Networks of Josephson junctions and their synchronization

Yurii N. Ovchinnikov[1,2] and Vladimir Z. Kresin[3]
[1]*L. Landau Institute of Theoretical Physics, Moscow 117334, Russia*
[2]*Max-Planck Institute for the Physics of Complex Systems, Dresden D-01187, Germany*
[3]*Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720, USA*
(Received 17 July 2013; published 5 December 2013)

One can demonstrate that a 1D Josephson network containing junctions with different tunneling resistances can be synchronized at frequencies, which are multiples of $2eV$, where V is the total dc voltage applied across the network. The appearance of such synchronization follows from the law of charge conservation and takes place if charge transfer is dominated by the Josephson channel. One can observe also a subharmonic structure. The result holds for cluster-based arrays as well as for the general case of a tunneling network.

DOI: 10.1103/PhysRevB.88.214504 PACS number(s): 74.50.+r, 73.22.−f, 85.30.Mn

I. INTRODUCTION

This paper is concerned with Josephson tunneling networks. It represents a continuation of our analysis[1,2] of networks formed by superconducting nanoparticles. Those papers[1,2] describe the transfer of dc current, while here we study the impact of an external voltage applied to the network. More specifically, the focus of this paper is on the possibility of network synchronization. It should be emphasized that the problem of synchronization is not specific to nanosystems but is of general importance and has attracted a lot of interest (see, e.g. Refs. 3–7).

As is known, a single junction radiates at the frequency $\omega = 2eV$, where V is the externally applied voltage. If the voltage has both dc and ac components,

$$U = V + v \sin \Omega t, \tag{1}$$

then at certain frequencies $\Omega = 2eV/n$, Shapiro steps can be observed.[8]

Consider a 1D Josephson tunneling network consisting of N junctions with an external dc voltage V applied to its ends. If the junctions are all equivalent, then the network radiates at the frequency $\omega = 2eV/N$.

In this paper, we focus on the more realistic case when the junctions are not equivalent. Then one might expect that, because of the nonuniform distribution of voltages across the network, there would appear a rather broad spectrum of radiated frequencies. However, the picture is more involved and on the whole represents a complex nonlinear problem. Nevertheless, it can be demonstrated, and this is the main goal of this paper that, despite the variation of junction tunneling resistances, the network can be synchronized at the frequency $\omega = 2eV$, where V is the total voltage. This follows from the law of charge conservation. The microscopic derivation of this statement will be presented below.

II. MAIN EQUATIONS

Consider a 1D tunneling network containing N junctions. With a total voltage V applied across it,

$$V = \sum_{n=1}^{N} V_n. \tag{2}$$

Here, V_n is the voltage across the nth junction.

As noted above, $V =$ const. As for the voltages $V_n (n = 1,\dots,N)$, they, generally speaking, depend on time so that $V_n \equiv V_n(t)$. The current j_n flowing through the nth junction has the form

$$j_n = j_n^0 \sin\left(2e\int_0^t V_n dt + \varphi_n\right) + (V_n/R_n) \quad j_n^0 \equiv j_{n;\max}. \tag{3}$$

The first term describes the Josephson current. The second term corresponds to single-particle current, i.e. the tunneling of thermal excitations [$\sim \exp(-\varepsilon_n/T)$]. This term is small at temperatures $T \ll \varepsilon_n$, that is, at low temperatures. The displacement current can also be incorporated (see below).

Because of charge conservation, currents flowing through neighboring junctions must be equal, that is, $j_s = j_{s+1}$ or

$$\frac{V_s}{R_s} + J_s^0 \sin\left(2e\int_0^t V_s dt + \varphi_s\right) = \frac{V_{s+1}}{R_{s+1}} + J_{s+1}^0 \sin\left(2e\int_0^t V_{s+1} dt + \varphi_{s+1}\right), \tag{4}$$

where $s = 1,\dots,N-1$.

Equations (2) and (4) are the main equations which form the basis for the analysis. As noted above, we are dealing with a complex and nonlinear problem, and our main goal is to demonstrate the possibility for the network to become synchronized. According to Eq. (3), the tunneling current has two components: (1) Josephson tunneling and (2) one-particle ("normal") component. The situation when charge transfer is dominated by the Josephson channel ($T \ll \varepsilon_n$) is of special interest.

III. SYNCHRONIZATION

Consider a Josephson tunneling network with an external dc voltage V applied across it. We focus on the most interesting case when charge transfer across the network is dominated by the Josephson channel. As indicated above, we consider the realistic situation of inequivalent Josephson junctions. We now demonstrate that such a network can be synchronized at frequencies determined by the external potential V. One also can show (see below) that the displacement current as well as the additional contribution from single-particle tunneling do not affect the synchronization.

To begin, let us analyze a network made up of only two junctions. This case is interesting both for its own sake and because it allows us to demonstrate the main physics of the synchronization phenomenon.

Then we have [see Eqs. (2) and (4)]

$$V = V_1 + V_2, \tag{5}$$

$$J_1^0 \sin\left(2e\int_0^t V_1 dt + \varphi_1\right) = J_2^0 \sin\left(2e\int_0^t V_2 dt + \varphi_2\right). \tag{6}$$

We consider the case of two different junctions. Assume that $J_1^0 > J_2^0$. With the use of Eqs. (5) and (6) can be written in the form

$$\eta \sin Z_1 = \sin(2eVt - Z_1 + \theta). \tag{7}$$

Here,

$$Z_1 \equiv Z_1(t) = 2e\int_0^t V_1 dt + \varphi_1; \quad \theta = \sum_{i=1}^{2}\varphi_i;$$

$$\eta = J_1^0/J_2^0 > 1.$$

As a result, we obtain

$$Z_1 \equiv 2e\int_0^t V_1 dt + \varphi_1 = \arctan\frac{\sin[2eV(t+t_0)]}{\eta + \cos[2eV(t+t_0)]}, \tag{8}$$

where $t_0 = \theta/2eV$.

One can see directly from Eq. (8) that the function $Z_1(t)$ and, correspondingly, the current j_1 [see Eqs. (3) and (6)] are periodic functions with frequency $\omega = 2eV$. Because of this, the quantity Z_1 defined by Eq. (8) can be expanded in a series:

$$Z_1 \equiv 2e\int_0^t V_1 dt + \varphi_1 = \sum_{n=1}^{\infty}\frac{B_n}{n\omega}\sin n\omega(t+t_0). \tag{9}$$

Here, $\omega = 2eV$ and

$$\frac{B_n}{n\omega} = \frac{1}{\pi}\int_0^{2\pi} dz \arctan[\sin z(\eta + \cos z)^{-1}]\sin nz. \tag{9$'$}$$

One also can see from Eq. (9) that

$$2eV_1(t) = \sum_{n=1}^{\infty} B_n \cos n\omega(t+t_0); \quad \omega = 2eV. \tag{10}$$

Therefore, the potential V_1, as well as $V_2 = V - V_1$, are time dependent.

One also can see that

$$Z_2 \equiv 2e\int_0^t V_2 dt + \varphi_2 = 2eV(t+t_0) - Z_1. \tag{11}$$

Here, Z_1 is determined by Eq. (8) and is described by the expansion of Eq. (9) with coefficients in Eq. (9′).

Therefore, the network consisting of two different junctions ($J_1^0 \neq J_2^0$) is synchronized at frequencies $\omega = 2eV, 4eV, \ldots$, which are multiples of the total external potential.

As was noted above, the potentials V_1 and V_2 depend on time, so that $V_1 \equiv V_1(t)$ and $V_2 \equiv V_2(t)$. At the same time $V_1 + V_2 = V = \text{const}$.

It is crucial that the average values of the potentials V_1 and V_2 are different. Indeed, one can see directly from the definitions in Eqs. (8) and (9), and the expression in Eq. (5) that $\langle V_1\rangle = 0$, whereas $\langle V_2\rangle = V$. Therefore, the law of charge conservation, Eq. (6), leads to a rather peculiar time-dependent potential distribution.

If the junctions are very different ($\eta \gg 1$), then, as follows from Eq. (9′), $B_1/\omega \simeq \eta^{-1}$ for $n = 1$, and $B_n \ll B_1$ for $n \neq 1$ (e.g. $B_2/\omega \approx -\eta^{-2}$). For $\eta = 1$, Eqs. (8) and (11) yield $V_1 = V_2 = V/2$.

Based on Eqs. (8) and (9), one can see that, if the external field has the general form in Eq. (1), it leads to an appearance of Shapiro steps. However, here, the Shapiro step picture is more complex relative to that for a single junction. Namely, in addition to the main steps at $\hbar\Omega = n\hbar\omega$ ($\hbar\omega = 2eV$), one also can observe a subharmonic structure. Indeed, for a single junction, the current is described by a simple sinusoidal time dependence. In the array case, the dependence is also periodic but more complicated [see Eqs. (8) and (9)]. In this case, one should perform an additional Fourier transform [Eqs. (9) and (10)]. This yields the picture of Shapiro steps with $2eV = \hbar\Omega n/n'$. The scenario is similar to that for superconducting microbridges (see, e.g. Ref. 9 and also the review in Ref. 10), where the current is also described by a periodic nonsinusoidal function. The I-V characteristics will display the usual Shapiro steps at $n = 1,2,\ldots$, but, in addition, one can observe a subharmonic structure corresponding to different values of n'.

In a similar way, one can consider a three-junction network. Here, we are dealing with the following equations:

$$\eta_{1;2}\sin Z_1 = \sin Z_2, \tag{12}$$

$$\eta_{2;3}\sin Z_2 = \sin Z_3, \tag{12$'$}$$

$$V = \sum_{i=1}^{3} V_i, \tag{12$''$}$$

where $Z_i = 2e\int_0^t V_i dt + \varphi_i; i = 1,2,3,\quad \eta_{r;s} = J_r^0/J_s^0;\ r; s \equiv 1,2,3$ [cf. Eq. (4)]; we assume that $j_1^0 > j_2^0 > j_3^0$. One can see that $Z_3 = 2eV(t+t_0) - Z_1 - Z_2$, $t_0 = \sum_{i=1}^{3}\varphi_i/2eV$.

Using Eqs. (12′) and (12″), we obtain

$$Z_2 = \arctan\frac{\sin[2eV(t+t_0) - Z_1]}{\eta_{2;3} + \cos[2eV(t+t_0) - Z_1]}. \tag{13}$$

With the help of this expression and Eq. (12), one can express the potential V_1 in terms of V:

$$\sin Z_1 = \eta_{3;1}\frac{\sin[2eV(t+t_0) - Z_1]}{\{1 + \eta_{3;2}^2 + 2\eta_{3;2}\cos[2eV(t+t_0) - Z_1]\}^{1/2}}. \tag{14}$$

Therefore, all the quantities Z_1, Z_2, Z_3 depend periodically on time with the frequency $\omega = 2eV$.

Once again, one can see that the system can be synchronized with an external potential V applied to the edges of the network. The main Shapiro steps again will be multiples of V ($\Omega = 2eVn, n = 1,2,\ldots$); there will be also a subharmonic structure, (cf. Ref. 10), see above. If $\eta_{3;2} \ll 1$, we obtain from the Eqs. (12):

$$Z_1 = \eta_{3;1}\sin 2eV(t+t_o); \quad Z_2 = \eta_{3;2}\sin 2eV(t+t_o);$$

$$Z_3 = 2eV(t+t_o). \tag{15}$$

Therefore, this network is indeed synchronized with the external voltage V. Note also that $\langle V_1\rangle = \langle V_2\rangle = 0$, whereas $\langle V_3\rangle = V$.

The derivations can easily be generalized to the case of N junctions. By analyzing the system

$$J_s^0 \sin\left(2e\int_0^t V_s dt + \varphi_s\right) = J_{s+1}^0 \sin\left(2e\int_0^t V_{s+1} dt + \varphi_{s+1}\right)$$
$$(s = 1,\ldots,N-1),$$

which is valid for the case when the Josephson channel is dominant, analogously to the derivation described above, one can demonstrate that all currents are periodic functions of the total external potential V. This leads to the corresponding synchronization and the appearance of Shapiro steps.

Consideration of the small additional contribution of single-particle tunneling [see Eqs. (3) and (4)] will not affect the synchronization picture. Indeed, consider again the system of two junctions. Based on Eq. (4), one can write [cf. Eq. (7)]

$$\eta \sin Z_1 = \sin[2eV(t+t_0) - Z_1] + S, \qquad (16)$$

where Z_1, t_0, η are defined by Eqs. (7) and (8) and

$$S = \left[(V - V_1)R_2^{-1} - V_1 R_1^{-1}\right]/J_2.$$

Since $S \ll 1$ in the low-temperature region ($T \ll \varepsilon_i$), one can treat this term as a perturbation. Correspondingly, one can write $V_1 = V_1^0 + V_1^1$, so that V_1^0 is described by Eqs. (9) and (10). After a simple calculation, we obtain

$$Z_1^1 \equiv 2e\int_0^t V_1 dt + \delta\varphi_1$$
$$= \tilde{S}\left\{\eta\cos Z_1^0 + \cos\left[2eV(t+t_0) - Z_1^0\right]\right\}^{-1}, \qquad (17)$$

where $f^0 = \cos[2eV(t+t_0) - Z_1^0], \tilde{S} = S + \delta\theta f^0$.

It is essential that the denominator on the right-hand side of Eq. (17) does not vanish at any value of V and t. Indeed, it vanishes only if $\tan Z_1^0 = -[\eta + \cos 2eV(t+t_0)]/\sin 2eV(t+t_0)$, which is incompatible with the solution in Eq. (8). Therefore, $Z_1^1 \ll 1$ and is a periodic function of the external potential V; this follows from Eqs. (9) and (14).

As a result, the conclusion that the system is synchronized remains valid. This is also unchanged when displacement currents $C_i(\partial V_i/\partial t)$ are taken into account. We assume that their contribution is also small relative to that of the Josephson channel. In other words, we assume that the parameter $(CV\omega/J_c^0) \ll 1$, which is perfectly realistic (see below). The treatment can be performed similarly to that for single-particle tunneling above. One can see that, in this case, the correction to $Z_{1;dc}$ [cf. Eq. (15)] has the form

$$Z_{1;dc} = (C_1 + C_2)J_2^{-1}\{[\eta + \cos 2eV(t+t_0)]\cos Z_1$$
$$+ \sin[2eV(t+t_0)]\sin Z_1\}^{-1}\partial V_1^0/\partial t \qquad (18)$$

and is periodic at $\omega = 2eV$.

IV. DISCUSSION

It has been demonstrated above that a 1D Josephson tunneling network can be synchronized at a frequency equal to $\omega = 2eV$, where V is the external voltage applied across the network. Correspondingly, Shapiro steps can be observed. Thus, the network response is similar to that of a single junction. The main condition which needs to be satisfied in order to attain the synchronization is that the Josephson tunneling channel must be dominant. That is, single-particle tunneling must be relatively small ($T \ll \varepsilon_i$, where ε_i is the energy gap).

The displacement current also should be relatively small. Such a situation is perfectly realistic. For example, if the network is characterized by the following parameter values: $V \simeq 2.2$ mV, $C \approx 0.5$ pF, $\omega \approx 5\times 10^2$ GHz, $J_c^0 \approx 3-5$ mA, then $(CV\omega/J_c^0) \ll 1$ ($\sim 10^{-1}$).

Network synchronization has been observed in a number of studies (see, e.g. Refs. 5–7). Measurements were performed mainly with 1D and 2D arrays where the junctions were similar or almost similar. For a 1D network of N such junctions, the synchronization occurs at $\omega = 2eV/N$. Here, we focus on the entirely different case when junctions are very dissimilar. It is remarkable that, in this case, one still can observe synchronization (and Shapiro steps) at frequencies which are multiples of $2eV$, where V is the total dc voltage across the network. The analytical treatment described above shows that this phenomenon occurs thanks to a peculiar voltage distribution, which is time dependent, and follows directly from the principle of charge conservation.

Synchronization of a pair of $YBa_2Cu_3O_{7-\delta}$ junctions was described in Refs. 11 and 12. Shapiro steps were observed when 12 GHz microwave radiation with sufficient power was applied. The *I-V* characteristics clearly display Shapiro steps if $V = \Omega/2en$, where V is the voltage across both junctions. We suggest that these steps correspond to the synchronization described above.

Let us also comment on the "giant" Josephson proximity effect observed in Ref. 13. The electrodes were formed out of $La_{0.85}Sr_{0.15}CuO_4$ films ($T_c \simeq 45$ K) and were separated by the underdoped LaCuO compound with $T_c' \simeq 25$ K; the width of the separating layer reached 200 Å (!). Despite such a large separation scale, the authors of Ref. 13 observed at $35\text{ K} > T > T_c'$ both dc and ac Josephson tunneling. According to our theory,[14,15] the separating layer contains superconducting regions embedded in a normal metallic matrix, hence we are dealing with Josephson tunneling through a network of such regions. Our analysis of the dc Josephson current[16] is in good agreement with the data.[13] According to Ref. 13, Shapiro steps can be observed, so the picture appears to be similar to that for a single junction. We believe that this observation reflects the synchronization described above. The superconducting state in the isolated regions persists up to $T_c^* \simeq 80$ K,[17] hence the condition $T < \varepsilon$ is satisfied.

As was noted in the Introduction, this paper represents a continuation of our previous work.[1,2] This study allows one to state that a cluster-based Josephson network is also capable of transferring an ac current synchronized at the frequency corresponding to the total voltage applied across the network.

ACKNOWLEDGMENTS

The authors are grateful to R. Dynes and S. Cybart for fruitful discussions. The research of Y.N.O. is supported by EOARD under Contract No. 097006.

[1] Y. N. Ovchinnikov and V. Z. Kresin, Phys. Rev. B **81**, 214505 (2010).
[2] Y. N. Ovchinnikov and V. Z. Kresin, Phys. Rev. B **85**, 064518 (2012).
[3] A. K. Jain, K. K. Likharev, J. E. Lukens, and J. E. Saurageau, Phys. Rep. **109**, 309 (1984).
[4] P. Hadley, M. R. Beasley, and K. Wiesenfeld, Phys. Rev. B **38**, 8712 (1988).
[5] A. B. Cawthorne, P. Barbara, S. V. Shitov, C. J. Lobb, K. Wiesenfeld, and A. Zangwill, Phys. Rev. B **60**, 7575 (1999).
[6] G. Filatrella, N. Pedersen, C. Lobb, and P. Barbara, Eur. Phys. J. B **34**, 3 (2003).
[7] S. Benz and C. Burroughs, Appl. Phys. Lett. **58**, 2162 (1991).
[8] S. Shapiro, Phys. Rev. Lett. **11**, 80 (1963).
[9] A. Dayem and J. Wiegland, Phys. Rev **155**, 419 (1967).
[10] D. Tilley and J. Tilley, *Superfluidity and Superconductivity* (Adam Hilger, Bristol, 1986).
[11] S. Cybart, Ph.D. thesis, University of California, San Diego (2005). The authors are grateful to S. Cybart for bringing our attention to this and the following publications.
[12] Ke Chen, S. Cybart, and R. Dynes, IEEE Trans. Appl. Supercond. **15**, 149 (2005).
[13] I. Bozovic, G. Logvenov, M. A. J. Verhoeven, P. Caputo, E. Goldobin, and M. R. Beasley, Phys. Rev. Lett. **93**, 157002 (2004).
[14] Y. N. Ovchinnikov, S. A. Wolf, and V. Z. Kresin, Phys. Rev. B **60**, 4329 (1999).
[15] V. Kresin, Y. Ovchinnikov, and S. Wolf, Phys. Rep. **431**, 231 (2006).
[16] V. Kresin, Y. Ovchinnikov, and S. Wolf, Appl. Phys. Lett. **83**, 722 (2003).
[17] I. Igushi, T. Yamagreshi, and A. Sagimoto, Nature **412**, 420 (2001).