Long-range random-field Ising model: Phase transition threshold and equivalence of short and long ranges

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The one-dimensional Ising model in a random field and with power-law decaying ferromagnetic bonds is studied at zero temperature. Comparing the scaling of the energy contributions of the ferromagnetic domain wall flip and of the random field *à la* Imry-Ma, a threshold value for the power ρ of the long-range interaction can be determined, beyond which no critical behavior occurs. The critical threshold value is $\rho_c = 3/2$, at a difference with the zero field model in which $\rho_c = 2$. This prediction is analyzed by numerical computation of the ground states below, at, and above this threshold value. The analogy between the critical behavior of long-range one-dimensional systems and in short-range D-dimensional systems is investigated. At the critical threshold value of ρ, corresponding to the lower critical dimension, numerical evidence is found for a zero temperature transition at a finite critical field. Possible finite size crossover effects are discussed in this case. Some implications for the critical behavior of spin glasses in a field are conjectured.

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I. INTRODUCTION

According to the well-known Imry-Ma argument,[1–3] the random field Ising model (RFIM) with nearest-neighbor interaction does not display any spontaneous magnetization in $D \leqslant 2$. Spontaneous magnetization is, instead, present in $D = 3$ where a rigorous result has shown the occurrence of a finite dimensional phase transition,[4,5] and numerical simulations[6–10] confirm this result. Further analysis by numerical simulations,[11,12] rigorous approach,[13] perturbation theory,[14] and RG transformations[15,16] have shown that $D = 2$ is, actually, the lower critical dimension, and no phase transition takes place in dimension two,[17,18] both for $T > 0$ and at $T = 0$. In the latter case the relevant variable is the strength of the random magnetic field, i.e., the square root of its variance, normalized to the ferromagnetic coupling. Renormalization group arguments show that the finite temperature transition is dominated by the zero temperature fixed point.

In the present paper we report on our investigation of the zero temperature critical behavior in a one-dimensional RFIM with long-range (LR) power-law decaying interaction. Given any two sites i and j at distance r_{ij} their interaction decays like

$$J_{ij} \sim r_{ij}^{-\rho} \ .$$

Our aim is twofold: (i) to characterize the behavior of the system at threshold value of the power $\rho = \rho_c$ above which the system does not undergo any phase transition; (ii) to gain insight about the correspondence between LR models with a certain power of the interaction decay and short-range (SR) models in a given dimension D in the presence of a field. We will extend to bond diluted systems the rigorous prediction[19] that the critical threshold value for the power corresponding to a lower critical dimension is $\rho_c = 1.5$ in the 1D RFIM. This is $\rho_c = 2$ in the 1D ferromagnetic model in the absence of a field, modeling the well-known Kondo problem.[20–22] Such a difference has direct consequences on the determination of the lower critical dimension in the presence of a field by means of the analogy between long-range (LR) and short-range (SR) systems.

A. SR↔LR connection with no field

We recall that a quantitative relationship can be established between the power-law ρ of the LR interaction decay in a 1D lattice and the dimension D of a SR system displaying the same critical behavior. The requirement that the renormalized coupling constant has the same scaling dimension leads to

$$\rho - 1 = \frac{2}{D}. \tag{1}$$

Below the upper critical dimension (UCD), though, i.e., for $\rho > \rho_{\rm mf}$,[23] such a relationship is not exact anymore. Moreover, it grossly fails at the lower critical dimension (LCD), $D = 1$ for the purely ferromagnetic model, predicting a $\rho_c = 3 > 2$ in a 1D LR chain. We can improve Eq. (1) by looking at the behavior of the renormalized space correlation function at criticality in the SR model: $C(r) \sim r^{-D+2-\eta_{\rm sr}(D)}$. Requiring that at the LCD the correlation function does not display any power-law critical decay, i.e., $D = 2 - \eta_{\rm sr}(D)$ and imposing the correct $\rho_c = 2$, Eq. (1) is modified as

$$\rho - 1 = \frac{2 - \eta_{\rm sr}(D)}{D}. \tag{2}$$

By construction it is exact at the LCD. The same relation holds for Heisenberg ferromagnets, where at the LCD ($D = 2$), $\eta_{\rm sr}(D) = 0$. Equation (2) has been first obtained, in the framework of spin glasses, by comparing the singular part of the free energy per spin in a LR system of $N = L^d$ spins and in a D-dimensional SR system with the same number of spins, $N = L^D$. The magnetic scaling exponents turn out to follow the relationship $y_h^{\rm lr} = y_h^{\rm sr}(D)/D$, being $2y_h = D + 2 - \eta$.[24,25] Since in LR models, both with and without quenched disorder, the two point vertex function is not renormalized and $\eta_{\rm lr} = 3 - \rho$[26–28] also in the infrared divergence regime, Eq. (2) is recovered.[24,25,29,30]

Equation (2) states that the critical behavior of the two models, i.e., the D-dim. SR and the 1D LR models, should be similar for all (ρ, D) couples between $(\rho_{\rm mf}, UCD)$ and (ρ_c, LCD) for $\rho < \rho_{\rm mf} = 3/2$ the system is in the mean-field regime.

In the SR Ising model this corresponds to $D > D_{\rm UCD} = 4$. As $D < D_{\rm UCD}$ infrared divergences occur in the vertex functions and a nonzero anomalous exponent. In $D = 3$, a good numerical estimate is $\eta = 0.031(5)$,[31] corresponding to $\rho = 1.656(2)$. In $D = 2$, Onsager solution yields $\eta_{\rm sr} = 1/4$ and the system is "critically equivalent" to the $\rho = 15/8$ LR model.

By direct inspection, it is known that no transition is present at $\rho > \rho_c = 2$. Exactly at $\rho = 2$, though, a phase transition does occur. This is the Kondo transition in 1D magnetic chains.[22] On the contrary, the SR 1D Ising chain does not display any critical point. This discrepancy is, actually, not unusual and it is due to a direct long interaction of interfaces in LR models. The critical behavior of the LR model at ρ_c and of the SR model exactly at the LCD is often different: In some instances no transition is present in the SR model, while a transition may be present in the corresponding LR model. Rigorous results for the LR RFIM, though, predict no zero temperature transition at $\rho = \rho_c = 1.5$.[17,18] We anticipate that, in the present work we actually find numerical evidence compatible with a $T = 0$ fixed point with logarithmic scaling in the LR model at $\rho = 1.5$. We will discuss the issue in Sec. III.

Equation (2) appears to hold also for systems with quenched bond disorder, the so-called spin glasses,[25] in which a rigorous result confirms $\rho_c = 2$.[32,33] Only the mean-field threshold value of ρ is modified, because the relevant interaction term at criticality, and, thus, the UCD is different: $\rho_{\rm mf}^{\rm sg} = 4/3$.[27,28]

B. SR↔LR connection in a field

As an external field is switched on a new critical fixed point arises that is different from the zero-field fixed point. This is true both for systems with and without quenched bond disorder. Lower and upper critical dimensions appear not to decrease in all known cases. In particular, the critical dimensions of the RFIM increase to become $D_{UCD} = 6$ and $D_{LCD} = 2$.

The extension of Eq. (2) to the random magnetic case requires some care. Different definitions of the exponent $\eta_{\rm sr}$ are indeed possible since connected and disconnected correlation functions decay differently and hyperscaling does not hold.[6,15,34–36] One defines an exponent $\bar{\eta}_{\rm sr}$ by the condition that the Fourier transform of spin-spin *disconnected* correlation behaves in momentum space as $k^{-4+\bar{\eta}_{\rm sr}}$, or equivalently in position space $C_{\rm sr}^{\rm disc}(r) \sim r^{-D+4-\bar{\eta}_{\rm sr}(D)}$, where the Schwartz-Soffer inequality holds: $\bar{\eta}_{\rm sr} \leqslant 2\eta_{\rm sr}$.[35] The difference between 2η and $\bar{\eta}$ decreases with the dimension,[9,10] eventually tending to zero at the LCD.

We now present our study of the 1D LR RFIM. First, we describe an Imry-Ma-like argument predicting $\rho_c = 1.5$. A rigorous proof of this threshold value for the fully connected version of the model can be found in Refs. 17 and 18 in the case of a continuous distribution of random fields. Further, we analyze the critical behavior of the model at $\rho \sim \rho_c$ by means of numerical computations of the ground state properties at $T = 0$ as a function of the strength of the ferromagnetic interaction J.

II. THE LONG-RANGE RFIM AND THE IMRY-MA ARGUMENT

The Hamiltonian of the LR 1D RFIM is

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j - \sum_i h_i s_i, \qquad (3)$$

where $s_i = \pm 1$, $J_{ij} = J|i-j|^{-\rho}$, and h_i is a random field with a Gaussian distribution of zero mean and variance h^2.

In an ordinary ferromagnet the cost to flip a domain of spins of length L grows like $L^{2-\rho}$. As the random field is switched on this will compete with the energy of the orientation along the field going like $L^{1/2}$. According to the argument developed by Imry and Ma for SR D-dimensional systems,[1] as $\rho > 1.5$ there will always be a size large enough for the field to destroy any ferromagnetic domain, and no long-range order can be established. The exponent value $\rho = 1.5$ should, therefore, be the analog of the LCD in nearest-neighbor interacting D-dimensional RFIM, i.e., $D = 2$.[1–5,9,10,17,18]

A. Lévy lattice

In order to validate this analytic prediction we performed numerical estimates of the ground state properties at $T = 0$ for the 1D RFIM on a Lévy lattice, that is a finite connectivity random graph equivalent to a fully connected LR model.[37] In this dilute graph two sites i and j are connected (i.e., $J_{ij} \neq 0$) with a probability

$$P(J_{ij} = J) = \frac{|i-j|^{-\rho}}{\sum_r r^{-\rho}}, \qquad (4)$$

where the sum runs over all possible distances realizable on the 1D chain of length L and such that the total number of bonds is independent from ρ and equal to zL, where z is the average spin connectivity. For ρ large enough one has a nearest-neighbor chain, whereas for $\rho = 0$ the distribution of the connectivities is Poissonian and the system corresponds to an Erdös-Rényi graph.

III. NUMERICAL RESULTS

Using the Minimum Cut algorithm (see, e.g., Refs. 9,38, and 39) of the Lemon Graph Library[40] we have computed thermodynamic observables on $T = 0$ ground states of Lévy graphs of different length, averaging over different realizations of random fields. The computation has been performed varying the ferromagnetic coupling magnitude J. The random-field mean square displacement is kept constant, $h = 1$.

First, we present the numerical results at $\rho = \rho_c = 1.5$ where we used sizes ranging from $L = 250$ to $L = 256\,000$. The number of samples of disorder are 10 000 for $L \leqslant 64\,000$, 5000 at $L = 128\,000$ and 2000 at $L = 256\,000$. For each sample we compute the ground state for 41 values of the J coupling in the interval [0.2 : 0.4].

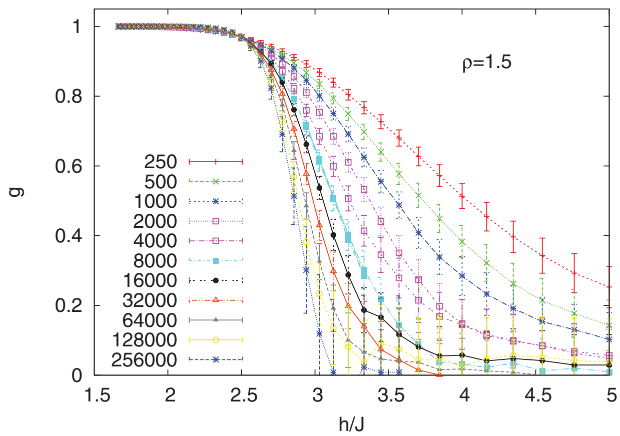

FIG. 1. (Color online) Finite size Binder cumulants versus the strength of the random field in J units at $\rho = 1.5$. The critical value estimate by FSS analysis is $(h/J)_c = 2.31(5)$.

To understand whether a critical behavior is there we study the finite size behavior of the Binder cumulant

$$g = \frac{1}{2}\left(3 - \frac{\overline{\langle s\rangle^4}}{\overline{\langle s\rangle^2}^2}\right). \tag{5}$$

If, in the thermodynamic limit, a phase transition occurs at a given critical field h_c, the Binder cumulant will be one (long-range order) for $h < h_c$ and zero for $h > h_c$. As L increases, we observe that the various Binder curves tend to a limiting curve, cf. Fig. 1, with a behavior that we will show compatible with a logarithmic scaling decay.

To estimate the critical point we look at the values of h/J that, for different sizes, yield the same g value. Specifically, in Fig. 2 we show the behavior of the limiting inverse critical field value $(J/h)_c$ as computed at every size for different fixed values of the Binder cumulant ($g = 0.3, 0.4, 0.5, 0.6, 0.7$) versus $(\log L)^{-1}$. In the $L \to \infty$ limit all curves are compatible with a multiple linear fit in $1/\ln L$ yielding the estimate $(J/h)_c = 0.433(9)$, or $(h/J)_c = 2.31(5)$.

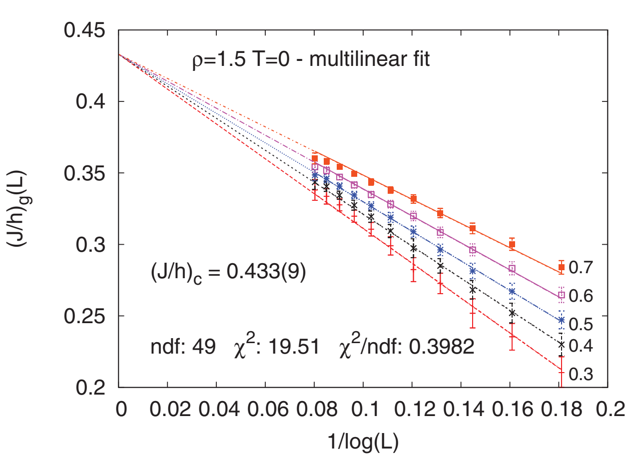

FIG. 2. (Color online) Multilinear interpolation for the J/h value at which finite size Binder cumulants significantly change value ($g = 0.3, 0.4, 0.5, 0.6$, and 0.7) with $J_L(\ln L|g) = J_c + b_g/\ln L$.

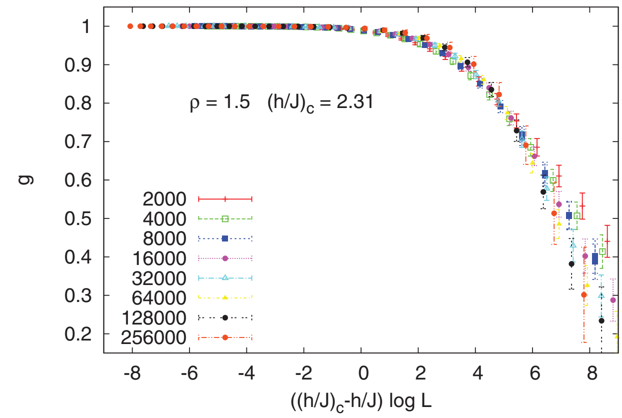

FIG. 3. (Color online) Rescaled Binder cumulant versus h/J at $\rho = 1.5$.

In Fig. 3 we plot the curves in the rescaled variable and observe a very good collapse in the critical region. To further characterize the transition we look at the behavior of magnetization momenta at the critical point. In Fig. 4 we thus present the behavior of the squared magnetization around the estimated critical value compatible with a logarithmic finite size scaling (FSS).

As a complementary analysis we perform a scaling analysis for the correlation length ξ of the two point disconnected correlation function $m_2 = \overline{\langle s\rangle\langle s\rangle}$. The infinite size limit of the zero temperature correlation length ξ_J at a given J/h value is estimated from the length parameter ratio ξ_J/ξ'_J, yielding the best data collapse of the curves $m_2(L/\xi_J)$ and $m_2(L/\xi'_J)$, shown in Fig. 5 for $J = 0.32, \ldots, 0.40$. The reference value $\xi_J = 1$ is taken in the large field limit, far from the transition point. Operatively, we take $J/h = 0.32$ as large field reference in the present case. In the inset of Fig. 5 we show that, for the simulated system sizes, ξ_J scales in J/h with an exponential law

$$\xi(J) \propto \exp\frac{A}{J/h - (J/h)_c}. \tag{6}$$

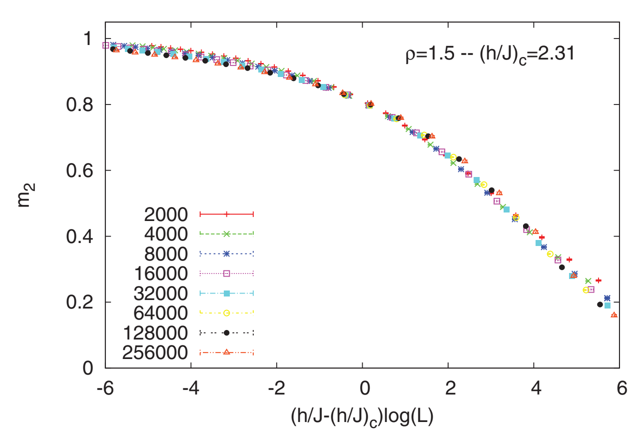

FIG. 4. (Color online) Rescaled $m_2 = \overline{\langle s\rangle^2}$ curves versus h/J at $\rho = 1.5$.

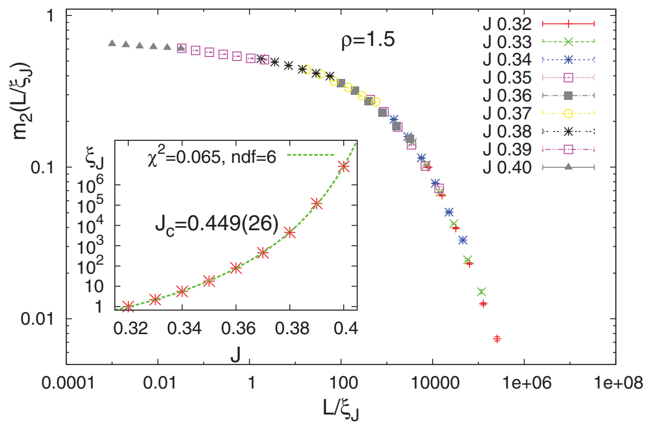

FIG. 5. (Color online) Disconnected correlation function scaling versus J/h at $\rho = 1.5$. Inset: exponential divergence for correlation length vs J/h at $J_c = 0.449(26)$.

A three parameter fit yields $(J/h)_c = 0.449(26)$ [$(h/J)_c = 2.23(13)$] consistent with the previous estimate by scaling analysis of the Binder cumulant.

A. Below and above ρ_c

We also present the behavior of the Binder cumulant for values of the power ρ slightly below and above ρ_c. For $\rho = 1.4$ and $\rho = 1.6$ we compute the ground states of systems of size between $L = 250$ and $L = 128\,000$ averaging over 10 000 disordered field configurations on 51 J values.

At $\rho = 1.4$, cf. Fig. 6, Binder curves cross each other at finite h/J and a FSS analysis of the crossing points yields a critical value $(h/J)_c = 3.23(7)$. Using the scaling property of the disconnected correlation function $\overline{\langle s\rangle^2} \sim L^{3-\bar\eta_{\rm lr}}$, with $\bar\eta_{\rm lr} = 2(3-\rho)$ (holding for $\rho \in [1,3/2]$), from the FSS of the crossing points we further obtain the estimate $(h/J)_c = 3.266(2)$ and from the FSS of the derivatives of $\overline{\langle s\rangle^2}$ we estimate $1/\nu = 0.316(9)$, cf. inset of Fig. 6. For $\rho = 1.6$, on the contrary, no crossing is observable and the nonzero Binder values continuously run away towards smaller and smaller fields, cf. Fig. 7, compatible with the claim of absence of transition above $\rho = 1.5$.

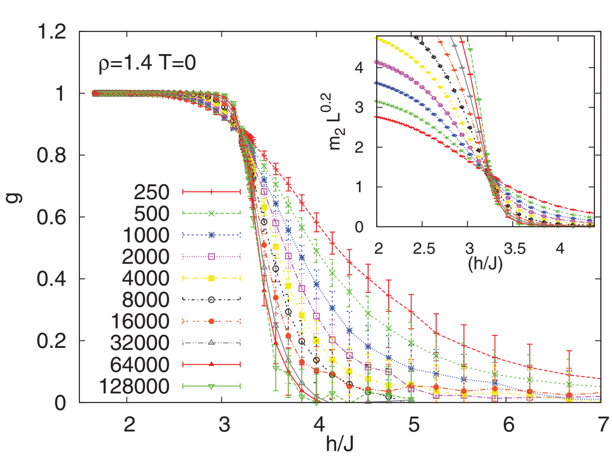

FIG. 6. (Color online) Finite size Binder cumulants at $\rho = 1.4$ for $L = 250,\ldots,128\,000$. The critical field estimate is $(h/J)_c = 3.23(7)$. Inset: scale invariant $m_2 L^{0.2} = \overline{\langle s\rangle^2} L^{\bar\eta - 3}$ vs h/J, $\bar\eta = 6 - 2\rho$.

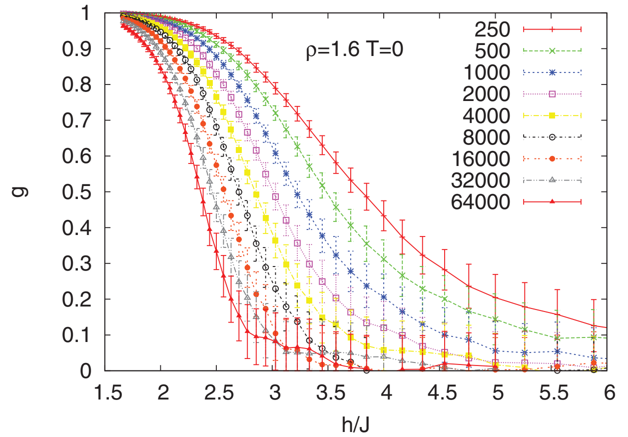

FIG. 7. (Color online) Finite size Binder cumulants at $\rho = 1.6$ for $L = 250,\ldots,64\,000$.

IV. CONCLUSION AND DISCUSSION

Equation (2) for Ising systems appears to be a fairly good approximation for what concerns the transition without field. With no field, at $\rho = 3/2$, Eq. (2) would predict a mean-field transition (corresponding to $D = 4$); non-mean-field transitions would be expected at $\rho = 1.6545$ [corresponding to $D = 3$, with $\eta_{\rm sr}(3) = 0.0364(5)$],[41] and $\rho = 1.875$ [corresponding to $D = 2$, $\eta_{\rm sr}(2) = 1/4$];[42] eventually, the "LCD"-equivalent exponent value corresponding to $D = 1$ [$\eta_{\rm sr}(1) = 1$] would be $\rho_c = 2$. Such predictions have been recently numerically investigated showing that the critical exponents at $\rho = 1.6546$ and 1.875 do not strictly correspond to, respectively, $2D$ and $3D$ critical exponents:[43] If for 3D, nearer to the mean-field threshold, numerical estimates are still consistent with each other, in 2D they appear not compatible anymore. A similar trend has been identified in LR versions of the ferromagnetic XY model[30] and of the Ising spin-glasses.[25]

When the random field is switched on and a new fixed point for the RG flow arises the situation changes. The mean-field threshold is now $\rho_{\rm mf} = 4/3$ (UCD = 6). In the present work we clearly see that the reference value of ρ for the critical threshold is also different from those obtained in the absence of a field: $\rho_c = 1.5$. We obtain such evidence by means of a numerical study of the zero temperature ground states of the RFIM on a Lévy lattice for system sizes ranging from 250 to 256 000 spins, confirming rigorous results obtained in fully connected lattices with continuous probability distribution of the fields.[17–19]

Contrarily to the prediction of Refs. 17 and 18 though, exactly at $\rho = 1.5$ a $T = 0$ fixed point is still present even if displaying a logarithmic scaling rather than a power-law one. Aizenman and Wehr, under the hypothesis of a continuous field distribution, indeed, predicted that the average magnetization over the random-field distribution would go to zero for any positive field at any temperature, including $T = 0$. In Refs. 17 and 18 the average magnetization was obtained from the

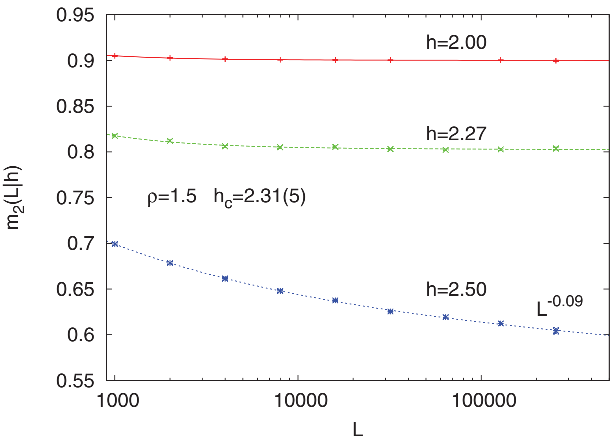

FIG. 8. (Color online) Average squared magnetization m_2 versus L at $\rho = 1.5$ for three values of the mean square displacement of the field. The lines are double power-law decay interpolations in the interval $L = 1000, \ldots, 256\,000$. Above $(h/J)_c = 2.31(5)$ m_2 decays to zero asymptotically with a power compatible with 0.09(5). For $h = 2.27$ and $h = 2.0$ no decay to zero is observed for the simulated sizes.

average of the derivative of the difference in the free energy between the + and the − boundary conditions.

We have carefully analyzed our data, in particular the behavior of the largest simulated sizes of the average squared magnetization (that is also the integrated two point correlation function), but we found no evidence for any decreasing with the size for the largest simulated sizes. As an illustrative instance we show in Fig. 8 the behavior of m_2 for h below the estimated h_c and h above it. Even though for $h > h_c$ $m_2(L)$ appears to decrease, for $h < h_c$ we were not able to detect any decreasing. Not even a $1/\ln\ln(L)$ decay. The difference is unlikely to be due to a discrete distribution of the random fields, since we use Gaussian normal random variables with a numerical discretization of the order of 10^{-5}. We can, then, hypothesize a finite size crossover at sizes much larger than the largest simulated size ($L = 256\,000$) to regime where m_2 decreases to zero with the size and the correlation length does not increase [like in Eq. (6)]. In the absence of an analytic upper bound for the scaling to zero of the magnetization though, the estimate of the crossover size appears unfeasible to us.

In the framework of the LR↔SR critical equivalence this is also in contrast to what happens in the 2D RFIM model where the zero temperature transition is at zero field. This would not be a strong issue though, since a similar LR↔SR mismatch takes place varying the temperature in the Ising model without field: No finite T transition takes place in $D = 1$ in the SR model but a Kondo, Kosterlitz-Thouless-like, finite temperature transition is there for the LR 1D chain at $\rho = \rho_c = 2$.

In the presence of a random field we can reformulate the "ρ-D" relationship Eq. (2) in terms of the anomalous exponent $\bar{\eta}_{\rm sr}(D)$, rather than $\eta_{\rm sr}(D)$. That is, we consider the most divergent correlation function at criticality in a SR system in dimension D: the disconnected one. At the lower critical dimension ($D = 2$), where $D - 4 + \bar{\eta}_{\rm sr}(D) = 0$, the threshold value of the power ρ has to be equal to the maximum one compatible with the existence of a transition: $\rho_c = 3/2$. This leads to

$$\rho - 1 = \frac{2 - \bar{\eta}_{\rm sr}(D)/2}{D} \qquad (7)$$

yielding the value of ρ corresponding to a SR model in D dimensions. As Eq. (2) in zero field, Eq. (7) is exact, at all events, at $D =$ UCD and LCD. Since in the latter case $\bar{\eta}_{\rm sr} \simeq 2\eta_{\rm sr}$, we notice that in this particular case Eq. (7) coincides with Eq. (2). For both Eqs. (2) and (7) the LCD equivalent value of ρ is the correct one: $\rho_c = 1.5$.

It is important to stress that a given value of ρ corresponds to completely different critical behaviors and to different dimensions of short-range critically equivalent systems if the field is present or absent. As an instance, $\rho = 1.5$ is the mean-field threshold in the Ising ferromagnetic model, corresponding to UCD $D = 4$, and it is the critical threshold in the RFIM, corresponding to LCD $D = 2$.

Does this relationship hold also in the presence of *random bonds*, besides *random fields*? The Imry-Ma argument is specific for the RFIM and cannot be exported to spin glasses because these more complicated systems lack any long-range order in the frozen phase. Unless one takes for granted the droplet ansatz of Fisher and Huse,[44] in whose framework the Imry-Ma argument can be straightforwardly implemented. It implies the absence of a spin-glass ground state as soon as an infinitesimal magnetic field is switched on. According to this ansatz the critical value of the long-range exponent is, thus, $\rho_c = \rho_{\rm mf} = 4/3$. This implication, however, turns out to be inconsistent with the outcome of numerical simulations in long-range systems for $\rho > \rho_{\rm mf}$.[45] Therefore, an unbiased quantitative estimate of the threshold value ρ_c^h corresponding to the SR LCD is beyond the reach of the analysis presented here.

In LR systems in a field, then, we believe that further investigation should be devoted to the occurrence of a spin-glass phase for values of $\rho > \rho_c^h = 3/2$ for which no ferromagnetic transition is present in the absence of bond disorder. Before drawing conclusions about short-range finite dimensional spin-glass systems relying too heavily on the SR↔LR relation Eq. (2) with $\rho_c = 2$, much caution should be taken in numerical data interpretation in LR spin-glass systems in the presence of a field, above all when ultrametricity[46] or lack of Almeida-Thouless transition[47] are tested at $\rho > 1.5$.

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[1] Y. Imry and S.-K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).
[2] J. Chalker, J. Phys. C **16**, 6615 (1983).
[3] D. Fisher, J. Frölich, and T. Spencer, J. Stat. Phys. **34**, 863 (1984).
[4] J. Bricmont and A. Kupiainen, Phys. Rev. Lett. **59**, 1829 (1987).
[5] J. Bricmont and A. Kupiainen, Commun. Math. Phys. **116**, 539 (1988).
[6] H. Rieger and A. P. Young, J. Phys. A **26**, 5279 (1993).
[7] J.-C. Anglès d'Auriac and N. Sourlas, Erophys. Lett. **39**, 473 (1997).
[8] N. Sourlas, Comp. Phys. Comm. **121-122**, 183 (1999).
[9] A. A. Middleton and D. S. Fisher, Phys. Rev. B **65**, 134411 (2002).
[10] A. K. Hartmann, Phys. Rev. B **65**, 174427 (2002).
[11] I. Morgenstern, K. Binder, and R. M. Hornreich, Phys. Rev. B **23**, 287 (1981).
[12] K. Binder, Z. Phys. B **50**, 343 (1983).
[13] J. Z. Imbrie, Phys. Rev. Lett. **53**, 1747 (1984).
[14] J. Villain, J. Phys. (France) **46**, 1843 (1985).
[15] A. Aharony and E. Pytte, Phys. Rev. B **27**, 5872 (1983).
[16] A. N. Berker, Phys. Rev. B **29**, 5243 (1984).
[17] M. Aizenman and J. Wehr, Phys. Rev. Lett. **62**, 2503 (1989).
[18] M. Aizenman and J. Wehr, Comm. Math. Phys. **130**, 489 (1990).
[19] M. Cassandro, E. Orlandi, and P. Picco, Comm. Math. Phys. **288**, 731 (2009).
[20] P. W. Anderson and G. Yuval, Phys. Rev. B **1**, 1522 (1970).
[21] P. W. Anderson and G. Yuval, J. Phys. C **4**, 607 (1971).
[22] J. Cardy, J. Phys. A **14**, 1407 (1981).
[23] $\rho_{\rm mf} = 3/2$ in the ordered ferromagnet, whose UCD = 4, and 4/3 in the RFIM and in the spin glass, where UCD = 6.
[24] D. Larson, H. G. Katzgraber, M. A. Moore, and A. P. Young, Phys. Rev. B **81**, 064415 (2010).
[25] R. A. Baños, L. A. Fernandez, V. Martin-Mayor, and A. P. Young, Phys. Rev. B **86**, 134416 (2012).
[26] M. E. Fisher, S.-K. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).
[27] G. Kotliar, P. W. Anderson, and D. L. Stein, Phys. Rev. B **27**, 602 (1983).
[28] L. Leuzzi, J. Phys. A: Math. Gen. **32**, 1417 (1999).
[29] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Phil. Mag. **91**, 1917 (2011).
[30] M. I. Berganza and L. Leuzzi, Phys. Rev. B **88**, 144104 (2013).
[31] G. S. Pawley, R. H. Swendsen, D. J. Wallace, and K. G. Wilson, Phys. Rev. B **29**, 4030 (1984).
[32] M. Campanino, E. Olivieri, and A. C. D. van Enter, Commun. Math. Phys. **108**, 1181 (1987).
[33] The LCD of a spin glass is 2.5 (Refs. 48 and 49). The relationship Eq. (2) thus implies $\eta(D=2.5) = -0.5$, reasonably compatible with known estimates of $\eta(D=3)$ and $\eta(D=4)$, see, e.g., Refs. 50–53.
[34] A. Aharony, Y. Imry, and S.-K. Ma, Phys. Rev. Lett. **37**, 1364 (1976).
[35] M. Schwartz and A. Soffer, Phys. Rev. Lett. **55**, 2499 (1985).
[36] A. J. Bray and M. A. Moore, J. Phys. C **18**, L927 (1985).
[37] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Phys. Rev. Lett. **101**, 107203 (2008).
[38] J.-C. Picard and M. Queyranne, Math. Prog. Study **13**, 8 (1980).
[39] A. K. Hartmann, Physica A **248**, 1 (1998).
[40] Lemon Graph Library (2011), URL http://lemon.cs.elte.hu.
[41] A. Pelissetto and E. Vicari, Phys. Rep. **368**, 549 (2002).
[42] L. Onsager, Phys. Rev. **65**, 117 (1943).
[43] M. C. Angelini, Ph.D. Thesis, Rome Sapienza University, Italy, 2013.
[44] D. S. Fisher and D. A. Huse, Phys. Rev. Lett. **56**, 1601 (1986).
[45] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Phys. Rev. Lett. **103**, 267201 (2009).
[46] H. G. Katzgraber, T. Jörg, F. Krzakala, and A. K. Hartmann, Phys. Rev. B **86**, 184405 (2012).
[47] H. G. Katzgraber and A. P. Young, Phys. Rev. B **72**, 184416 (2005).
[48] S. Franz, P. Parisi, and M. A. Virasoro, J. Phys. I (France) **4**, 1657 (1994).
[49] S. Boettcher, Phys. Rev. Lett. **95**, 197205 (2005).
[50] T. Jorg, Phys. Rev. B **73**, 224431 (2006).
[51] M. Hasenbusch, F. Parisen Toldin, A. Pelissetto, and E. Vicari, Phys. Rev. E **78**, 011110 (2008).
[52] T. Jorg, H. G. Katzgraber, and F. Krzakala, Phys. Rev. Lett. **100**, 197202 (2008).
[53] R. Alvarez Baños, A. Cruz, L. A. Fernandez, J. M. Gil-Narvion, A. Gordillo-Guerrero, M. Guidetti, D. Iñiguez, A. Maiorano, E. Marinari, V. Martin-Mayor, J. Monforte-Garcia, A. M. Sudupe, D. Navarro, G. Parisi, S. Perez-Gaviro, J. J. Ruiz-Lorenzo, S. F. Schifano, B. Seoane, A. Tarancon, P. Tellez, R. Tripiccione, and D. Yllanes, PNAS **109**, 6452 (2012).