Theory of defects in Abelian topological states

Maissam Barkeshli, Chao-Ming Jian, and Xiao-Liang Qi

Department of Physics, Stanford University, Stanford, California 94305, USA

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The structure of extrinsic defects in topologically ordered states of matter is host to a rich set of universal physics. Extrinsic defects in $2 + 1$-dimensional topological states include linelike defects, such as boundaries between topologically distinct states, and pointlike defects, such as junctions between different line defects. Gapped boundaries in particular can themselves be *topologically* distinct, and the junctions between them can localize topologically protected zero modes, giving rise to topological ground-state degeneracies and projective non-Abelian statistics. In this paper, we develop a general theory of point defects and gapped line defects in $2 + 1$-dimensional Abelian topological states. We derive a classification of topologically distinct gapped boundaries in terms of certain maximal subgroups of quasiparticles with mutually bosonic statistics, called Lagrangian subgroups. The junctions between different gapped boundaries provide a general classification of point defects in topological states, including as a special case the twist defects considered in previous works. We derive a general formula for the quantum dimension of these point defects and a general understanding of their localized "parafermion" zero modes and we define a notion of projective non-Abelian statistics for them. The critical phenomena between topologically distinct gapped boundaries can be understood in terms of a general class of quantum spin chains or, equivalently, "generalized parafermion" chains. This provides a way of realizing exotic $1 + 1$D generalized parafermion conformal field theories in condensed-matter systems.

I. INTRODUCTION

One of the most fundamental discoveries in condensed-matter physics has been the understanding of topologically ordered states of matter.[1,2] Topologically ordered states are gapped many-body states that possess quasiparticle excitations with fractional statistics and fractional charges, topology-dependent ground-state degeneracies, different patterns of long-range entanglement,[3,4] and many other exotic characteristics. The most common topological states seen experimentally are the fractional quantum Hall (FQH) states. There is also increasing support from numerical simulations and experiments[5–12] that topologically ordered states are found in frustrated magnets.

Recently, a new direction in the study of topologically ordered states, called twist defects, or extrinsic defects, has attracted increasing research interest.[13–32] An extrinsic defect is a pointlike or linelike defect either in a topological state or on the interface between two topologically distinct states, which leads to new topological properties that are absent in the topological state without the defect. A crucial distinction between extrinsic defects and the more familiar quasiparticle excitations is that the former are not deconfined excitations of the system; rather, the energy cost for separating point defects, for instance, will generally depend either logarithmically or linearly on their separation.

A simple example of an extrinsic defect is the "genon" defined in Refs. 13 and 14. As shown in Fig. 1, a branch-cut line is introduced in a bilayer topological state, across which the two layers are exchanged.[20] The genon in this case is defined as the branch-cut point where the branch-cut line ends. From its definition one can see that a bilayer system with genons is topologically equivalent to a single-layer system on a Riemann surface.[13,20] Using such a mapping, the topological properties of genons such as quantum dimension and (projective) braiding statistics can be studied

systematically.[14] Even when the topological state in each layer is an Abelian theory, the genons have non-Abelian statistics. This has led to a recent experimental proposal for synthesizing a wide variety of possible topological qubits using the simplest Abelian bilayer FQH states.[21] It has also been shown that the braiding statistics of genons can allow for universal topological quantum computation (TQC) even in cases where the host topological state without the genons is not by itself universal for TQC.[14]

Extrinsic defects with the same type of non-Abelian statistics as some of the genons studied in Refs. 13 and 21 have also been proposed in other physical systems. These include lattice defects in certain exactly solvable Z_N rotor models,[19,22] FQH states in proximity with superconductivity (SC), and fractional quantum spin Hall (FQSH) states in proximity with SC and ferromagnetism (FM).[15–17] The latter FQSH proposals are generalizations of earlier proposals of realizing Majorana zero modes on the boundary of the quantum spin Hall insulator.[33] In these FQSH realizations, the extrinsic defect is a point on the boundary of the system where the boundary condition changes, while the rest of the boundary is in a gapped state. For Abelian states, the extrinsic defects realized in the superconducting proximity proposals reviewed above can be mapped to genons in suitable bilayer Abelian states.[14]

In this paper, we develop a general theory of extrinsic defects in Abelian topological states, which generalize the extrinsic defects reviewed above to the most generic possible form. For two-dimensional topological states, there are two general forms of extrinsic defects. These are line defects, which separate two different or identical topological states, and point defects, which may live in a topological state (such as genons) or live on line defects (such as the FM/SC domain wall on the FQSH edge). We demonstrate that all extrinsic defects can be mapped to *boundary defects*, i.e., boundary lines of topological states with point defects separating different
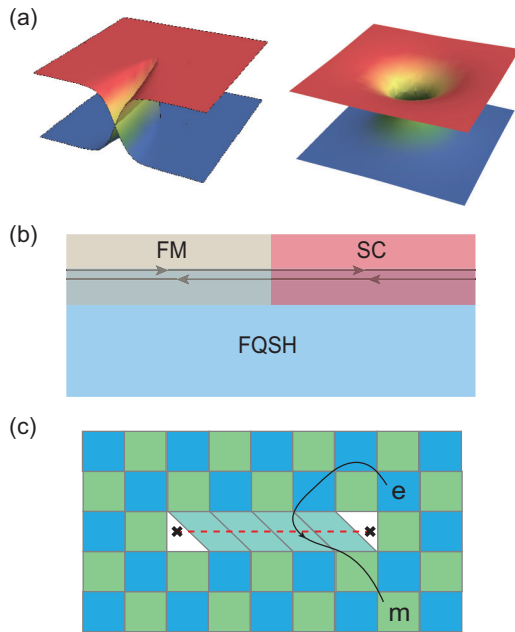
FIG. 1. (Color online) Examples of point defects studied previously in the literature. (a) Genons in bilayer systems.[13,14] (b) Domain walls between ferromagnetic and superconducting backscattering at the edge of a fractional quantum spin Hall (FQSH) state.[15–17] (c) Lattice dislocations in solvable models of Z_N topological order.[18,19]

boundary regions. Generically, the boundary lines may be gapped or gapless. The gapped cases are of interest for us since they support point defects that have nontrivial topological properties.

We prove a classification of topologically distinct line defects of general Abelian topological states, extending the results of previous works.[25,28–32,34] In particular, it was proven recently in Ref. 28 that a gapped boundary in an Abelian state is determined by a "Lagrangian subgroup" which consists of certain maximal subsets of topological quasiparticles that have trivial self and mutual statistics. In this paper, we prove that *every* such Lagrangian subgroup corresponds to a topologically distinct gapped edge and that Lagrangian subgroups therefore provide a classification of topologically distinct gapped edges. Assuming that the notion of topological boundary conditions studied in Ref. 32 is equivalent to gapped boundaries of local Hamiltonians, this proves the classification that was conjectured in Ref. 32. Our proof is constructive, in the sense that given an arbitrary Lagrangian subgroup M, we show explicitly which *local* operators to add to the edge theory to gap the edge in a way that corresponds to M.

We further show that the nontrivial point defects on the boundary are then classified by the domain wall between gapped edges corresponding to different Lagrangian subgroups. We compute the quantum dimension for generic point defects, which demonstrates that the point defects are non-Abelian, and we develop an understanding of the zero modes that are topologically localized to the point defects. These "generalized parafermion" zero modes vastly generalize the well-known Majorana fermion zero modes that are currently under intense theoretical and experimental investigation.[35] Although it is generally not possible to geometrically braid

the point defects that live on the boundary, we demonstrate that effective "braiding" operations can be realized in general by quasiparticle tunneling processes between pairs of defects. Such braiding operations are topologically robust unitary transformations of the topologically degenerate states. We show that they can always be mapped to the braiding of genons in a bilayer system.

We also studied the quantum phase transitions between different gapped boundary states realized on the same line defect. Interestingly, the transition between two different types of boundary states M and M', corresponding to two different Lagrangian subgroups, can be realized by nucleation of a periodic array of M' regions in M. The domain walls between M and M' regions define a periodic array of lattice defects, each of which supports non-Abelian zero modes. This approach allows us to describe the quantum phase transition by a quantum spin chain that characterizes the coupling between the topological zero modes. Alternatively, the spin chain can be formulated as a "generalized parafermion chain," and we expect that their phase transitions may be described by generalized parafermion conformal field theories.

We would like to further clarify the relation of our work with some previous works in the literature. Gapped edges have been considered in recent years in several works. Reference 30 constructed a set of gapped edges for quantum double models, which are microscopic models of topologically ordered states described at low energies by an emergent discrete gauge theory. These are restricted to time-reversal invariant bosonic systems. It is not clear whether that construction provides a complete classification of all possible gapped edges for those models. Subsequently, Ref. 25 developed a systematic microscopic analysis of gapped edges for a class of exactly soluble bosonic lattice models—the Levin-Wen models[36]—which pertain to both Abelian and non-Abelian states of time-reversal and parity symmetric bosonic systems. Reference 32 studied "topological boundary conditions" of Abelian Chern-Simons theory for bosonic systems and conjectured that they are classified by Lagrangian subgroups. As pointed out in Ref. 28, it is not clear whether topological boundary conditions are equivalent to gapped boundaries of local Hamiltonians; Ref. 28 further proved that the existence of a Lagrangian subgroup is a necessary and sufficient condition for when an Abelian topological phase of bosons or fermions, realized by a local Hamiltonian, can possibly admit a gapped edge. Reference 27 also studied topological boundary conditions in topological quantum field theories (TQFTs) from a mathematical point of view, utilizing the framework of category theory. However, we would like to emphasize that it is not clear whether the topological boundary conditions of TQFTs that are classified in Ref. 27 are equivalent to gapped edges of local Hamiltonians, which are the focus of this paper and of Ref. 28.

We note that a portion of the results discussed in this paper were also reported by us in a recent shorter treatment.[37]

The rest of the paper is organized as follows. In Sec. II, we briefly review the formalism for characterizing topological order and the Abelian Chern-Simons theory framework for characterizing all Abelian topological states. In Sec. III, we introduce in more detail the notion of extrinsic line and point defects in topological states and discuss the mapping of all such defects to boundary defects of topological states. In

Sec. IV, we discuss gapped boundary defects of topological states, and prove the classification of line defects in terms of Lagrangian subgroups referred to above. In Sec. V, we study the topological properties of point defects as domain walls between topologically distinct gapped edges. We derive a general formula for their quantum dimension and a general understanding of the localized "parafermion" zero modes on the domain walls, and we discuss their non-Abelian braiding statistics. In Sec. VI, we discuss the critical phenomena between topologically distinct gapped edges; we show that this can be mapped onto the physics of a generalized quantum spin chain, or, equivalently, a "generalized parafermion" chain. We conclude with a discussion in Sec. VII.

II. CHARACTERIZATION OF TOPOLOGICAL ORDER

General topologically ordered states in $2 + 1$ dimensions are characterized by the topological properties of a set of topologically nontrivial quasiparticle excitations, $\{\gamma_i\}$, for $i = 1, \ldots, N_{qp}$, where N_{qp} is the number of quasiparticles. When two quasiparticles are observed from far away, they in general behave like a superposition of single quasiparticle states. This is described by the fusion rules $\gamma_i \times \gamma_j = \sum_k N_{ij}^k \gamma_k$. Second, when two quasiparticles γ_i, γ_j wind around each other, a phase $e^{i\theta_{ij}^k}$ is obtained, which depends on the fusion channel k. θ_{ij}^k is referred to as the braid statistics of the quasiparticles. When a particle is spun around itself by 2π, it generically gains a nontrivial phase $e^{i\theta_i}$. $\theta_i = 0$ for bosons and π for fermions. For a topological phase where the microscopic degrees of freedom are all bosons, θ_i is topologically well-defined modulo 2π. In contrast, when the microscopic degrees of freedom also contain fermions, then θ_i is topologically well defined only modulo π. The braiding, fusion rules, and spins must satisfy some consistency conditions, which we do not review here.[38–40]

A topological state is "Abelian" when quasiparticles at fixed locations do not induce additional topological ground-state degeneracies. This is equivalent to the condition that $N_{ij}^k = 1$ for only one value of k, and $N_{ij}^k = 0$ otherwise.

A systematic description of all Abelian topological states is given by Abelian Chern-Simons (CS) theory,[1,41] described by the Lagrangian density

$$\mathcal{L}_{\text{CS}} = \frac{1}{4\pi} K_{IJ} \epsilon^{\mu\nu\lambda} a_\mu^I \partial_\nu a_\lambda^J, \tag{1}$$

where a^I for $I = 1, \ldots, \text{rank}(K)$ are compact U(1) gauge fields; K is a nonsingular, integer symmetric matrix; and μ, ν, λ are $2 + 1$-dimensional space-time indices. The topologically nontrivial quasiparticles are described by integer vectors $\boldsymbol{l}$, where two integer vectors $\boldsymbol{l}$ and $\boldsymbol{l}'$ describe topologically equivalent quasiparticles if $\boldsymbol{l}' = \boldsymbol{l} + K\boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is an integer vector. Therefore, the integer lattice in $\text{rank}(K)$ dimensions, modulo this equivalence relation, defines a discrete group consisting of the quasiparticles, with the number of topologically distinct quasiparticles given by $|\text{Det} K|$. The exchange statistics of a quasiparticle labeled by $\boldsymbol{l}$ is given by $\theta_{\boldsymbol{l}} = \pi \boldsymbol{l}^T K^{-1} \boldsymbol{l}$, and the mutual statistics of two quasiparticles $\boldsymbol{l}$, $\boldsymbol{l}'$ is $\theta_{\boldsymbol{l}\boldsymbol{l}'} = 2\pi \boldsymbol{l}^T K^{-1} \boldsymbol{l}'$. $\theta_{\boldsymbol{l}\boldsymbol{l}'}$ is defined modulo 2π, while $\theta_{\boldsymbol{l}}$ is defined modulo 2π for a topological phase of bosons, and modulo π if the microscopic Hamiltonian includes fermions. Vectors $K\boldsymbol{\Lambda}$

describe local particles, which are always bosons or fermions. If all diagonal elements of K are even integers (referred to as K being even), then all local particles are bosons, and the theory describes a topological phase of bosons; otherwise, we say K is odd, and the microscopic degrees of freedom must contain fermions (possibly in addition to bosons).

Different K matrices can specify equivalent topological states if they have the same quasiparticle content. For example, the transformation $K \to W^T K W$, for W an integer matrix with $|\text{Det} W| = 1$, yields a different K matrix, but describing the same topological order. Alternatively, consider extending the K matrix as

$$K' = \begin{pmatrix} K & 0 \\ 0 & K_0 \end{pmatrix}, \tag{2}$$

where K_0 is an even-dimensional matrix with unit determinant and zero signature (equal number of positive and negative eigenvalues). Since $|\text{Det} K_0| = 1$, extending K to K' in this way does not add any additional topological quasiparticles. Therefore, K' and K also describe the same topological order, as the group of quasiparticles and their statistics is the same.

Equation (1) possesses gapless edge states described by a $1 + 1$D chiral Luttinger liquid theory,[1]

$$\mathcal{L}_{\text{edge}} = \frac{K_{IJ}}{4\pi} \partial_x \phi_I \partial_t \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J, \tag{3}$$

where V_{IJ} is a positive-definite "velocity" matrix. The numbers of left- and right-moving bosons, n_L and n_R, respectively, are set by the number of positive and negative eigenvalues of K, respectively. The electron operators Ψ_I and quasiparticle operators χ_I on the edge are given by

$$\Psi_I = e^{i K_{IJ} \phi_J}, \quad \chi_I = e^{i \boldsymbol{l}^T \phi}, \tag{4}$$

where $\boldsymbol{l}$ is an integer vector describing the quasiparticles. When Ψ_I has integer scaling dimension, the "electron" is a boson, and if it is half integer, it is a fermion.

The Lagrangian (3) is gapless. When an Abelian topological state admits a gapped edge, it can be obtained by adding additional backscattering terms in Eq. (3) to generate an energy gap in the edge theory.

III. LINE AND POINT DEFECTS

A general line defect in a topological state is a one-dimensional boundary between two topological states, A_1 and A_2 (see Fig. 2). In some cases, such as when A_1 and A_2 have gapless edge states with differing chiral central charges,
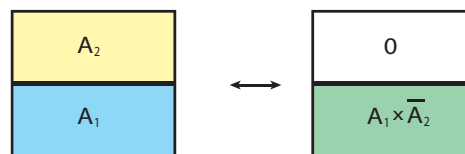

FIG. 2. (Color online) A line defect can be considered to be a domain wall between two kinds of topological phases, A_1 and A_2. By folding one side over onto the other, this can be mapped to an edge between $A_1 \times \bar{A}_2$ and the trivial gapped state, which we label "0". Under general conditions, the line defect will either host topologically protected gapless edge states or be fully gapped.

the boundary possesses topologically protected gapless edge states. In other cases, assuming certain criteria[28,42] that we review below are met, it is possible for the boundary to be gapped. In this paper, we only consider line defects that correspond to gapped boundaries.

The topological phases A_1 and A_2 do not necessarily have to be distinct: If $A_1 = A_2 = A$, there can still be many different kinds of line defects. These correspond to situations where quasiparticles are permuted among themselves as they cross the boundary, in a way which preserves their topological quantum numbers.[14,25,30] Such line defects are "invisible," in the sense that braiding and fusion of quasiparticles on either side of the line defect yield the same results.

In order to understand the properties of general boundaries, it is helpful to apply a folding process, which has been employed previously in Refs. 30 and 32, and which we review here. In order to understand the boundary between A_1 and A_2, for concreteness we can consider A_1 and A_2 on a sphere or plane and then fold back A_2 onto A_1 to obtain a boundary between the topological phase $A_1 \times \bar{A}_2$ and the topologically trivial gapped phase, which we label "0". $\bar{A}_2$ denotes the parity-reversed copy of A_2, which is necessary since the folding operation changes the parity of the state that is being folded because one of the directions is being reversed. Therefore, to study line defects, it suffices to consider all possible boundaries between generic topological phases and the trivial gapped phase.

Given the possibility of different kinds of gapped edges between topological phases, it is also possible to have domain walls and junctions between them (see Fig. 3). These pointlike defects can localize exotic topological zero modes, giving rise to topological ground-state degeneracies, and projective non-Abelian statistics. In the special case where the line defects separate the same topological phase on either side, the point defects are "twist defects":[14] As a quasiparticle encircles the defect, it gets permuted by a symmetry of the topological quantum numbers. Various examples of this have been studied previously in the literature.[13,15–19,22] In some of these physical realizations, such as those in Ref. 13, it is possible that the ground-state energy of the system in the presence of the defects depends logarithmically on the separation between them, as
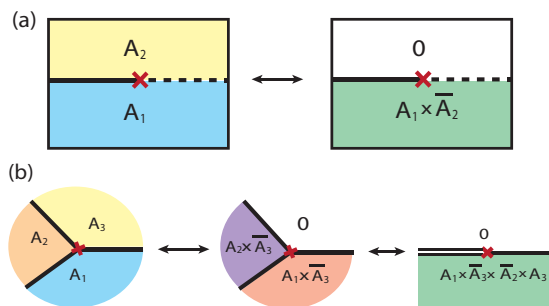

FIG. 3. (Color online) (a) A domain wall between two different kinds of gapped edges separating topological phases A_1 and A_2. By folding A_2 over, this can be mapped to a domain wall on the boundary separating $A_1 \times \bar{A}_2$ and the trivial gapped state, "0". (b) A junction where multiple gapped edges meet is also a possible type of point defect. On an infinite plane, by applying the folding trick multiple times, this can also be mapped to a domain wall on the boundary separating a topological phase and 0.

opposed to the more general linear energy cost for separating generic point defects.

Using the folding process, we can also understand point defects by mapping them to domain walls between different gapped edges separating a topological phase and 0 (see Fig. 3). In the case of junctions where multiple edges meet [Fig. 3(b)], then we may apply the folding, in conjunction with deformations of the location of the edges, several times in order to map the original configuration onto a domain wall between different gapped edges separating a topological phase and 0. We note that this folding does not directly apply if the whole system is on a spatial manifold of arbitrary topology or for arbitrary configurations of point defects, but it is useful to understand the topological behavior of each point defect by considering it in isolation on a plane and then applying the folding process.

Therefore, in what follows we need only focus on gapped boundaries between a generic Abelian topological phase and the trivial phase 0 and domain walls between different such gapped boundaries.

IV. CLASSIFICATION OF LINE DEFECTS

A. Review of null vectors and Lagrangian subgroups

As a starting point of our discussion, we review the concepts of null vectors and Lagrangian subgroups discussed in Refs. 28 and 42, which are the basic tools to obtain our new results. We consider a generic Abelian topological phase, characterized by an Abelian CS theory with generic K matrix. The edge theory is described by (3). Backscattering terms can be added on the edge, with the restriction that they be *local* operators on the edge. For systems involving fermions as the microscopic local degrees of freedom, the backscattering terms must also conserve fermion number modulo 2, known as the *fermion parity* symmetry. Therefore, the allowed backscattering terms on the edge are of the form

$$\delta H_b = \sum_i \alpha_i(x) \cos\left[\boldsymbol{\Lambda}_i^T K \phi + \theta_i(x) \right], \tag{5}$$

where $\boldsymbol{\Lambda}_i$ are integer vectors, $\boldsymbol{\Lambda}_i^T K \boldsymbol{\Lambda}_i$ is even to ensure that the cosine terms are bosonic operators (i.e., have integer scaling dimension), and $\alpha_i(x)$ and $\theta_i(x)$ are spatially varying functions.[43] When the number of left and right movers are unequal, $n_L \neq n_R$, the edge states cannot be fully gapped. When $n_L = n_R = N$, i.e., there are an equal number of counterpropagating modes, it is possible but not guaranteed that the edge be fully gapped, *even in the absence of any symmetry.* In fact, it has been shown that (3) can be fully gapped if and only if there exist N linearly independent vectors $\{\boldsymbol{\Lambda}_i\}$ satisfying[28,42]

$$\boldsymbol{\Lambda}_i^T K \boldsymbol{\Lambda}_j = 0. \tag{6}$$

This is a highly nontrivial constraint; for example, as discussed in Ref. 28, the $\nu = 2/3$ FQH edge, described by the K matrix $K = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$ does not admit such null vectors and therefore the edge cannot be gapped, *even when particle number conservation is broken*. On the other hand, the edge of the $\nu = 8/9$ FQH state described by $K = \begin{pmatrix} 1 & 0 \\ 0 & -9 \end{pmatrix}$ can be gapped if particle number conservation is broken.

The fact that such a set of null vectors $\{\boldsymbol{\Lambda}_i\}$ can cause an energy gap on the edge can be seen as follows. We perform a transformation $\phi = W\phi'$, such that ϕ' has the commutation relations of a usual N-channel Luttinger liquid, $[\phi_i'(x),\phi_j'(y)] = \pm\delta_{ij}i\pi\,\mathrm{sgn}(x-y)$; ϕ_i' for $i = 1,\ldots,N$ can be chosen to be the left movers, while for $i = N+1,\ldots 2N$ they are the right movers. Under this condition, $\cos(\boldsymbol{\Lambda}_i^T K\phi)$ becomes a conventional backscattering term $\cos(\phi_i' \pm \phi_{i+N}')$. It follows that the ground states can be characterized by the classical minima of the cosine terms: $\boldsymbol{\Lambda}_i^T K\phi = 2\pi n_i$, for $n_i \in \mathbb{Z}$.

The gapped boundary induced by the backscattering terms (5) can be understood as a one-dimensional "condensate" of certain topological particles. The different components of the integer-valued vector $K\boldsymbol{\Lambda}_i$ may have common factors. Denote $K\boldsymbol{\Lambda}_i = c_i\boldsymbol{m}_i$ with $c_i \in \mathbb{Z}$ and $\boldsymbol{m}_i$ the minimal integer vector with no common factor in its components. Since $\boldsymbol{\Lambda}_i^T K\phi = 2\pi n_i$ obtains a classical value on the edge, so does the quasiparticle operator $e^{i\boldsymbol{m}_i^T \phi}$, which satisfies $\langle e^{i\boldsymbol{m}_i^T \phi}\rangle = e^{i2\pi n_i/c_i}$. Therefore, the quasiparticle $\boldsymbol{m}_i$ is also condensed on the boundary line.

Taking this point of view, a gapped boundary can be generically viewed as a particle condensate. The condensed quasiparticles form a subgroup M of the group of all particles, with the group multiplication defined by particle fusion. For the particle condensate to be defined consistently, the subgroup M must satisfy the following conditions:

(1) $e^{i\theta_{mm'}} = 1$ for all $\boldsymbol{m}, \boldsymbol{m}' \in M$, and

(2) $e^{i\theta_{lm}} \neq 1$ for at least one $\boldsymbol{m} \in M$, if $\boldsymbol{l} \notin M$.

For bosonic states (K even), we also have $e^{i\theta_m} = 1$ for all $\boldsymbol{m} \in M$. The subgroup M has been referred to as a "Lagrangian subgroup."[27,28,32]

The first condition requires that every two particles in M are mutually bosonic, so that they can be condensed simultaneously. The second condition requires that all other quasiparticles not in M are confined after the condensation of M. Consequently, the resulting state has no topologically nontrivial quasiparticle excitations that can propagate along the edge.

Following the discussion above, one can see that null vectors in backscattering terms can be related to the condensation of a Lagrangian subgroup on the edge, but the two are not obviously equivalent. Particles $\boldsymbol{m} \in M$ in a Lagrangian subgroup are not necessarily null vectors. Gapped edges that correspond to different Lagrangian subgroups M are clearly topologically distinct; Ref. 28 showed that every gapped edge corresponds to a choice of M and that every system with at least one Lagrangian subgroup has at least one type of gapped edge. Here, we strengthen this result by proving that *every* Lagrangian subgroup M corresponds to a gapped edge that condenses M. This shows that Lagrangian subgroups can *classify* gapped edges.

Before presenting the proof of the classification in the next section, we discuss some other useful properties of Lagrangian subgroups. The condensation of the quasiparticle set M along the edge means that a local operator can annihilate the quasiparticles in M at the boundary, but not in the bulk. As we show in the following section, this generally also implies that the operators in the edge theory corresponding to quasiparticles in M acquire a nonzero expectation value. This condensation
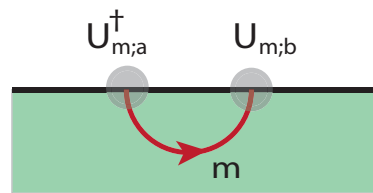

FIG. 4. (Color online) Depiction of the process $W_{\boldsymbol{m}}(\gamma)$, which creates a quasiparticle $\boldsymbol{m}$ at location a on the edge with a local operator. The quasiparticle propagates along a path γ in the bulk, and is annihilated at b with a local operator.

at the edge introduces a new process (Fig. 4), whereby the system can start in the ground state, a quasiparticle can be created at a point on the edge by a local operator, propagate through the bulk, and then get annihilated by another local operator at a different point on the edge, no matter how far apart the two points on the edge are.[28] We let this process be described by a quasiparticle line operator $W_{\boldsymbol{m}}(\gamma)$, where γ is the path of the quasiparticle $\boldsymbol{m}$. In the effective field theory,

$$W_{\boldsymbol{m}}(\gamma) = U_{\boldsymbol{m};a}\,e^{i\boldsymbol{m}^T \int_\gamma a\cdot dl}\,U_{\boldsymbol{m};b}^\dagger, \tag{7}$$

where $e^{i\boldsymbol{m}^T \int_\gamma a\cdot dl}$ is the Wilson line operator describing the propagation of the quasiparticle $\boldsymbol{m}$ through the bulk, while U_{ma} and $U_{mb}^\dagger$ are the local operators on the edge that annihilate/create the quasiparticle $\boldsymbol{m}$ at points a, b, respectively. This process leaves the system in its ground state,

$$W_{\boldsymbol{m}}(\gamma)|\psi\rangle = |\psi\rangle, \tag{8}$$

where $|\psi\rangle$ is a ground state of the system. These processes play a fundamental role in the proof that every gapped edge condenses a Lagrangian subgroup.[28] As we will show, they similarly play a fundamental role in the analysis of junctions between different gapped edges.

It is helpful to understand the role of the Lagrangian subgroup in cases where there is a boundary between two topological states, A_1 and A_2. In the previous section, we discussed the folding process, where we can consider the boundary between A_1 and A_2 as a boundary between $A_1 \times \bar{A}_2$ and the trivial state. A gapped interface between A_1 and A_2 is folded to a gapped boundary of $A_1 \times \bar{A}_2$. If the gapped boundary corresponds to a Lagrangian subgroup M, the condensed particles $\boldsymbol{m} \in M$ live in $A_1 \times \bar{A}_2$. In other words, $\boldsymbol{m}$ is a pair of quasiparticles $\boldsymbol{m} = (\boldsymbol{q}_1, \bar{\boldsymbol{q}}_2)$ with $\boldsymbol{q}_1$ and $\bar{\boldsymbol{q}}_2$ a quasiparticle in A_1 and $\bar{A}_2$, respectively. If quasiparticle $(\boldsymbol{q}_1, \bar{\boldsymbol{q}}_2)$ is condensed at the boundary, in the unfolded picture that means $\boldsymbol{q}_1$ and $\bar{\boldsymbol{q}}_2$ can be brought to the boundary and annihilate each other. Consequently, $\boldsymbol{q}_1$ can cross the boundary and become $\boldsymbol{q}_2$, the antiparticle of $\bar{\boldsymbol{q}}_2$. Therefore, in this case, each Lagrangian subgroup M specifies a consistent set of transmission, reflection, and absorption processes that happens at the interface.

In the special case of twist defects,[14] $A_1 = A_2 = A$. A Lagrangian subgroup of $A \times \bar{A}$ contains the pairs $(\boldsymbol{q}_1, \bar{\boldsymbol{q}}_2)$ with $\boldsymbol{q}_1, \boldsymbol{q}_2 \in A$. Therefore, each Lagrangian subgroup defines a mapping $\boldsymbol{q}_1 \to \boldsymbol{q}_2$ in theory A. The property that all $(\boldsymbol{q}_1, \bar{\boldsymbol{q}}_2)$ in the Lagrangian subgroup are mutually bosonic is equivalent to the condition that the mapping $\boldsymbol{q}_1 \to \boldsymbol{q}_2$ preserves the braiding and fusion rules of A. Therefore, we

correctly reproduced the known fact that twist defects are are in one-to-one correspondence with the symmetries of the topological quantum numbers of A.[14]

B. From Lagrangian subgroups to backscattering terms

In this section, we sketch the proof that there is a one-to-one correspondence between Lagrangian subgroups M and sets of null vectors $\{\mathbf{\Lambda}_i\}$ which can be used to gap the edges and condense the set M on the boundary. An important condition for this proof to be valid is that it is allowed to couple the edge state to topologically trivial one-dimensional states. This section expands the discussion in Ref. 37 recently presented by the authors. Since our proof is constructive, our analysis provides a way to systematically construct the local backscattering terms in the edge theory that condense any given Lagrangian subgroup M.

We develop the argument for this in two steps. In the first step, we prove that a Lagrangian subgroup defines a backscattering term on the boundary if it is generated by a set of null quasiparticles (defined below). In the second step, we prove that every Lagrangian subgroup can be generated by a set of null quasiparticles, as long as it is possible to introduce purely one-dimensional edge degrees of freedom which couple to the topological edge state.

1. From null quasiparticles to backscattering terms

Consider a $2N \times 2N$ K matrix with zero signature, and a Lagrangian subgroup M generated by N linearly independent $2N$-component integer vectors $\{\mathbf{m}_i\}$. We assume that $\mathbf{m}_i$ are null quasiparticles, which satisfy the condition

$$\mathbf{m}_i^T K^{-1} \mathbf{m}_j = 0, \quad i,j = 1, \dots, N. \qquad (9)$$

Then, we define

$$\mathbf{\Lambda}_i = c_i K^{-1} \mathbf{m}_i, \qquad (10)$$

where $c_i \in \mathbb{Z}$ is the minimal integer such that $\mathbf{\Lambda}_i$ is an integer vector. The $\{\mathbf{\Lambda}_i\}$ defined this way satisfy (6), and, therefore,

$$\delta H = g \sum_{i=1}^{N} \cos\left(\mathbf{\Lambda}_i^T K \phi\right) \qquad (11)$$

can generate an energy gap in the edge states, by pinning the argument of the cosine terms,

$$\mathbf{\Lambda}_i^T K \phi = c_i \mathbf{m}_i^T \phi = 2\pi n_i, \qquad (12)$$

where n_i is an integer. This directly implies that

$$\left\langle e^{i \mathbf{m}_i^T \phi} \right\rangle = e^{i 2\pi n_i / c_i} \neq 0. \qquad (13)$$

Therefore, the edges are gapped and the Lagrangian subgroup M is condensed on the boundary.

2. From general Lagrangian subgroups to null quasiparticles

Now we need to prove that every Lagrangian subgroup M can be represented by N linearly independent vectors $\{\mathbf{m}_i\}$ which satisfy (9). Naively, for a given K matrix this is not true. For example, consider $K = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$, which describes Z_4 topological order. This has a Lagrangian subgroup generated by $\mathbf{m}_1^T = (2,0)$, $\mathbf{m}_2^T = (0,2)$ which does not satisfy (9). It turns out the statement that we actually need is the following.

Lemma. Suppose we have a set of vectors $\{\mathbf{m}_i\}$, $i = 1, \dots, N_M$ (where N_M is not necessarily equal to N), which generate a Lagrangian subgroup: i.e., $\mathbf{m}_i^T K^{-1} \mathbf{m}_j \in \mathbb{Z}$, $\mathbf{m}_i^T K^{-1} \mathbf{l} \notin \mathbb{Z} \forall \mathbf{l} \notin M$, and $\mathbf{m}_i^T K^{-1} \mathbf{m}_i$ is even for K even. Then there exists a K' which is topologically equivalent to K, such that $\dim(K') = 2N'$, and a set of N'-component vectors, $\{\mathbf{m}_i'\}$, for $i = 1, \dots, N'$, which satisfy $\mathbf{m}_i'^T K'^{-1} \mathbf{m}_j' = 0$, and which generate the same Lagrangian subgroup M.

The proof of this is somewhat technical and is presented in an Appendix. The main idea is that one can define K' of the form

$$K' = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & \mathbb{I} \\ 0 & \mathbb{I} & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} K & 0 & 0 \\ 0 & \mathbb{I} & 0 \\ 0 & 0 & -\mathbb{I} \end{pmatrix}, \qquad (14)$$

with $\mathbb{I}$ an $N \times N$ identity matrix. The two forms should be applied to the K's describing a boson theory or a fermion theory, respectively. Since $|\text{Det} K'| = |\text{Det} K|$, the new blocks do not introduce any new particle types, and K' and K are topologically equivalent. Every generator $\mathbf{m}_i$ of the Lagrangian subgroup is mapped to a higher dimensional vector $\{\mathbf{m}_i'\}$ by simply expanding the K matrix to include additional counterpropagating topologically trivial edge states. Although the added topologically trivial degrees of freedom do not change the topological properties of quasiparticle $\mathbf{m}_i$, it can change the inner product of $\mathbf{m}_i'$. A suitable choice can always be made to satisfy $\mathbf{m}_i'^T K'^{-1} \mathbf{m}_j' = 0$. In the next section, we discuss a number of explicit examples of this.

The above lemma, taken together with step 1 above, proves that every Lagrangian subgroup M of an Abelian topological phase corresponds to a gapped edge where M is condensed. Therefore, the Lagrangian subgroups provide a topological classification of gapped edges. Edges corresponding to different Lagrangian subgroups clearly cannot be adiabatically connected to each other without closing the energy gap in the edge states.

C. Examples

Let us begin with a simple example. Consider two independent time-reversed copies of a $1/m$-Laughlin FQH state, described by the K matrix $K = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}$. The edge of this state can be terminated by a charge-conserving backscattering term,

$$H_b = \frac{g}{2}(\Psi_{eL}^\dagger \Psi_{eR} + \text{H.c.}) = g \cos[m(\phi_L + \phi_R)], \qquad (15)$$

or by SC,

$$H_{sc} = \frac{g}{2}(\Psi_{eL}^\dagger \Psi_{eR}^\dagger + \text{H.c.}) = g \cos[m(\phi_L - \phi_R)]. \qquad (16)$$

In the first case, the Lagrangian subgroup generated by $(1,1)$ is condensed on the edge; in the second case, the Lagrangian subgroup generated by $(1, -1)$ is condensed on the edge.

Now let us consider a more nontrivial example that illustrates the necessity of the lemma introduced in the previous section. Consider $K = \begin{pmatrix} 9 & 0 \\ 0 & -9 \end{pmatrix}$. This has a Lagrangian subgroup

generated by $m_1^T = (3,0)$ and $m_2^T = (0,3)$. Now we define

$$K' = \begin{pmatrix} K & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{17}$$

and $m_1'^T = (3,0,0,1)$ and $m_2'^T = (0,3,1,0)$. Again, in the absence of any symmetries, K' is topologically equivalent to K and $m_i'^T K'^{-1} m_j' = 0$. Thus, the backscattering terms associated with this Lagrangian subgroup are given by $\sum_{i=1}^2 \cos(\Lambda_i^T K' \phi)$, with $\Lambda_i = 3K^{-1} m_i'$.

Another example is given by the mutual CS theory describing the Z_4 toric code model.[44] Let $K = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$, and $m_1^T = (2,0)$ and $m_2^T = (0,2)$. We define

$$K' = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{18}$$

and $m_1'^T = (2,0,0,1)$ and $m_2'^T = (0,2,-1,0)$. Here K' is topologically equivalent to K, and $m_i'^T K'^{-1} m_j' = 0$. Now the backscattering terms associated with this Lagrangian subgroup are given by $\sum_{i=1}^2 \cos(\Lambda_i^T K' \phi)$, with $\Lambda_i = 2K^{-1} m_i'$.

Let us now consider more generally the Z_N toric code model, described by $K = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$. For any set of integers r,t such that $rt = N$, there is a Lagrangian subgroup generated by the quasiparticles $(r,0)$ and $(0,t)$. In other words, every distinct divisor of N yields a different Lagrangian subgroup. The number of Lagrangian subgroups is therefore equal to the number of divisors of N. For example, when N is prime, there are two Lagrangian subgroups, corresponding to whether the "electric" particles $(1,0)$ are condensed, or whether the "magnetic" ones $(0,1)$ are condensed.[34] When $N = \prod_i p_i^{s_i}$, where the p_i are all distinct prime numbers and s_i are their multiplicities, there are $\prod_i (1 + s_i)$ Lagrangian subgroups and therefore $\prod_i (1 + s_i)$ topologically distinct boundaries.

In the following discussion, we provide a microscopic lattice model construction of these different gapped edges of the Z_N toric code. We use a specific construction of this phase, called the Z_N plaquette model.[19,45,46] The degrees of freedom consist of N states on each of the sites of a square lattice. The Hamiltonian with the gapped edge contains two terms:

$$H_{\text{Total}} = H_{\text{Bulk}} + H_{\text{Edge}}, \tag{19}$$

where

$$H_{\text{Bulk}} = -\sum_p (\mathcal{O}_p + \text{H.c.}). \tag{20}$$

The sum is over all plaquettes of the square lattice, and the plaquette operator $\mathcal{O}_p$, as shown in Fig. 5(a), is defined as

$$\mathcal{O}_p = T_1 U_2 T_3^\dagger U_4^\dagger, \tag{21}$$

where T_i and U_i are $N \times N$ matrices satisfying $U_i T_i = T_i U_i e^{i\frac{2\pi}{N}}$ and $T_i^N = U_i^N = 1$. The N states at each site form an N-dimensional irreducible representation of this algebra. Since all the plaquette operators $\mathcal{O}_p$ commute with each other, the ground state is the common eigenvector of all $\mathcal{O}_p$'s with the real part of the eigenvalue maximized. If the real part of the eigenvalue of $\mathcal{O}_p$ on a red (blue) plaquette is not maximized,
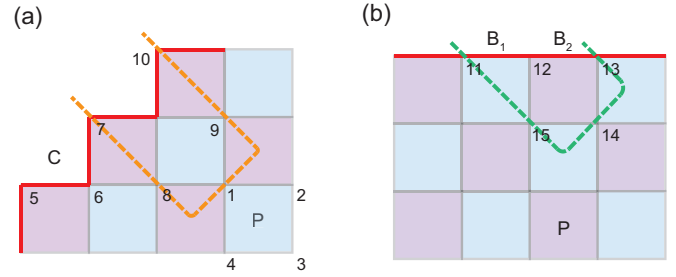

FIG. 5. (Color online) (a) Lattice model of Z_N toric code with gapped boundary (red line) that corresponds to the Lagrangian subgroup generated by $(1,0)$ and $(0,N)$. The Wilson line (the orange dash line) shows that the quasiparticle $(1,0)$ and its antiparticle $(-1,0)$ can be created together from the vacuum and annihilated at different locations of the boundary. (b) The Z_N toric code with gapped boundary (red line) that corresponds to the Lagrangian subgroup generated by $(r,0)$ and $(0,t)$. The Wilson line (the green dash line) shows that the quasiparticle $(0,t)$ and its antiparticle $(0,-t)$ can be created together from the vacuum and annihilated at different locations of the boundary.

the state contains an electric (magnetic) quasiparticle at that plaquette. The choice of gapped edge is set by the nature of H_{Edge}. For a gapped edge that corresponds to the Lagrangian subgroup generated by $(1,0)$ and $(0,N)$, we consider the physical edge in Fig. 5(a) indicated by the red line, with

$$H_{\text{Edge}} = -\sum_C (\mathcal{V}_C + \text{H.c.}), \tag{22}$$

which is a sum over all "corner operators" on the edge [see Fig. 5(a)]. The corner operator at the corner C is defined as

$$\mathcal{V}_C = U_7 T_6^\dagger U_5^\dagger. \tag{23}$$

With this definition, all terms in $H_{\text{Total}} = H_{\text{Bulk}} + H_{\text{Edge}}$ commute with each other. The ground state will be a common eigenvector of all $\mathcal{O}_p$'s and $\mathcal{V}_C$'s with the real parts of all the eigenvalues maximized. By counting the number of constraints from this and the total number of degrees of freedom, we find there are only a finite number of ground states with gapped excitations, which implies a gapped edge. Now we want to show that the edge indeed corresponds to the Lagrangian subgroup generated by $(1,0)$ and $(0,N)$. We can consider the process in which an electric particle-antiparticle pair is created in the bulk and then annihilated at different locations on the edge. An example of this process is described by the Wilson line operator indicated by the orange dashed line in Fig. 5(a):

$$W_e = U_7 U_8 T_1^\dagger U_9^\dagger U_{10}^\dagger. \tag{24}$$

Notice that W_e can be written as a product of plaquette operators and corner operators. Thus, W_e leaves the ground state invariant, which means that the electric particle is condensed on the edge. Moreover, in this construction, $(0,N)$ is a trivial particle, and therefore is already "condensed" on the edge. Thus, the model we write here describes the Z_N toric code with a gapped edge corresponding to the Lagrangian subgroup generated by $(1,0)$ and $(0,N)$.

More generally, for the gapped edge that corresponds to the Lagrangian subgroup generated by $(r,0)$ and $(0,t)$, we consider

the physical edge in Fig. 5(b) indicated by the red line, with

$$H_{\text{Edge}} = -\sum_B (\mathcal{R}_B + \text{H.c.}),\tag{25}$$

which is a sum over all "bond operators" on the edge as shown in Fig. 5(b). If the bond is the edge of a magnetic plaquette, say bond B_1, we define

$$\mathcal{R}_{B_1} = (T_{12}^\dagger U_{11}^\dagger)^t,\tag{26}$$

while the bond operator on the edge of an electric plaquette, say bond B_2, is defined as

$$\mathcal{R}_{B_2} = (T_{13}^\dagger U_{12}^\dagger)^r.\tag{27}$$

Again, all terms in $H_{\text{Total}} = H_{\text{Bulk}} + H_{\text{Edge}}$ commute with each other. The ground state is a common eigenvector of all $\mathcal{O}_p$'s and $\mathcal{R}_B$'s with the real parts of all the eigenvalues maximized. By a similar analysis as above, the Hamiltonian H_{Total} produces a gapped edge. Now we want to show that the particles $(r,0)$ and $(0,t)$ are condensed on this edge. We consider the process in which the pair $(0,t)$ and $(0,-t)$ is created from the vacuum, and then annihilated at different locations on the edge. This process can, for example, be described by the Wilson line operator W_{tm} denoted in Fig. 5(b) by the green dashed line:

$$W_{tm} = (U_{11}U_{15}T_{14}^\dagger U_{13}^\dagger)^t.\tag{28}$$

Notice that W_{tm} can be written as a product of plaquette operators and bond operators. Thus, W_{tm} leaves the ground state invariant, which means that the particle $(0,t)$ is condensed on the edge of the ground state. A parallel analysis can be performed for the $(r,0)$ particle to show that it also condenses at the edge. Therefore, this lattice construction corresponds to the Lagrangian subgroup generated by $(r,0)$ and $(0,t)$.

V. CLASSIFICATION OF POINT DEFECTS

Now let us consider junctions where various gapped edges meet at a point. As shown in Fig. 3, using the folding process, on an infinite plane such junctions can always be mapped to domain walls between two different gapped edges. Therefore, here we need only to focus on domain walls between two different gapped edges in order to understand the essential topological properties of generic point defects.

Thus, consider two kinds of gapped edges associated with two different Lagrangian subgroups M and M'. In order to understand basic properties such as quantum dimension, zero modes, and non-Abelian statistics, we consider the system on the disk geometry, with $2n$ well-separated domain walls separating the two gapped edges. We refer to the edges where a Lagrangian subgroup M is condensed as an M edge, and similarly for edges where M' is condensed.

A. Topological degeneracies: Quantum dimension

The essential feature of the existence of domain walls between multiple gapped edges is the introduction of novel line operators, $W_m(a_i)$ and $W_{m'}(b_i)$, where $m \in M$ and $m' \in M'$, with paths a_i and b_i that can intersect only once (see Fig. 6). The definition of these line operators, for the case of a single gapped edge, is discussed in Sec. IV A and depicted in Fig. 4.
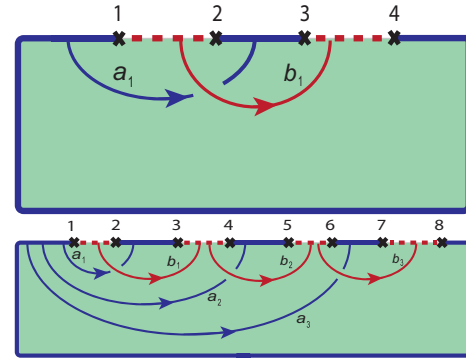

FIG. 6. (Color online) (Top panel) A topological phase on a disk topology, with the trivial gapped state outside of the disk. The solid blue (dashed red) lines on the edge correspond to an M (M') edge. The black crosses indicate the domain walls between the two kinds of gapped edges. The existence of the domain walls implies the possibility of noncommuting Wilson line operators associated with quasiparticles of the two Lagrangian subgroups M and M'. (Bottom panel) The existence of many domain walls between gapped edges corresponding to Lagrangian subgroups M and M' implies the possibility of many noncommuting Wilson line operators associated with quasiparticles of M and M'. A convenient set of paths, $\{a_i\}$ and $\{b_i\}$, is shown, which shows that for n pairs of domain walls, there are $n-1$ copies of the algebra (32).

Since $\boldsymbol{m}$ and $\boldsymbol{m}'$ have fractional mutual statistics, these operators do not commute with each other,

$$W_{\boldsymbol{m}}(a_i)W_{\boldsymbol{m}'}(b_j) = W_{\boldsymbol{m}'}(b_j)W_{\boldsymbol{m}}(a_i)e^{\delta_{ij}2\pi i \boldsymbol{m}^T K^{-1}\boldsymbol{m}'},\tag{29}$$

where a_i and b_i are the paths shown in Fig. 6(b). Since these line operators leave the system in its ground-state subspace, the ground states must form a representation of this algebra. The dimension of the smallest irreducible representation of this algebra is generally larger than one, from which we can conclude that the domain walls must introduce topological ground-state degeneracies into the system.

These Wilson line operators are generalizations of the Wilson loop operators in the presence of twist defects,[13,14,21] where the defects introduce novel noncontractible loops that lead to a nontrivial loop algebra and therefore a topological ground-state degeneracy.

In what follows, we provide a general formula for the ground-state degeneracy, which allows us to obtain the quantum dimension, or effective number of degrees of freedom, of each domain wall.

1. Calculation of quantum dimension

In the case where we have one pair of point defects, there is no nontrivial line algebra induced by the defects, and therefore the defects do not induce any topological degeneracy. Therefore, we will begin with the case where we have two pairs of point defects (Fig. 6).

In order to compute the topological ground-state degeneracy, we must find the dimension of the smallest irreducible representation of (29). To do this, it is convenient to first write the quasiparticles in terms of the generators of the Lagrangian

subgroups M and M'. That is,

$$\boldsymbol{m} = \sum_{i=1}^{N} q_i \boldsymbol{m}_i, \quad \boldsymbol{m}' = \sum_{i=1}^{N} q_i' \boldsymbol{m}_i', \tag{30}$$

where q_i and q_i' are integers, and $\{\boldsymbol{m}_i\}$, $\{\boldsymbol{m}_i'\}$, for $i = 1, \ldots, N$ are the generators of M and M', respectively. Here we use the N null vectors described in Sec. IV B for the generators of the Lagrangian subgroups, where $\dim(K) = 2N$. For simplicity we relabel the line operators

$$A_{\vec{q}} = W_{\sum_i q_i \boldsymbol{m}_i}(a_1), \quad B_{\vec{q}'} = W_{\sum_i q_i' \boldsymbol{m}_i'}(b_1), \tag{31}$$

where $\vec{q}$ and $\vec{q}'$ above are N-component integer vectors.

In this notation, the line algebra is

$$A_{\vec{q}} B_{\vec{q}'} = B_{\vec{q}'} A_{\vec{q}} e^{2\pi i \vec{q}^T R \vec{q}'}, \tag{32}$$

where R is an $N \times N$ matrix:

$$R_{ij} = \boldsymbol{m}_i^T K^{-1} \boldsymbol{m}_j'. \tag{33}$$

The A's and B's all commute with each other, and

$$A_{\vec{q}} A_{\vec{q}'} = A_{\vec{q}+\vec{q}'}, \quad B_{\vec{q}} B_{\vec{q}'} = B_{\vec{q}+\vec{q}'}. \tag{34}$$

Note also that

$$\begin{aligned}
A_{\vec{q}} &= 1, \quad \text{if} \quad R^T \vec{q} \in \mathbb{Z}^N, \\
B_{\vec{q}'} &= 1, \quad \text{if} \quad R \vec{q}' \in \mathbb{Z}^N.
\end{aligned} \tag{35}$$

This is because such $A_{\vec{q}}$ and $B_{\vec{q}'}$ will commute with all operators in the algebra and can therefore be represented as the identity in the ground-state subspace.

The smallest irreducible representation of this algebra can be obtained by diagonalizing one set, such as $A_{\vec{q}}$, and having $B_{\vec{q}'}$ act as the ladder operators,

$$A_{\vec{q}}|\vec{\alpha}\rangle = e^{2\pi i \vec{q} \cdot \vec{\alpha}}|\vec{\alpha}\rangle, \quad B_{\vec{q}'}|\vec{\alpha}\rangle = |\vec{\alpha} + R\vec{q}'\rangle, \tag{36}$$

where $\vec{\alpha}$ is an N-component rational-valued vector. The fact that the eigenvalues must be phases follows from (34), and the fact that the $\vec{\alpha}$ are rational-valued follows from (35). Equation (35) also implies

$$|\vec{\alpha}\rangle = |\vec{\alpha} + R\vec{q}'\rangle, \quad \text{if} \quad R\vec{q}' \in \mathbb{Z}^N. \tag{37}$$

Therefore, the number of ground states can be obtained by counting the number of independent possible values of $|\vec{\alpha}\rangle$. The ladder operators $B_{\vec{q}}$ define states associated with the lattice $R\mathbb{Z}^N$, subject to the equivalence (37). Therefore, R can be viewed as generating a lattice of rational-valued vectors, where each state corresponds to a point on the lattice, and two states are equivalent if they differ by an integer-valued vector.

The number of such states can be computed as follows. We consider the lattice

$$\Gamma = \{R\vec{\Lambda}' + \vec{\Lambda} : \vec{\Lambda}' \in \mathbb{Z}^N, \vec{\Lambda} \in \mathbb{Z}^N\}. \tag{38}$$

Γ is an N-dimensional lattice, which can be generated by a matrix:

$$\Gamma = \tilde{R}\mathbb{Z}^N. \tag{39}$$

Now observe that $\mathbb{Z}^N$ is itself a sublattice of Γ. Therefore, the unit cell of $\mathbb{Z}^N$, which has unit volume, must contain an integer number D of unit cells of Γ. This implies that the volume of each unit cell of Γ is $1/D$: $|\mathrm{Det}\tilde{R}| = 1/D$. Each unit cell can be

associated with one state, and the inequivalent states all exist inside the unit sublattice of $\mathbb{Z}^N$. Thus, there are D states, so the dimension of the smallest irreducible representation of (29) is D. In Sec. V A 3, we provide a number of concrete examples of this calculation.

In the case where we have $2n$ domain walls, there are $n - 1$ independent copies of the above algebra [see Fig. 6(b)], and therefore the ground-state degeneracy is D^{n-1}. This implies that the quantum dimension of each defect is

$$d = \sqrt{D} = (|\mathrm{Det}\tilde{R}|)^{-1/2}. \tag{40}$$

We note that the topological degeneracy studied above is exact in the limit that the defects are infinitely far apart. If the defects are separated by a finite distance ℓ, then these topologically degenerate states will obtain an energy splitting proportional to $e^{-\ell/\xi}$, where ξ is the correlation length of either the edge states or the bulk, depending on which one is larger. This is due to the fact that a finite separation of the defects allows the instanton processes associated with the Wilson line operators to appear in the Hamiltonian and thus to split the energy of the degenerate states. The amplitude for these terms decays exponentially as $e^{-\ell/\xi}$, due to the energy gap of the system. This is similar to the case of topological degeneracies of topological states on closed surfaces, where the finite system size induces exponentially small splittings among the degenerate ground states.[47] This mechanism for inducing an energy gap among the degenerate ground states will be important for the discussions of Secs. V D and VI.

2. General discussion of quantum dimension

The quantum dimension and associated ground-state degeneracy calculated above is topologically robust and derived from the fractional statistics of the bulk quasiparticles. Therefore, it does not include any possible additional topological degeneracies that may arise from purely one-dimensional physics. In the absence of any symmetries, it has been proven that bosons in one dimension cannot give rise to any topological degeneracies.[48] Therefore, the result above fully captures the quantum dimension for domain walls in bosonic systems.

In contrast, fermionic systems in one dimension, in the absence of any symmetry, have a Z_2 topological classification.[49–51] There can be domain walls between different gapped 1D fermionic systems that localize Majorana fermion zero modes, which have a quantum dimension of $\sqrt{2}$. These Majorana fermion zero modes are protected by fermion parity symmetry: If the system is coupled to a gapless reservoir of fermions, the fermion parity symmetry of the edge system will be broken, and the topological degeneracy associated with the Majorana modes will be split. Therefore, for topological states that include at least one fermion species, the quantum dimension of the defect is

$$d_{\text{defect}} = d_{1D} d_{\text{bulk}}, \tag{41}$$

where d_{bulk} is the Wilson line algebra contribution of (40), and d_{1D} is either $\sqrt{2}$ or 1, depending on whether the purely one-dimensional fermion physics has an extra Majorana zero mode, protected by fermion parity. d_{1D} is independent of the Lagrangian subgroup and depends more precisely on the backscattering Hamiltonian of the edge theory. It is possible, in

principle, to compute d_{1D}, although this requires information beyond just the Lagrangian subgroups M and M' and requires knowledge of the precise Hamiltonian which is generating the energy gap on the edge.

We conclude that the Lagrangian subgroups can fully classify gapped edges only "modulo one-dimensional physics." For bosonic systems, the Lagrangian subgroups are expected to provide a full classification, while for fermionic systems, it is possible that two gapped edges correspond to the same Lagrangian subgroup, but cannot be adiabatically connected to each other. If we define the equivalence by allowing arbitrary one-dimensional degrees of freedom to be added to the edges, then of course the Lagrangian subgroups provide a full classification even for fermionic systems.

We note that here we have computed the topological degeneracy by studying the representations of the line operators that begin and end on the edge. This is a generalization to boundary domain walls of the loop algebra approach to computing the quantum dimension of twist defects, used in Ref. 14. An equivalent way to compute the topological degeneracy is directly within the edge theory, by studying the ground states of the chiral Luttinger liquid theory (3), with different kinds of backscattering terms of the form (11). Such an analysis was presented in special cases in Refs. 13–17. The main idea can be understood simply as follows. The edge Hamiltonian can be written as

$$\delta H_{\text{edge}} = g \sum_{i=1}^{N} \begin{cases} \cos\left(c_i \boldsymbol{m}_i^T \phi\right) & \text{if} \quad x \in M \quad \text{region,} \\ \cos\left(c_i' \boldsymbol{m}_i'^T \phi\right) & \text{if} \quad x \in M' \quad \text{region.} \end{cases} \quad (42)$$

Recall that $\{\boldsymbol{m}_i\}$ and $\{\boldsymbol{m}_i'\}$ are the N generators of the $2N$-dimensional matrix K, and c_i, c_i' are the smallest integers such that $c_i K^{-1} \boldsymbol{m}_i$, $c_i' K^{-1} \boldsymbol{m}_i'$ are integer vectors, as explained in Sec. IV B. The cosine terms in both domains cannot simultaneously acquire their classical minimum values. From (11) to (13), we see this would require $\langle e^{i \boldsymbol{m}^T \phi(x)} \rangle \neq 0$ for x in the M-gapped region and $\langle e^{i \boldsymbol{m}'^T \phi(x)} \rangle \neq 0$ for x in the M'-gapped region, which is not possible, in general, because these two operators generically do not commute at different points in space, and therefore they cannot be simultaneously diagonalized. Picking only the operators in one region to be fully diagonalized then leads to a topological degeneracy that grows exponentially with the number of domain walls. Such an understanding is useful for more detailed computations using the edge theory.

3. Examples

Here we review some examples, taken from previous studies,[13–20,22] using the general framework developed here in terms of the Wilson line algebra.

First, let us consider the proposals of Refs. 15–17, which consider a FQH state with K matrix,

$$K = \begin{pmatrix} N & 0 \\ 0 & -N \end{pmatrix}. \quad (43)$$

The two Lagrangian subgroups M and M' considered are generated by $\boldsymbol{m}_1 = (1,1)$ and $\boldsymbol{m}_1' = (1,-1)$, respectively. When these vectors are used as backscattering terms in the edge theory, the former can be interpreted physically as a normal

(charge-conserving) backscattering between counterpropagating edge states, while the latter physically corresponds to superconductivity-induced backscattering. The resulting line algebra (29) in this special case becomes $W_{\boldsymbol{m}_1}(a_i) W_{\boldsymbol{m}_1'}(b_j) = W_{\boldsymbol{m}_1'}(b_j) W_{\boldsymbol{m}_1}(a_i) e^{\delta_{ij} 2\pi i 2/N}$. When N is odd, corresponding to fermionic FQH states, this gives a quantum dimension $d_{\text{bulk}} = \sqrt{N}$, which agrees with the result $d = \sqrt{2}\sqrt{N}$ found in Refs. 15–17. As discussed in the previous section, the additional factor of $\sqrt{2}$ originates from the purely one-dimensional fermionic physics that gives rise to an additional Majorana fermion zero mode. When N is even, we have $d_{\text{bulk}} = \sqrt{N/2}$. The result $d = \sqrt{2N} = 2\sqrt{N/2}$ of Refs. 15–17, for N even, has an additional factor of 2 relative to d_{bulk}. This is again due to purely one-dimensional bosonic physics, where there is an additional boson parity symmetry of the model that was considered, which leads to additional ground-state degeneracies. Breaking this boson parity symmetry will lead to a topologically robust $\sqrt{N/2}$ quantum dimension.

As a second example, let us consider Z_2 twist defects of the state described by[13,14,20]

$$K = \begin{pmatrix} p & q \\ q & p \end{pmatrix}. \quad (44)$$

The special case $p = 0$ describes the case studied in Refs. 18 and 19. The twist defects have the property that a quasiparticle described by the vector (q_1, q_2) is transformed into (q_2, q_1) upon encircling the twist defect. Such twists were shown to have a quantum dimension $\sqrt{|p-q|}$. Through the folding process, we obtain a matrix

$$\tilde{K} = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}. \quad (45)$$

The twist defect then maps to a domain wall between gapped edges corresponding to two Lagrangian subgroups M and M'. M contains quasiparticles that are generated by $\{\boldsymbol{m}_1, \boldsymbol{m}_2\} = \{(1,0,-1,0)^T, (0,1,0,-1)^T\}$ and M' contains quasiparticles that are generated by $\{\boldsymbol{m}_1', \boldsymbol{m}_2'\} = \{(1,0,0,-1)^T, (0,1,-1,0)^T\}$. In this basis, the algebra of Wilson line operators is described by (32), with

$$R = \frac{1}{p-q} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (46)$$

Forming the lattice $\Gamma = \tilde{R} \mathbb{Z}^2$ [see Eq. (38)], we obtain

$$\tilde{R} = \frac{1}{p-q} \begin{pmatrix} 1 & 0 \\ -1 & p-q \end{pmatrix}. \quad (47)$$

The quantum dimension of the defects is therefore $\sqrt{|\text{Det}\tilde{R}|^{-1}} = \sqrt{|p-q|}$, in agreement with previous calculations.[13,14,20]

Finally, let us consider the boundary of Z_N topological states, described by

$$K = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}. \quad (48)$$

As discussed in Sec. IV C, two simple kinds of gapped edges correspond to condensation of either electric or magnetic particles. These correspond to Lagrangian subgroups M and M' generated by $\boldsymbol{m}_1 = (1,0)^T$ and $\boldsymbol{m}_1' = (0,1)^T$, respectively. The resulting Wilson line algebra is $W_{\boldsymbol{m}_1}(a_i) W_{\boldsymbol{m}_1'}(b_j)$

$= W_{\boldsymbol{m}'_1}(b_j)W_{\boldsymbol{m}_1}(a_i)e^{\delta_{ij}i2\pi/N}$. Therefore, the domain wall between these two kinds of gapped edges has a quantum dimension $\sqrt{N}$.

B. Localized toplogical zero modes

A physical consequence of the existence of the domain walls is the presence of a topologically robust nonzero density of states at zero energy for a subgroup L of the quasiparticles with fractional statistics. This subgroup L is defined as follows:

$$L = \{\boldsymbol{m} + \boldsymbol{m}' | \boldsymbol{m} \in M, \boldsymbol{m}' \in M'\}. \tag{49}$$

The fractional statistics of the quasiparticles in L derives from the fractional mutual statistics between $\boldsymbol{m}$ and $\boldsymbol{m}'$. It also is useful to define the subset of quasiparticles

$$\begin{aligned} \tilde{L} &\equiv L \backslash (M \cup M') \\ &= \{\boldsymbol{m} + \boldsymbol{m}' | \boldsymbol{m} \in M, \boldsymbol{m}' \in M', \boldsymbol{m} + \boldsymbol{m}' \notin M, M'\}. \end{aligned} \tag{50}$$

$\tilde{L}$ simply contains the nontrivial quasiparticles in L that do not belong to M or M'. We show that the zero-energy density of states of the quasiparticles in $\tilde{L}$ is exponentially localized to the domain wall.

To see this, first recall that the gapped regions introduce the processes $W_{\boldsymbol{m}}(a_i)$, $W_{\boldsymbol{m}'}(b_i)$ [see Fig. 6(b)], which leave the system in the ground-state subspace. If we consider bringing the starting points of the paths near the domain walls, and pushing the end points out to infinity, we have the process $W_{\boldsymbol{m}+\boldsymbol{m}'}(a) = \lim_{a_\pm \to a} W_{\boldsymbol{m}}(a_+)W_{\boldsymbol{m}'}(a_-)$ (see Fig. 7). $a_\pm$ are the paths that start infinitesimally to the left/right of the domain wall and go out to infinity, and a is the limiting path obtained by fusing a_+ and a_- together in such a way that they both start at the domain wall. The quasiparticle $\boldsymbol{l}$, which consists of the fusion of $\boldsymbol{m}$ with $\boldsymbol{m}'$, can therefore be created at the position of the domain wall, propagate through the bulk, and be annihilated at a different defect, keeping the system in the ground-state subspace. Away from the domain wall, $\boldsymbol{l}$ cannot be absorbed or emitted, as long as $\boldsymbol{l} \notin M, M'$ because it necessarily includes a nontrivial particle from both Lagrangian subgroups, and only one Lagrangian subgroup is condensed on either side of the domain wall. This directly implies that the anyons of the form $\boldsymbol{l} = \boldsymbol{m} + \boldsymbol{m}'$ have a nonzero density of states, at zero energy, localized to the domain walls, as long as $\boldsymbol{l} \notin M, M'$ (i.e., $\boldsymbol{l} \in \tilde{L}$). These zero modes are topologically
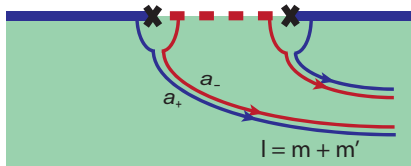

FIG. 7. (Color online) The domain walls between gapped edges labeled by Lagrangian subgroups M and M' allow for a process where an anyonic quasiparticle labeled $\boldsymbol{l} = \boldsymbol{m} + \boldsymbol{m}'$, with $\boldsymbol{m} \in M$, $\boldsymbol{m}' \in M'$, can be emitted from one domain wall and be absorbed at another. Schematically, the $\boldsymbol{m}$ quasiparticle is absorbed/emitted on one side of the domain wall, while the $\boldsymbol{m}'$ quasiparticle is absorbed/emitted on the other side. At the domain wall both of these processes can occur together, allowing for the anyon $\boldsymbol{l} = \boldsymbol{m} + \boldsymbol{m}'$ to be absorbed/emitted at the domain wall. This directly implies that $\boldsymbol{l}$ has a nonzero density of states at zero energy, localized exponentially to the domain wall.
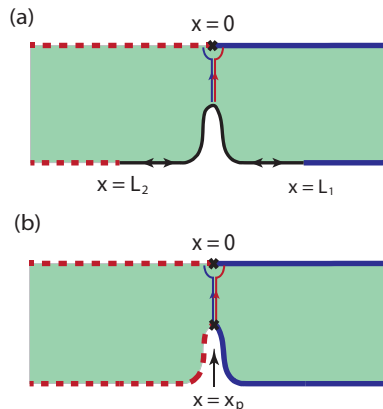

FIG. 8. (Color online) Possible geometries for probing the quasiparticle zero-energy density of states localized at the domain wall. (a) There is a gapless edge in the region $x \in [L_1, L_2]$, from which a quasiparticle can tunnel into the zero mode localized at the domain wall. (b) There can be a second domain wall at $x = x_p$, where the two defects at x_p and 0 can be arbitrarily far apart along the edge. A subset of the quasiparticles can be absorbed/emitted at zero energy between domain walls that are well separated along the edge.

robust, as they are protected by the topological nature of the gapped edges on either side of the domain wall.

1. The zero mode in quasiparticle density of states

In order to understand this more concretely in the edge theory let us consider a domain wall at $x = 0$ between M and M' edges. Furthermore, let us suppose that there is some region where the edge is gapless, between $L_1 < x < L_2$, which we use to "probe" the defect [Fig. 8(a)]. Equivalently, the gapless region can be considered to shrink to a point [Fig. 8(b)], in which case we are considering the tunneling of quasiparticles from one defect to another.

More concretely, we consider the following backscattering terms on the edge:

$$\delta H_{\text{edge}} = g \sum_{i=1}^{N} \begin{cases} \cos\left(c_i \boldsymbol{m}_i^T \phi\right) & \text{for } x \in [0, L_1], \\ \cos\left(c_i' \boldsymbol{m}_i'^T \phi\right) & \text{for } x \in [x_L, 0], [L_2, x_R], \end{cases} \tag{51}$$

where $x_L < 0$ and $x_R > L_2$. Recall that $\{\boldsymbol{m}_i\}$ and $\{\boldsymbol{m}_i'\}$ are the generators of M and M', respectively. In the regions $x < x_L$ and $x > x_R$, we do not specify the nature of the edge, except to assume that it allows topologically degenerate sectors due to the domain wall at $x = 0$. For concreteness let us suppose all of the regions are infinitely long: $x_L \to -\infty$ and $L_1, L_2, x_R \to \infty$. Furthermore, in the case where $L_1 < x < L_2$ is a gapless region of the edge theory, let us assume $|L_1 - L_2| \to \infty$ in order for the edge to not have a finite-size gap in this region. Alternatively, if $|L_1 - L_2| \to 0$, we can assume this point is the location of a second domain wall at $x = L_1 = L_2 \equiv x_p$ [Fig. 8(b)].

The classical minima in the M-gapped regions are set by $\boldsymbol{m}_i^T \phi = 2\pi\alpha_i/c_i$ for integer α_i. The topologically degenerate sectors therefore can be (partially) labeled by $|\vec{\alpha}\rangle$, such that $e^{i\boldsymbol{m}_i^T \phi(x)}|\vec{\alpha}\rangle = e^{i2\pi\alpha_i/c_i}|\vec{\alpha}\rangle$ for $0 < x < L_1$. Note that in this basis, we cannot simultaneously diagonalize all of the $e^{i\boldsymbol{m}'^T\phi(x)}$

for $x_L < x < 0$, because of the nontrivial commutation relation between $e^{i m^T \phi(y)}$ and $e^{i m'^T \phi(x)}$ at separate points x, y. Also note that this is a partial labeling of the states because we have not specified the nature of the edge for $x < x_L$ and $x > x_R$.

Now consider the quasiparticle operator

$$\chi_l(x) = e^{i l^T \phi(x)}, \qquad (52)$$

for $l \in \tilde{L}$. That is, for $l = m + m'$, with $m \in M$ and $m' \in M'$, such that $l \notin M, M'$. We also consider m and m' to be generated by the null quasiparticle vectors $\{m_i\}$. Consider the correlation function

$$G_{l;\vec{\alpha}}(x,t) = \langle \vec{\alpha}|\chi_l(x_p,t)\chi_l^\dagger(x,t)\chi_l(x,0)\chi_l^\dagger(x_p,0)|\vec{\alpha}\rangle, \qquad (53)$$

where $|\vec{\alpha}\rangle$ labels the different ground states under consideration. We include the operator $\chi_l^\dagger(x_p,0)$, which creates the quasiparticle l at $x = x_p$, in order to be able to fully describe the system using only the Hilbert space in the long-wavelength theory of the edge, without reference to the bulk. $\chi_l(x)$ alone is not a physical, gauge-invariant process on the edge. The point x_p can be considered to be located in the region of a gapless edge or at another domain wall.

We can show that, at low frequencies, $\omega \ll g$, where g is the scale of the energy gap of the edge states,

$$G_{l;\vec{\alpha}}(x,\omega) = \int dt\, G_{l;\vec{\alpha}}(x,t)e^{i\omega t} \sim e^{-|x|/\xi}\delta(\omega), \qquad (54)$$

where $\xi \propto 1/g$ is the correlation length of the gapped edge states. This directly implies that the quasiparticles of the form $l = m + m'$, such that $l \notin M, M'$, have a nonzero density of states, at zero energy, exponentially localized to the domain wall. To see this, we insert a complete set of states,

$$1 = \sum_a |a\rangle\langle a| = \sum_{\vec{\beta} \in G} |\vec{\beta}\rangle\langle\vec{\beta}| + \sum_{n \in E} |n\rangle\langle n|, \qquad (55)$$

where we have split the formal sum over all states in the edge theory into those in the ground-state subspace, labeled G, and the excited states, labeled E. Thus,

$$G_{l;\vec{\alpha}}(x,t) = \sum_{\vec{\beta} \in G} |\langle\vec{\beta}|\chi_l(x)\chi_l^\dagger(x_p)|\vec{\alpha}\rangle|^2 + \cdots. \qquad (56)$$

The $\cdots$ represents the sum over excited states, which can be neglected at frequencies much smaller than the energy gap in the edge states. Now consider the matrix elements

$$|\langle\vec{\alpha}|\chi_l(x)\chi_l^\dagger(x_p)|\vec{\beta}\rangle| = |\langle\vec{\alpha}|e^{i m'^T(\phi(x)-\phi(x_p))}|\vec{\beta}\rangle|, \qquad (57)$$

where we have assumed, without loss of generality, that $x > 0$, so that $e^{i m^T \phi(x)}$ are diagonalized in the $\vec{\alpha}$ basis and thus simply contribute an unimportant U(1) phase factor to the matrix elements.

Furthermore, observe that the operators $e^{i m'^T[\phi(0)-\phi(x_p)]}$, for $m' \in M'$ act as ladder operators that connect the different ground states $|\vec{\alpha}\rangle$. To see this explicitly consider the commutation relations for $x > 0$,

$$[m^T \phi(x), m'^T[\phi(0) - \phi(x_p)]] = i2\pi\, m^T K^{-1} m', \qquad (58)$$

which implies that

$$e^{i m'^T[\phi(0)-\phi(x_p)]}|\vec{\alpha}\rangle = |\vec{\gamma}\rangle, \qquad (59)$$

where $\gamma_i = \alpha_i + m'^T(c_i K^{-1} m_i)$. Therefore, we can write

$$|\vec{\alpha}\rangle = e^{-i m'^T[\phi(0)-\phi(x_p)]}|\vec{\gamma}\rangle. \qquad (60)$$

Equation (57) thus simplifies to

$$|\langle\vec{\alpha}|\chi_l(x)\chi_l^\dagger(x_p)|\vec{\beta}\rangle| = |\langle\vec{\gamma}|e^{-i m'^T \phi(0)}e^{i m'^T \phi(x)}|\vec{\beta}\rangle|. \qquad (61)$$

The operators inside the expectation value simply introduce a kink in the uniform boson configuration at position x. It is clear that this equal-time correlation function must decay exponentially,

$$|\langle\vec{\gamma}|e^{-i m'^T \phi(0)}e^{i m'^T \phi(x)}|\vec{\beta}\rangle| \propto \delta_{\vec{\gamma}\vec{\beta}}e^{-x/\xi}, \quad x > 0, \qquad (62)$$

because the edge states are gapped for $x > 0$ and $x < 0$, with a correlation length $\xi \propto 1/g$. The Kronecker δ, $\delta_{\vec{\gamma}\vec{\beta}}$, arises because the M edge extends to infinity, while the kink is created at a finite distance x away from the location of the domain wall.

We can repeat a slightly modified argument to obtain an exponential decay for $x < 0$ as well. This proves (54) and therefore that the quasiparticles in $\tilde{L}$ have a nonzero density of states at zero energy, exponentially localized to the domain wall.

We see that the physical origin of this nonzero density of states is simply that $\chi_l(0)\chi_l^\dagger(x_p)$ can take one degenerate ground state to another: The domain wall can absorb/emit quasiparticles of the form $l = m + m'$, while staying in the ground-state subspace. The $\delta(\omega)$ arises because, aside from the degenerate ground-state subspace, there is a gap of order g to any other states, and therefore for $\omega \ll g$ there will be no spectral weight aside from the δ function at $\omega = 0$.

While the χ_l for $l \in \tilde{L}$ have a nonzero density of states at zero energy localized to the domain wall, the zero-energy density of states is not localized if $l \in M$ or M'. In contrast, if $l \notin L$, then there will be a vanishing density of states at zero energy everywhere, because such quasiparticle operators do not act in the degenerate ground-state subspace, even when applied at the domain wall.

In the fermionic case, where K has at least one odd element along the diagonal, there is an additional subtlety. As stated in Sec. IV B, we can always pick a choice of K and a basis $\{m_i\}$ for the Lagrangian subgroup M, such that $m_i^T K^{-1} m_j = 0$. While m_i and $m_i + K\Lambda$, for Λ an integer vector, describe the same topological quasiparticle, they may differ in fermion parity: One may be a boson while the other is a fermion. In this case, $\chi_l(x)$ has a robust zero-energy density of states only if $l = m + m'$, with m, m' both bosons. Otherwise, creating the quasiparticle l on the edge will require also adding an additional local fermion, which will not be guaranteed to have a nonzero density of states at zero energy, unless there happens to be an additional Majorana fermion zero mode due to purely one-dimensional physics, as discussed in Sec. V A 2. Indeed, the above analysis assumed explicitly that one can choose a basis where either $e^{i m^T \phi(x)}$ or $e^{i m'^T \phi(x)}$ can acquire nonzero expectation values for $x > 0$ and $x < 0$, respectively, which is only possible for bosonic operators.

2. Generalized parafermion zero-mode algebra

Now let us consider a set of domain walls between the M and M'-gapped regions, at the positions x_i. Based on the above analysis, the quasiparticle operators $\chi_l(x)$ have a nonzero local

density of states at zero energy at x_i for quasiparticles $\boldsymbol{l} \in L$. We define the zero-mode operators (see Fig. 7),

$$\gamma_{li} = \lim_{\epsilon \to 0^+} e^{i\boldsymbol{m}^T \phi(x_i+\epsilon)} e^{i\boldsymbol{m}'^T \phi(x_i-\epsilon)}, \tag{63}$$

for $\boldsymbol{l} = \boldsymbol{m} + \boldsymbol{m}'$, if $x_i + \epsilon$ is an M-gapped region and $x_i - \epsilon$ is an M'-gapped region. For the reverse scenario, the roles of $\boldsymbol{m}$ and $\boldsymbol{m}'$ are interchanged above. This regularization is important in order to properly define the commutation relations between different γ_{li} on the same domain wall.

Note that the operators γ_{li} are zero-mode operators in the sense that they can be created/absorbed at the domain walls at zero energy, as long as they are correspondingly absorbed/created somewhere else. That is, $\gamma_{li}^\dagger \gamma_{lj}$ always keep the edge in its ground-state subspace and therefore commute with the Hamiltonian:

$$[H_{\text{edge}}, \gamma_{li}^\dagger \gamma_{lj}] = 0. \tag{64}$$

In general, the γ_{li} individually either commute or anticommute with H_{edge}, since they are mutually local.

The zero modes $\gamma_{l;i}$ satisfy the following algebra:

$$\gamma_{l;i}\gamma_{\bar{l};j} = \gamma_{\bar{l};j}\gamma_{l;i}e^{i\pi \boldsymbol{l}^T K^{-1}\bar{\boldsymbol{l}}\operatorname{sgn}(i-j)}, \quad i \neq j, \tag{65}$$

$$\gamma_{l;i}\gamma_{\bar{l};i} = \gamma_{l+\bar{l};i}e^{\pm i\pi \boldsymbol{m}' K^{-1}\bar{\boldsymbol{m}}}$$

$$= \gamma_{\bar{l};i}\gamma_{l;i}e^{\pm i\pi(\boldsymbol{m}'^T K^{-1}\bar{\boldsymbol{m}} - \boldsymbol{m}K^{-1}\bar{\boldsymbol{m}}')}, \tag{66}$$

where $\boldsymbol{l} = \boldsymbol{m} + \boldsymbol{m}'$ and $\bar{\boldsymbol{l}} = \bar{\boldsymbol{m}} + \bar{\boldsymbol{m}}'$, with $\boldsymbol{m}, \bar{\boldsymbol{m}} \in M$ and $\boldsymbol{m}', \bar{\boldsymbol{m}}' \in M'$. The $\pm$ sign in the latter equation depends on whether the domain wall has an M edge to the left and an M' edge to the right, or vice versa, and can be obtained using the point-splitting regularization defined in Eq. (63). Each operator $\gamma_{l;i}$ has a finite order: $\gamma_{n_l l}$ will commute with the whole algebra for some integer n_l and therefore can be represented in the algebra by the identity operator.

Equations (65) and (66) define a *generalized parafermion* algebra. The simplest version of this algebra, which consists of a single parafermion generator,[52–54] appears for the zero modes localized to the topological defects considered in Refs. 13–17, 19, and 21–23.

C. Mapping to genons

The Wilson line algebra (29) induced by the defects is similar to the Wilson loop algebra of an Abelian CS theory on a high genus surface. In the latter case, the Wilson loop algebra describes the quasiparticle propagation along noncontractible cycles of the high genus surface. Here we show that (29) is exactly equivalent to the Wilson loop algebra of some Abelian CS theory on a high genus surface. Consequently, we refer to such boundary defects as *genons* in the sense of topological equivalence.

Let us begin by considering the following algebra for the Wilson lines,

$$A_{\vec{q}} B_{\vec{q}'} = B_{\vec{q}'} A_{\vec{q}} e^{i2\pi \vec{q}^T \tilde{R} \vec{q}'}, \tag{67}$$

where recall that $\vec{q}$ and $\vec{q}'$ are N-component integer vectors. From the definition of $\tilde{R}$ [see Eq. (38) and subsequent discussion], we can see that this algebra is equivalent to

the original algebra (32) up to a possible relabeling of the operators.

Here we would like to show that one can always choose a basis of the Lagrangian subgroups M and M' so that $\tilde{R}^{-1}$ is a diagonal integer matrix. To see that it is integer, observe that the lattice Γ includes every integer vector $\vec{\Lambda} \in \mathbb{Z}^N$. Therefore, $\forall \vec{\Lambda} \in \mathbb{Z}^N$; there must exist $\vec{\Lambda}' \in \mathbb{Z}^N$ such that $\tilde{R}\vec{\Lambda}' = \vec{\Lambda}$. In other words, for every $\vec{\Lambda} \in \mathbb{Z}^N$, $\tilde{R}^{-1}\vec{\Lambda}$ is an integer vector, which implies that $\tilde{R}^{-1}$ must be integer.

To see that $\tilde{R}^{-1}$ can be diagonal, we use the following theorem of linear algebra. If A is an integer $N \times N$ matrix, then there exist integer matrices S and T with unit determinant such that $A' = SAT$ is a diagonal integer matrix. A' is known as the *Smith normal form* of A.[55] Applying this to $\tilde{R}$ implies that one can always find a basis of Lagrangian subgroups for M and M' such that $\tilde{R}$ will be diagonal.

Therefore, if we define $\tilde{K} \equiv \tilde{R}^{-1}$, then (67) can be interpreted as the Wilson loop algebra of a U(1)^N CS theory on a torus, characterized by the matrix $\tilde{K}$, where $A_{\vec{q}}$ and $B_{\vec{q}}$ are represented as

$$A_{\vec{q}} = e^{iq_I \oint_a a_I \cdot dl}, \quad B_{\vec{q}} = e^{iq_I \oint_b a_I \cdot dl}. \tag{68}$$

Here a_I are the U(1) gauge fields of this U(1)^N CS theory, a and b are the noncontractible loops on the torus, and $a + b$ is the noncontractible loop that encircles both a and b once.

As is well known, such a theory has a topological degeneracy given by $|\text{Det } \tilde{K}| = |\text{Det } \tilde{R}|^{-1} = D$. Therefore, with $2n$ domain walls, there are $n - 1$ copies of the above algebra, and therefore the algebra corresponds to the Wilson loop algebra of the associated U(1)^N CS theory with matrix $\tilde{K}$ on a genus $g = n - 1$ surface.

We show in Sec. V D that one can also define a notion of braiding of these domain walls, which correspond to modular transformations, or Dehn twists, of the corresponding Abelian CS theory on the genus $g = n - 1$ surface. This dramatically generalizes earlier results about twist defects.[14] Since the ground-state degeneracy and braiding of the defects in these cases can be understood in terms of the properties of an Abelian CS theory on a high genus surface, we refer to these defects as *genons*.

We would also like to note that the defect and antidefect are topologically equivalent. In the basis in which $\tilde{R}$ is symmetric, the Lagrangian subgroups M and M' can be mapped to each other, with a quasiparticle $\boldsymbol{m} = q_i \boldsymbol{m}_i \in M$ being mapped to $\boldsymbol{m}' = q_i \boldsymbol{m}_i' \in M'$. Using this mapping, the on-site commutation relations of the parafermion zero modes (66) are the same for defects and antidefects. This is consistent with the case of Z_2 twist defects studied previously,[14] where defects and antidefects are equivalent to each other.

D. Projective non-Abelian statistics

We have seen that the domain walls of the gapped edge states localize topologically protected zero modes and give rise to topological degeneracies. This raises the question of whether it is possible to "braid" these defects, that is, to carry out topologically protected unitary transformations in the degenerate subspace.

Since the defects generally exist as domain walls on the boundary of a topological phase, it is not possible to

geometrically braid the defects and return to the original configuration of the system. However, it has been shown that in some cases it is possible to define a notion of braiding through a different approach that involves tuning the tunneling of quasiparticles between the domain walls.[14–16,56,57]

In particular, let us consider bringing the edges close together, in order to induce quasiparticle tunneling between either the domain walls or the gapped domains. This leads to a Hamiltonian that acts on the degenerate ground-state subspace, opening the possibility of finding a closed path in Hamiltonian space that successfully carries out an adiabatic non-Abelian Berry phase on the ground-state subspace. Such non-Abelian Berry phases can, under certain conditions, be topologically protected up to an overall phase. This yields the possibility of defining a notion of projective non-Abelian statistics for the defects.

In what follows, we show that it is, in general, possible to generate the following topologically protected transformations. Consider an array of point defects, labeled $1, \ldots, 2n$, separating M and M' edges. Pick any neighboring pair of defects, such as 1 and 2 for convenience, and define the loops $\{a_i\}$ and $\{b_i\}$, as shown in Fig. 6(b). Without loss of generality, we suppose that defect 1 has an M-gapped edge to the left and an M'-gapped edge to the right, and vice versa for defect 2.

We show that we can generate a topologically protected braiding transformation B_{12}, which is a unitary transformation in the Hilbert space of topologically degenerate states, and has the following action on the Wilson line operators $\{W_{\boldsymbol{m}}(a_i)\}$ and $\{W_{\boldsymbol{m}'}(b_i)\}$,

$$
\begin{aligned}
B_{12}: W_{\boldsymbol{m}}(a_1) &\to e^{i\theta_{\boldsymbol{m}}} W_{\boldsymbol{m}}(a_1), \\
W_{\boldsymbol{m}'}(b_1) &\to e^{i\theta'_{\boldsymbol{m}}} W_{\boldsymbol{m}}^{\dagger}(a_1) W_{\boldsymbol{m}'}(b_1),
\end{aligned} \tag{69}
$$

while the rest of the Wilson line operators are left invariant. The phases $e^{i\theta_{\boldsymbol{m}}}$ and $e^{i\theta'_{\boldsymbol{m}}}$ are defined below, in Eq. (81). Equation (69) requires a canonical pairing between quasiparticles $\boldsymbol{m} \in M$ and $\boldsymbol{m}' \in M'$, which we explain below. In the mapping to the high genus surface (see Sec. V C), the above braiding transformations can simply be understood as Dehn twists, or modular transformations, of the genus g surface. This generalizes the result found in Ref. 14 in the context of twist defects to a more general class of point defects of an Abelian topological phase.

In order to derive the above result, we first define the following zero-mode Hamiltonian:

$$
H_{ab} = \sum_{i=1}^{N} \left(t_{l_i} \gamma_{l_i;a}^{\dagger} \gamma_{l_i;b} + \text{H.c.} \right). \tag{70}
$$

Here the quasiparticle $\boldsymbol{l}_i$ is defined by

$$
\boldsymbol{l}_i = \boldsymbol{m}_i + \boldsymbol{m}_i', \quad i = 1, \ldots, N, \tag{71}
$$

where $\{\boldsymbol{m}_i\}$ and $\{\boldsymbol{m}_i'\}$ for $i = 1, \ldots, N$ are the generators of M and M', in the basis defined by Eq. (67) with $\tilde{R}$ diagonal. Note that this basis pairs every generator $\boldsymbol{m}_i$ of M with a generator $\boldsymbol{m}_i'$ of M' and therefore induces a natural pairing between every $\boldsymbol{m} \in M$ and $\boldsymbol{m}' \in M'$. In the notation of (67), this is the statement that for every N-dimensional vector $\vec{q}$, we define an operator $A_{\vec{q}} \equiv W_{\sum_i q_i \boldsymbol{m}_i}(a_1)$, which is associated to Wilson lines of particles in M, and an operator $B_{\vec{q}} \equiv W_{\sum_i q_i \boldsymbol{m}_i'}(b_1)$, which is associated with Wilson lines of particles in M'. Recall
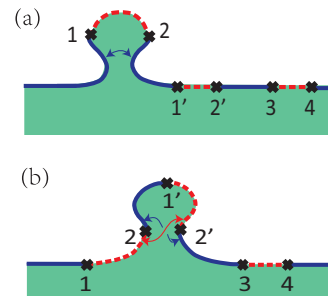

FIG. 9. (Color online) (a) Bringing different gapped edges together can induce tunneling of quasiparticles associated with the corresponding Lagrangian subgroups. H_{12} can, for example, be realized using the geometry shown, with the double arrows indicated tunneling of an $\boldsymbol{m} \in M$ quasiparticle between M-gapped edges (blue solid lines). (b) Bringing different defects (such as 2 and 2') in close proximity can induce tunneling of quasiparticles with zero modes localized to the defects. Double arrows indicate tunneling of zero modes between defects.

that N is half the dimension of the K matrix: $\dim(K) = 2N$, and, following the discussion of Sec. IV B, we are picking N generators $\{\boldsymbol{m}_i\}$ of M, and similarly for M'. Note that the individual terms in the sum, $\gamma_{l_i;a}^{\dagger} \gamma_{l_i;b}$, commute with each other.

When a and b are nearest neighbors, e.g., $b = a + 1$, the sum in $H_{a,a+1}$ effectively runs over all possible zero modes. This is equivalent to a sum over all Wilson line operators that connect the gapped region to the left of the defect at x_a and the region to the right of the defect at x_{a+1}. Physically, this can be achieved by bringing the edges in close physical proximity, as in Fig. 9(a).

In contrast, when a and b are not nearest neighbors, then we see that H_{ab} only sums over a restricted set of zero modes. For the braiding we define below to be topologically robust, it is crucial that the only terms which appear in the sum consist of quasiparticles generated by those of the form $\boldsymbol{l}_i = \boldsymbol{m}_i + \boldsymbol{m}_i'$. Couplings of any other zero modes must be exponentially suppressed. In general, zero modes at x_a and x_b can be coupled by bringing the point defects at x_a and x_b in close proximity in order to induce the relevant quasiparticle tunneling [see Fig. 9(b)]. However, in order to suppress the tunneling of quasiparticles that are not generated by $\{\boldsymbol{l}_i\}$, there may need to be additional geometric or energetic constraints. An example of such a geometric constraint occurs in the examples studied in Refs. 15 and 16, where physically the system is not folded (in the sense of Sec. III), and bringing well-separated, non-neighboring defects together will only allow a subset of the zero modes to tunnel with appreciable amplitude from one defect to another through the bulk.

In order to braid the defects 1 and 2, we introduce another pair of domain walls, 1' and 2' (see Fig. 10), which are initially coupled via $H_{1'2'}$ and therefore do not contribute to the initial ground-state degeneracy. Subsequently, we consider the following path in Hamiltonian space:

$$
H(\tau) = \begin{cases} H_{2 \to 1'} = (1 - \tau) H_{1'2'} + \tau H_{22'}, & \tau \in [0,1], \\ H_{2 \to 1'} = (2 - \tau) H_{22'} + (\tau - 1) H_{12}, & \tau \in [1,2]. \end{cases} \tag{72}
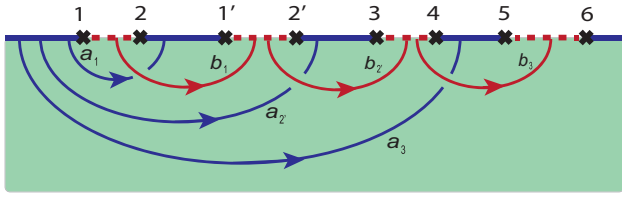$$

FIG. 10. (Color online) For the braiding between defects 1 and 2, we introduce another ancillary pair of defects $1'$ and $2'$. The degrees of freedom at 1 can then first be transferred to $2'$, and then those at 2 can be transferred to $1'$. After the process, 1 and 2 are coupled and can be annihilated, and $1'$ and $2'$ are relabeled 1 and 2, respectively. This effectively exchanges the degrees of freedom at 1 and 2.

Focusing on the D^2-dimensional Hilbert space associated with the defects 1, 2, $1'$, and $2'$ (see Sec. V A for a definition of D), we can show that $H_{1'2'}$, $H_{22'}$, and H_{12} will have a ground-state degeneracy of D, with a gap to the next excited states, assuming certain special values of the hopping coefficients t_{l_i} are avoided (see Appendix B for a detailed explanation). Therefore, $H(\tau)$ will typically have D ground states throughout the process; if there are any additional accidental degeneracies along the way, it is possible to choose a slightly different path in Hamiltonian space that avoids them without modifying $H(0)$, $H(1)$, and $H(2)$. Therefore, the system is in the D-dimensional ground-state subspace throughout the entire adiabatic process.

The first process $H_{2 \to 1'}$ can be thought of as transferring the zero modes from 2 to $1'$, while the second process $H_{1 \to 2'}$ can be thought of as transferring the zero modes from 1 to $2'$. After the process is over, we again have two uncoupled domain walls, $1'$ and $2'$. Therefore, $1'$, $2'$ after the process play the role of 1, 2 before the process. Therefore, if we rename $1'$ and $2'$ to be the new 1 and 2, we can see that 1 and 2 are effectively exchanged.

To be more explicit, we define the following physical operators:[14–16]

$$\mathcal{O}_{l;1} = \gamma_{l;2}\gamma_{l;1'}^{\dagger}\gamma_{l;2'}\gamma_{l;\infty}^{\dagger}, \quad \mathcal{O}_{l;2} = \gamma_{l;1}^{\dagger}\gamma_{l;2}\gamma_{l;2'}^{\dagger}\gamma_{l;\infty}. \quad (73)$$

Here we have included the operator $\gamma_{l;\infty}$, which represents a quasiparticle operator at some reference defect (or, alternatively, infinitely far away), which is necessary to ensure that $\mathcal{O}_{l;1}$ and $\mathcal{O}_{l;2}$ are physical operators that act on the Hilbert space of the edge theory.

The action of $H(\tau)$ on the ground-state subspace can be understood by noticing that these operators commute with the two processes:

$$[H_{2 \to 1'}, \mathcal{O}_{l_i;1}] = 0, \quad [H_{1 \to 2'}, \mathcal{O}_{l_i;2}] = 0, \quad (74)$$

for l_i, $i = 1, \ldots, N$, of the form described in Eq. (71).

The ground states of H_{ab} have definite eigenvalues for the operators $\gamma_{l_i;a}^{\dagger}\gamma_{l_i;b}$. For $(a,b) = (1,2),(1',2'),(2,2')$, we have

$$\gamma_{l_i;a}^{\dagger}\gamma_{l_i;b}|k; \vec{\alpha}_{ab}\rangle = e^{i2\pi(\vec{\alpha}_{ab})_i}|k; \vec{\alpha}_{ab}\rangle. \quad (75)$$

$k = 1, \ldots, D$ parametrize the D ground states of H_{ab}, while $\vec{\alpha}_{ab}$, which depend on k, characterize the eigenvalues of $\gamma_{l_i;a}^{\dagger}\gamma_{l_i;b}$.

Defining $\mathcal{P}(\tau)$ to be the projection onto the ground-state subspace of $H(\tau)$, we find:

$$\mathcal{P}(0)\mathcal{O}_{l_i;1}\mathcal{P}(0) = e^{i2\pi(\vec{\alpha}_{1'2'})_i}\gamma_{l_i;2}\gamma_{l_i;\infty}^{\dagger},$$
$$\mathcal{P}(1)\mathcal{O}_{l_i;1}\mathcal{P}(1) = e^{-i2\pi(\vec{\alpha}_{22'})_i}e^{i\pi l_i^T K^{-1}l_i}\gamma_{l_i;2'}\gamma_{l_i;1'}^{\dagger}\gamma_{l_i;2'}\gamma_{l_i;\infty}^{\dagger}. \quad (76)$$

This implies that after the first process, $\tau = 0 \to 1$, the zero mode $\gamma_{l_i;2}$ transforms into $e^{-2\pi i[(\vec{\alpha}_{1'2'})_i + (\vec{\alpha}_{22'})_i]}$ $e^{i\pi l_i^T K^{-1}l_i}\gamma_{l_i;2'}\gamma_{l_i;1'}^{\dagger}\gamma_{l_i;2'}$.

Similarly,

$$\mathcal{P}(1)\mathcal{O}_{l_i;2}\mathcal{P}(1) = e^{-i2\pi(\vec{\alpha}_{22'})_i}e^{i\pi l_i^T K^{-1}l_i}\gamma_{l_i;1}^{\dagger}\gamma_{l_i;\infty},$$
$$\mathcal{P}(2)\mathcal{O}_{l_i;2}\mathcal{P}(2) = e^{i2\pi(\vec{\alpha}_{12})_i}\gamma_{l_i;2'}^{\dagger}\gamma_{l_i;\infty}, \quad (77)$$

which implies that after the second process, $\tau = 1 \to 2$, $\gamma_{l_i;1}^{\dagger} \to e^{-i\pi l_i^T K^{-1}l_i}e^{i2\pi[(\vec{\alpha}_{12})_i + (\vec{\alpha}_{22'})_i]}\gamma_{l_i;2'}^{\dagger}$. Also, we note that

$$[\gamma_{l_i;2'}\gamma_{l_i;1'}^{\dagger}\gamma_{l_i;2'}\gamma_{l_i;\infty}^{\dagger}, H_{1 \to 2'}] = 0, \quad (78)$$

so that the result of the first process is not affected by the second process.

At the beginning of this process, we had the zero modes at x_1 and x_2 while the ones at x_1' and x_2' were coupled and therefore did contribute to the ground-state degeneracy. At the end of the process, we ended up with the zero modes at x_1' and x_2' while the ones at x_1 and x_2 are coupled. The final system is equivalent to the original system, and so we can just relabel $\gamma_{l_i;1'}$ and $\gamma_{l_i;2'}$ as $\gamma_{l_i;1}$ and $\gamma_{l_i;2}$, respectively.

The above results imply that the braiding B_{12} has the following action on the zero-mode operators:

$$B_{12}^{\dagger}\gamma_{l_i;2}B_{12} = e^{-2\pi i[(\vec{\alpha}_{1'2'})_i + (\vec{\alpha}_{22'})_i]}e^{i\pi l_i^T K^{-1}l_i}\gamma_{l_i;2}\gamma_{l_i;1}^{\dagger}\gamma_{l_i;2},$$
$$B_{12}^{\dagger}\gamma_{l_i;1}B_{12} = e^{-i2\pi[(\vec{\alpha}_{12})_i + (\vec{\alpha}_{22'})_i]}e^{i\pi l_i^T K^{-1}l_i}\gamma_{l_i;2}. \quad (79)$$

In terms of the Wilson lines,

$$W_{\boldsymbol{m}_i}(a_1) = \gamma_{l_i;1}^{\dagger}\gamma_{l_i;2}, \quad W_{\boldsymbol{m}_i'}(b_1) = \gamma_{l_i;2}^{\dagger}\gamma_{l_i;3}, \quad (80)$$

we obtain the result (69) for the action of B_{12}, with the phases

$$e^{i\theta_{\boldsymbol{m}}} = e^{2\pi i \vec{q} \cdot (\vec{\alpha}_{12} - \vec{\alpha}_{1'2'})}, \quad e^{i\theta_{\boldsymbol{m}}'} = e^{2\pi i \vec{q} \cdot (\vec{\alpha}_{1'2'} + \vec{\alpha}_{22'})}, \quad (81)$$

where the N-component integer vector $\vec{q}$ is defined by $\boldsymbol{m} = \sum_{i=1}^{N} q_i \boldsymbol{m}_i$.

Using these results, we can explicitly derive the braid matrix, in a given basis. As explained in Sec. V A, we can label the ground states associated with the defects 1, 2 in terms of eigenvalues $\vec{\alpha}$ of $A_{\vec{q}} \equiv W_{\sum_i q_i \boldsymbol{m}_i}(a_1)$: $A_{\vec{q}}|\vec{\alpha}\rangle = e^{2\pi i \vec{q} \cdot \vec{\alpha}}|\vec{\alpha}\rangle$. In this basis, $B_{\vec{q}} \equiv W_{\sum_i q_i \boldsymbol{m}_i'}(b_1)$ act as ladder operators, so that

$$|\vec{\alpha}\rangle = B_{\vec{q}}|0\rangle, \quad (82)$$

where

$$\vec{\alpha} = \tilde{R}\vec{q} \quad (83)$$

[see Eq. (38) for a definition of $\tilde{R}$]. Therefore, the braid matrix is

$$B_{12}^{\dagger}|\vec{\alpha}\rangle = B_{12}^{\dagger}B_{\vec{q}}|0\rangle = e^{i\phi}B_{12}^{\dagger}B_{\vec{q}}B_{12}|0\rangle$$
$$= e^{i\phi + i\theta_{\boldsymbol{m}}'}A_{\vec{q}}^{\dagger}B_{\vec{q}}|0\rangle = e^{i\phi + \theta_{\boldsymbol{m}}'}e^{-i2\pi \vec{q}^T \tilde{R}\vec{q}}|\alpha\rangle. \quad (84)$$

Here $e^{i\phi}$ is an undetermined phase, associated with the eigenvalue $B_{12}^{\dagger}|0\rangle$, and $\boldsymbol{m}$ is defined by $\vec{q}$: $\boldsymbol{m} = \sum_i q_i \boldsymbol{m}_i$.

Therefore, in this basis,

$$(B_{12}^\dagger)_{\vec{q}\vec{q}'} = \delta_{\vec{q}\vec{q}'} e^{i\phi} e^{i2\pi i \vec{q} \cdot (\vec{\alpha}_{1'2'} + \vec{\alpha}_{22'})} e^{-i2\pi \vec{q}^T \tilde{R} \vec{q}}, \qquad (85)$$

where $\vec{q}$ and $\vec{q}'$ are related to the eigenvalues $\vec{\alpha}$ and $\vec{\alpha}'$ via $\tilde{R}$, as shown in Eq. (83).

The above sequence of Hamiltonians can be thought of as inducing interactions between the defects, which leads to the quasiparticle tunneling between the defects. As discussed in Ref. 57, adiabatically tuning such interactions is equivalent to performing a sequence of pairwise projections on the subspace associated with different pairs of defects. Various sequences of such projections can lead to a nontrivial unitary transformation on the original space of ground states and effectively carries out a braiding process. Since the time-dependent Hamiltonian simply realizes these projection operators, it is clear that small deformations of the path in Hamiltonian space will not affect the result except up to an overall phase, as long as $H(\tau)$, when considered in the Hilbert space of the defects 1, 2, 1′, 2′, has a D-dimensional ground-state degeneracy throughout entire the process, with a finite gap to other excited states. Therefore, the resulting non-Abelian Berry phase is *topologically protected*. Since the overall phase is not topological, we refer to this as a *projective* realization of non-Abelian statistics. That the braiding operations indeed satisfy projectively the defining relations of the braid group follows from the fact that we have shown they can be mapped to modular transformations of an Abelian CS theory on a high genus surface, which are known to form a projective representation of the braid group.

VI. CRITICAL PHENOMENA BETWEEN GAPPED EDGE STATES

A defining property of topologically distinct gapped edges is that it is not possible to adiabatically tune from one to the other without closing the energy gap on the edge. This raises the question of whether we can understand the critical phenomena between different gapped edges in terms of the topological properties of the different gapped edges.

Let us focus on the transition between two different kinds of gapped edges, labeled by Lagrangian subgroups M and M'. One way to understand a transition between the M edge and the M' edge is as follows. Consider starting with the M edge and nucleating N pairs of point defects that enclose the M' edge, as shown in Fig. 11. Next, the size of the M' regions is increased, until the M regions shrink to zero. This process can be described by the following Hamiltonian, which acts in the topologically degenerate subspace of the N pairs of point defects,

$$H_{\text{edge}} = \sum_{i=1}^{N} (A_i + B_i),$$

$$A_i = \sum_{\boldsymbol{m} \in M} t_{\boldsymbol{m}} W_{\boldsymbol{m}}(c_{2i-1}) + \text{H.c.}, \qquad (86)$$

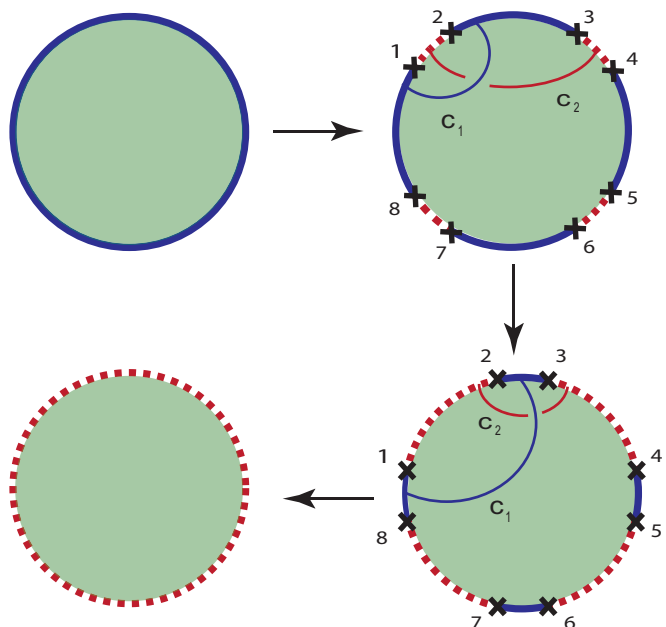$$B_i = \sum_{\boldsymbol{m}' \in M'} \tilde{t}_{\boldsymbol{m}'} W_{\boldsymbol{m}'}(c_{2i}) + \text{H.c.},$$

FIG. 11. (Color online) Starting with a gapped edge corresponding to a Lagrangian subgroup M, we can nucleate pairs of domain walls enclosing gapped regions corresponding to a Lagrangian subgroup M', separate them, and reannihilate them in pairs, leaving the M' gapped edge. In the M phase, the dominant quasiparticle tunnelings are along the c_{2i-1} paths enclosing the M' regions. In the M' phase, the dominant quasiparticle tunnelings are along the c_{2i} paths enclosing the M regions.

where $i = 2N + 1$ and $i = 1$ label the same domain wall. The paths c_i are defined to enclose the defects i and $i + 1$, as shown in Fig. 11.

Here we have included only Wilson lines that enclose one pair of defects to obtain a model Hamiltonian that can describe the transitions between different edge states. A Hamiltonian that is more physically realistic for a given microscopic setup may also include longer-range tunneling terms.

In the limit where the first sum dominates, $t_{\boldsymbol{m}} \gg t_{\boldsymbol{m}'} \forall \boldsymbol{m} \in M, \boldsymbol{m}' \in M'$, the degeneracy is lifted due to $\boldsymbol{m}$ quasiparticles tunneling around the M' domains, which corresponds to the case where the M' regions are small and the edge is in the M phase. On the other hand, in the limit where the second sum dominates, $\tilde{t}_{\boldsymbol{m}'} \gg t_{\boldsymbol{m}}$, the degeneracy is lifted due to $\boldsymbol{m}'$ quasiparticles tunneling around the M domains, which corresponds to the case where the M regions are small and the edge is in the M' phase.

H_{edge} above simply describes a 1D generalized quantum spin chain. One way to define this mapping to a spin chain is as follows. Every "site" i of the quantum spin chain can be associated with a pair $(2i, 2i + 1)$ of neighboring defects. Furthermore, each site i of the spin chain is assigned a D-dimensional Hilbert space, associated with the finite-dimensional representation of the algebra

$$W_{\boldsymbol{m}}(a_i) W_{\boldsymbol{m}'}(b_i) = W_{\boldsymbol{m}'}(b_i) W_{\boldsymbol{m}}(a_i) e^{2\pi i \boldsymbol{m}^T K^{-1} \boldsymbol{m}'}, \qquad (87)$$

which we studied in Sec. V A [see Fig. 6(b) for a definition of the paths $\{a_i\}, \{b_i\}$]. Note that the operators $W_{\boldsymbol{m}'}(c_{2i}) = W_{\boldsymbol{m}'}(b_i)$ [see Figs. 6(b) and 11]. Also note that for the disk

topology shown in Fig. 11, only $N-1$ pairs of defects are independent, due to the relations $W_{\boldsymbol{m}'}(c_{2N}) = \prod_{i=1}^{N-1} W_{\boldsymbol{m}'}(c_{2i})$ and $W_{\boldsymbol{m}}(c_{2N-1}) = \prod_{i=1}^{N-1} W_{\boldsymbol{m}}(c_{2i-1})$.

The A_i and B_i in Eq. (86) satisfy

$$
\begin{aligned}
[A_i, A_j] &= [B_i, B_j] = 0, \\
[A_i, B_j] &= 0 \quad \text{if} \quad j \neq i, i-1, \\
[A_i, B_j] &\neq 0 \quad \text{if} \quad j = i, i-1.
\end{aligned}
\tag{88}
$$

The latter nonvanishing commutation relation can be found by using the algebra of the Wilson line operators $W_{\boldsymbol{m}}(c_{2i-1})$ and $W_{\boldsymbol{m}'}(c_{2i})$. The $\{A_i\}$ can be viewed as the nearest-neighbor spin exchange interactions of a spin chain, while the $\{B_i\}$ act in analogy to transverse fields.

Alternatively, since the Wilson line operators are represented in the edge theory as bilinears in the parafermion zero modes $\gamma_{l;i}$, it is also possible to write the above Hamiltonian as a "generalized parafermion chain." Specifically, in the edge theory,

$$
\begin{aligned}
W_{\boldsymbol{m}}(c_i) &= \gamma_{\boldsymbol{m}+\boldsymbol{m}';i}^{\dagger} \gamma_{\boldsymbol{m}+\boldsymbol{m}';i+1} \\
&= \frac{1}{|M'|} \sum_{\boldsymbol{m}' \in M'} \gamma_{\boldsymbol{m}+\boldsymbol{m}';i}^{\dagger} \gamma_{\boldsymbol{m}+\boldsymbol{m}';i+1},
\end{aligned}
\tag{89}
$$

for i odd. Here $|M|$ and $|M'|$ denote the number of elements in the Lagrangian subgroups M and M', respectively. The first line above holds for any $\boldsymbol{m}' \in M'$, justifying the sum in the second line. Similarly,

$$
\begin{aligned}
W_{\boldsymbol{m}'}(c_i) &= \gamma_{\boldsymbol{m}+\boldsymbol{m}';i}^{\dagger} \gamma_{\boldsymbol{m}+\boldsymbol{m}';i+1} \\
&= \frac{1}{|M|} \sum_{\boldsymbol{m} \in M} \gamma_{\boldsymbol{m}+\boldsymbol{m}';i}^{\dagger} \gamma_{\boldsymbol{m}+\boldsymbol{m}';i+1},
\end{aligned}
\tag{90}
$$

for i even. Therefore,

$$
\begin{aligned}
A_i &= \sum_{\substack{\boldsymbol{m} \in M \\ \boldsymbol{m}' \in M'}} \bar{t}_{\boldsymbol{m}} \gamma_{\boldsymbol{m}+\boldsymbol{m}';2i-1}^{\dagger} \gamma_{\boldsymbol{m}+\boldsymbol{m}';2i} + \text{H.c.}, \\
B_i &= \sum_{\substack{\boldsymbol{m}' \in M' \\ \boldsymbol{m} \in M}} \bar{t}_{\boldsymbol{m}'} \gamma_{\boldsymbol{m}+\boldsymbol{m}';2i}^{\dagger} \gamma_{\boldsymbol{m}+\boldsymbol{m}';2i+1} + \text{H.c.},
\end{aligned}
\tag{91}
$$

where $\bar{t}_{\boldsymbol{m}} = t_{\boldsymbol{m}}/|M|$ and $\bar{t}_{\boldsymbol{m}'} = t_{\boldsymbol{m}'}/|M'|$. The algebra of the "parafermion" operators γ_{li} was given in Eqs. (65) and (66).

As we noted previously, the commutation relations of zero modes on defects and "antidefects" (i.e., the defects labeled $2i$ and $2i+1$) are equivalent. Therefore, the above Hamiltonian has an enhanced symmetry, associated with translating by one defect site, $\gamma_{l,i} \rightarrow \gamma_{l,i+1}$, and replacing $\bar{t}_{\boldsymbol{m}} \leftrightarrow \bar{t}_{\boldsymbol{m}'}$. Such self-duality can lead to powerful constraints on the phase diagram of the parafermion chains. In some simple cases,[14–16] the properties of such parafermion chains[54,58,59] are well understood, and the critical points can, for example, give rise to parafermion conformal field theories (CFTs).[53] For example, for the simple case of the FM/SC domain wall at the fractional quantum spin Hall edge,[15,16] the edge theory (86) describes Z_m parafermions.[54] For $m = 2,3$, we know that there is only one phase transition which occurs at the self-dual point of the model, so that the self-dual Hamiltonian $\bar{t}_{\boldsymbol{m}} = \bar{t}_{\boldsymbol{m}'}$ must be at the critical point.

We note that another way of studying the critical phenomena associated with different gapped edges is simply to consider the theory obtained by uniformly and simultaneously adding both sets of backscattering terms associated with M and M' edges:

$$
\delta H_{\text{edge}} = \sum_{i=1}^{N} \left[\lambda \cos\left(c_i \boldsymbol{m}_i^T \boldsymbol{\phi}\right) + \lambda' \cos\left(c_i' \boldsymbol{m}_i'^T \boldsymbol{\phi}\right) \right].
\tag{92}
$$

When $\lambda \gg \lambda' \gg 1$, the edge will condense the Lagrangian subgroup M, and when $\lambda' \gg \lambda \gg 1$ it will condense the Lagrangian subgroup M'. The critical phenomena associated with the intermediate regime will generally be described by exotic conformal field theories. In some simple cases, the behavior of such competing backscattering terms are known,[60] and give rise to parafermion CFTs, just as predicted by starting with the parafermion chains described above.

It would be interesting to develop a more general understanding of the possible critical phenomena of both the generalized parafermion chains and the competing backscattering terms in Eq. (92). Based on the simplest examples, it is natural to expect that the critical phenomena will generally be described by a suitable class of generalized parafermion CFTs, possibly including those described in Refs. 53, 61, and 62.

We note that the arguments given above also imply that as long as the domain wall between two kinds of gapped edges does not localize a topological zero mode, then this process of nucleating and reannihilating domain walls to go from one kind of edge to the other never closes the edge energy gap. Therefore, two gapped edges can always be adiabatically connected if the domain wall between them does not localize a topological zero mode. This suggests that Lagrangian subgroups provide a complete topological classification of gapped edges in the absence of any symmetries, modulo any additional purely one-dimensional topological physics in fermionic systems.

VII. CONCLUSION

In this paper, we have studied gapped boundaries between Abelian topological states, and the point defects arising from junctions among different gapped edges, vastly generalizing the twist defects studied in previous works.[13–20,22–24] Using the folding process, we argued that line defects can always be mapped to gapped boundaries between a generic topological state and the trivial gapped state, while junctions between different line defects can be understood in terms of domain walls between different classes of gapped edges separating a generic topological state and the trivial gapped state.

Using Abelian CS theory and its associated edge theory, we have proven, for topological phases of both fermions and bosons, that every Lagrangian subgroup M corresponds to a gapped edge where M is condensed. Edges corresponding to different Lagrangian subgroups are topologically distinct, in the sense that there is no way to adiabatically tune from one to the other without closing the energy gap on the edge. The physical meaning of M is to determine how quasiparticles are transmitted/reflected at the line defect. When the line defect separates two identical topological phases, M simply determines how quasiparticles are permuted by a symmetry of the topological order as they cross the line defect.

For a special class of topologically ordered states, the Z_N toric code models, we found that the gapped boundary conditions correspond to different factorizations of N. We also provided an explicit microscopic lattice model realization of each boundary condition.

We proposed a topological classification of point defects, which are domain walls between different gapped edges and are characterized by two Lagrangian subgroups, M and M', associated with the groups of quasiparticles that are condensed on either side of the domain wall. We have shown that the point defects necessarily localize topologically robust zero modes and give rise to topological ground-state degeneracies. We have derived a formula for the quantum dimension of such point defects by studying the algebra of line operators that end on the edges. We also showed that the localized zero modes are associated with a nonzero density of states at zero energy, localized to the domain wall, for subsets of quasiparticles of the form $l = m + m'$, where $m \in M$, $m \in M'$, $l \notin M, M'$.

We found that the defects can all be mapped to genons. This is defined by the property that the Wilson line algebra that determines their topological ground-state degeneracy is equivalent to the Wilson loop algebra of some Abelian CS theory on a high genus surface, with the genus proportional to the number of defects. This vastly generalizes the kinds of defects that can be understood as genons, from certain classes of twist defects[14] to more general point defects.

While the domain walls cannot be braided geometrically because they exist on a line defect, we found that we can define a notion of projective non-Abelian braiding statistics. The interactions between the edges and domain walls can be used to realize topologically protected unitary transformations on the ground-state subspace. The overall phase of these transformations is not topologically protected, which is why it is referred to as *projective* non-Abelian braiding statistics. By studying the action of this braiding on the Wilson line operators that define the ground-state subspace, we found that the projective non-Abelian statistics can be understood in terms of Dehn twists, or modular transformations, of the corresponding Abelian CS theory on the high genus surface, generalizing earlier results.[14]

For bosonic topologically ordered states, the topological zero modes occur only at domain walls between gapped edges associated with different Lagrangian subgroups. Therefore, for bosons, in the absence of any symmetries, Lagrangian subgroups fully classify the topologically distinct gapped edges.

For fermionic states, due to the possibility of Majorana zero modes protected by fermion parity symmetry in one dimension, two gapped edges can be topologically distinct even if they correspond to the same Lagrangian subgroup. Therefore, the Lagrangian subgroups classify gapped edges of fermionic systems, up to this additional Z_2 topological classification arising from purely one-dimensional physics.

Finally, we pointed out that the critical phenomena between topologically distinct gapped edges can be understood in terms of a generalized quantum spin chain or, equivalently, a generalized parafermion chain. The properties of such chains are understood only in the simplest cases.

The study here raises many interesting questions for future research. These include understanding how to extend these results to non-Abelian topological states, developing a deeper understanding of the critical phenomena between different edge states, studying the interplay of this purely topological physics with global symmetries,[63] understanding the collective phenomena of the non-Abelian point defects,[59] and extending results to higher dimensional topological states.

Another interesting direction is to develop a theory where the extrinsic defects are dynamical degrees of freedom rather than static defects. In the simpler cases of twist defects,[14] it was shown that the defects can become deconfined non-Abelian excitations of a non-Abelian topological phase if the symmetry associated with the twist defect is gauged, and the topological properties of the resulting non-Abelian state were studied in some simple cases.[14,20,64,65] A systematic generalization of these results may also clarify the relation between extrinsic defects and intrinsic quasiparticle excitations.

Noted added in proof. Recently, we became aware that the result that every Lagrangian subgroup M can correspond to a gapped edge where M is condensed was independently found by Levin and included in an updated version of Ref. 28; our work uses results from an early version of Ref. 28.

APPENDIX A: PROOF OF LEMMA

Here we prove the lemma presented in Sec. IV B. To prove this, we focus on two cases independently. In the first case, K has only even entries along the diagonals, which describes topological phases where the microscopic degrees of freedom only consist of bosons. In the second case, K can have odd entries along the diagonals, which is appropriate when the microscopic degrees of freedom have at least one species of fermions.

We note that the proof below was also presented by us recently in a shorter treatment in Ref. 37; it is included here to make the paper self-contained. Our proof closely follows and builds upon the argument in Appendix A1 of Ref. 28 (as our work was being completed, we became aware that a similar improved result was included in a revised draft of Ref. 28 in Appendix A3).

1. Proof for K even

Let us first consider the case where K is an even matrix, meaning that its diagonal entries are all even. Note that K is also an integer symmetric nonsingular matrix with vanishing signature. Consider the lattice

$$\Gamma = \{m + K\Lambda : m \in M, \Lambda \in \mathbb{Z}^{2N}\}. \tag{A1}$$

Γ is a $2N$-dimensional integer lattice and can be written as $\Gamma = U\mathbb{Z}^{2N}$, where U is a $2N$-dimensional integer matrix. Define

$$P = U^T K^{-1} U. \tag{A2}$$

P is an even integer symmetric matrix with unit determinant and nonvanishing signature.[28] The fact that it is an even integer matrix follows because the columns of U generate the Lagrangian subgroup M, and these all have bosonic mutual and self-statistics by definition. The fact that it is symmetric and has vanishing signature follows from the fact that K is symmetric and has vanishing signature. Finally, P has unit determinant for the following reason. Consider any integer vector $\Lambda \in \mathbb{Z}^{2N}$ and any noninteger $2N$-component vector, x. By definition of the Lagrangian subgroup, $\Lambda^T P x$ must be noninteger, which implies that $P x$ must be noninteger. This then implies that if $P x$ is integer for any $2N$-component vector x, then x must be integer, which in turn implies that P^{-1} is integer. P and P^{-1} can both be integer if and only if P has unit determinant.

Since P is an even symmetric integer matrix with vanishing signature and unit determinant, it follows from a mathematical theorem[66] that it is always possible to find a $SL(2N;\mathbb{Z})$ transformation W such that

$$W^T P W = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \tag{A3}$$

where I is an $N \times N$ identity matrix. Thus, we consider a transformed theory:

$$\begin{aligned} \tilde{U} &= W^T U W, \\ \tilde{K} &= W^T K W, \\ \tilde{P} &= W^T P W = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \end{aligned} \tag{A4}$$

Since $W \in SL(2N;\mathbb{Z})$, $\tilde{K}$ and K describe topologically equivalent theories. Clearly, the columns of $\tilde{U}$ generate the Lagrangian subgroup M. Let $\tilde{u}_i$ denote the ith column of $\tilde{U}$. Let us extend $\tilde{K}$ to a $4N \times 4N$ matrix K', which is composed of $\tilde{K}$ and N copies of $\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ along the block diagonal entries,

$$K' = \begin{pmatrix} \tilde{K} & & & \\ & \tau_x & & \\ & & \tau_x & \\ & & & \ddots \end{pmatrix}, \tag{A5}$$

where the rest of the entries are zero. Again, K' describes the same topological order as K. Now we define

$$\begin{aligned} m_1'^T &= (\tilde{u}_1^T, 0, 1, 0, 0, \ldots, 0, 0), \\ m_2'^T &= (\tilde{u}_{N+1}^T, -1, 0, 0, 0, \ldots, 0, 0), \\ m_3'^T &= (\tilde{u}_2^T, 0, 0, 0, 1, \ldots, 0, 0), \\ m_4'^T &= (\tilde{u}_{N+2}^T, 0, 0, -1, 0, \ldots, 0, 0), \\ &\vdots \\ m_{2N-1}'^T &= (\tilde{u}_N^T, 0, 0, \ldots, 0, 1), \\ m_{2N}'^T &= (\tilde{u}_{2N}^T, 0, 0, \ldots, -1, 0). \end{aligned} \tag{A6}$$

Since the additional components added in K' are all trivial degrees of freedom, the $2N$ vectors $\{m_i'\}$ still generate the same Lagrangian subgroup M. It is easy to see that

$$m_i'^T K'^{-1} m_j' = 0. \tag{A7}$$

This proves the lemma for K even. In practice, in most cases of interest it is easy to find N columns of $\tilde{U}$ that generate M and that satisfy $\tilde{u}_i^T K^{-1} \tilde{u}_j = 0$, so the above extension to a $4N$-dimensional K matrix will not be necessary.

2. Proof for K odd

Let us now consider the case where K is odd (i.e., it has at least one odd element along the diagonal). As before, we define the matrix U, and $P = U^T K^{-1} U$. Now, P is an integer symmetric, nonsingular matrix with unit determinant, nonvanishing signature, and at least one odd element along the diagonal. Under these conditions, it is always possible to find $W \in SL(2N;\mathbb{Z})$ such that[66]

$$W^T P W = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \tag{A8}$$

where I is an $N \times N$ identity matrix. Thus, we consider a transformed theory:

$$\begin{aligned} \tilde{U} &= W^T U W, \\ \tilde{K} &= W^T K W, \\ \tilde{P} &= W^T P W = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}. \end{aligned} \tag{A9}$$

Again, the original Lagrangian subgroup is generated by the columns of $\tilde{U}$.

Let $\tilde{u}_i$ denote the ith column of $\tilde{U}$, and $\tilde{u}_{i\pm} = \tilde{u}_i \pm \tilde{u}_{N+i}$, for $i = 1, \ldots, N$.

Now, as in the case where K is even, let us extend the K matrix to a $4N \times 4N$ matrix K', which is composed of K and now with N copies of $\tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ along the block diagonal entries,

$$K' = \begin{pmatrix} \tilde{K} & & & \\ & \tau_z & & \\ & & \tau_z & \\ & & & \ddots \end{pmatrix}, \tag{A10}$$

where the rest of the entries are zero. In the absence of any symmetries, K' describes the same topological order as K. Now we define

$$\begin{aligned} m_1'^T &= (\tilde{u}_{1+}^T, 1, 1, 0, 0, \ldots, 0, 0), \\ m_2'^T &= (\tilde{u}_{1-}^T, -1, 1, 0, 0, \ldots, 0, 0), \\ m_3'^T &= (\tilde{u}_{2+}^T, 0, 0, 1, 1, \ldots, 0, 0), \\ m_4'^T &= (\tilde{u}_{2-}^T, 0, 0, -1, 1, \ldots, 0, 0), \\ &\vdots \\ m_{2N-1}'^T &= (\tilde{u}_{N+}^T, 0, 0, \ldots, 1, 1), \\ m_{2N}'^T &= (\tilde{u}_{N-}^T, 0, 0, \ldots, -1, 1). \end{aligned} \tag{A11}$$

Since the additional components added to K' are all trivial degrees of freedom, the $2N$ vectors $\{m_i'\}$ still generate the same Lagrangian subgroup M. It is easy to see that

$$m_i'^T K'^{-1} m_j' = 0. \tag{A12}$$

This proves the lemma for K odd.

APPENDIX B: GROUND-STATE DEGENERACY OF DEFECT-COUPLING HAMILTONIANS H_{ab}

Here we explain the claim made in Sec. V D, that the Hamitonians H_{12}, $H_{1'2'}$, and $H_{22'}$, defined in Eq. (70), have D degenerate ground states in the Hilbert space defined by defects 1,2,1′, 2′ (see Fig. 10).

First, we observe that the Hilbert space defined by the defects 1,2,1′, 2′ is D^2-dimensional, and forms a representation of the following Wilson line algebra:

$$W_{\boldsymbol{m}}(a_i)W_{\boldsymbol{m}'}(b_i) = W_{\boldsymbol{m}'}(b_i)W_{\boldsymbol{m}}(a_i)e^{\delta_{ij}i2\pi \boldsymbol{m}^T K^{-1}\boldsymbol{m}'}, \quad (B1)$$

for $i,j = 1,2,1',2'$, and $\boldsymbol{m} \in M$, $\boldsymbol{m}' \in M'$. This forms two independent copies of the same algebra, studied in Sec. V A, each with a D-dimensional irreducible representation. The D^2 states can each be labeled by the eigenvalues of $W_{\boldsymbol{m}}(b_i)$:

$$W_{\boldsymbol{m}}(b_i)|\vec{\alpha}_1\vec{\alpha}_2\rangle = e^{2\pi i \vec{q}\cdot\vec{\alpha}_i}|\vec{\alpha}_1\vec{\alpha}_2\rangle, \quad (B2)$$

where $\boldsymbol{m} = \sum_{i=1}^{N} q_i \boldsymbol{m}_i$ and $\vec{\alpha}_i$ is a rational-valued N-component vector.

Let us begin by considering

$$H_{12} = \sum_{i=1}^{N} t_i \gamma_{l_i;1}^{\dagger}\gamma_{l_i;2} + \text{H.c.}$$
$$= \sum_{i=1}^{N} t_i W_{\boldsymbol{m}_i}(b_1) + \text{H.c.} \quad (B3)$$

H_{12} is therefore diagonal in the basis of (B2) and given by

$$H_{12}|\vec{\alpha}_1\vec{\alpha}_2\rangle = \sum_{i=1}^{N} 2|t_i|\cos[2\pi(\vec{\alpha}_1)_i + \phi_i]|\vec{\alpha}_1\vec{\alpha}_2\rangle, \quad (B4)$$

where $t_i = |t_i|e^{i\phi_i}$. We see that H_{12} is independent of the D distinct eigenvalues $\vec{\alpha}_2$. Furthermore, unless there are

accidental degeneracies, generic choices of ϕ_i will give distinct energies to the different eigenvalues $\vec{\alpha}_1$, and there will be a unique choice of $\vec{\alpha}_1$ that minimizes the energy. Therefore, H_{12} generically has D distinct ground states.

The analysis for $H_{1'2'}$ is identical, except now there will be a unique choice of $\vec{\alpha}_2$ which minimizes the energy, while there are D degenerate states associated with the possible values of $\vec{\alpha}_1$.

Now let us consider Hamiltonians of the form $H_{11'}$ and $H_{22'}$. First consider $H_{11'}$:

$$H_{11'} = \sum_{i=1}^{N} t_i \gamma_{l_i;1}^{\dagger}\gamma_{l_i;1'} + \text{H.c.}$$
$$= \sum_{i=1}^{N} t_i W_{\boldsymbol{m}_i}(a_1)W_{\boldsymbol{m}_i}(b_1) + \text{H.c.} \quad (B5)$$

Let us define the operators

$$\overline{W}_{\boldsymbol{m}}(a_1) \equiv W_{\boldsymbol{m}}(a_1)W_{\boldsymbol{m}'}(b_1),$$
$$\overline{W}_{\boldsymbol{m}}(a_2) \equiv W_{\boldsymbol{m}}(a_2), \quad (B6)$$
$$\overline{W}_{\boldsymbol{m}'}(b_i) \equiv W_{\boldsymbol{m}'}(b_i).$$

These operators satisfy the same algebra as in (B1):

$$\overline{W}_{\boldsymbol{m}}(a_i)\overline{W}_{\boldsymbol{m}'}(b_i) = \overline{W}_{\boldsymbol{m}'}(b_i)\overline{W}_{\boldsymbol{m}}(a_i)e^{\delta_{ij}i2\pi \boldsymbol{m}^T K^{-1}\boldsymbol{m}'}. \quad (B7)$$

In terms of these operators, $H_{11'}$ becomes

$$H_{11'} = \sum_{i=1}^{N} t_i \overline{W}_{\boldsymbol{m}_i}(a_1) + \text{H.c.} \quad (B8)$$

Now we can directly take over the analysis of H_{12} to find that $H_{11'}$ also generically has D ground states. By symmetry, the result follows for $H_{22'}$ as well.

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