\chi(m)$ is related to the Hall viscosity and hall torque viscosity of the spontaneous QAH phase. Moreover, the presence of this Berry phase term makes the dynamic critical exponent of the effective theory of the nematic fields to be $z = 2$.

The first term of the effective action $\mathcal{L}_{\text{eff}}[\mathbf{Q}]$ of Eq. (5.5) is of first order in time derivatives, reflecting the spontaneous breaking of time-reversal invariance in the (spontaneous) QAH phase and, hence, is *odd* under time reversal. This term can be regarded as a Berry phase of the time evolution of the nematic order parameter field. Maciejko and collaborators⁴⁶ have shown that it is possible to rewrite the effective field theory of the nematic order parameter field as a nonlinear sigma model whose target space is a hyperbolic space, a coset of $SO(2,1)$. The form of our Berry phase term is consistent with the one discussed by Maciejko and collaborators⁴⁶ in the limit $\mathbf{Q} \ll 1$, which we have used here.

Before we discuss the phases of this theory and the behavior of the nematic degrees of freedom, it is worth to comment on the symmetries of the effective Lagrangian of Eq. (5.5). As it is apparent, this effective Lagrangian is invariant under a global rotation of the nematic order parameter field (modulo π). This symmetry is the result of setting $t = t'$ in the lattice Hamiltonian of Eq. (2.1) and of the fact that we kept only the lowest terms in momenta in the long wavelength theory of the fermions of Eq. (2.3). On the other hand, if $t \neq t'$,

the effective low-energy theory has a lower C_4 symmetry. At the level of the nematic order parameter, this is equivalent to an Ising symmetry (of rotations by $\pi/2$). The same type of symmetry breaking is obtained in the corrections to Eq. (2.3) of order p^4 (or higher) in the effective low-energy Hamiltonian of the fermions. The net effect of these corrections is nominally irrelevant operators which break the continuous $O(2)$ symmetry down to a discrete (Ising) symmetry.

A. The isotropic QAH phase

In the isotropic QAH phase and to lowest order in the $1/N$ expansion, we find that the stiffnesses are

$$\kappa_1 = \frac{1}{12\pi|m|}, \quad \kappa_2 = 0. \quad (5.8)$$

Hence, in the isotropic phase, the terms of the effective action that depend on the spatial gradients, after an integration by parts, can be written in the form

$$-\kappa_1 \text{Tr}[\mathbf{Q}\mathbf{K}\mathbf{Q}] = \kappa_1 [(\nabla \cdot \mathbf{M})^2 + (\nabla \times \mathbf{M})^2]. \quad (5.9)$$

Hence the two Frank constants are equal in the isotropic phase.

It is straightforward to see that the nematic modes are gapped in the isotropic phase and that their gap vanishes at the quantum phase transition. Again, provided the explicit lattice symmetry breaking effects, we discussed above can be neglected, the spectrum of nematic modes will be gapped but degenerate.

B. The nematic QAH phase

However, in the nematic QAH phase where, the rotational symmetry is spontaneously broken. This has two consequences. One is that instead of a single Frank constant (stiffness) we now find two:

$$\kappa_1 = \frac{1}{12\pi|m|}, \quad \kappa_2 = \frac{|\bar{\mathbf{M}}|}{16\pi m^2}, \quad (5.10)$$

where \mathbf{Q} represents now the *fluctuations* of the nematic order parameter in the nematic QAH phase, $|\bar{\mathbf{M}}|$ is the expectation value of the nematic field in the $N \rightarrow \infty$ limit and is given in Eq. (3.36). By symmetry, the Frank stiffness κ_2 is an odd function of the magnitude of the nematic order parameter $|\bar{\mathbf{M}}|$. Thus, provided we restrict ourselves to the vicinity of the transition, in Eq. (5.10) we may keep only the leading (linear) term.

Hence, as expected, in the nematic QAH phase there are two Frank constants, and the spatial terms of the effective Lagrangian for the nematic fluctuations now becomes (also after an integration by parts)

$$\begin{aligned} & -\kappa_1 \text{tr}(\mathbf{Q}\mathbf{K}\mathbf{Q}) - \kappa_2 \text{tr}(\sigma_1 \mathbf{Q}\mathbf{K}'\mathbf{Q}) \\ & = (\kappa_1 + \kappa_2)(\nabla \cdot \mathbf{M})^2 + (\kappa_1 - \kappa_2)(\nabla \times \mathbf{M})^2, \end{aligned} \quad (5.11)$$

which is the generally expected form for the energy of nematic fluctuations.^{80,83} A similar result generally holds in other electronic nematic phases.⁷⁹

The other consequence is that there is a gapless Goldstone mode of the spontaneously broken symmetry. Again, if the microscopic theory only has a discrete C_4 invariance the

Goldstone modes is gapped but the gap can be small if the explicit symmetry breaking is weak.

C. Critical behavior

We will now discuss briefly the critical behavior. By examining the effective Lagrangian of Eq. (5.5), we see that the nematic order parameter field has scaling dimension 1, i.e., $[Q] = l^{-1}$ (where l is a length scale) or $\Delta_Q = 1$. This scaling follows from the presence of the Berry phase term in the effective Lagrangian. Incidentally, the main effect of the Berry phase term is to make the two components of the nematic order parameter field to be canonically conjugate pairs. From the fact that the order parameter has scaling dimension $\Delta_1 = 1$, it follows that the scaling dimension of the quartic term of the effective Lagrangian has dimension $\Delta_4 = 4$ and that the effective coupling constant can be made dimensionless [by absorbing the Berry phase $\chi(m)$ in a rescaling of the nematic field]. This is consistent with the fact that the dynamical exponent is $z = 2$ and the dimensionality of space is $d = 2$. Hence the effective dimension is $d + z = 4$, hence the quartic term of the Lagrangian is superficially marginal at the nematic quantum critical free field point, $r = 0$. Thus this theory appears to behave much in the same way as conventional (relativistically invariant) ϕ^4 quantum field theory of four space-time dimensions.

Just as in conventional ϕ^4 theory, the quartic term is also marginally irrelevant at the free field fixed point with $z = 2$. Provided this assumption (which we have not verified) is correct, we deduce that the quantum critical behavior is that of the effective classical theory, of Eq. (5.5), with logarithmic corrections to scaling. On the other hand, if the quartic term were to be marginally relevant, it would turn this quantum phase transition into a fluctuation-induced first-order transition.

Finally, this theory has a finite-temperature thermodynamic phase transition at a T_c at which the nematic order is lost. If the symmetry is $O(2)$, then we expect a conventional nematic continuous (Kosterlitz-Thouless) phase transition. On the other hand, if the symmetry is broken (microscopically) down to a discrete Ising (\mathbb{Z}_2) symmetry, the finite-temperature transition would be in the 2D Ising universality class.

VI. TRANSVERSE DISSIPATIONLESS RESPONSE TO SHEAR STRESS: HALL TORQUE VISCOSITY IN THE QUANTUM ANOMALOUS HALL STATE

Quantum Hall fluids and other two-dimensional systems with broken time-reversal invariance such as Chern insulators, show a variety of dissipationless responses to external fields which do not exist in normal fluids. In a system with broken time-reversal invariance due either to an external perpendicular magnetic field or to topologically nontrivial band structures, an in-plane electric field induces a Hall current which is perpendicular to the applied field and has a Hall conductance, which is precisely determined by the topological properties of these fluids. Similarly, in a two-dimensional system with broken time reversal invariance and parity, by shearing the system in one direction a momentum transfer is induced

in the perpendicular direction. As a result, the stress tensor has an antisymmetric component that is proportional to the shear rate. The associated transport coefficient is the Hall viscosity.^{48–50,52,84}

While the resulting Hall conductance is dimensionless and universal (in units of e^2/h), the Hall viscosity has units of length^{-2} . If the system is Galilean invariant (which is the case, to a good approximation, in the 2DEG in AlAs-GaAs heterostructures and quantum wells) then the length scale is supplied by the magnetic length and, in this sense, the Hall viscosity is also universal.⁵² On the other hand, in the case of topological Chern insulators, although there is a finite Hall viscosity in general it is the sums of a nonuniversal term (which is determined by microscopic physics) and an essentially universal term.⁵⁰

In this section, we will first derive an expression of the Hall viscosity for the system at hand, a Chern insulator originating from an instability of a system with a quadratic band crossing. Here, we will show that the Hall viscosity is related to both the Hall conductivity of the QAH phase and with the coefficient χ of the Berry phase term obtained in Eq. (4.11). We will also see how this is related to the concept of Hall torque viscosity which we introduce below.

For a parity violating system, such as the quantum Hall fluids of 2DEGs, a change in the background metric g_{ij} of the surface on which the electron fluid resides modifies the definition of the momentum of the electrons through their coupling to the metric. A consequence of the breaking of time reversal and parity (either explicit or spontaneous) the effective field theory of the weak perturbation of the metric contains a term which is odd under parity and time reversal. Such Chern-Simons-type terms are first order in time derivatives, and their coefficient is the Hall viscosity.

On the other hand, the fermion field of the system we are interested in is a theory of two-component spinors and it is not Galilean invariant. A system of spinors, such as the one given in the effective long wavelength Hamiltonian of Eq. (2.3), is defined with respect to a frame of orthonormal two-component vectors e^a (with $a = 1,2$) tangent to the two-dimensional space. Microscopically, these vectors are tied to the local geometry of the underlying two-dimensional lattice. Thus, under a lattice deformation (which includes local rotations), these local frames, which following tradition we will call zweibeins, accordingly change slowly.

Let us now suppose that we rotate the “spinor frame” of the fermion field, i.e., that we make a local change of basis of the spinors. A global change of basis with a rotation axis normal to the plane is a symmetry since it is equivalent to a rotation of the space axis. However, spinor rotations about arbitrary axis and/or under a local change of basis, i.e., a change of the local frame, are not symmetries of the system. As a result of such transformations the system generally experiences a torque viscosity that is perpendicular to the axis of rotation. In what follows, we will be interested in adiabatic changes in the frames of the spinors and in the Berry phase terms they induce.

We will now show that the coefficient $\chi(m)$ of the effective action of the nematic order parameter fields is related to the Hall viscosity in the QAH phase.^{48–50,52} An excellent discussion of the Hall viscosity can be found in the recent

work of Hughes, Leigh, and Parrikar⁸⁴ whose methods we use here.

In order to represent the local deformations of the space, one couples the frames (the zweibeins) directly to the covariant derivative. However, in our case, there is an orbital degree of freedom and an analog of a spin connection is required. The long-wavelength Lagrangian for the free fermions on the undistorted lattice is

$$\mathcal{L} = \bar{\psi}_a(x) [i\gamma_0\partial_0 - \gamma_1(\partial_1^2 - \partial_2^2) - \gamma_2 2\partial_1\partial_2 + M_0] \psi_a(x). \quad (6.1)$$

In this section, we will discuss the behavior of the Hall viscosity and the Hall torque viscosity in the isotropic QAH in the $N \rightarrow \infty$ limit. In this limit, and in this phase, the nematic order parameter field has vanishing expectation value and does not contribute. However, its fluctuations do contribute (to order $1/N$) to the corrections at small but finite momenta of these quantities.

By adding the background distortion connecting between real space (or momentum) and orbital space, the new Lagrangian, which now depends explicitly on the frame fields $e^a(x)$, becomes

$$\mathcal{L} = \bar{\psi}_\alpha(x) (i\gamma_0\partial_0 - T_a^{ij} e_k^a \gamma_k \partial_i \partial_j + M_0)_{\alpha,\beta} \psi_\beta(x), \quad (6.2)$$

where $a = 1,2$, $\alpha, \beta = 1,2$, and $i, j, k = 1,2$. As before, we have set $T_1 = \sigma_z$ and $T_2 = \sigma_x$. The metric tensor of the 2D distorted space is $g_{ij} = e_i^a e_j^a$. For a system on a flat metric, i.e., an undistorted lattice, the frame vectors are $e_i^a = \delta_i^a$ and, in this case, $g_{ij} = \delta_{ij}$, and the Lagrangian of Eq. (6.2) reduces to our original free fermion Lagrangian of Eq. (6.1).

Here, we will be interested in shear distortions and rotations, which are area-preserving diffeomorphisms. We can parametrize the frame fields e^a as follows:

$$e_1^1 - 1 = -(e_2^2 - 1) = e_1, \quad e_2^1 = e_1^2 = e_2. \quad (6.3)$$

Under this distortion, the free-fermion Lagrangian becomes

$$\begin{aligned} \mathcal{L} = & \bar{\psi}(x) [i\gamma_0\partial_0 - \gamma_1(\partial_1^2 - \partial_2^2) - \gamma_2 2\partial_1\partial_2 - M_0] \psi(x) \\ & + \bar{\psi}(x) [-e_1\gamma_1(\partial_1^2 - \partial_2^2) + e_1\gamma_2 2\partial_1\partial_2] \psi(x) \\ & + \bar{\psi}(x) [-e_2\gamma_2(\partial_1^2 - \partial_2^2) - e_2\gamma_1\partial_1\partial_2] \psi(x), \end{aligned} \quad (6.4)$$

where $e_1(x)$ and $e_2(x)$ are two slowly varying functions of space and time.

After integrating-out the fermion field, the effective theory of the frame fields e^a contains a parity-violating term, which appears to the first order time derivatives. In momentum and frequency space, it has the form

$$\begin{aligned} S_{\text{eff}}[e_i] = & \int \frac{d\omega}{2\pi} \frac{d^2p}{(2\pi)^2} i\eta(\mathbf{p},\omega) \omega \epsilon^{ij} e_i(\mathbf{p},\omega) e_j(-\mathbf{p},-\omega) \\ & + \dots, \end{aligned} \quad (6.5)$$

where $\eta(\mathbf{p},\omega)$ is given by

$$\eta(\mathbf{p},\omega) = \frac{1}{i\omega} \epsilon^{ij} \frac{\delta^2 S}{\delta e_i(\mathbf{p},\omega) \delta e_j(-\mathbf{p},-\omega)}. \quad (6.6)$$

In what follows, we will only be interested in the adiabatic regime. Thus we will take the limit $\omega \rightarrow 0$. In this limit, we can expand $\eta(\mathbf{p},0) = \eta(\mathbf{p})$ in powers of the momentum \mathbf{p} . In the

isotropic QAH phase, $\eta(\mathbf{p})$ can only be a function of \mathbf{p}^2 . To lowest orders, we obtain

$$\eta(\mathbf{p}) = \eta(0) + \eta_1 \mathbf{p}^2 + \eta_2 \mathbf{p}^4 + \dots, \quad (6.7)$$

where $\mathbf{p}^4 = (\mathbf{p}^2)^2$, etc. For symmetry reasons, only powers even powers of the momentum are allowed to enter in this expansion.

On the other hand, in the nematic QAH insulating phase, in addition to an isotropic component of the form of Eq. (6.7) there is an anisotropic piece. Close to the quantum critical point the anisotropic piece of the term quadratic in momenta is a linear function of the expectation value of the nematic order parameters and has the form (up to a constant prefactor) $(p_1^2 - p_2^2)M_1 + 2p_1 p_2 M_2$. Similar considerations apply to the higher-order terms in the expansion in momenta.

The zeroth-order coefficient $\eta(0)$ in Eq. (6.7) is the Hall viscosity η ,

$$\begin{aligned} \eta &= \int \frac{d^2k}{(2\pi)^2} \frac{m(k_1^2 + k_2^2)^2}{[k_0^2 - (k_1^2 + k_2^2)^2 - m^2 - i\epsilon]^2} \\ &= \frac{m}{16\pi} \ln\left(\frac{2\Lambda^2}{m}\right) - \frac{m}{16\pi}, \end{aligned} \quad (6.8)$$

which depends both on the magnitude *and* the sign of the mass m . Notice that the Hall viscosity, as expected, has units of m , or what is the same units of length^{-2} . The Hall viscosity $\eta = \eta(0)$ can also be computed from the correlation function of the stress tensor, $\langle T_i^a T_j^b \rangle$.⁵⁰

The coefficient η_2 for the term $O(\mathbf{p}^4)$ in the expansion of Eq. (6.7) is proportional to the coefficient $\chi(m)$ appearing in the Berry phase term in effective nematic theory,

$$\begin{aligned} \eta_2 &= \epsilon^{ij} \frac{1}{i\omega} \frac{\delta^2 S}{\delta[p^2 e_i(p)] \delta[p^2 e_j(-p)]} \\ &\propto \int \frac{d^3k}{(2\pi)^3} \frac{m}{[k_0^2 - (k_1^2 + k_2^2)^2 - m^2 - i\epsilon]^2}. \end{aligned} \quad (6.9)$$

Hence we find that

$$\begin{aligned} \eta_2 &\propto \lim_{\omega \rightarrow 0} \lim_{p \rightarrow 0} \epsilon^{bc} \frac{1}{i\omega} \frac{\delta^2 S_{\text{eff}}(\mathbf{M})}{\delta Q_{ab}(p) \delta Q_{ac}(-p)} \\ &= \chi(m). \end{aligned} \quad (6.10)$$

Actually, the coefficient of the \mathbf{p}^2 term of the expansion is proportional to the Hall conductance,

$$\begin{aligned} \eta_1 &= \epsilon^{ij} \frac{1}{i\omega} \frac{\delta^2 S}{\delta[p e_i(p)] \delta[p e_j(-p)]} \\ &\propto \int \frac{d^3k}{(2\pi)^3} \frac{m(k_1^2 + k_2^2)}{[k_0^2 - (k_1^2 + k_2^2)^2 - m^2 - i\epsilon]^2}. \end{aligned} \quad (6.11)$$

Hence, we also find that

$$\eta_1 \propto \lim_{\omega \rightarrow 0} \lim_{p \rightarrow 0} \epsilon^{ij} \frac{1}{i\omega} \frac{\delta^2 S_{\text{eff}}(\mathbf{M})}{\delta A_i(p) \delta A_j(-p)} = \frac{1}{4} \sigma_{xy}. \quad (6.12)$$

Unlike the Hall conductivity, the Hall viscosity is not a topological response as it does depend on microscopic details of the fermionic system. Furthermore, if we were to include the nematic field in Eq. (6.1), even in the isotropic phase its

fluctuations to order $1/N$ modify the values of η_1 and η_2 but do not affect the value of the Hall viscosity η . In this sense, the relationship between χ , σ_{xy} and η_1, η_2 is not universal. Moreover, in the nematic phase, the coefficients η_1 and η_2 become tensors, reflecting the nematic nature of the phase.

Now we come to the Hall torque viscosity. As in most (but not all) Chern insulators, the fermion field of the quadratic band crossing model is a two component spinor which labels the two different bands. In the case of the checkerboard model, the spinor labels can be traced back to the two-sublattice structure of the lattice. Suppose we now rotate the ‘‘spinor frame’’ of the fermion by an $SU(2)$ unitary transformation of the form

$$\Psi'_\alpha(x) = [e^{i(-\theta_2 \sigma_x + \theta_1 \sigma_z)}]_{\alpha\beta} \Psi_\beta(x). \quad (6.13)$$

The rotation axis of this transformation lies on the xz plane. Suppose now that we consider an infinitesimal rotation angle so that we can expand the rotation matrix to lowest order in θ ,

$$\Psi' = \Omega \Psi, \quad \Omega = \begin{pmatrix} 1 - i\theta_2 & i\theta_1 \\ i\theta_1 & 1 + i\theta_2 \end{pmatrix}. \quad (6.14)$$

This is not a symmetry transformation of the Lagrangian. Indeed, upon this rotation of the spinor frame, the Lagrangian Eq. (6.1) changes as follows:

$$\begin{aligned} \mathcal{L} &= \bar{\psi}'(x) [i\gamma_0 \partial_0 - \gamma_1 (\partial_1^2 - \partial_2^2) - \gamma_2 2\partial_1 \partial_2 - m] \psi'(x) \\ &\quad - \bar{\psi}'(x) [\theta_1 (\partial_1^2 - \partial_2^2) + \theta_2 2\partial_1 \partial_2 + m\theta_1 \gamma_1 \\ &\quad + m\theta_2 \gamma_2] \psi'(x). \end{aligned} \quad (6.15)$$

As we can see, the last two terms generated by a rotation of the spinor frame have exactly the same form as the nematic order parameter. In addition, the spinor rotation also mixes with the time-reversal symmetry breaking mass term (albeit with terms which are quadratic in spatial derivatives).

It is straightforward to obtain the effective action for the spinor rotation angles in the adiabatic regime. Similarly to the calculation that we did for the Hall viscosity, here too, we find an antisymmetric term which is first order in time derivatives,

$$\mathcal{L}(\theta) = -\eta^s \epsilon^{ij} \theta_i \partial_0 \theta_j + \dots, \quad (6.16)$$

where η_s is the torque viscosity and we find it to be

$$\eta_s = -\frac{m}{16\pi} \ln\left(\frac{2\Lambda^2}{m}\right) + \frac{m}{8\pi}. \quad (6.17)$$

This result shows the existence of a dissipationless transport property, namely, the Hall torque viscosity, which is the response of the action under an adiabatic rotation of the spinor frame.

By analogy with the stress-energy tensor for a metric distortion, here we can define the torque $\langle S \rangle$ for the rotation of the spinor frame,

$$\langle S_i \rangle = \frac{\delta S}{\delta \theta_i} = A^{ij} \partial_0 \theta_j + B^{ij} \theta_j + \dots. \quad (6.18)$$

The second term yields the linear response between the torque and the time derivative of the rotation angle (the angular velocity). The rank tensor A^{ij} is the torque viscosity. In a time-reversal and parity invariant fluid, this viscosity tensor is symmetric, indicating the rotation entails an energy cost and, furthermore, in general it is a dissipative response. However,

in a system of spinors with broken parity and time-reversal invariance, such as QAH phase of our system, the tensor A^{ij} must have an antisymmetric part which is odd under parity. Thus, when we rotate the spinor frame in the QAH phase, there is a torque viscosity η^s , which is not parallel but perpendicular to the direction of the rotation. This dissipationless rotation response is a unique signature of parity-violating phase of a system with spinor degrees of freedom.

In Chern insulators, the spinor and orbital degrees of freedom are locked to each other. In the case of a Dirac (Weyl) fermion, the spinor polarization is locked with the direction of propagation of the state (the momentum). In our case, the spinor polarization is locked instead with quadrupole moment of the momentum of the state. In this way, a rotation in spinor space induces a momentum current and vice versa.

A consequence of these observations is that there must be a relation between the Hall viscosity and Hall torque viscosity. To see what the relation is, let us compare the stress tensor with the spinor torque. Let us compute the rate of change of the action under an infinitesimal change of the frame fields, parametrized by e_1 and e_2 , respectively [defined in Eq. (6.3)], and compare that with the torque. We obtain

$$\begin{aligned} T_{ij} + T_{ji} &= \frac{\delta S}{\delta e_1} = -\bar{\psi} [2\partial_1 \partial_2 \gamma_1 + (\partial_1^2 - \partial_2^2) \gamma_2] \psi, \\ T_{ii} - T_{jj} &= \frac{\delta S}{\delta e_2} = -\bar{\psi} [-2\partial_1 \partial_2 \gamma_2 + (\partial_1^2 - \partial_2^2) \gamma_1] \psi, \end{aligned} \quad (6.19)$$

and

$$\begin{aligned} S_1 &= \frac{\delta S}{\delta \theta_1} = \bar{\psi} [(\partial_1^2 - \partial_2^2) + m\gamma_1] \psi, \\ S_2 &= \frac{\delta S}{\delta \theta_2} = \bar{\psi} (2\partial_1 \partial_2 + m\gamma_2) \psi. \end{aligned} \quad (6.20)$$

After some simple algebra, it is easy to check the equivalence between spin rotation torque and the stress tensor,

$$\begin{aligned} T_{ij} + T_{ji} &= -(S_1 \gamma_2 + S_2 \gamma_1), \\ T_{ii} - T_{jj} &= -(S_1 \gamma_1 - S_2 \gamma_2). \end{aligned} \quad (6.21)$$

As a result, if we subtract the antisymmetric parts from both the stress tensor correlator and of the torque correlator, we obtain

$$-\left\langle \frac{\delta^2 S}{\delta \theta_1 \delta \theta_2} \right\rangle + \left\langle \frac{\delta^2 S}{\delta M_1 \delta M_2} \right\rangle = \left\langle \frac{\delta^2 S}{\delta e_1 \delta e_2} \right\rangle. \quad (6.22)$$

This identity implies the following linear relation between Hall viscosity η , the Hall torque viscosity η^s , and the Berry phase χ coefficient in our effective theory,

$$-\eta^s + 4\chi = \eta. \quad (6.23)$$

Thus the Berry phase term that was obtained from the effective theory for the nematic order parameter field measures the difference of Hall viscosity and Hall torque viscosity. We should note that the expressions for χ , η , and η_s given, respectively, in Eqs. (5.7), (6.8), and (6.17), obey this relation exactly.

The validity of these results is not restricted to the particular Chern insulator we studied here. The Hall torque viscosity is a universal property in all kinds of QAH phases. In systems

in which the fermions arise from several orbitals, the fermion operator in the effective action is a multicomponent spinor. Suppose that the system has a nonvanishing Chern number, and hence that it is in a QAH state. If we rotate the spinor frame, the torque viscosity tensor, which is the linear response coefficient between torque and the angular velocity of the spinor rotation, must always include an antisymmetric part resulting from the parity violation in the fermion system.

As an example, let us choose the case of a Dirac (Weyl) fermion. Suppose we rotate the spinor frame in a similar way as in Eq. (6.14). After this rotation which, again is not a symmetry transformation, the Lagrangian changes to

$$\begin{aligned} \mathcal{L} &= \bar{\psi}'(x)(i\gamma_0 p_0 - \gamma_1 p_1 - \gamma_2 p_2 - m)\psi'(x) \\ &\quad + \bar{\psi}'(x)(\theta_1 p_1 + \theta_2 p_2 + m\theta_1 \gamma_1 + m\theta_2 \gamma_2)\psi'(x). \end{aligned} \quad (6.24)$$

In the case of a Dirac (Weyl) fermion, the rotation metric couples both with the current and momentum. If we integrate-out the fermion, we would also get a Hall torque viscosity term

$$\mathcal{L}(\theta) = \frac{(-m\Lambda + 4m^2)}{8\pi} \epsilon^{ij} \theta_i \partial_0 \theta_j + \dots \quad (6.25)$$

For a Dirac fermion, the spin is locked with linear momentum. Therefore the equivalence between a spinor rotation and momentum current is expected and, hence, there is a similar relation between Hall viscosity and Hall torque viscosity.

VII. CONCLUSIONS

In this paper, we presented a theory of the Mott quantum anomalous Hall state in the vicinity of its transition to a nematic QAH state. Our theory was developed in the context of a theory of spinless fermions, which, at the free fermion level, has a quadratic band crossing. A main result of this work is the effective field theory of Secs. IV and V in which we derived the effective action for the hydrodynamic gauge fields a_μ and b_μ (which represent the charge currents) and the nematic order parameter field \mathbf{M} . The gauge theory sector is dominated by two topological terms, the BF term and the Chern-Simons term. The effective action of the nematic fields was found to contain a Berry phase term whose parity and time-reversal odd coefficient χ controls the dynamics. In particular, the effective dynamical exponent of the nematic fields is $z = 2$, consistent with the results of Maciejko *et al.* developed in the context of the fractional quantum Hall states.⁴⁶ We also found that the nematic fields couple to the gauge field a_μ as a spatial metric. Our results clarify the role of geometric degrees of freedom in systems that exhibit the quantum Hall effect. We expect that these results should also apply to the case of the fractional quantum Hall effect and we will discuss these results elsewhere.⁵⁴

In this work, we considered the transition from the QAH phase to a nematic QAH phase (which is a continuous transition). It is also possible to instead consider different regime of coupling constants in which the leading instability from the QBC is to a nematic semimetal followed by a first order transition to the nematic QAH.⁵³ However, in this case, the theory that we presented here does not strictly apply since the transition would now be first order. Nevertheless, the structure of our main results will still hold. A direct instability

from the free QBC system to a nematic QAH phase does not seem to occur naturally.

In Sec. VI, we investigated the relation between the coefficient χ of the Berry phase of the nematic fields and the Hall viscosity η of the spinors, which measures the transverse response to a local change of the spinor frame. Here, we found that the complete picture requires the introduction of the concept of the torque Hall viscosity η_s , which is related to the fact that for a system of spinors a deformation of the underlying space requires the introduction of a spin connection. This effect is associated with the kinematics of spinors. Although it is always present in multicomponent fermionic systems, it takes a different form for Dirac fermions and in this model with a quadratic band crossing (with unit Chern number). In particular, we found that these three coefficients obey a universal linear relation given in Eq. (6.23). Nevertheless, these features are generic properties.

Our results are of interest in several systems accessible to experiment. One such system is bilayer graphene, which has two (almost exact) quadratic band crossings in the Brillouin zone. They are almost exact in that their quadratic band crossing is not protected by symmetry. However, it is “protected” by the chemistry (and physics) of the orbitals of carbon, which renders their parity-even gaps extremely small (and negligible in practice). This is a point that has been investigated at length in the literature.^{57–59} However, in the case of bilayer graphene, it is necessary to include the spin degrees of freedom (which we suppressed here). This leads to a more complex (and interesting!) phase diagram,^{53,58,64} which deserves further exploration.

In the transport experiments of Xia *et al.*³⁷ on the 2DEG, in the first Landau level, a large nematic susceptibility is seen in the longitudinal resistivities at finite temperature with a weak in-plane magnetic field. The results presented elsewhere in this paper predict a similar behavior for the longitudinal resistivity at finite temperature in the QAH-nematic phase.

Other systems of great interest for which these results may be relevant are the topological crystalline insulators.⁶⁰ Systems of these type have surface states (protected by mirror symmetry) which to a good approximation are described (at the level of the band structure) by a low-energy Hamiltonian with two quadratic band crossings. In materials such as $\text{Pb}_{1-x}\text{Sn}_x\text{Se}$ and $\text{Pb}_{1-x}\text{Sn}_x\text{Te}$, these crossings which are expected to occur at the X points on the edges of the surface Brillouin zone have been seen in ARPES and STM experiments.^{67,68,85–87} However, each quadratic crossing is found to be split into a pair of gapless Dirac cones. Although there are material-specific symmetry breaking effects that can explain these findings,⁶¹ it is also possible that the splittings may be driven by correlation effects, as in the case of the nematic semimetal phase discussed in Ref. 53. Nevertheless, it is possible that these materials (or a close relative of them) may also exhibit a spontaneous quantum anomalous Hall phase such as the one discussed here (based on the work of Ref. 53) and that the physics that we discussed here in detail may apply there too. Other materials in which these ideas may be relevant are the pyrochlore iridates.^{69,70,88}

One of the motivations of this work, as we stated above, was to explore the interplay between the topological sector of these

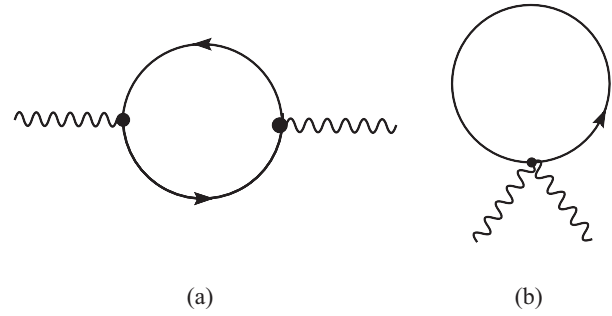


FIG. 1. One-loop self-energy diagrams for the hydrodynamic gauge field a_μ .

systems and the more microscopic “geometric” degrees of freedom. This issue was raised originally in the context of the experiments of Xia *et al.* in fractional quantum Hall states in the first Landau level of the 2DEG³⁷ and has motivated several important theoretical developments.^{39,40,43,46} Much of that work (see, e.g. Ref. 44) has focused on the role of geometric changes at the microscopic level (i.e., at the length scale of the magnetic length). However, as we showed in this paper, these “geometric” degrees of freedom can be self-organized into nematic order parameter fields whose fluctuations may manifest at even long length scales and hence may trigger a quantum phase transition of a nematic topological phase. In a separate publication,⁵⁴ we will show how the ideas presented here extend to the case of the 2DEG in the fractional quantum Hall regime.

ACKNOWLEDGMENTS

We would like to thank Kai Sun, Taylor Hughes, Rob Leigh, Joseph Maciejko, Mike Mulligan, Chetan Nayak, Steve Kivelson, Benjamin Hsu, Shivaji Sondhi, Chen Fang, Gil Cho, and Matthew Gilbert for helpful discussions. Y-Z would like to thank Bo Yang for intuitive advice. This work is supported in part by the National Science Foundation through the Grant DMR-1064319 at the University of Illinois.

APPENDIX A: THE CALCULATION OF THE EFFECTIVE GAUGE THEORY

To obtain the effective action of the gauge fields $S[a_\mu]$, we need to compute the one loop self-energy diagrams shown in Figs. 1(a) and 1(b). Let $G(p)$ be the fermion propagator of the quadratic band dispersion Chern insulator with mass m , i.e., in the isotropic QAH phase given in Eq. (3.23) with $\mathbf{M} = 0$, we can write the one-loop correction to the action $S^{(2)}[a_\mu]$ of the gauge fields in the standard form

$$S^{(2)}[a_\mu] = \frac{N}{2} \int \frac{d^3 p}{(2\pi)^3} a_\mu(-p) \Pi_{\mu\nu}(p) a_\nu(p). \quad (\text{A1})$$

$\Pi_{\mu\nu}(p)$ is the polarization operator, which is the sum of two contributions:

$$\Pi_{\mu\nu}(p) = \Pi_{\mu\nu}^{(1)}(p) + \Pi_{\mu\nu}^{(2)}(p). \quad (\text{A2})$$

$\Pi_{\mu\nu}^{(1)}(p)$ is the diagram shown in Fig. 1(a) and is given by

$$\Pi_{\mu\nu}^{(1)}(p) = i \int \frac{d^3k}{(2\pi)^3} \text{tr}[G_0(p+k)\mathcal{J}_\mu(2k+p)G_0(k)\mathcal{J}_\nu(2k+p)], \quad (\text{A3})$$

$$\mathcal{J}_0(2k+p) = \gamma_0, \quad (\text{A4})$$

$$\mathcal{J}_1(2k+p) = \gamma_1(2k_1+p_1) + \gamma_2(2k_2+p_2), \quad (\text{A5})$$

$$\mathcal{J}_2(2k+p) = -\gamma_1(2k_2+p_2) + \gamma_2(2k_1+p_1). \quad (\text{A6})$$

So the polarization tensor $\Pi_{\mu\nu}^{(1)}(p)$ has the expression

$$\begin{aligned} \Pi_{\mu\nu}^{(1)}(p) = i \int \frac{d^3k}{(2\pi)^3} \text{tr} \left\{ \frac{(p_0+k_0)\gamma_0 - [(p_1+k_1)^2 - (p_2+k_2)^2]\gamma_1 - 2(p_1+k_1)(p_2+k_2)\gamma_2 + m}{(p_0+k_0)^2 - [(p_1+k_1)^2 + (p_2+k_2)^2] - m^2 - i\epsilon} \right. \\ \left. \times \mathcal{J}_\mu(2k+p) \frac{k_0\gamma_0 - (k_1^2 - k_2^2)\gamma_1 - 2k_1k_2\gamma_2 + m}{k_0^2 - (k_1^2 + k_2^2) - m^2 - i\epsilon} \mathcal{J}_\nu(2k+p) \right\}. \end{aligned} \quad (\text{A7})$$

As we are only concerned with the long-wave-length behavior, we expand momentum p in $G_0(p+k)$ by order and only keep $O(p^2)$.

$\Pi_{\mu\nu}^{(2)}(p)$ is given by the diagram shown in Fig. 1(b) and is given by

$$\Pi_{\mu\nu}^{(2)}(p) = -i \int \frac{d^3k}{(2\pi)^3} \text{tr}[G_0(k)\mathbf{T}_{ij}], \quad (\text{A8})$$

where $G_0(p)$ is the Feynman propagator of the isotropic QAH phase given in Eq. (4.2) and

$$T_{11} = \gamma_1, \quad T_{22} = -\gamma_1, \quad T_{12} = \gamma_2, \quad T_{21} = \gamma_2. \quad (\text{A9})$$

In Eq. (A8), the indices $\mu, \nu = i, j$ act only on the spatial components. Here, we have to trace over all the matrix indices involved. Since there is either γ_1 or γ_2 in the expression for T_{ij} [see Eq. (A9)], the only nonvanishing contribution to the trace of

$$G_0(k) = \frac{k_0\gamma_0 - (k_1^2 - k_2^2)\gamma_1 - 2k_1k_2\gamma_2 + m}{k_0^2 - (k_1^2 + k_2^2) - m^2 - i\epsilon} \quad (\text{A10})$$

should also include γ_1 or γ_2 . However, these contributions have factors of $k_1^2 - k_2^2$ or k_1k_2 in the numerator and hence cancel out when we perform the after momentum integration. Thus we have

$$\Pi_{\mu\nu}^{(2)}(p) = 0. \quad (\text{A11})$$

The full one-loop polarization $\Pi_{\mu\nu}(p)$ is explicitly transverse. The resulting action $S[a_\mu]$ is gauge-invariant and is a sum of a parity-odd Chern-Simons term and a parity-even Maxwell term. The proof of gauge invariance is presented in Appendix C.

To obtain the leading coupling between nematic field and gauge field $S[a_\mu, \mathbf{M}]$, we need to calculate three-leg one-loop

diagrams shown in Figs. 2(a) and 2(b),

$$\begin{aligned} S[a_\mu, \mathbf{M}] \\ = \frac{N}{3} \int \frac{d^3p}{(2\pi)^3} \Pi_{\mu\nu,i}(p_1, p_2) a_\mu(-p_1 - p_2) a_\nu(p_1) M_i(p_2). \end{aligned} \quad (\text{A12})$$

There are two diagrams with nonvanishing value, so $\Pi_{\mu\nu,i}$ are composed of two parts, which are included in Figs. 2(a) and 2(b),

$$\Pi_{\mu\nu,i}(p_1, p_2) = \Pi_{\mu\nu,i}^{(1)}(p_1, p_2) + \Pi_{\mu\nu,i}^{(2)}(p_1), \quad (\text{A13})$$

where p_1 and p_2 are, respectively, the energy-momenta of the gauge field a_ν and of the nematic field M_i . Notice that $\Pi_{\mu\nu,i}^{(2)}(p_1)$ is only defined for $\mu, \nu = 1, 2$.

The one-loop three-legged diagram of Fig. 2(a) is

$$\begin{aligned} \Pi_{\mu\nu,i}^{(1)} = -i \int \frac{d^3k}{(2\pi)^3} \text{tr}[G_0(k-p_2)\mathcal{J}_\mu(2k-p_2+p_1) \\ \times G_0(k+p_1)\mathcal{J}_\nu(2k+p_1)G_0(k)\gamma_i]. \end{aligned} \quad (\text{A14})$$

Note that here the greek symbol index labels the gauge field, while the latin symbol index labels the nematic field. The latin symbols only run for spatial index. The one-loop diagram of

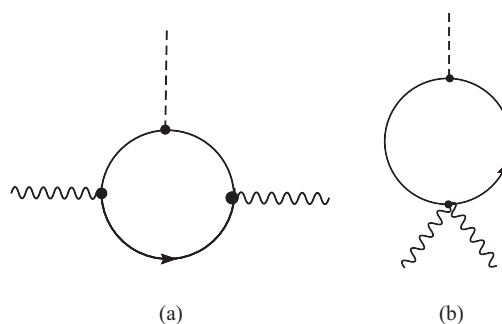


FIG. 2. One-loop contributions to the vertex of the gauge field a_μ and the nematic order parameter field \mathbf{M} .

Fig. 2(b) has the expression (for $\mu = j$ and $\nu = k$)

$$\Pi_{jk,i}^{(2)}(p_2) = i \int \frac{d^3k}{(2\pi)^3} \text{tr}[G_0(-p_2 + k)T_{jk}G_0(k)\gamma_i], \quad (\text{A15})$$

where T_{jk} is given in Eq. (A9).

Here, we expanded the momentum p_i by order and found that the leading coupling term is the interplay between the nematic field and Maxwell term, which is parity even. This is quite obvious. Since the gauge field enters quadratically in

$$\begin{aligned} \Pi_{\mu\nu,i}^{(1)}(p_1) &= -i \int \frac{d^3k}{(2\pi)^3} \text{tr}[G_0(k)\mathcal{J}_\mu(2k + p_1)G_0(k + p_1)\mathcal{J}_\nu(2k + p_1)G_0(k)\gamma_i] \\ &= -i \int \frac{d^3k}{(2\pi)^3} \text{tr}[\mathcal{J}_\mu(2k + p_1)G_0(k + p_1)\mathcal{J}_\nu(2k + p_1)G_0(k)\gamma_i G_0(k)]. \end{aligned} \quad (\text{A16})$$

If it is odd in p_1 , the first terms in the products

$$\mathcal{J}_\mu(2k + p_1)G_0(k + p_1)\mathcal{J}_\nu(2k + p_1)G_0(k) \quad (\text{A17})$$

being even and symmetric in the momentum k , should include a Levi-Civita tensor. In this sense, to obtain a nonvanishing value after trace, the $\gamma_i G_0(k)$ term should not contribute any Gamma matrix. As a result, it would involve with $k_1^2 - k_2^2$, which makes the whole polarization tensor vanish after integration.

Upon expanding in derivatives of the nematic field p_2 to the $O(p_2)$ order, we have

$$\begin{aligned} \Pi_{\mu\nu,i}^{(1)} &= -i \int \frac{d^3k}{(2\pi)^3} \text{tr}[G_0(k)\mathcal{J}_\mu(-p_2)G_0(k + p_1)\mathcal{J}_\nu(2k + p_1)G_0(k)\gamma_i] \\ &\quad - i \int \frac{d^3k}{(2\pi)^3} \text{tr} \left[\frac{F(p_2, k)}{k_0^2 - (k_1^2 + k_2^2)^2 - m^2 - i\epsilon} \mathcal{J}_\mu(2k + p_1)G_0(k + p_1)\mathcal{J}_\nu(2k + p_1)G_0(k)\gamma_i \right]. \end{aligned} \quad (\text{A18})$$

Here, $F(p_2, k)$ is a function that is linear in p_2 and odd in k .

If it is odd in p_1 , the second term of Eq. (A18),

$$\mathcal{J}_\mu(2k + p_1)G_0(k + p_1)\mathcal{J}_\nu(2k + p_1)G_0(k), \quad (\text{A19})$$

includes a Levi-Civita tensor and is even and symmetric in k . However, $F(p_2, k)$ is odd in k and the integral vanishes. For the first term of Eq. (A18), if $\mu = 0$, $\mathcal{J}_0(-p_2)$ does not depend on p_2 , this term is still of zeroth-order in p_2 , and vanishes as we showed before. Otherwise, if it is odd in p_1 , it is also odd in k and the integral vanishes. Thus, to lowest order, there is no parity-odd coupling between the nematic field \mathbf{M} and the gauge field a_μ .

APPENDIX B: THE CALCULATION OF THE EFFECTIVE NEMATIC ACTION

The only one-loop diagram that contributes is the self-energy of the effective field theory of the nematic order

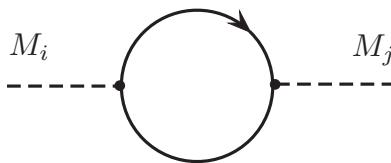


FIG. 3. One-loop self-energy diagram for the nematic order parameter field.

these diagrams, the leading gauge-invariant terms can only be the Chern-Simons term and Maxwell term. Since this theory is not Lorentz invariant, terms like $B\nabla \cdot E$ are allowed. We can ignore them as they are of higher order in derivatives than the Maxwell term. The Chern-Simons term is topological and as such it does not depend on the metric of the space-time. Thus the only most relevant coupling should be the Maxwell term. This can also be seen from the polarization tensor.

If we expand derivatives of nematic field p_2 in the polarization tensor by order, to the $O(1)$ order, we have

parameter is shown in Fig. 3 and it is given by the expression

$$\begin{aligned} S_{\text{eff}}(\mathbf{M}) &= -\frac{N}{2} \int d^3x d^3y \text{tr}[G_0(x - y)\boldsymbol{\gamma} \cdot \mathbf{M}(y)G_0(y - x)\boldsymbol{\gamma} \\ &\quad \cdot \mathbf{M}(x)] - \int d^3x \frac{N}{2g} \mathbf{M}^2(x) \\ &= -\frac{N}{2} \int \frac{d^3p}{(2\pi)^3} M_i(-p)\Gamma_{ij}(p)M_j(p) \\ &\quad - \frac{N}{2g} \int \frac{d^3p}{(2\pi)^3} |\mathbf{M}(p)|^2, \end{aligned} \quad (\text{B1})$$

where $\Gamma_{ij}(p)$ is the one-loop kernel

$$\begin{aligned} \Gamma_{ij}(p) &= \int \frac{d^3k}{(2\pi)^3} \text{tr}[\gamma_i G_0(p + k)\gamma_j G_0(k)] \\ &= \epsilon^{ij} p_0 \int \frac{d^3k}{(2\pi)^3} \frac{m}{(k_0^2 - (k_1^2 + k_2^2)^2 - m^2 - i\epsilon)^2} \\ &\quad + \delta_{ij} \int \frac{d^3k}{(2\pi)^3} \frac{m^2 - k_0^2}{(k_0^2 - (k_1^2 + k_2^2)^2 - m^2 - i\epsilon)^2} \\ &\quad + O(p^2). \end{aligned} \quad (\text{B2})$$

The first term is odd in the frequency p_0 and contributes to the Berry phase term. The second term, which is even in the

frequency p_0 , contributes to the mass term of the nematic order parameter field and thus contains the information of the

critical coupling constant for the quantum phase transition to the nematic phase.

APPENDIX C: SHORT PROOF ON GAUGE INVARIANCE OF THE POLARIZATION TENSOR

To verify the gauge invariance of the effective field theory, we sketch here a proof on the gauge invariance of the polarization tensor. For the one-loop gauge field self-energy diagrams shown in Figs. 1(a) and 1(b). For gauge invariance to hold, the polarization tensor should obey the transversality (conservation) condition

$$\Pi_{\mu\nu} p^\nu = 0. \tag{C1}$$

Since the theory we start with is not Lorentz-invariant, the polarization tensor here decomposes into two parts, one of which, called $\Pi_{\mu\nu}^{(1)}$, comes from the linear terms of the gauge field a_μ of the Lagrangian, while $\Pi_{\mu\nu}^{(2)}$ comes from the terms which are quadratic in this gauge field,

$$\Pi_{\mu\nu} = \Pi_{\mu\nu}^{(1)} + \Pi_{\mu\nu}^{(2)}. \tag{C2}$$

We already showed in Appendix A that the second piece vanishes, $\Pi_{\mu\nu}^{(2)} = 0$. Hence we only have to prove that

$$\Pi_{\mu\nu}^{(1)} p^\nu = 0. \tag{C3}$$

Explicitly, the left-hand side of this equation is equal to

$$\Pi_{\mu\nu} p^\nu = \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(p+k) \mathcal{J}_\mu(2k+p) G_0(k) \mathcal{J}_\nu(2k+p) p_\nu \right]. \tag{C4}$$

Using the following Ward identity (whose validity is elementary to check):

$$\mathcal{J}_\nu(2k+p) p^\nu = p_0 \gamma_0 + (p_1^2 - p_1^2 + 2p_1 k_1 - 2p_2 k_2) \gamma_1 + 2(p_1 p_2 + p_1 k_2 + p_2 k_1) \gamma_2 = G_0^{-1}(p+k) - G_0^{-1}(k), \tag{C5}$$

we can write Eq. (C4) in the form

$$\begin{aligned} \Pi_{\mu\nu} p^\nu &= \text{tr} \left\{ \int \frac{d^3k}{(2\pi)^3} G_0(p+k) \mathcal{J}_\mu(2k+p) G_0(k) [G_0^{-1}(p+k) - G_0^{-1}(k)] \right\} \\ &= \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(p+k) \mathcal{J}_\mu(2k+p) \right] - \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k) \mathcal{J}_\mu(2k+p) \right]. \end{aligned} \tag{C6}$$

For $\mu = 0$, $\mathcal{J}_0 = \gamma_0$, we find

$$\text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(p+k) \gamma_0 \right] - \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k) \gamma_0 \right] = 0, \tag{C7}$$

and for $\mu = 1$, we get

$$\begin{aligned} \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(p+k) \mathcal{J}_1(2k+p) \right] - \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k) \mathcal{J}_1(2k+p) \right] &= -\text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(p+k) (p_x \gamma_1 + p_y \gamma_2) \right] \\ &= -\text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k) (p_x \gamma_1 + p_y \gamma_2) \right] = 0. \end{aligned} \tag{C8}$$

Similarly, for $\mu = 2$, we also get

$$\begin{aligned} \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(p+k) \mathcal{J}_2(2k+p) \right] - \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k) \mathcal{J}_2(2k+p) \right] &= -\text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(p+k) (-p_y \gamma_1 + p_x \gamma_2) \right] \\ &= -\text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k) (-p_y \gamma_1 + p_x \gamma_2) \right] = 0. \end{aligned} \tag{C9}$$

Thus the polarization tensor of the gauge field a_μ , the one-loop diagram of Fig. 1(a), is transverse and, hence, the action of a_μ is gauge invariant.

We now turn to the gauge invariance of the coupling between the gauge field a_μ and the nematic order parameter field \mathbf{M} . The lowest-order contribution to this coupling in the $1/N$ expansion is given by the Feynman diagrams shown in Figs. 2(a) and 2(b). These diagrams contribute to the effective action in the form

$$S[a_\mu, \mathbf{M}] = \frac{N}{3} \int \frac{d^3p}{(2\pi)^3} \Pi_{\mu\nu i}(p_1, p_2) a_\mu(-p_1 - p_2) a_\nu(p_1) M_i(p_2). \tag{C10}$$

Invariance under a gauge transformation $a_\mu + \partial_\mu \theta$ requires that this new polarization tensor, $\Pi_{\mu\nu i}(p_1, p_2)$, should obey the following rule:

$$\begin{aligned} & \Pi_{\mu\nu, i}^1(p_1, p_2) p_1^\nu a_\mu(-p_1 - p_2) M_i(p_2) + \Pi_{\tau\sigma, j}^1(p_1, p_2) (-p_1^\tau - p_2^\tau) a_\sigma(p_1) M_j(p_2) \\ & + \Pi_{\alpha\beta, k}^2(p_1, p_2) (-p_1^\alpha - p_2^\alpha) T_{\alpha\beta} a_\beta(p_1) M_k(p_2) + \Pi_{\alpha\beta, k}^2(p_1, p_2) p_1^\beta T_{\alpha\beta} a_\alpha(-p_1 - p_2) M_k(p_2) = 0, \end{aligned} \quad (C11)$$

where $\Pi_{\mu\nu, i}^1(p_1, p_2)$ is given by

$$\begin{aligned} \Pi_{\mu\nu, i}^1(p_1, p_2) p_1^\nu &= \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_\mu G_0(k + p_1) \mathcal{J}_\nu p_\nu G_0(k) \gamma_i \right] \\ &= \text{tr} \left\{ \int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_\mu G_0(k + p_1) [G_0^{-1}(k + p_1) - G_0^{-1}(k)] G_0(k) \gamma_i \right\} \\ &= \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_\mu G_0(k) \gamma_i \right] - \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_\mu G_0(k + p_1) \gamma_i \right]. \end{aligned} \quad (C12)$$

For $\mu = 0$, $\mathcal{J}_0 = \gamma_0$, we get

$$\begin{aligned} \Pi_{\mu\nu, i}^1(p_1, p_2) p_1^\nu &= \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \gamma_0 G_0(k) \gamma_i \right] - \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \gamma_0 G_0(k + p_1) \gamma_i \right] \\ &= \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \gamma_0 G_0(k) \gamma_i \right] - \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \gamma_0 G_0(k + p_1) \gamma_i \right] \\ &= \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \gamma_0 G_0(k) \gamma_i \right] - \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2 - p_1) \gamma_0 G_0(k) \gamma_i \right]. \end{aligned} \quad (C13)$$

Likewise, for $\sigma = 0$, $\mathcal{J}_0 = \gamma_0$, we obtain

$$\begin{aligned} \Pi_{\tau\sigma, j}^1(p_1, p_2) (-p_1^\tau - p_2^\tau) &= \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_\tau p_\tau G_0(k + p_1) \mathcal{J}_\sigma G_0(k) \gamma_j \right] \\ &= \text{tr} \left\{ \int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) [G_0^{-1}(k - p_2) - G_0^{-1}(k + p_1)] G_0(k + p_1) \mathcal{J}_\sigma G_0(k) \gamma_j \right\} \\ &= \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k + p_1) \mathcal{J}_\nu G_0(k) \gamma_j \right] - \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_\sigma G_0(k) \gamma_j \right]. \end{aligned} \quad (C14)$$

Hence

$$\Pi_{\tau\sigma, j}^1(p_1, p_2) (-p_1^\tau - p_2^\tau) = \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k + p_1) \gamma_0 G_0(k) \gamma_j \right] - \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \gamma_0 G_0(k) \gamma_j \right]. \quad (C15)$$

It is easy to check that for each $\nu = \tau$, $\mu = \sigma = 0$,

$$\Pi_{\mu\nu, i}^1(p_1, p_2) p_1^\nu a_\mu(-p_1 - p_2) M_i(p_2) = -\Pi_{\tau\sigma, j}^1(p_1, p_2) (-p_1^\tau - p_2^\tau) a_\sigma(p_1) M_j(p_2). \quad (C16)$$

For $\mu = 1$, we get

$$\begin{aligned} \Pi_{\mu\nu, i}^1(p_1, p_2) p_1^\nu &= \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_1(2k - p_2 + p_1) G_0(k) \gamma_i \right] \\ &\quad - \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_1(2k - p_2 + p_1) G_0(k + p_1) \gamma_i \right] \\ &= \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_1(2k - p_2 + p_1) G_0(k) \gamma_i \right] \\ &\quad - \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2 - p_1) \mathcal{J}_1(2k - p_2 - p_1) G_0(k) \gamma_i \right] \\ &= \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_1(2k - p_2) G_0(k) \gamma_i \right] + \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2) \mathcal{J}_1(p_1) G_0(k) \gamma_i \right] \\ &\quad - \text{tr} \left[\int \frac{d^3 k}{(2\pi)^3} G_0(k - p_2 - p_1) \mathcal{J}_1(2k - p_2 - p_1) G_0(k) \gamma_i \right] \end{aligned} \quad (C17)$$

and for $\sigma = 1$,

$$\begin{aligned}
 \Pi_{\tau\sigma,j}^1(p_1, p_2)(-p_1^\tau - p_2^\tau) &= \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k+p_1) \mathcal{J}_1(2k+p_1) G_0(k) \gamma_j \right] - \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k-p_2) \mathcal{J}_1(2k+p_1) G_0(k) \gamma_j \right] \\
 &= \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k+p_1) \mathcal{J}_1(2k+p_1) G_0(k) \gamma_j \right] \\
 &\quad - \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k-p_2) \mathcal{J}_1(2k-p_2+p_2+p_1) G_0(k) \gamma_j \right] \\
 &= \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k+p_1) \mathcal{J}_1(2k+p_1) G_0(k) \gamma_j \right] - \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k-p_2) \mathcal{J}_1(2k-p_2) G_0(k) \gamma_j \right] \\
 &\quad + \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k-p_2) \mathcal{J}_1(-p_2-p_1) G_0(k) \gamma_j \right]. \tag{C18}
 \end{aligned}$$

After some algebra, it could be checked that the rest of the terms after a gauge transformation are

$$\begin{aligned}
 &\Pi_{1v,i}^1(p_1, p_2) p_1^v a_1(-p_1-p_2) M_i(p_2) + \Pi_{\tau 1,i}^1(p_1, p_2)(-p_1^\tau - p_2^\tau) a_1(p_1) M_i(p_2) \\
 &= 2 \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k-p_2) \mathcal{J}_1(-p_2-p_1) G_0(k) \gamma_i \right] a_1(p_1) M_i(p_2). \tag{C19}
 \end{aligned}$$

This contribution is canceled by the ‘‘tadpole+nematic’’ diagram of Fig. 2(b). Indeed, up to a gauge transformation, the extra terms generated in this diagram are

$$\begin{aligned}
 &\Pi_{\alpha 1,k}^2(p_1, p_2)(-p_1^\alpha - p_2^\alpha) T_{\alpha 1} a_1(p_1) M_k(p_2) + \Pi_{1\beta,k}^2(p_1, p_2) p_1^\beta T_{1\beta} a_1(-p_1-p_2) M_k(p_2) \\
 &= -2 \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k-p) (\gamma_1 p'_x + \gamma_2 p'_y) G_0(k) \gamma_k \right] a_1(-p-p') M_k(p) \\
 &= -2 \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k-p) \mathcal{J}_1(p') G_0(k) \gamma_k \right] a_1(-p-p') M_k(p), \tag{C20}
 \end{aligned}$$

which exactly cancels the offending terms.

In the case of the three-legged diagram, which defines the tensor $\Pi_{2v,i}^1(p_1, p_2)$, we also obtain the same condition for $\mu = \sigma = 2$. The remaining terms, after a gauge transformation, are

$$\begin{aligned}
 &\Pi_{2v,i}^1(p_1, p_2) p_1^v a_2(-p_1-p_2) M_i(p_2) + \Pi_{\tau 2,i}^1(p_1, p_2)(-p_1^\tau - p_2^\tau) a_2(p_1) M_i(p_2) \\
 &= 2 \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k-p_2) \mathcal{J}_2(-p_2-p_1) G_0(k) \gamma_i \right] a_2(p_1) M_i(p_2). \tag{C21}
 \end{aligned}$$

This contribution is canceled by the extra terms in the tadpole+nematic diagram, Fig. 2(b), after the gauge transformation

$$\begin{aligned}
 &-2 \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k-p) (\gamma_2 p'_x - \gamma_1 p'_y) G_0(k) \gamma_k \right] a_2(p') M_k(p) \\
 &= -2 \text{tr} \left[\int \frac{d^3k}{(2\pi)^3} G_0(k-p) \mathcal{J}_2(p') G_0(k) \gamma_k \right] a_2(-p-p') M_k(p). \tag{C22}
 \end{aligned}$$

So the polarization tensor is transverse and the action is gauge-invariant (as it should be).

¹K. von Klitzing, G. Dorda, and M. Pepper, *Phys. Rev. Lett.* **45**, 494 (1980).

²D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Phys. Rev. Lett.* **48**, 1559 (1982).

³R. B. Laughlin, *Phys. Rev. B* **23**, 5632 (1981).

⁴D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, *Phys. Rev. Lett.* **49**, 405 (1982).

⁵Q. Niu, D. J. Thouless, and Y.-S. Wu, *Phys. Rev. B* **31**, 3372 (1985).

⁶R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).

⁷F. D. M. Haldane, *Phys. Rev. Lett.* **51**, 605 (1983).

⁸B. I. Halperin, *Phys. Rev. Lett.* **52**, 1583 (1984).

⁹X. G. Wen and Q. Niu, *Phys. Rev. B* **41**, 9377 (1990).

¹⁰S. C. Zhang, T. H. Hansson, and S. Kivelson, *Phys. Rev. Lett.* **62**, 82 (1989).

¹¹A. López and E. Fradkin, *Phys. Rev. B* **44**, 5246 (1991).

¹²J. Fröhlich and T. Kerler, *Nucl. Phys. B* **354**, 369 (1991).

- ¹³X.-G. Wen and A. Zee, *Phys. Rev. B* **46**, 2290 (1992).
- ¹⁴X. G. Wen, *Adv. Phys.* **44**, 405 (1995).
- ¹⁵H. C. Jiang, H. Yao, and L. Balents, *Phys. Rev. B* **86**, 024424 (2012).
- ¹⁶D. S. Rokhsar and S. A. Kivelson, *Phys. Rev. Lett.* **61**, 2376 (1988).
- ¹⁷R. Moessner and S. L. Sondhi, *Phys. Rev. Lett.* **86**, 1881 (2001).
- ¹⁸B. A. Bernevig, T. L. Hughes, and S. C. Zhang, *Science* **314**, 1757 (2006).
- ¹⁹M. König, S. Steffen, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X. L. Qi, and S. C. Zhang, *Science* **318**, 766 (2007).
- ²⁰L. Fu, C. L. Kane, and E. J. Mele, *Phys. Rev. Lett.* **98**, 106803 (2007).
- ²¹M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).
- ²²M. Z. Hasan and J. E. Moore, *Annu. Rev. Condens. Matter Phys.* **2**, 55 (2011).
- ²³F. D. M. Haldane, *Phys. Rev. Lett.* **61**, 2015 (1988).
- ²⁴T. Neupert, L. Santos, C. Chamon, and C. Mudry, *Phys. Rev. Lett.* **106**, 236804 (2011).
- ²⁵D. N. Sheng, Z. C. Gu, K. Sun, and L. Sheng, *Nat. Commun.* **2**, 389 (2011).
- ²⁶N. Regnault and B. A. Bernevig, *Phys. Rev. X* **1**, 021014 (2011).
- ²⁷L. Cincio and G. Vidal, *Phys. Rev. Lett.* **110**, 067208 (2013).
- ²⁸S. L. Sondhi, A. Karlhede, S. A. Kivelson, and E. H. Rezayi, *Phys. Rev. B* **47**, 16419 (1993).
- ²⁹T.-L. Ho, *Phys. Rev. Lett.* **73**, 874 (1994).
- ³⁰Y. P. Shkolnikov, S. Misra, N. C. Bishop, E. P. De Poortere, and M. Shayegan, *Phys. Rev. Lett.* **95**, 066809 (2005).
- ³¹D. A. Abanin, S. A. Parameswaran, S. A. Kivelson, and S. L. Sondhi, *Phys. Rev. B* **82**, 035428 (2010).
- ³²M. P. Lilly, K. B. Cooper, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, *Phys. Rev. Lett.* **82**, 394 (1999).
- ³³W. Pan, R. R. Du, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, *Phys. Rev. Lett.* **83**, 820 (1999).
- ³⁴K. B. Cooper, M. P. Lilly, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, *Phys. Rev. B* **65**, 241313 (2002).
- ³⁵E. Fradkin and S. A. Kivelson, *Phys. Rev. B* **59**, 8065 (1999).
- ³⁶E. Fradkin, S. A. Kivelson, E. Manousakis, and K. Nho, *Phys. Rev. Lett.* **84**, 1982 (2000).
- ³⁷J. Xia, V. Cvicek, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, *Phys. Rev. Lett.* **105**, 176807 (2010).
- ³⁸J. Xia, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, *Nat. Phys.* **7**, 845 (2011).
- ³⁹M. Mulligan, C. Nayak, and S. Kachru, *Phys. Rev. B* **82**, 085102 (2010).
- ⁴⁰M. Mulligan, C. Nayak, and S. Kachru, *Phys. Rev. B* **84**, 195124 (2011).
- ⁴¹L. Balents, *Europhys. Lett.* **33**, 291 (1996).
- ⁴²K. Musaelian and R. Joynt, *J. Phys.: Condens. Matter* **8**, L105 (1996).
- ⁴³F. D. M. Haldane, *Phys. Rev. Lett.* **107**, 116801 (2011).
- ⁴⁴B. Yang, Z. Papić, E. H. Rezayi, R. N. Bhatt, and F. D. M. Haldane, *Phys. Rev. B* **85**, 165318 (2012).
- ⁴⁵R.-Z. Qiu, F. D. M. Haldane, X. Wan, K. Yang, and S. Yi, *Phys. Rev. B* **85**, 115308 (2012).
- ⁴⁶J. Maciejko, B. Hsu, S. A. Kivelson, Y. J. Park, and S. L. Sondhi, *Phys. Rev. B* **88**, 125137 (2013).
- ⁴⁷D. G. Barci and E. Fradkin, *Phys. Rev. B* **83**, 100509 (2011).
- ⁴⁸J. E. Avron, R. Seiler, and P. G. Zograf, *Phys. Rev. Lett.* **75**, 697 (1995).
- ⁴⁹N. Read, *Phys. Rev. B* **79**, 045308 (2009).
- ⁵⁰T. L. Hughes, R. G. Leigh, and E. Fradkin, *Phys. Rev. Lett.* **107**, 075502 (2011).
- ⁵¹A. Nicosolis and D. T. Son, [arXiv:1103.2137](https://arxiv.org/abs/1103.2137).
- ⁵²C. Hoyos and D. T. Son, *Phys. Rev. Lett.* **108**, 066805 (2012).
- ⁵³K. Sun, H. Yao, E. Fradkin, and S. A. Kivelson, *Phys. Rev. Lett.* **103**, 046811 (2009).
- ⁵⁴Y. You and E. Fradkin (in preparation).
- ⁵⁵E. Ardonne, P. Fendley, and E. Fradkin, *Ann. Phys. (NY)* **310**, 493 (2004).
- ⁵⁶S. Raghu, X.-L. Qi, C. Honerkamp, and S.-C. Zhang, *Phys. Rev. Lett.* **100**, 156401 (2008).
- ⁵⁷R. Nandkishore and L. Levitov, *Phys. Rev. B* **82**, 115124 (2010).
- ⁵⁸O. Vafek and K. Yang, *Phys. Rev. B* **81**, 041401(R) (2010).
- ⁵⁹Y. Lemonik, I. Aleiner, and V. I. Fal'ko, *Phys. Rev. B* **85**, 245451 (2012).
- ⁶⁰L. Fu, *Phys. Rev. Lett.* **106**, 106802 (2011).
- ⁶¹C. Fang, M. J. Gilbert, S.-Y. Xu, B. A. Bernevig, and M. Z. Hasan, *Phys. Rev. B* **88**, 125141 (2013).
- ⁶²K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, *Phys. Rev. Lett.* **106**, 236803 (2011).
- ⁶³W.-F. Tsai, C. Fang, H. Yao, and J. Hu, [arXiv:1112.5789](https://arxiv.org/abs/1112.5789).
- ⁶⁴O. Vafek, *Phys. Rev. B* **82**, 205106 (2010).
- ⁶⁵K. G. Wilson, *Phys. Rev. D* **2**, 1478 (1970).
- ⁶⁶A. A. Abrikosov, *Z. Eksp. Theor. Fiz.* **66**, 1443 (1974) [*Sov. Phys. JETP* **39**, 709 (1974)].
- ⁶⁷S.-Y. Xu, C. Liu, N. Alidoust, M. Neupane, D. Qian, Belopolski, J. D. Denlinger, Y. J. Wang, H. Lin, L. A. Wray *et al.*, *Nature Commun.* **3**, 1192 (2012).
- ⁶⁸A. Gyenis, I. K. Drozdov, S. Nadj-Perge, O. B. Jeong, J. Seo, I. Pletikosić, T. Valla, G. D. Gu, and A. Yazdani, *Phys. Rev. B* **88**, 125414 (2013).
- ⁶⁹B.-J. Yang and Y. B. Kim, *Phys. Rev. B* **82**, 085111 (2010).
- ⁷⁰W. Witczak-Krempa and Y. B. Kim, *Phys. Rev. B* **85**, 045124 (2012).
- ⁷¹W. Witczak-Krempa, G. Chen, Y. B. Kim, and L. Balents, [arXiv:1305.2193](https://arxiv.org/abs/1305.2193).
- ⁷²A. M. Chan, T. L. Hughes, S. Ryu, and E. Fradkin, *Phys. Rev. B* **87**, 085132 (2013).
- ⁷³E. Fradkin and F. A. Schaposnik, *Phys. Lett. B* **338**, 253 (1994).
- ⁷⁴J. C. Le Guillou, E. Moreno, F. A. Schaposnik, and C. Núñez, *Phys. Lett. B* **409**, 257 (1997).
- ⁷⁵C. P. Burgess, C. A. Lütken, and F. Quevedo, *Phys. Lett. B* **336**, 18 (1994).
- ⁷⁶G. Y. Cho and J. E. Moore, *Ann. Phys.* **326**, 1515 (2011).
- ⁷⁷J. Fröhlich and A. Zee, *Nucl. Phys. B* **364**, 517 (1991).
- ⁷⁸M. Moshe and J. Zinn-Justin, *Phys. Rept.* **385**, 69 (2003).
- ⁷⁹V. Oganesyan, S. A. Kivelson, and E. Fradkin, *Phys. Rev. B* **64**, 195109 (2001).
- ⁸⁰P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge University Press, Cambridge, UK, 1995).
- ⁸¹S. Deser, R. Jackiw, and S. Templeton, *Phys. Rev. Lett.* **48**, 975 (1982).
- ⁸²A. N. Redlich, *Phys. Rev. D* **29**, 2366 (1984).
- ⁸³P. G. de Gennes and J. Prost, *The Physics of Liquid Crystals* (Oxford Sci./Clarendon, Oxford, UK, 1993).
- ⁸⁴T. L. Hughes, R. G. Leigh, and O. Parrikar, *Phys. Rev. D* **88**, 025040 (2013).

- ⁸⁵T. H. Hsieh, H. L., J. W. Liu, W. H. Duan, A. Bansil, and L. Fu, [Nat. Commun. **3**, 982 \(2012\)](#).
- ⁸⁶P. Dziawa, B. J. Kowalski, K. Dybko, R. Buczko, A. Szczerbakow, M. Szot, E. Łusakowska, T. Balasubramanian, B. M. Wojek, M. H. Berntsen *et al.*, [Nat. Mater. **11**, 1023 \(2012\)](#).
- ⁸⁷Y. Okada, M. Serbyn, H. Lin, D. Walkup, W. W. Zhou, C. Dhital, M. Neupane, S. Y. Xu, Y. J. Wang, R. Sankar *et al.*, [Science **341**, 1496 \(2013\)](#).
- ⁸⁸X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, [Phys. Rev. B **83**, 205101 \(2011\)](#).