Schwinger-Dyson renormalization group

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We use the Schwinger-Dyson equations as a starting point to derive renormalization flow equations. We show that Katanin's scheme arises as a simple truncation of these equations. We then give the full renormalization group equations up to third order in the irreducible vertex. Furthermore, we show that to the fifth order, there exists a functional of the self-energy and the irreducible four-point vertex whose saddle point is the solution of Schwinger-Dyson equations.

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I. INTRODUCTION

Renormalization group (RG) and Schwinger-Dyson equation (SDE) hierarchies are widely used methods to study the correlation functions of quantum field theories. The SDEs are a hierarchy of integral equations[1,2] for the Green functions obtained, e.g., by integration by parts in the functional integral representation. The simplest truncation of this hierarchy corresponds to the Hartree-Fock equations and the next level also includes loop corrections to the two-fermion scattering. Depending on the model, these equations may require renormalization subtractions to be well-defined. Even after that, the equations are typically singular integral equations, allowing for solutions that exhibit symmetry breaking or other drastic changes compared to the bare system. The *functional* RG approach (for reviews, see, e.g., Refs. 3–5) starts by introducing a scale parameter and a modification of the theory that regularizes, i.e., smoothes out singularities, in the propagator. The RG flow is obtained when the regulator is gradually removed by taking a limit of the scale parameter Λ (in this paper we use the standard Wilsonian convention that the Λ corresponds to an infrared regulator which is decreased to generate the flow, and eventually sent to zero, $\Lambda \to 0$). The resulting RG equation is a functional differential equation which becomes a hierarchy of equations in the usual expansion in the fields. Instead of a self-consistency, as in the SDE, it describes a flow. This regularizing effect in the above procedure (often corresponding to a localization in position space) is the main reason for the good mathematical properties of the RG equations. Terms that lead to symmetry breaking or other singular behavior develop early on in the flow, without any need for assumptions on their type, and gradually grow, so that their effect can be taken into account scale by scale.

It is useful to make the relation between the two approaches as explicit as possible. In the first part of this paper, we formulate RG equations based on the SDE hierarchy, and then relate Katanin's truncation[6] of the RG hierarchy for the one-particle irreducible vertex functions to a particular truncation of the RG derived from the SDE. This is important because Katanin's truncation scheme has been shown[6,7] to generate a flow that automatically satisfies certain self-consistency equations exactly, and because this scheme has been used extensively in RG calculations, both for fermionic and for spin systems.[5] We then also exhibit the higher-order terms in the SDE-RG.

The full SDE hierarchy encodes all analytic and combinatorial properties of the vertex functions, and retains the symmetries of the original action. Truncations of this hierarchy usually violate Ward identities and conservation laws. In the RG approach, the same problem arises, but in addition, the regularization may violate some symmetries explicitly, so that the restoration of Ward identities in the limit where the regulator is removed requires proof. The most important example of this are theories with local gauge symmetries, e.g., QED.[8,9] A general theory of conserving approximations was developed by Baym and Kadanoff[10,11] in the context of many-body theory, and later also used in high-energy quantum field theory.[12] An essential object there is the Luttinger-Ward (LW) functional, which expresses the grand canonical potential as a function of the bare vertex and the full propagator of the theory. The field equations are obtained by a stationarity condition as the propagator is varied. Similar constructions using the self-energy as the variational parameter instead of the propagator were introduced by Potthoff.[13,14]

It is a natural question whether there is a scale-dependent variant of this functional, in which both the full scale-dependent propagator and the effective two-particle vertex (instead of the bare one) appear, and we here define a functional which has these properties, in terms of an expansion in powers of the effective two-particle vertex. In principle, the iteration of the RG equations in their integral form (in a procedure generalizing the derivations in Ref. 7) automatically generates such an expansion; however, the so obtained expression for the functional involves integrals over intermediate scales, similarly to the Brydges-Kennedy formula,[15] which provides an explicit solution to Polchinski's equation.

Another generalization of the LW functional was given by the Lund group where the bare interaction is replaced by the screened interaction using the Bethe-Salpeter equation. The resulting functional is variational in both parameters.[16,17] In Sec. IV we derive from the SDEs a functional of the self-energy and the irreducible four-point vertex, which is local in the RG flow parameter, at least up to the fifth order in the effective vertex. The stationary points of this functional satisfy the scale-dependent SDE.

The scope of this paper is theoretical, i.e., we focus on the question of the precise relation between SDEs and functional RG equations in general, and on the problem of generalizing the Luttinger-Ward functional, and we postpone the study of specific models to further publications. Since the RG in Katanin's truncation has had significant practical success already and since this truncation appears as a particular approximation to the hierarchy derived here, the main question for applications is how the rearrangement of terms affects practical calculations and how much additional information can be obtained from the higher-loop terms appearing at each truncation level. We shall discuss some aspects of this, as well as the potential use of these equations for solving self-consistency equations, at the end of the paper.

II. SCHWINGER-DYSON EQUATIONS AND RENORMALIZATION GROUP

Consider a lattice fermion system described by Grassmann fields ψ, $\overline{\psi}$ and the action

$$\mathcal{S}[\psi,\overline{\psi}] = -(\overline{\psi},C^{-1}\psi) - V[\psi,\overline{\psi}], \tag{1}$$

where C is the propagator of the noninteracting system and V is a two-body interaction of the general form

$$V[\psi,\overline{\psi}] = \frac{1}{4}\sum_{x_1,x_2,x_1',x_2'} v_{x_1',x_2',x_1,x_2}\overline{\psi}_{x_1'}\overline{\psi}_{x_2'}\psi_{x_2}\psi_{x_1}. \tag{2}$$

In applications, the labels x are often composed of several variables. In momentum space with a single-particle basis, one has $x = (k_0,\mathbf{k},\sigma)$, where k_0 is the Matsubara frequency, $\mathbf{k}$ is the momentum, and σ denotes the spin orientation. Depending on the representation, there might be prefactors such as volume or temperature, which are not shown here. v is antisymmetric under independent exchange of its first two and last two arguments. The bilinear form (f,g) is defined as the sum $\sum_x f(x)g(x)$.

The generating functional of the connected Green functions is given by

$$\mathcal{G}[\eta,\overline{\eta}] = -\ln\int d\mu_C e^{V[\psi,\overline{\psi}]}e^{(\overline{\eta},\psi)+(\overline{\psi},\eta)}, \tag{3}$$

where $d\mu_C := \mathcal{N}\prod_x d\psi_x d\overline{\psi}_x e^{(\overline{\psi},C^{-1}\psi)}$, with a normalization constant $\mathcal{N}$. The connected m-particle Green function is obtained from the generator $\mathcal{G}$ by differentiating with respect to the sources $\eta,\overline{\eta}$ and evaluating for vanishing sources:

$$\begin{aligned} &G^{(2m)}_{x_1,\ldots,x_m;x_1',\ldots,x_m'} \\ &= (-1)^m \frac{\partial^{2m}\mathcal{G}[\eta,\overline{\eta}]}{\partial\overline{\eta}_{x_1}\cdots\partial\overline{\eta}_{x_m}\partial\eta_{x_m'}\cdots\partial\eta_{x_1'}}\bigg|_{\eta=\overline{\eta}=0} \\ &= -\langle\psi_{x_1}\cdots\psi_{x_m}\overline{\psi}_{x_m'}\cdots\overline{\psi}_{x_1'}\rangle_c. \end{aligned} \tag{4}$$

Here $\langle\cdots\rangle_c$ stands for the connected average of the expression between the brackets. The *effective action* is the Legendre transform of $G[\eta,\overline{\eta}]$,

$$\Gamma[\psi,\overline{\psi}] = (\overline{\eta},\psi) + (\overline{\psi},\eta) + G[\eta,\overline{\eta}], \tag{5}$$

with $\psi = -\frac{\partial\mathcal{G}}{\partial\overline{\eta}}$ and $\overline{\psi} = \frac{\partial\mathcal{G}}{\partial\eta}$. It generates the one-particle irreducible (1PI) Green functions $\Gamma^{(2m)}_{x_1,\ldots,x_m;x_1',\ldots,x_m'}$.

By integration by parts,

$$\int d\mu_C\psi_x F[\psi,\overline{\psi}] = -\sum_y C_{x,y}\int d\mu_C\frac{\partial}{\partial\overline{\psi}_y}F[\psi,\overline{\psi}], \tag{6}$$

every correlation function obeys a Schwinger-Dyson equation

$$\begin{aligned} &\int d\mu_C\psi_{x_1}\cdots\psi_{x_m}\overline{\psi}_{x_m'}\cdots\overline{\psi}_{x_1'}e^{V[\psi,\overline{\psi}]} \\ &= -\sum_y C_{x_1,y}\int d\mu_C\frac{\partial}{\partial\overline{\psi}_y}\psi_{x_2}\cdots\psi_{x_m}\overline{\psi}_{x_m'}\cdots\overline{\psi}_{x_1'}e^{V[\psi,\overline{\psi}]}. \end{aligned} \tag{7}$$

The correlation function on the right-hand side is in general disconnected, but can be expressed by standard cumulant formulas in terms of the connected Green functions, which are in turn given by a standard expansion in trees that have the full propagator $G := G^{(2)}$ associated with lines and the 1PI vertex functions to the vertices . For $m = 1$, Eq. (7) gives an equation for G,

$$\begin{aligned} G_{x,x'} = C_{x,x'} &- \sum_{z_1\cdots z_4} C_{x,z_1}G_{z_4,z_2}v_{z_1,z_2,z_3,z_4}G_{z_3,x'} \\ &- \frac{1}{2}\sum_{\substack{z_1\cdots z_4\\ y_1\cdots y_4}}\big(C_{x,z_1}v_{z_1,z_2,z_3,z_4}G_{z_3,y_1}G_{z_4,y_2}\Gamma^{(4)}_{y_1,y_2,y_3,y_4} \\ &\times G_{y_4,z_2}G_{y_3,x'}\big), \end{aligned} \tag{8}$$

After rewriting in terms of the self-energy, one obtains $\Sigma = C^{-1} - G^{-1}$ is given by

$$\begin{aligned} \Sigma_{x,x'} = &-\sum_{z_2,z_4} G_{z_4,z_2}v_{x,z_2,x',z_4} \\ &-\frac{1}{2}\sum_{\substack{z_2\cdots z_4\\ y_1\cdots y_3}} v_{x,z_2,z_3,z_4}G_{z_3,y_1}G_{z_4,y_2}\Gamma^{(4)}_{y_1,y_2,x',y_4}G_{y_4,z_2}. \end{aligned} \tag{9}$$

The SD equation for $m = 2$ gives the four-point vertex (two-particle vertex) as

$$\begin{aligned} &\Gamma^{(4)}_{x_1,x_2;x_1',x_2'} \\ &= v_{x_1,x_2;x_1',x_2'} + \frac{1}{2}\sum_{z_1\cdots z_4} v_{x_1,x_2;z_1,z_2}G_{z_1,z_3}G_{z_2,z_4}\Gamma^{(4)}_{z_3,z_4;x_1',x_2'} \\ &\quad -\Bigg(\sum_{z_1\cdots z_4} v_{x_1,z_1;x_3,z_2}G_{z_4,z_1}G_{z_2,z_3}\Gamma^{(4)}_{z_3,x_2;z_4,x_2'} - (x_3 \leftrightarrow x_4)\Bigg) \\ &\quad +\frac{1}{2}\sum_{z_1\cdots z_6} v_{z_1,x_1;z_2,z_3}G_{z_3,z_5}G_{z_2,z_4}G_{z_6,z_1}K_{z_4,z_5,x_2,z_6,x_1',x_2'} \\ &\quad +\frac{1}{2}\sum_{z_1\cdots z_6} v_{z_1,x_1;z_2,z_3}G_{z_3,z_5}G_{z_2,z_4}G_{z_6,z_1}\Gamma^{(6)}_{z_4,z_5,x_2,z_6,x_1',x_2'}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} K_{x_1,x_2,x_3;x_1',x_2',x_3'} := 9\,\mathbb{S}_3\sum_{z_1,z_2}\Gamma^{(4)}_{x_2,x_3;x_1',z_1}G_{z_1,z_2}\Gamma^{(4)}_{x_1,z_2;x_2',x_3'} \\ + \sum_{z_1,z_2}\Gamma^{(4)}_{x_1,x_2;x_1',z_1}G_{z_1,z_2}\Gamma^{(4)}_{z_2,x_3;x_2',x_3'}, \end{aligned} \tag{11}$$

and the antisymmetrization operator $\mathbb{S}_m$ projects functions $f_{x_1,\ldots;x_1',\ldots,x_m'}$ to their totally antisymmetric part,

$$\begin{aligned} \mathbb{S}_m f_{x_1,\ldots,x_m;x_1',\cdots,x_m'} = \frac{1}{(m!)^2}\sum_{\pi,\pi'\in\mathcal{P}_m}\mathrm{sgn}(\pi)\mathrm{sgn}(\pi') \\ \times f_{x_{\pi(1)},\ldots,x_{\pi(m)},x'_{\pi'(1)},\ldots,x'_{\pi'(m)}}. \end{aligned} \tag{12}$$

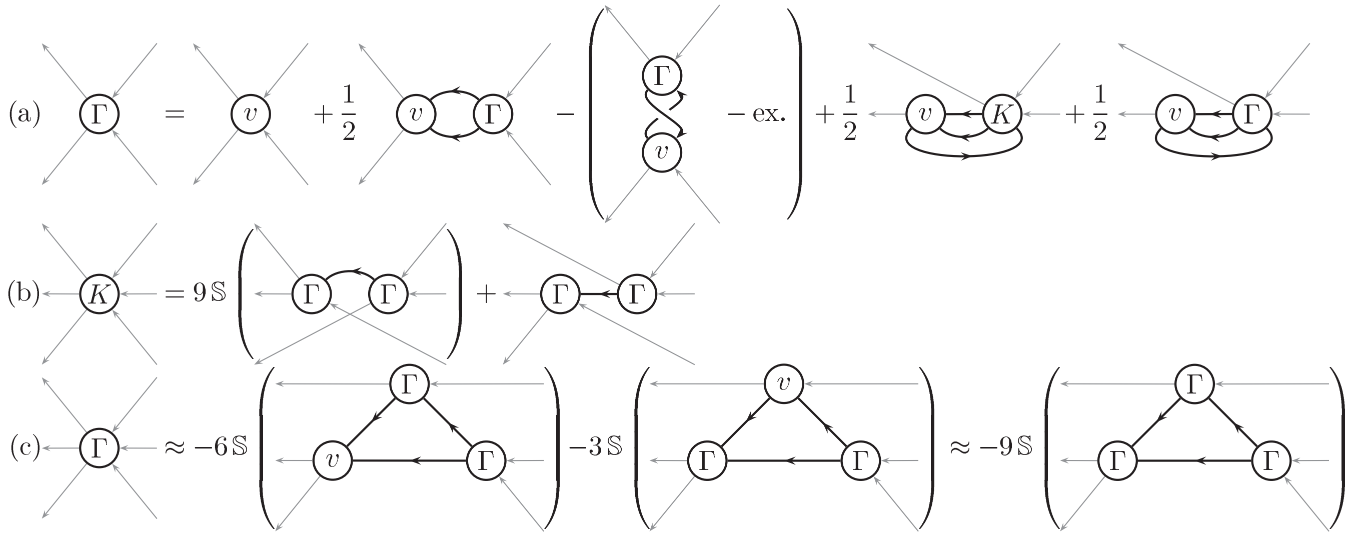
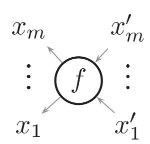


FIG. 1. The diagrammatic representations of the SD equation (10) is given in (a), (b) represents Eq. (11), and (c) corresponds to Eq. (20).

The second term in Eq. (11) cancels a contribution from the first term which would otherwise lead to a reducible diagram in Eq. (10). In a graphical representation where a vertex f is depicted by

$$f_{x_1,\ldots,x_m;x_1',\ldots,x_m'} \simeq \tag{13}$$



and where heavy lines represent propagators G,

$$G_{x_1;x_1'} \simeq x_1 \longleftarrow x_1', \tag{14}$$

Eq. (10) takes the form shown in Fig. 1.

If $C = C_\Lambda$ depends on a parameter Λ, the SD equations (9) and (10) determine Λ-dependent self-energies Σ_Λ and two-particle vertices $\Gamma^{(4)}_\Lambda$, etc. We assume that C depends differentiably on Λ and that the bare interaction V remains independent of Λ. Following the standard convention in fRG studies of condensed-matter systems, we arrange things such that for some value Λ_0 of Λ (the "starting scale"), the vertex functions are given by the bare ones, and the full correlation functions are recovered as $\Lambda \to 0$. Moreover, we assume that Λ is introduced in a way that has a regularizing effect, so that Σ_Λ and $\Gamma^{(2m)}_\Lambda$ are differentiable functions of Λ as well, and derivatives with respect to Λ can be exchanged with the summations occurring in the SDE. Note that this is an assumption on the *solution* of the hierarchy, which will in general contain singular functions in the limit $\Lambda \to 0$, hence checking it is important and nontrivial. However, for the standard momentum space cutoff RG, it has been proven,[18,19] and this proof extends to any RG flow that imposes a sufficient regularization on C, in particular the temperature RG flow,[20] flows with a frequency cutoff, or the Ω regularization.[21] Thus the assumption is satisfied in a large class of flows, for which the SDEs hold at every scale Λ.

We have chosen to make C, but not V, in (1) [(3)] depend on Λ because we want to draw a connection between SDE and standard functional RG flows, and this choice of Λ dependence is the same as in the derivation of the functional RG equation in Refs. 3,5, and 19. One can think of many other useful ways in which a parameter Λ could be introduced in the SDE, and also in the interaction (or only there). A natural way to check the differentiability assumption is then to truncate the SDE hierarchy at successive levels, and within each truncation verify the differentiability conditions by analysis of the right-hand side of the flow equation.

To avoid overloading the notation, we drop the subscript Λ from $\Gamma^{(4)}_\Lambda$, Σ_Λ, and G_Λ. The derivative of Eq. (10) with respect to Λ (denoted here by a dot and written out explicitly only in the particle-particle channel),

$$\begin{aligned}
&\dot\Gamma^{(4)}_{x_1,x_2;x_1',x_2'} \\
&\quad = \frac{1}{2}\sum_{z_1\cdots z_4} v_{x_1,x_2;z_1,z_2}\frac{d}{d\Lambda}\left(G_{z_1,z_3}G_{z_2,z_4}\right)\Gamma^{(4)}_{z_3,z_4;x_1',x_2'} \\
&\qquad + \frac{1}{2}\sum_{z_1\cdots z_4} v_{x_1,x_2;z_1,z_2}G_{z_1,z_3}G_{z_2,z_4}\dot\Gamma^{(4)}_{z_3,z_4;x_1',x_2'} + \cdots,
\end{aligned} \tag{15}$$

gives rise to terms on the right-hand side where only propagators are differentiated and ones where $\dot\Gamma^{(4)}$ appears. At each order in an expansion in $\Gamma^{(4)}$, it is possible to eliminate v and $\dot\Gamma^{(4)}$ from the right-hand side of Eq. (15) by substituting v from Eq. (10) and iterating Eq. (15). This results in

$$\begin{aligned}
\dot\Gamma^{(4)}_{x_1,x_2;x_1',x_2'} &= \frac{1}{2}\sum_{z_1\cdots z_4}\Gamma^{(4)}_{x_1,x_2;z_1,z_2}\left(\frac{d}{d\Lambda}G_{z_1,z_3}G_{z_2,z_4}\right)\Gamma^{(4)}_{z_3,z_4;x_1',x_2'} \\
&\quad - (\text{ph.} - \text{ex.}) + O(\Gamma^{(4)})^3.
\end{aligned} \tag{16}$$

Taking the Λ derivative of Eq. (9) gives

$$\begin{aligned}
\dot\Sigma_{x_1,x_1'} &= -\sum_{z_2,z_4}\dot G_{z_4,z_2}v_{x,z_2,x',z_4} - \frac{1}{2}\frac{d}{d\Lambda}\sum_{\substack{z_2\cdots z_4\\ y_1\cdots y_3}} v_{x,z_2,z_3,z_4} \\
&\quad \times G_{z_3,y_1}G_{z_4,y_2}\Gamma^{(4)}_{y_1,y_2,x',y_4}G_{y_4,z_2}.
\end{aligned} \tag{17}$$

Consider the system of equations obtained by dropping the second term in this equation, which leaves

$$\dot{\Sigma}_{x_1,x_1'} = -\sum_{z_2,z_4} \dot{G}_{z_4,z_2} v_{x,z_2,x',z_4}, \tag{18}$$

and combining it with the modification of (16) where all third-order terms in $\Gamma^{(4)}$ are dropped.

In the first term in (18), v could be replaced by $\Gamma^{(4)}$ up to orders $(\Gamma^{(4)})^2$, but as it stands, this term can be directly integrated, hence combined naturally with the resummation of the four-point function implied by keeping only one of the three terms in (16). This is at the basis of recovering self-consistent ladder summations.[6,7] Because $\dot{G} = G\dot{\Sigma}G + S$, where S is the single-scale propagator appearing in the standard RG equations for the irreducible vertex functions,[3,5] we see that by substituting for v in (17) by reinserting the SDE for $\Gamma^{(4)}$, the standard 1PI equation for the self-energy,

$$\dot{\Sigma}_{x_1,x_1'} = -\sum_{z_2,z_4} S_{z_4,z_2} \Gamma^{(4)}_{x,z_2,x',z_4}, \tag{19}$$

is obtained when terms of third and higher order in $\Gamma^{(4)}$ are dropped. At first glance, this may be surprising, because the error terms seem to be of second order in $\Gamma^{(4)}$. However, the calculation shows that the second-order terms cancel. This must be so by a more general argument: because the two hierarchies are equivalent, their expansions in $\Gamma^{(4)}$ must coincide at every order in $\Gamma^{(4)}$. Thus, provided the right-hand side equation for $\dot{\Gamma}^{(4)}$ contains all second-order terms in $\Gamma^{(4)}$, the error made in $\dot{\Sigma}$ is at least of third order. The combination of (16) (with terms of third and higher order in $\Gamma^{(4)}_\Lambda$ omitted) and (19) is Katanin's truncation of the RG hierarchy. Within the 1PI RG hierarchy, (19) has no additional terms of higher order in $\Gamma^{(4)}$. Equation (17) contains no approximations, hence it can be used as a replacement of (19) in the 1PI hierarchy, without introducing any additional approximation.

III. HIGHER-ORDER CONTRIBUTIONS TO THE SELF-CONSISTENT FLOW EQUATIONS

Higher-order contributions can be computed in a similar way by taking into account the higher-order terms in the SD equations.

The six-point vertex in Eq. (10) can itself be expressed in terms of the interaction v, the four-point, the six-point, and the eight-point vertices. At lowest order one obtains from Eq. (7) for $m = 3$,

$$\begin{aligned}\Gamma^{(6)}_{x_1,x_2,x_3;x_1',x_2',x_3'} &\approx \mathbb{S}\Bigg(-6\sum_{z_1\cdots z_6} v_{x_1,x_2;z_1,z_2}\Gamma^{(4)}_{z_3,x_3;z_4,x_3'}\Gamma^{(4)}_{z_5,z_6;x_1',x_2'}G_{z_1,z_5}G_{z_2,z_3}G_{z_4,z_6}\\ &\quad -3\sum_{z_1\cdots z_6}\Gamma^{(4)}_{x_1,x_2;z_1,z_2}v_{z_3,x_3;z_4,x_3'}\Gamma^{(4)}_{z_5,z_6;x_1',x_2'}G_{z_1,z_5}G_{z_2,z_3}G_{z_4,z_6}\Bigg)\\ &\approx -9\mathbb{S}\sum_{z_1\cdots z_6}\Gamma^{(4)}_{x_1,x_2;z_1,z_2}\Gamma^{(4)}_{z_3,x_3;z_4,x_3'}\Gamma^{(4)}_{z_5,z_6;x_1',x_2'}G_{z_1,z_5}G_{z_2,z_3}G_{z_4,z_6}.\end{aligned} \tag{20}$$

In the last step we have replaced v by $\Gamma^{(4)} + O(\Gamma^{(4)})^2$ according to Eq. (10). Following this procedure, we obtain an equation $\Gamma^{(4)} - v = \cdots$, where the right-hand side consists of diagrams involving both the vertex $\Gamma^{(4)}$ and v. In this case, v can be eliminated from the right-hand side by means of iteration. This leads to a self-consistent equation $\Gamma^{(4)} - v = \Theta$, as follows. Denote the particle-particle bubble propagator by Π,

$$(\Pi_G)_{x_1,x_2;x_1',x_2'} := G_{x_1,x_1'}G_{x_2,x_2'}, \tag{21}$$

and the particle-hole bubble propagator by Υ,

$$(\Upsilon_G)_{x_1,x_2;x_1',x_2'} := G_{x_1,x_2'}G_{x_1',x_2}, \tag{22}$$

and define

$$(f\circ g)_{x_1,x_2;x_3,x_4} := \sum_{z_1,z_2} f_{x_1,x_2,z_1,z_2}g_{z_1,z_2,x_3,x_4} \tag{23}$$

and

$$(f * g)_{x_1,x_2;x_3,x_4} := \sum_{z_1,z_2} f_{z_1,x_2,z_2,x_4}g_{x_1,z_1,x_3,z_2}. \tag{24}$$

In this notation,

$$\begin{aligned}\Gamma^{(4)} - v = \Theta := \mathbb{S}\big\{&\tfrac{1}{2}\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)} - 2\Gamma^{(4)}*\Upsilon_G*\Gamma^{(4)}\\ &-\tfrac{1}{4}\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\\ &-2\Gamma^{(4)}*\Upsilon_G*\Gamma^{(4)}*\Upsilon_G*\Gamma^{(4)}\\ &+\tfrac{1}{8}\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\\ &-2\Gamma^{(4)}*\Upsilon_G*\Gamma^{(4)}*\Upsilon_G*\Gamma^{(4)}*\Upsilon_G*\Gamma^{(4)}\\ &-4Q(G,\Gamma^{(4)})\big\} + O(\Gamma^{(4)})^5.\end{aligned} \tag{25}$$

At this order, up to the last term, Θ consist of particle-particle and particle-hole ladder diagrams. The last term $Q(G,\Gamma^{(4)})$ is given by

$$\begin{aligned}&[Q(G,\Gamma^{(4)})]_{x_1,x_2;x_3,x_4}\\ &:= \sum_{z_1\cdots z_{12}}\Gamma^{(4)}_{x_1,z_1;z_2,z_3}\Gamma^{(4)}_{z_4,x_2;z_5,z_6}\Gamma^{(4)}_{x_7,z_8;x_3,z_9}\Gamma^{(4)}_{z_{10},z_{11};z_{12},x_4}\\ &\quad\times G_{z_5,z_1}G_{z_3,z_{10}}G_{z_2,z_7}G_{z_9,z_4}G_{z_6,z_{11}}G_{z_{12},z_8}.\end{aligned} \tag{26}$$

Equation (25) is interesting by itself and will be used in the next section to construct a functional whose stationary points are solutions of the SDEs.

The flow equation for the vertex $\Gamma^{(4)}$ is now given by $\dot{\Gamma}^{(4)} = \frac{d}{d\Lambda}\Theta$. Derivatives of $\Gamma^{(4)}$ which appear on the right-hand side can be eliminated by iterating the result. The interaction v does not appear in the flow equation, since it was assumed to be independent of Λ. It serves as the initial condition for the integration of $\dot{\Gamma}^{(4)}$. In the limit $\Lambda \to \infty$ where all fluctuations are suppressed, $\Gamma^{(4)} = v$. Although Eq. (25) has a rather simple structure, the process of resubstituting $\dot{\Gamma}^{(4)}$ when it appears on the right-hand side mixes and proliferates the terms. The fourth-order corrections are too long to be presented here. Up to third order, the flow equation is given by

$$\begin{aligned}
\dot{\Gamma}_{x_1,x_2;x_3,x_4} = \mathbb{S}_{x_1,x_2;x_3,x_4}\Bigg(&\sum_{z_1\cdots z_4}\Gamma_{x_1,x_2;z_1,z_2}\Gamma_{z_3,z_4;x_3,x_4}\dot{G}_{z_1,z_3}G_{z_2,z_4} - 4\sum_{z_1\cdots z_4}\Gamma_{z_1,x_1;x_3,z_2}\Gamma_{x_2,z_3;z_4,x_4}\dot{G}_{z_4,z_1}G_{z_2,z_3}\\
&+4\sum_{z_1\cdots z_8}\Gamma_{z_1,x_1;x_3,z_2}\Gamma_{z_3,x_2;z_4,z_5}\Gamma_{z_6,z_7;z_8,x_4}G_{z_8,z_1}G_{z_2,z_3}G_{z_4,z_6}\dot{G}_{z_5,z_7}\\
&+8\sum_{z_1\cdots z_8}\Gamma_{z_1,x_1;x_3,z_2}\Gamma_{z_3,x_2;z_4,z_5}\Gamma_{z_6,z_7;z_8,x_4}G_{z_4,z_1}G_{z_2,z_6}G_{z_8,z_3}\dot{G}_{z_5,z_7}\\
&+8\sum_{z_1\cdots z_8}\Gamma_{z_1,x_1;x_3,z_2}\Gamma_{z_3,x_2;z_4,z_5}\Gamma_{z_6,z_7;z_8,x_4}G_{z_4,z_1}G_{z_2,z_6}\dot{G}_{z_8,z_3}G_{z_5,z_7}\\
&+2\sum_{z_1\cdots z_8}\Gamma_{x_1,z_1;z_2,z_3}\Gamma_{z_4,x_2;z_5,z_6}\Gamma_{z_7,z_8;x_3,x_4}\dot{G}_{z_6,z_1}G_{z_2,z_4}G_{z_3,z_8}G_{z_5,z_7}\\
&-4\sum_{z_1\cdots z_8}\Gamma_{x_1,z_1;z_2,x_3}\Gamma_{x_2,z_3;z_4,x_4}\Gamma_{z_5,z_6;z_7,z_8}G_{z_7,z_1}\dot{G}_{z_2,z_5}G_{z_8,z_3}G_{z_4,z_6}\\
&-4\sum_{z_1\cdots z_8}\Gamma_{x_1,z_1;z_2,x_3}\Gamma_{x_2,z_3;z_4,x_4}\Gamma_{z_5,z_6;z_7,z_8}\dot{G}_{z_7,z_1}G_{z_2,z_5}G_{z_8,z_3}G_{z_4,z_6}\\
&+2\sum_{z_1\cdots z_8}\Gamma_{x_2,x_1;z_1,z_2}\Gamma_{z_3,z_4;x_3,z_5}\Gamma_{z_6,z_7;z_8,x_4}G_{z_1,z_4}G_{z_2,z_6}G_{z_8,z_3}\dot{G}_{z_5,z_7}\Bigg).
\end{aligned} \tag{27}$$

The first term remains unaffected by the antisymmetrization operator and is the same as in Eq. (16). The result can in principle be extended to any order, but the computational effort grows rapidly. This RG scheme can also be derived from the 1PI scheme by replacing the single scale propagator by the $\dot{G}$, and taking care of the difference up to the desired order in the irreducible vertex.[7]

IV. A STATIONARY POINT FORMULATION OF THE SCHWINGER-DYSON EQUATIONS

We return to Eq. (25), set $\Gamma^{(4)} - v = \Theta$, and study $\Theta := \Theta(G,\Gamma^{(4)})$ as a functional depending on $\Gamma^{(4)}$ and G. For the solution of the SDE, G itself depends on C and Σ, so that the equations for G and $\Gamma^{(4)}$ are really coupled, but we now consider $\Gamma^{(4)}$ and G as two independent variables. To avoid confusion, the solutions of the SDE will be hatted from now on, i.e., denoted as $\hat{G}$ and $\hat{\Gamma}^{(4)}$. The functional Θ can be written as a gradient with respect to $\Gamma^{(4)}$. The integrability of Θ is a nontrivial property and rather interesting. It allows us to formulate the Schwinger-Dyson equations in term of a stationary point problem, as will be shown below.

For a four-point function f, define $\mathcal{C}$ as the operations

$$\mathcal{C}(f) = \sum_{x,y} f_{x,y;y,x}, \tag{28}$$

which consists of closing the diagram and results in a scalar. Then the SD equation (25) is equivalent to $\frac{d}{d\Gamma^{(4)}}\mathcal{F}_1(G,\Gamma^{(4)}) = 0$ with

$$\begin{aligned}
&\mathcal{F}_1(G,\Gamma^{(4)})\\
&= \tfrac{-1}{4}\mathcal{C}\big\{\tfrac{1}{2}(\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}) - (\Gamma^{(4)}\circ\Pi_G\circ v)\\
&\quad - \tfrac{1}{6}(\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)})\\
&\quad - \tfrac{2}{3}(\Gamma^{(4)} * \Upsilon_G * \Gamma^{(4)} * \Upsilon_G * \Gamma^{(4)})\\
&\quad + \tfrac{1}{16}(\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)})\\
&\quad - \tfrac{1}{2}(\Gamma^{(4)} * \Upsilon_G * \Gamma^{(4)} * \Upsilon_G * \Gamma^{(4)} * \Upsilon_G * \Gamma^{(4)})\\
&\quad - \tfrac{1}{40}(\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\circ\Pi_G\circ\Gamma^{(4)}\\
&\quad \times \circ\,\Pi_G\circ\Gamma^{(4)})\\
&\quad - \tfrac{2}{5}(\Gamma^{(4)} * \Upsilon_G * \Gamma^{(4)} * \Upsilon_G * \Gamma^{(4)} * \Upsilon_G\\
&\quad \times * \Gamma^{(4)} * \Upsilon_G * \Gamma^{(4)})\\
&\quad + \tfrac{4}{5}(Q(G,\Gamma^{(4)})\circ\Pi_G\circ\Gamma^{(4)})\big\} + O(\Gamma^{(4)})^6.
\end{aligned} \tag{29}$$

Note that the components of the gradient with respect to $\Gamma^{(4)}$ are already antisymmetric. More precisely, we restrict $\Gamma^{(4)}$ to have the desired antisymmetry, meaning that the components of $\Gamma^{(4)}$ are not independent. The total derivative of a functional $F(\Gamma^{(4)})$ with respect to $\Gamma^{(4)}_{x_1,x_2;x_3,x_4}$ is then given by

$$\begin{aligned}
\frac{dF(\Gamma^{(4)})}{d\Gamma^{(4)}_{x_1,x_2;x_3,x_4}} &= \frac{\partial F(\Gamma^{(4)})}{\partial\Gamma^{(4)}_{x_1,x_2;x_3,x_4}} - \frac{\partial F(\Gamma^{(4)})}{\partial\Gamma^{(4)}_{x_2,x_1;x_3,x_4}}\\
&\quad - \frac{\partial F(\Gamma^{(4)})}{\partial\Gamma^{(4)}_{x_1,x_2;x_4,x_3}} + \frac{\partial F(\Gamma^{(4)})}{\partial\Gamma^{(4)}_{x_2,x_1;x_4,x_3}}.
\end{aligned} \tag{30}$$

The factor 1/4 in Eq. (29) is the same as the 1/4 hidden in the definition of the antisymmetrization operator in Eq. (25). The stationary point of $\mathcal{F}_1(\hat{G},\Gamma^{(4)})$ is already a solution of the Schwinger-Dyson equation (25). In the next step we want to use $\mathcal{F}_1$ to define a new functional $\mathcal{F}(\Sigma,\Gamma^{(4)})$ whose stationary point is a solution of both Eqs. (25) and (9). Consider G as a function of the self-energy Σ [since $G = (C^{-1} - \Sigma)^{-1}$], and let $\tilde{\Gamma}^{(4)}(G)$ denote a solution of the equation $\frac{d}{d\Gamma^{(4)}}\mathcal{F}_1(G,\Gamma^{(4)}) = 0$ for a given G. We take the derivative of $\mathcal{F}_1(G(\Sigma),\tilde{\Gamma}^{(4)}(G))$ with respect to Σ and make the following helpful observation:

$$\left(\frac{d}{d\Sigma_{x_1',x_1}}\mathcal{F}_1\right)(G(\Sigma),\tilde{\Gamma}^{(4)}[G(\Sigma)]) = -\frac{1}{2}\sum_{\substack{z_1\cdots z_4\\ y_1\cdots y_4}}\big(G_{x_1,z_1}v_{z_1,z_2,z_3,z_4}G_{z_3,y_1}G_{z_4,y_2}\tilde{\Gamma}^{(4)} \times [G(\Sigma)]_{y_1,y_2,y_3,y_4}G_{y_4,z_2}G_{y_3,x_1'}\big) + O(\Gamma^{(4)})^6. \quad (31)$$

The right-hand side looks very similar to the last term of Eq. (8). If we define $\mathcal{F}_2$ as

$$\mathcal{F}_2[G(\Sigma)] = -\sum_{z_1,z_2} G_{z_2,z_1}(C^{-1})_{z_1,z_2} - \frac{1}{2}\sum_{z_1,\ldots,z_4} v_{z_1,z_2;z_3,z_4} \times G_{z_3,z_1}G_{z_4,z_2} + \ln(\det G), \quad (32)$$

and add this $\Gamma^{(4)}$-independent term to $\mathcal{F}_1$, the stationary point of

$$\mathcal{F}(\Sigma,\Gamma^{(4)}) := \mathcal{F}_1(G(\Sigma),\Gamma^{(4)}) + \mathcal{F}_2[G(\Sigma)] \quad (33)$$

with respect to Σ at $\Gamma^{(4)} = \tilde{\Gamma}^{(4)}$ is a solution of Eq. (9). Since $\mathcal{F}_2$ is independent of $\Gamma^{(4)}$, we conclude that the solution of the Schwinger-Dyson equations (9) and (25) is a stationary point of $\mathcal{F}$,

$$\frac{d}{d\Sigma}\mathcal{F}(\Sigma,\Gamma^{(4)}) = 0, \quad \frac{d}{d\Gamma^{(4)}}\mathcal{F}(\Sigma,\Gamma^{(4)}) = 0. \quad (34)$$

The diagrammatic representation of $\mathcal{F}$ is shown in Fig. 2. The solution of Eq. (34) satisfies

$$\Sigma_{x_1;x_1'} = \frac{d\Phi}{dG_{x_1';x_1}} \quad (35)$$

with $\Phi = -\frac{1}{2}vG^2 + \mathcal{F}_1$, hence the self-energy is Φ-derivable. The stationarity of $\mathcal{F}$ at the true solution opens up—in principle—variational methods for the search of solutions for G, which, as mentioned in Ref. 14, need not start close to the free fermion propagator $G_0 = C$. Indeed, the functional $\mathcal{F}$ is a polynomial in G and $\Gamma^{(4)}$, hence it can be studied on a rather general domain of definition for G and $\Gamma^{(4)}$. One should keep in mind, however, that the *derivation* of $\mathcal{F}$ does assume that an expansion in $\Gamma^{(4)}$ makes sense and that $G^{-1} - G_0^{-1}$ is regular enough for a Dyson self-energy to be defined. This is not special to our derivation; the same remark applies in general to derivations of Luttinger-Ward-type functionals by diagrammatic methods.

V. CONCLUSION

We have derived RG equations by making the bare propagator in the functional integral defining the theory dependent on a parameter Λ. (We have discussed Λ as a scale parameter, but that specific choice is not needed in the derivation.) This makes the standard SDE hierarchy of equations Λ-dependent, and taking the derivative with respect to Λ gives a hierarchy of RG flow equations. An approximation to second order in the Λ-dependent vertex $\Gamma_\Lambda^{(4)}$ allows us to rederive Katanin's truncation of the usual RG hierarchy for the 1PI vertex functions. By solving the hierarchy iteratively in a systematic expansion in $\Gamma_\Lambda^{(4)}$, we then constructed an analog of the Luttinger-Ward functional, to a fixed order in an expansion in graphs with vertices $\Gamma_\Lambda^{(4)}$ and full propagators G_Λ.

Because Λ is introduced in the same way here as in the derivation of the functional RG (see, e.g., Refs. 3 and 5), the infinite hierarchy has exactly the same combinatorial and analytical content as the infinite 1PI hierarchy. However, the two hierarchies are not identical, an essential difference being

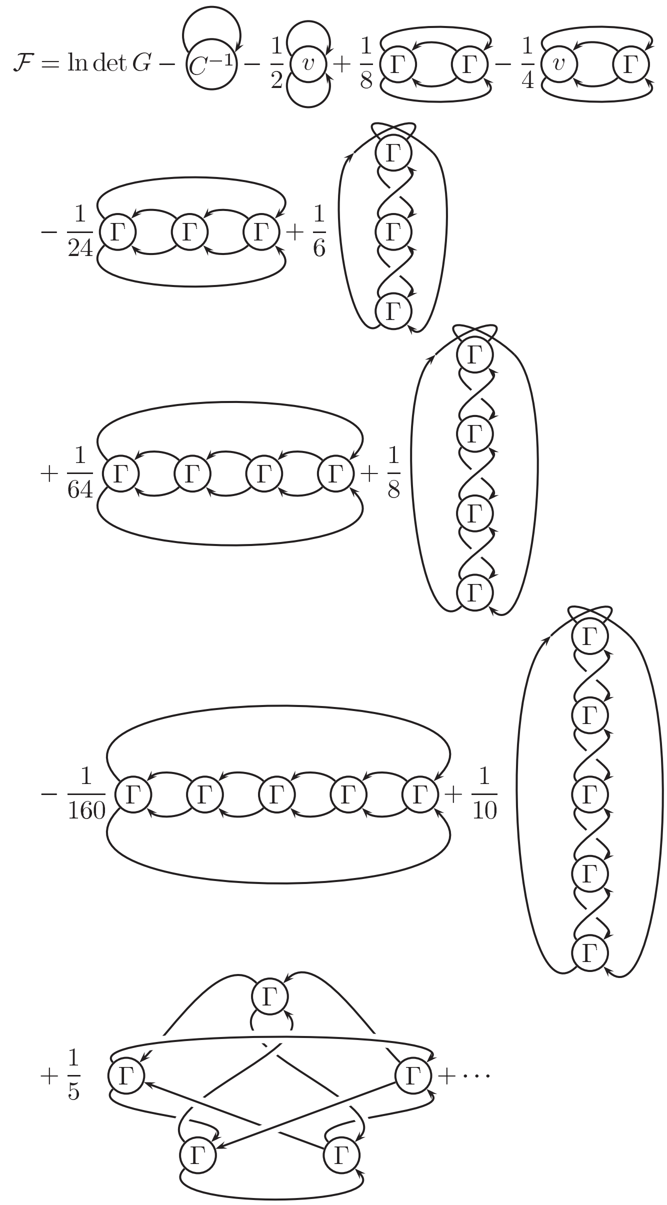


FIG. 2. The diagrammatic representations of the functional $\mathcal{F}$ defined in Eq. (33).

that the bare vertex v appears in every equation of the hierarchy based on the SDE. It is only in an iteration of substitutions that v gets replaced by the scale-dependent vertex $\Gamma_\Lambda^{(4)}$ everywhere, so that the equations take the form of the 1PI RG equations after expanding in $\Gamma_\Lambda^{(4)}$.

To generalize the LW functional to one for scale-dependent quantities, one first has to decide upon which vertex functions this functional is to depend, since at any intermediate scale Λ, all the higher irreducible vertex functions are nonzero. We have constructed an extension that is minimal in that only $\Gamma_\Lambda^{(4)}$ appears as an independent variable besides the full propagator G_Λ (or the self-energy Σ_Λ). Obviously, the higher irreducible vertex functions must be expanded in $\Gamma_\Lambda^{(4)}$ to achieve this. Thus this "minimal" extension is possible only in terms of an expansion in $\Gamma_\Lambda^{(4)}$; otherwise, the higher irreducible functions $\Gamma_\Lambda^{(\geqslant 6)}$ would appear and require separate equations. Compared to the form of the standard LW functional, the explicit terms involving $\det G$ and $\mathrm{Tr}(C^{-1}G)$ are the same, as is the structure of the higher terms as an expansion in vacuum graphs. The combinatorial factors differ from those of the LW functional due to the just mentioned substitution of $\Gamma_\Lambda^{(\geqslant 6)}$ by expressions involving G and $\Gamma_\Lambda^{(4)}$. However, as a functional of G and $\Gamma^{(4)}$, it is universal, i.e., given by a diagrammatical expansion with fixed weight factors. The renormalization group flow leads to the stationary points of $\mathcal{F}$.

That a bare vertex v appears in every equation may be advantageous in the numerical application to specific models, e.g., in the two-dimensional models on which the RG has been extensively used for instability analyses, simply because the bare vertex is known explicitly and in many cases has better properties than the effective vertex function $\Gamma_\Lambda^{(4)}$. For instance, in the Hubbard model, the bare vertex is local in space, which restricts summations in a position-space implementation of the RG equations (and equivalently, the bare vertex is independent of momenta and frequencies in Fourier/Matsubara space). We expect that (27) (or its analog where v is kept explicitly) is tractable in one-dimensional models or two-dimensional patch models, and that it will be useful for understanding deviations from the lower-order approximations. A significant other difference from the 1PI flow equations is that the flow equation obtained from (9) by Λ-differentiation contains a two-loop diagram of "sunset" type, so that the self-energy obtained from this flow contains a nontrivial frequency dependence even if a frequency-independent approximation is used for $\Gamma_\Lambda^{(4)}$. The tadpole-like equation in the 1PI hierarchy does not have this property.

It is a natural idea to use such flow equations also to solve self-consistency equations, as an alternative to iterative solutions. While we do not find it likely that using a flow equation will reduce the numerical cost generically, we do expect that some issues of convergence of iterations encountered in the unscaled SDE (due to the singular nature of the integral equations) may be avoided in an RG flow toward the solution of these equations, so that the method may be useful for that purpose.

The same strategy of deriving RG equations from the SDE can be employed for theories where the interaction between the fermions is mediated by boson exchange, and where both the fermionic and the bosonic propagator are made to be Λ-dependent.

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