Typical versus averaged overlap distribution in spin glasses: Evidence for droplet scaling theory

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We consider the statistical properties over disordered samples $(\mathcal{J})$ of the overlap distribution $P_{\mathcal{J}}(q)$ which plays the role of an order parameter in spin glasses. We show that near zero temperature (i) the *typical* overlap distribution is exponentially small in the central region of $-1 < q < 1$: $P^{\text{typ}}(q) = e^{\overline{\ln P_{\mathcal{J}}(q)}} \sim e^{-\beta N^{\theta}\phi(q)}$, where θ is the droplet exponent defined here with respect to the total number N of spins (in order to consider also fully connected models in which the notion of length does not exist); (ii) the rescaled variable $v = -[\ln P_{\mathcal{J}}(q)]/N^{\theta}$ remains an $O(1)$ random positive variable describing sample-to-sample fluctuations; (iii) the averaged distribution $\overline{P_{\mathcal{J}}(q)}$ is nontypical and dominated by rare anomalous samples. Similar statements hold for the cumulative overlap distribution $I_{\mathcal{J}}(q_0) \equiv \int_0^{q_0} dq\, P_{\mathcal{J}}(q)$. These results are derived explicitly for the spherical mean-field model with $\theta = 1/3$, $\phi(q) = 1 - q^2$, and the random variable v corresponds to the rescaled difference between the two largest eigenvalues of Gaussian orthogonal ensemble random matrices. Then we compare numerically the typical and averaged overlap distributions for the long-ranged one-dimensional Ising spin glass with random couplings decaying as $J(r) \propto r^{-\sigma}$ for various values of the exponent σ, corresponding to various droplet exponents $\theta(\sigma)$, and for the mean-field Sherrington-Kirkpatrick model (corresponding formally to the $\sigma = 0$ limit of the previous model). Our conclusion is that future studies on spin glasses should measure the *typical* values of the overlap distribution $P^{\text{typ}}(q)$ or of the cumulative overlap distribution $I^{\text{typ}}(q_0) = e^{\overline{\ln I_{\mathcal{J}}(q_0)}}$ to obtain clearer conclusions on the nature of the spin-glass phase.

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I. INTRODUCTION

In the statistical physics of quenched disordered systems, where each disordered sample $(\mathcal{J})$ is characterized by its partition function

$$Z_{\mathcal{J}} = \sum_{\mathcal{C}} e^{-\beta E_{\mathcal{J}}(\mathcal{C})} = e^{-\beta F_{\mathcal{J}}(\beta)}, \tag{1}$$

it has been realized from the very beginning[1] that the quenched free-energy

$$\overline{\ln Z_{\mathcal{J}}} = -\beta \overline{F_{\mathcal{J}}} \tag{2}$$

is typical, i.e., is representative of the physics in almost all samples $(\mathcal{J})$, whereas the averaged partition function $\overline{Z_{\mathcal{J}}}$ can be nontypical, especially at low temperature, because it can be dominated by very rare disordered samples $(\mathcal{J})$. Correlation functions are, from this point of view, very similar to partition functions: the averaged correlation can be very different from the typical correlation. It is very clear in one-dimensional spin systems,[2,3] where correlation functions can be written as a product of random numbers, but it is also true for higher-dimensional models.[4–8] More generally, for each observable, it is very important to be aware of the possible differences between typical and averaged values, and to have a clear idea of the distribution over samples.

In the field of classical spin glasses (see, for instance, Refs. 9–11), there has been an ongoing debate on the nature of the spin-glass phase between the droplet scaling theory,[12–14] which is based on real-space renormalization ideas (explicit real-space renormalization for spin glasses has been studied in detail within the Migdal-Kadanoff approximation[15]), and the alternative replica-symmetry-breaking scenario[16] based on the mean-field fully connected Sherrington-Kirkpatrick model.[17] The questions under debate include the presence of the number of ground states (two or many),[18–20] the properties of the overlap,[21–29] the statistics of excitations,[30,31] the structure of state space,[32] the absence or presence of an Almeida-Thouless line in the presence of a magnetic field,[33–38] etc. In particular, one of the standard observables to discriminate between the droplet and the replica theories has been the *averaged* overlap distribution $\overline{P_{\mathcal{J}}(q)}$. In the present paper, we show that this *averaged* overlap distribution $\overline{P_{\mathcal{J}}(q)}$ is actually nontypical and is governed by rare disordered samples, whereas the *typical* overlap distribution

$$P^{\text{typ}}(q) \equiv e^{\overline{\ln P_{\mathcal{J}}(q)}} \tag{3}$$

is in full agreement with the droplet scaling theory. Our conclusion is that it does not seem to be a good idea to use a nontypical observable such as the *averaged* overlap distribution $\overline{P_{\mathcal{J}}(q)}$ to elucidate the physics of spin glasses, and that future studies should focus on the *typical* overlap distribution to obtain clear conclusions. Note that two recent studies have also proposed to study other statistical properties of the overlap distribution $P_{\mathcal{J}}(q)$ than the averaged value, namely the statistics of peaks[39] or the median over samples of cumulative overlap distribution.[40] We hope that the numerical measure of the *typical* overlap distribution, which is a much simpler observable, will give even clearer evidence for the droplet scaling theory.

The paper is organized as follows. In Sec. II, we discuss the general properties of the overlap distribution. In Sec. III, we derive explicit results for the spherical mean-field model. In Sec. IV, we present numerical results for the one-dimensional long-ranged spin glass with random couplings decaying as $J(r) \propto r^{-\sigma}$ for various values of the exponent σ. In Sec. V, we show numerical results for the mean-field Sherrington-Kirkpatrick (SK) model (corresponding formally to the $\sigma = 0$ limit of the previous model). Our conclusions are summarized in Sec. VI. Appendix A contains a brief reminder on the physical meanings of the droplet exponent θ, whereas Appendix B briefly recalls the replica prediction for the distribution of the cumulative overlap distribution.

II. OVERLAP DISTRIBUTION IN A GIVEN DISORDERED SAMPLE

A. Notations

Let us consider a general spin-glass model containing N spins $S_i = \pm 1$ and random couplings $\mathcal{J} \equiv \{J_{ij}\}$,

$$H_{\mathcal{J}} = -\sum J_{ij} S_i S_j. \tag{4}$$

The partition function associated with the disordered sample $\mathcal{J} \equiv \{J_{ij}\}$ reads

$$Z_{\mathcal{J}}^{\text{single}}(\beta) = \sum_{\{S_i=\pm 1\}} e^{\beta \sum J_{ij} S_i S_j}. \tag{5}$$

We use here the notation "single" to stress that this partition function contains a single "copy" of spins, in contrast to partition functions concerning "two copies" of spins that we will introduce below. To characterize the spin-glass "order", one introduces the overlap

$$Q = \sum_{i=1}^{N} S_i^{(1)} S_i^{(2)} \tag{6}$$

between two independent copies of spins ($S_i^{(1)} = \pm 1, S_i^{(2)} = \pm 1$) in the same disordered sample $\mathcal{J} \equiv \{J_{ij}\}$. The parameter Q of Eq. (6) can take the $(N+1)$ discrete values $-N, -N+2, \ldots, N-2, N$, so that for a large system it is convenient to consider the rescaled overlap

$$q \equiv \frac{Q}{N}, \tag{7}$$

which remains in the interval $-1 \leqslant q \leqslant 1$.

B. Overlap distribution as a ratio of partition functions

The probability distribution of the overlap introduced in Eq. (6) can be written as the ratio of two partition functions concerning the two copies,

$$\mathcal{P}_{\mathcal{J}}(Q) = \frac{\mathcal{Z}_{\mathcal{J}}(\beta; Q)}{\mathcal{Z}_{\mathcal{J}}(\beta)}. \tag{8}$$

The numerator of Eq. (8) represents the partition function of two copies in the same disorder constrained to a given overlap Q [Eq. (6)],

$$\mathcal{Z}_{\mathcal{J}}(\beta; Q) \equiv \sum_{\{S_i^{(1)}=\pm 1\}} \sum_{\{S_i^{(2)}=\pm 1\}} e^{\beta \sum J_{ij}(S_i^{(1)} S_j^{(1)} + S_i^{(2)} S_j^{(2)})} \times \delta_{Q, \sum_{i=1}^{N} S_i^{(1)} S_i^{(2)}}. \tag{9}$$

The denominator is the full partition function of the two copies in the same disorder, with no constraint on the overlap, so that it factorizes into the product of two partition functions concerning a single copy [Eq. (5)],

$$\mathcal{Z}_{\mathcal{J}}(\beta) \equiv \sum_{\{S_i^{(1)}=\pm 1\}} \sum_{\{S_i^{(2)}=\pm 1\}} e^{\beta \sum J_{ij}(S_i^{(1)} S_j^{(1)} + S_i^{(2)} S_j^{(2)})} = \left[Z_{\mathcal{J}}^{\text{single}}(\beta)\right]^2. \tag{10}$$

The fact that the overlap distribution $\mathcal{P}_{\mathcal{J}}(Q)$ is a ratio of two partition functions [Eq. (8)] yields that its logarithm $\ln \mathcal{P}_{\mathcal{J}}(Q)$ corresponds to a difference of two free energies,

$$\ln \mathcal{P}_{\mathcal{J}}(Q) = \ln \mathcal{Z}_{\mathcal{J}}(\beta; Q) - \ln \mathcal{Z}_{\mathcal{J}}(\beta). \tag{11}$$

Since averaged free energies are known to be typical [see the Introduction around Eq. (2)], the typical overlap distribution defined as

$$\ln P^{\text{typ}}(Q) \equiv \overline{\ln \mathcal{P}_{\mathcal{J}}(Q)} = \overline{\ln \mathcal{Z}_{\mathcal{J}}(\beta; Q)} - \overline{\ln \mathcal{Z}_{\mathcal{J}}(\beta)} \tag{12}$$

will be representative of most samples, whereas the averaged value $P^{\text{av}}(Q)$ obtained by averaging directly the ratio of partition functions of Eq. (8),

$$P^{\text{av}}(Q) \equiv \overline{\mathcal{P}_{\mathcal{J}}(Q)} = \overline{\left(\frac{\mathcal{Z}_{\mathcal{J}}(\beta; Q)}{\mathcal{Z}_{\mathcal{J}}(\beta)}\right)}, \tag{13}$$

can be dominated by nontypical disordered samples, especially at very low temperature, as we now discuss.

C. Behavior near zero temperature

Exactly at zero temperature, the single-copy partition function of Eq. (5) will be dominated by the ground-state energy $E_{\mathcal{J}}^{(\text{GS})}$ corresponding to the two ground states related by a global flip of all the spins ($\{S_i^{(\text{GS})}\}$ and $\{-S_i^{(\text{GS})}\}$),

$$Z_{\mathcal{J}}^{\text{single}}(\beta) \underset{T\to 0}{\simeq} 2 e^{-\beta E_{\mathcal{J}}^{(\text{GS})}}. \tag{14}$$

The two-copies partition function of Eq. (9) will also be dominated by the cases in which each of the two copies is in either of the two ground states, so that it reads

$$\mathcal{Z}_{\mathcal{J}}(\beta; Q) \underset{T\to 0}{\simeq} e^{-2\beta E_{\mathcal{J}}^{(\text{GS})}} (2\delta_{Q,N} + 2\delta_{Q,-N}). \tag{15}$$

The overlap distribution of Eq. (8) has thus the following expected zero-temperature limit in each sample:

$$\mathcal{P}_{\mathcal{J}}^{T=0}(Q) = \tfrac{1}{2}(\delta_{Q,N} + \delta_{Q,-N}). \tag{16}$$

To obtain the dominant contribution near zero temperature at a given overlap value $Q \neq \pm N$, we may consider that one of the two copies (say $S_i^{(1)}$) is in one of the ground states [say $(S_i^{(\text{GS})})$] in Eq. (9): then to obtain a given overlap Q, the second copy $S_i^{(2)}$ must have

$$n \equiv \frac{N - Q}{2} \tag{17}$$

spins different from the first copy ($S_i^{(2)} = -S_i^{(1)}$) and $N - n = (N+Q)/2$ spins identical to the first copy ($S_i^{(2)} = S_i^{(1)}$),

$$\mathcal{Z}_{\mathcal{J}}(\beta; Q = N - 2n) \simeq 4 \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant N} e^{-\beta E_{\mathcal{J}}(i_1, \ldots, i_n)}$$

$$E_{\mathcal{J}}(i_1, \ldots, i_n) \equiv -\sum_{ij} J_{ij} S_i^{(2)} S_j^{(2)} \left(\prod_{k=1}^{n} \delta_{S_{i_k}^{(2)}, -S_{i_k}^{\text{GS}}}\right) \times \prod_{i \neq (i_1, \ldots, i_n)} \delta_{S_{i_k}^{(2)}, S_{i_k}^{\text{GS}}}. \tag{18}$$

The ratio of Eq. (8) for $Q \neq \pm N$ will thus have for the leading contribution

$$\mathcal{P}_{\mathcal{J}}(Q = N - 2n) \simeq \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant N} e^{-\beta[E_{\mathcal{J}}(i_1,\ldots,i_n) - E_{\mathcal{J}}^{\mathrm{GS}}]}, \tag{19}$$

which represents the partition function of excitations of a given size n. Near zero temperature, one further expects that in each given sample, the overlap distribution will be dominated by the biggest of these contributions,

$$\mathcal{P}_{\mathcal{J}}(Q = N - 2n) \simeq e^{-\beta E_{\mathcal{J}}^{\min}(n)}, \tag{20}$$

where

$$E_{\mathcal{J}}^{\min}(n) \equiv \min_{1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant N} [E_{\mathcal{J}}(i_1,\ldots,i_n) - E_{\mathcal{J}}^{\mathrm{GS}}] \tag{21}$$

represents the minimal energy cost $[E_{\mathcal{J}}(i_1,\ldots,i_n) - E_{\mathrm{GS}}]$ among all excitations involving the flipping of exactly $n = \frac{N-Q}{2}$ spins with respect to the ground state.

So we expect that the typical overlap has the following leading behavior near zero temperature:

$$\ln \mathcal{P}^{\mathrm{typ}}(Q) \equiv \overline{\ln \mathcal{P}_{\mathcal{J}}(Q)} \simeq -\beta \overline{E_{\mathcal{J}}^{\min}\left(n = \frac{N-Q}{2}\right)}. \tag{22}$$

D. Relation with the droplet scaling theory

The probability distribution $P_{\mathcal{J}}(q)$ of the rescaled variable $q = Q/N$ of Eq. (7) reads near zero temperature [Eq. (20)]

$$P_{\mathcal{J}}(q) = N\mathcal{P}_{\mathcal{J}}(Q) \simeq e^{-\beta E_{\mathcal{J}}^{\min}(n = N\frac{(1-q)}{2})}. \tag{23}$$

In the central region $-1 < q < 1$, the number $n = N\frac{(1-q)}{2}$ of spins is extensive in the total number N of spins of the disordered sample. According to the droplet scaling theory,[12–14] the droplet exponent θ describes the scaling of the energy "optimized excitations" with respect to their size (see Appendix A), so that we expect the scaling

$$\ln P_{\mathcal{J}}(q) \simeq -\beta E_{\mathcal{J}}^{\min}\left(n = N\frac{(1-q)}{2}\right) \simeq -\beta N^{\theta} v, \tag{24}$$

where v is a positive random variable of order $O(1)$. In particular, the corresponding typical value is exponentially small,

$$\ln \mathcal{P}^{\mathrm{typ}}(q) \equiv \overline{\ln \mathcal{P}_{\mathcal{J}}(q)} \simeq -\beta N^{\theta}, \tag{25}$$

whereas the averaged value will be governed by the rare samples having an anomalous small variable $v \leqslant T/N^{\theta}$. This analysis leads to a power-law decay with respect to the size N,

$$P^{\mathrm{av}}(q) \equiv \overline{P_{\mathcal{J}}(q)} \propto N^{-x}, \tag{26}$$

where the exponent x depends on the behavior of the probability distribution of the variable v near the origin $P(v \to 0)$, as well as on possible prefactors in front of the exponential factor of Eq. (23). For short-ranged spin-glass models, the standard droplet scaling theory[12–14] predicts a finite weight at the origin $P(v = 0) > 0$ for the variable v, and no size prefactors, so that the exponent x takes the simple value given by the droplet exponent,

$$x_{\mathrm{simple}} = \theta. \tag{27}$$

However, it is clear that these are two additional properties with respect to the analysis of the typical behavior. For instance, in the quantum random transverse-field Ising chain,[7] equivalent to the two dimensional classical McCoy-Wu model,[6] the typical correlation function decays as $C_{\mathrm{typ}}(r) = e^{\overline{\ln C(r)}} \sim e^{-r^{\theta}u}$ with the simple droplet exponent $\theta = 1/2$, whereas the *averaged* correlation decays as the power law $\overline{C(r)} \propto r^{-x}$ with the nontrivial exponent $x = (3 - \sqrt{5})/2$.[7] In summary, we feel that the exponential typical decay of Eq. (25) is a very robust conclusion of the droplet scaling theory, whereas the power-law decay with $x_{\mathrm{simple}} = \theta$ of the averaged value is based on additional hypotheses that are less general (see, for instance, Sec. III concerning the spherical model where the variable v does not have a finite weight near the origin [Eq. (59)]).

E. Cumulative overlap distribution in each sample

It is convenient to consider also the cumulative overlap distribution

$$I_{\mathcal{J}}(q_0) \equiv \int_0^{q_0} dq\, P_{\mathcal{J}}(q) = \sum_{Q=0}^{Nq_0} \mathcal{P}_{\mathcal{J}}(Q). \tag{28}$$

Near zero temperature, the leading contribution of Eq. (19) yields

$$I_{\mathcal{J}}(q_0) \simeq \sum_{n=\frac{N(1-q_0)}{2}}^{\frac{N}{2}} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant N} e^{-\beta[E_{\mathcal{J}}(i_1,\ldots,i_n) - E_{\mathcal{J}}^{\mathrm{GS}}]}, \tag{29}$$

which represents the partition function over excitations containing n flipped spins with respect to the ground state, where n is in the interval $\frac{N(1-q_0)}{2} \leqslant n \leqslant \frac{N}{2}$. The important point is that the minimal value $\frac{N(1-q_0)}{2}$ is also system-sized. So from the point of view of the droplet scaling theory, the minimal energy cost of these system-size excitations in each sample will lead to the same scaling as Eq. (24),

$$\ln I_{\mathcal{J}}(q_0) \simeq -\beta E_{\mathcal{J}}^{\min}\left(N\frac{(1-q_0)}{2} \leqslant n \leqslant \frac{N}{2}\right) \simeq -\beta N^{\theta} v, \tag{30}$$

where v is a positive random variable of order $O(1)$. As a consequence, the typical value $I^{\mathrm{typ}}(q_0)$ will be exponentially small,

$$\ln I^{\mathrm{typ}}(q_0) \equiv \overline{\ln I_{\mathcal{J}}(q_0)} \simeq -\beta N^{\theta}. \tag{31}$$

On the contrary, within the replica theory,[16] the typical value remains finite for $N = +\infty$ [see Eq. (B6) of Appendix B], i.e., roughly speaking, this corresponds to a vanishing droplet exponent $\theta = 0$.

Again, Eq. (30) yields that the averaged value $I^{\mathrm{av}}(q_0)$ of the cumulative distribution will be governed by the rare samples having an anomalous small variable v and will decay as a power law as Eq. (26).

III. FULLY CONNECTED SPHERICAL SPIN-GLASS MODEL

In this section, we consider the fully connected spherical spin-glass model introduced in Ref. 41 defined by the Hamiltonian

$$H_{\mathcal{J}} = -\sum_{1\leqslant i<j\leqslant N} J_{i,j} S_i S_j = -\frac{1}{2}\sum_{i\neq j} J_{i,j} S_i S_j, \qquad (32)$$

where the random couplings $J_{i,j} = J_{j,i}$ are drawn with the Gaussian distribution,

$$P(J_{ij}) = \sqrt{\frac{N}{2\pi}} e^{-\frac{N J_{ij}^2}{2}}, \qquad (33)$$

and where the spins are not Ising variables $S_i = \pm 1$ but are instead continuous variables $S_i \in]-\infty, +\infty[$ submitted to the global constraint

$$\sum_{i=1}^{N} S_i^2 = N \qquad (34)$$

so that the partition function for a given sample reads

$$Z_{\mathcal{J}}^{\text{single}}(\beta) = \left(\prod_{i=1}^{N}\int_{-\infty}^{+\infty} dS_i\right) e^{\frac{\beta}{2}\sum_{i\neq j} J_{i,j}S_iS_j}\,\delta\left(N - \sum_{i=1}^{N} S_i^2\right). \qquad (35)$$

A. Ground-state energy in each sample

The random couplings J_{ij} form a random Gaussian symmetric matrix $\tilde{J}$ of size N. Let us introduce its N eigenvalues in the order

$$\lambda_1 > \lambda_2 > \cdots > \lambda_N \qquad (36)$$

and the corresponding basis of eigenvectors $|\lambda_p\rangle$ to have the spectral decomposition

$$\tilde{J} = \sum_{p=1}^{N} \lambda_p |\lambda_p\rangle\langle\lambda_p|. \qquad (37)$$

Writing the spin vector in this new basis,

$$|S\rangle = \sum_{i=1}^{N} S_i|i\rangle = \sum_{p=1}^{N} S_{\lambda_p}|\lambda_p\rangle, \qquad (38)$$

the partition function of Eq. (35) becomes

$$Z_{\mathcal{J}}^{\text{single}}(\beta) = \left(\prod_{p=1}^{N}\int_{-\infty}^{+\infty} dS_{\lambda_p}\right) e^{\frac{\beta}{2}\sum_{p=1}^{N}\lambda_p S_{\lambda_p}^2}\,\delta\left(N - \sum_{p=1}^{N} S_{\lambda_p}^2\right). \qquad (39)$$

The ground state is now obvious: to maximize the argument of the exponential, one needs to put the maximal possible weight in the first possible maximal eigenvalue λ_1 [Eq. (36)] and zero weight in all other eigenvalues λ_p with $p = 2,3,\ldots,N$,

$$\begin{aligned} S_{\lambda_p\neq\lambda_1}^{\text{GS}} &= 0, \\ S_{\lambda_1}^{\text{GS}} &= \pm\sqrt{N - \sum_{p=2}^{N}\left(S_{\lambda_p}^{\text{GS}}\right)^2} = \pm\sqrt{N}. \end{aligned} \qquad (40)$$

So Eq. (39) has for the leading exponential term

$$Z_{\mathcal{J}}^{\text{single}}(\beta) \propto e^{\frac{\beta}{2}\lambda_1 (S_{\lambda_p}^{\text{GS}})^2} = e^{\beta\frac{N\lambda_1}{2}} \qquad (41)$$

and the ground-state energy is simply determined by the first eigenvalue λ_1,

$$E_{\mathcal{J}}^{\text{GS}}(N) \simeq \frac{\ln Z_{\mathcal{J}}^{\text{single}}(\beta)}{(-\beta)} = -N\frac{\lambda_1}{2}. \qquad (42)$$

The statistics of the largest eigenvalue λ_1 of random Gaussian symmetric matrices (Gaussian orthogonal ensemble) is known to be given by

$$\lambda_1 = 2\left(1 + \frac{u}{2N^{2/3}}\right), \qquad (43)$$

where the value 2 corresponds to the boundary of the semicircle law that emerges in the thermodynamic limit $N \to +\infty$, and where u is a random variable of order $O(1)$ distributed with the Tracy-Widom distribution.[42] The ground-state energy thus reads

$$E_{\mathcal{J}}^{\text{GS}}(N) = -N\frac{\lambda_1}{2} = -N - N^{1/3}\frac{u}{2}. \qquad (44)$$

In summary, the extensive term is nonrandom, and the next subleading term is of order $N^{1/3}$ and random, distributed with the Tracy-Widom distribution, as already mentioned in Ref. 43. Within the general analysis of the statistics of the ground-state energy recalled in the Appendix [Eqs. (A6) and (A7)], this means that the spherical model has for a droplet exponent and for a fluctuation exponent the same simple value,

$$\theta^{\text{sph}} = \tfrac{1}{3}, \quad \mu^{\text{sph}} = \tfrac{1}{3}. \qquad (45)$$

B. Overlap distribution in each sample

To analyze the overlap distribution in a given sample, we analyze similarly the two-copies partition function [Eq. (9)],

$$\begin{aligned} \mathcal{Z}_{\mathcal{J}}(\beta;Q) = &\left(\prod_{i=1}^{N}\int_{-\infty}^{+\infty} dS_i^{(1)}\int_{-\infty}^{+\infty} dS_i^{(2)}\right) \\ &\times e^{\frac{\beta}{2}\sum_{i\neq j} J_{i,j}(S_i^{(1)}S_j^{(1)} + S_i^{(2)}S_j^{(2)})} \\ &\times \delta\left(N - \sum_{i=1}^{N}(S_i^{(1)})^2\right)\delta\left(N - \sum_{i=1}^{N}(S_i^{(2)})^2\right) \\ &\times \delta\left(Q - \sum_{i=1}^{N} S_i^{(1)}S_i^{(2)}\right). \end{aligned} \qquad (46)$$

Using the basis of eigenvectors of the matrix $\tilde{J}$ of the couplings [Eq. (37)], Eq. (46) becomes

$$\begin{aligned} \mathcal{Z}_{\mathcal{J}}(\beta;Q) = &\left(\prod_{p=1}^{N}\int_{-\infty}^{+\infty} dS_{\lambda_p}^{(1)}\int_{-\infty}^{+\infty} dS_{\lambda_p}^{(2)}\right) \\ &\times e^{\frac{\beta}{2}\sum_{p=1}^{N}\lambda_p[(S_{\lambda_p}^{(1)})^2 + (S_{\lambda_p}^{(2)})^2]} \\ &\times \delta\left(N - \sum_{p=1}^{N}(S_{\lambda_p}^{(1)})^2\right)\delta\left(N - \sum_{p=1}^{N}(S_{\lambda_p}^{(2)})^2\right) \\ &\times \delta\left(Q - \sum_{p=1}^{N} S_{\lambda_p}^{(1)}S_{\lambda_p}^{(2)}\right). \end{aligned} \qquad (47)$$

To obtain the leading behavior near zero temperature, we may consider that one of the copy (say $S^{(1)}$) is in one of the two ground states [Eq. (40)],

$$S^{(1)}_{\lambda_p \neq \lambda_1} = S^{GS}_{\lambda_p \neq \lambda_1} = 0, \quad S^{(1)}_{\lambda_1} = S^{GS}_{\lambda_1} = \sqrt{N}. \tag{48}$$

Then the component $S^{(2)}_{\lambda_1}$ of the second copy on the first eigenvector is completely fixed by the overlap Q,

$$S^{(2)}_{\lambda_1} = \frac{Q}{\sqrt{N}}. \tag{49}$$

The best that we can do for the second copy is thus to put all the remaining weight on the second eigenvalue, and zero weight on higher eigenvalues $p = 3,4,\ldots,N$,

$$\begin{aligned} S^{(2)}_{\lambda_p<\lambda_2} &= 0, \\ S^{(2)}_{\lambda_2} &= \sqrt{N - \left(S^{(2)}_{\lambda_1}\right)^2} = \sqrt{N - \frac{Q^2}{N}}. \end{aligned} \tag{50}$$

Then the leading exponential term of Eq. (47) reads

$$\begin{aligned} \mathcal{Z}_{\mathcal{J}}(\beta;Q) &\propto e^{\frac{\beta}{2}\{\lambda_1[(S^{(1)}_{\lambda_1})^2+(S^{(2)}_{\lambda_1})^2]+\lambda_2(S^{(2)}_{\lambda_2})^2\}} \\ &= e^{\frac{\beta}{2}[\lambda_1(N+\frac{Q^2}{N})+\lambda_2(N-\frac{Q^2}{N})]}. \end{aligned} \tag{51}$$

The leading behavior of the denominator of Eq. (10) reads using Eq. (41)

$$\mathcal{Z}_{\mathcal{J}}(\beta) = \left[Z^{\text{single}}_{\mathcal{J}}(\beta)\right]^2 \propto e^{\beta N \lambda_1} \tag{52}$$

so the overlap distribution of Eq. (8) reads near zero temperature

$$\mathcal{P}_{\mathcal{J}}(Q) \propto \frac{\mathcal{Z}_{\mathcal{J}}(\beta;Q)}{[Z_{\mathcal{J}}(\beta)]^2} \propto e^{-\frac{\beta}{2}N(\lambda_1-\lambda_2)(1-\frac{Q^2}{N^2})}, \tag{53}$$

i.e., in the rescaled variable $q = Q/N$,

$$P_{\mathcal{J}}(q) = N\mathcal{P}_{\mathcal{J}}(Q = Nq) \propto e^{-\frac{\beta}{2}N(\lambda_1-\lambda_2)(1-q^2)}. \tag{54}$$

The difference between the two largest eigenvalues reads[44,45]

$$\lambda_1 - \lambda_2 = \frac{v}{N^{2/3}}, \tag{55}$$

where v is a positive random variable of order $O(1)$, whose distribution can be obtained from the joint distribution of (λ_1,λ_2)[44] [here we need the Gaussian orthogonal ensemble (GOE) case, but see Ref. 45 for the neighboring case of Gaussian unitary ensemble (GUE) matrices]. Plugging Eq. (55) yields the final result

$$P_{\mathcal{J}}(q) \propto e^{-\frac{\beta}{2}N^{1/3}(1-q^2)v}. \tag{56}$$

In particular, the typical value decays exponentially in $N^\theta = N^{1/3}$ in the whole central region $-1 < q < 1$,

$$\ln P^{\text{typ}}(q) \equiv \overline{\ln P_{\mathcal{J}}(q)} \propto -\frac{\beta}{2}N^{1/3}(1-q^2)\overline{v}, \tag{57}$$

and the appropriate rescaled variable is

$$v = \left(-\frac{\ln P_{\mathcal{J}}(q)}{\frac{\beta}{2}N^{1/3}(1-q^2)}\right), \tag{58}$$

which is the $O(1)$ positive random variable of Eq. (55) for GOE matrices. In the Gaussian random matrix ensembles, it is well known that there exists a level repulsion between nearest-neighbor eigenvalues as a consequence of the delocalized character of eigenstates, with the following power law for the distribution $P(v)$ of the variable v of Eq. (55) near the origin $v \to 0$:

$$P(v) \underset{v\to 0}{\propto} v^a, \tag{59}$$

where $a = 1$ for GOE ($a = 2$ for GUE). This is different from the finite weight $P(v = 0) > 0$ expected in short-ranged spin-glass models. As a consequence, the power-law decay of the averaged value in the spherical model,

$$P^{\text{av}}(q) \equiv \overline{P_{\mathcal{J}}(q)} \propto N^{-x_{\text{sph}}}, \tag{60}$$

will be different from the simple value of Eq. (27), and should be instead

$$x_{\text{sph}} = (1+a)\theta = \tfrac{2}{3}. \tag{61}$$

C. Cumulative overlap distribution in each sample

In each sample $\mathcal{J}$, the cumulative overlap distribution will inherit from Eq. (54) the same exponential decay with respect to the size,

$$\begin{aligned} I_{\mathcal{J}}(q_0) &\equiv \int_0^{q_0} dq\, P_{\mathcal{J}}(q) \propto e^{-\frac{\beta}{2}N(\lambda_1-\lambda_2)(1-q_0^2)} \\ &= e^{-\frac{\beta}{2}N^{1/3}(1-q_0^2)v}, \end{aligned} \tag{62}$$

where v is the positive random variable of order $O(1)$ of Eq. (55).

IV. ONE-DIMENSIONAL LONG-RANGED ISING SPIN GLASS

A. Model

The one-dimensional long-ranged Ising spin glass[46] is defined by the Hamiltonian

$$H_{\mathcal{J}} = -\sum_{1\leqslant i<j\leqslant N} J_{ij}S_iS_j, \tag{63}$$

where the N spins $S_i = \pm$ lie equidistantly on a ring, so that the distance between the two spins S_i and S_j reads

$$r_{ij} = \frac{N}{\pi}\sin\left(|j-i|\frac{\pi}{N}\right). \tag{64}$$

The couplings are chosen to decay with some power law of this distance,

$$J_{ij} = c_N(\sigma)\frac{\epsilon_{ij}}{r^\sigma_{ij}}, \tag{65}$$

where ϵ_{ij} are random Gaussian variables of zero mean $\overline{\epsilon} = 0$ and unit variance $\overline{\epsilon^2} = 1$. The constant $c_N(\sigma)$ is defined by the condition

$$1 = \sum_{j\neq 1}\overline{J^2_{1j}} = c^2_N(\sigma)\sum_{j\neq 1}\frac{1}{r^{2\sigma}_{1j}}. \tag{66}$$

It is important to distinguish the two regimes:

(i) For $0 \leqslant \sigma < 1/2$, there is an explicit size rescaling of the couplings,

$$c_N(\sigma) \propto N^{\sigma-\frac{1}{2}}, \tag{67}$$

as in the Sherrington-Kirkpatrick mean-field model that corresponds to the case $\sigma = 0$.

(ii) For $\sigma > 1/2$, there is no size rescaling of the couplings,

$$c_N(\sigma) = O(1). \tag{68}$$

The limit $\sigma = +\infty$ corresponds to the short-ranged one-dimensional model. There exists a spin-glass phase at low temperature for $\sigma < 1$.[46] The critical point is mean-field-like for $\sigma < 2/3$, and non-mean-field-like for $2/3 < \sigma < 1$.[46]

In summary, this model allows us to interpolate continuously between the one-dimensional short-ranged model ($\sigma = +\infty$) and the Sherrington-Kirkpatrick mean-field model ($\sigma = 0$), and is much simpler to study numerically than hypercubic lattices as a function of the dimension d. This is why this model has attracted a lot of interest recently[47–55] (there also exists a diluted version of the model[56]).

B. Measure of the droplet exponent $\theta(\sigma)$

Since we wished to evaluate minimal excitation energies such as Eq. (21), we have chosen to work, in each disordered sample, by exact enumeration of the 2^N spin configurations for small sizes $6 \leqslant N \leqslant 24$. The statistics over samples has been obtained, for instance, with the following numbers $n_s(N)$ of disordered samples:

$$\begin{aligned} n_s(L \leqslant 12) &= 2 \times 10^8; \ldots; \\ n_s(L = 16) &= 10^7; \\ \ldots n_s(L = 22) &= 10^5; \\ n_s(L = 24) &= 2 \times 10^4. \end{aligned} \tag{69}$$

1. *The droplet exponent as a stiffness exponent*

The droplet exponent $\theta(\sigma)$ as a function of σ has been measured via Monte Carlo simulations on sizes $L \leqslant 256$ in Ref. 47 from the difference of the ground-state energy between periodic and antiperiodic boundary conditions in each sample [see the Appendix around Eq. (A4) for more explanations],

$$E_{\mathcal{J}}^{\mathrm{GS(P)}} - E_{\mathcal{J}}^{\mathrm{GS(AP)}} = N^{\theta} u, \tag{70}$$

where u is an $O(1)$ random variable of zero mean (with a probability distribution symmetric in $u \to -u$). In this context, "antiperiodic" means the following prescription:[47] for each disordered sample (J_{ij}) considered as "periodic", the "antiperiodic" consists in changing the sign $J_{ij} \to -J_{ij}$ for all pairs (i,j) where the shortest path on the circle goes through the bond $(L,1)$. We have followed exactly the same procedure, and our results via exact enumeration on much smaller sizes $6 \leqslant L \leqslant 24$ for the three values of σ we have considered are actually close to the values given in Ref. 47,

$$\begin{aligned} \theta(\sigma = 0.1) &\simeq 0.3, \\ \theta(\sigma = 0.62) &\simeq 0.24, \\ \theta(\sigma = 0.75) &\simeq 0.17. \end{aligned} \tag{71}$$

We refer the reader to Ref. 47 for other values of σ.

2. *Statistics over samples of the ground-state energy*

We have also studied the statistics of the ground-state energy over samples [see the Appendix around Eqs. (A6) and (A7) for more explanations]. We find that the correction to extensivity of the averaged ground-state energy [see Eq. (A6)],

$$\overline{E_{\mathcal{J}}^{\mathrm{GS}}(N)} \simeq N e_0 + N^{\theta_{\mathrm{shift}}} e_1 + \cdots, \tag{72}$$

is governed by the droplet exponent measured in Eq. (71) from Eq. (70),

$$\theta_{\mathrm{shift}}(\sigma) = \theta(\sigma), \tag{73}$$

as expected in general within the droplet scaling theory [Eq. (A8)].

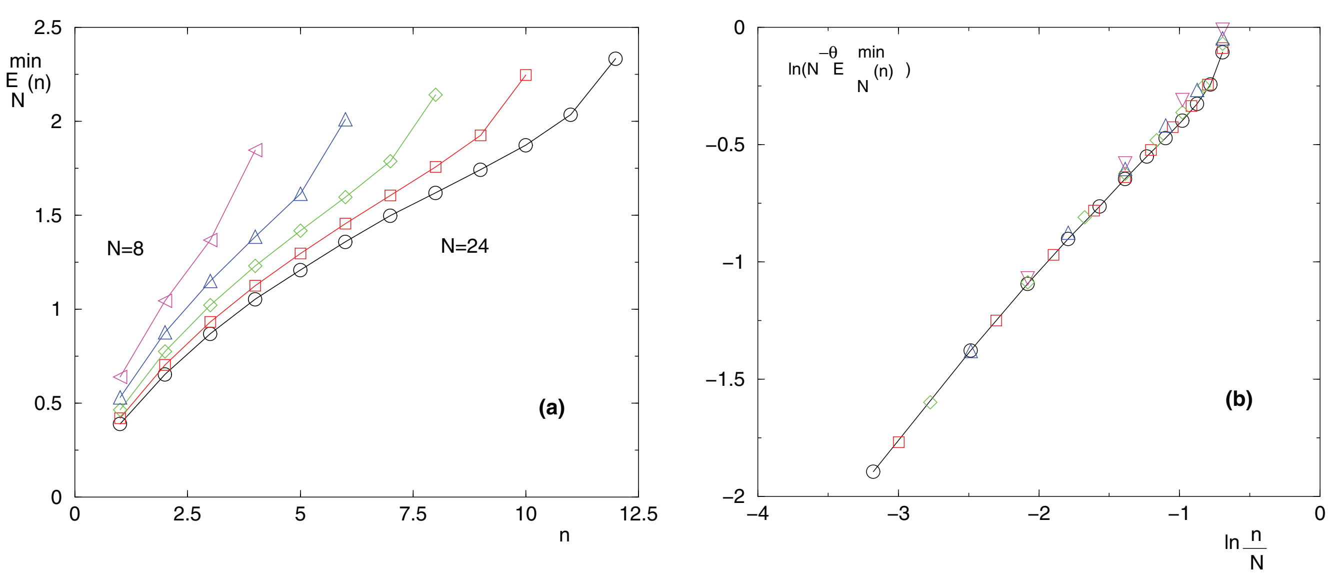

FIG. 1. (Color online) Statistics of the minimal excitation energy involving n spins [Eq. (75)] in the one-dimensional long-ranged model of power $\sigma = 0.1$: (a) Average over samples $E_N^{\min}(n) \equiv \overline{E_{\mathcal{J}}^{\min}(n)}$ as a function of n for sizes $N = 8$, 12, 16, 20, and 24. (b) Data collapse obtained by testing the scaling form of Eq. (77): $\ln[N^{-\theta} E_N^{\min}(n)]$ as a function of $\ln \frac{n}{N}$ with $\theta \simeq 0.3$.

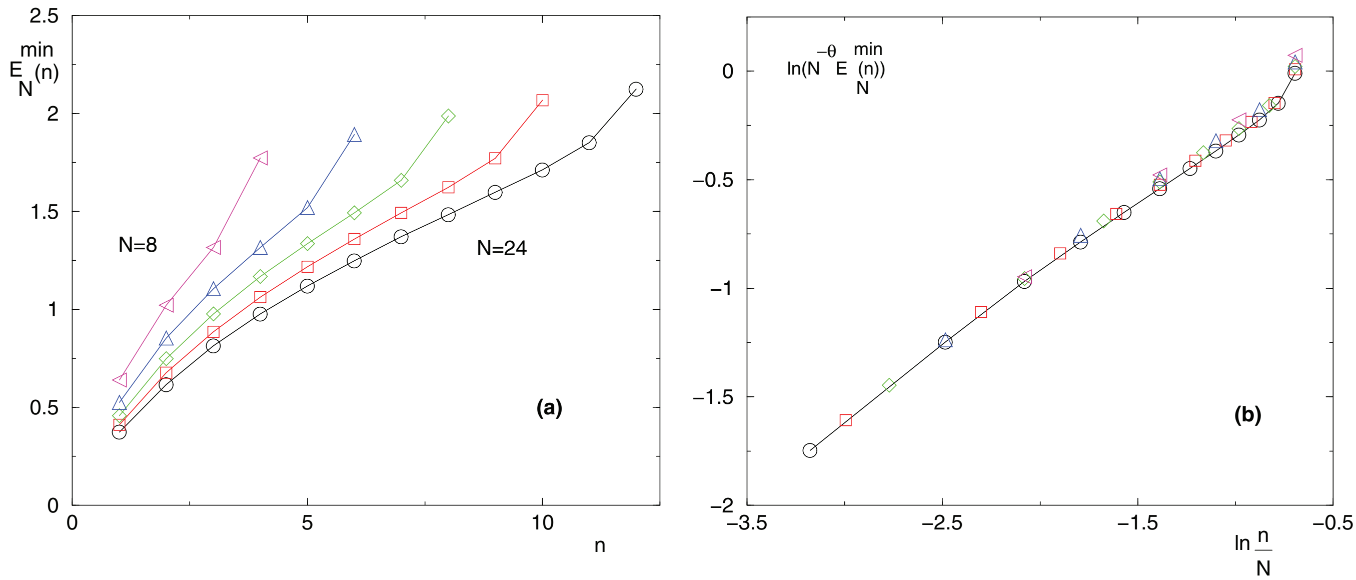

FIG. 2. (Color online) Statistics of the minimal excitation energy involving n spins [Eq. (75)] in the one-dimensional long-ranged model of power $\sigma = 0.62$: (a) Average over samples $E_N^{\min}(n) \equiv \overline{E_{\mathcal{J}}^{\min}(n)}$ as a function of n for sizes $N = 8$, 12, 16, 20, and 24. (b) Data collapse obtained by testing the scaling form of Eq. (77): $\ln[N^{-\theta} E_N^{\min}(n)]$ as a function of $\ln \frac{n}{N}$ with $\theta \simeq 0.24$.

We have also measured the fluctuation exponent μ of Eq. (A7),

$$\begin{aligned} \mu(\sigma = 0.1) &\simeq 0.3, \\ \mu(\sigma = 0.62) &\simeq 0.35, \\ \mu(\sigma = 0.75) &\simeq 0.4. \end{aligned} \tag{74}$$

The last two values are in agreement with Ref. 47, whereas the first value is larger than the value $\mu(\sigma = 0.1) \simeq 0.25$ of Ref. 47. We refer the reader to Ref. 47 for other values of σ.

3. Minimal energy of fixed-size excitations in a given sample

We have measured in each sample $\mathcal{J}$ the minimal energy cost $[E_{\mathcal{J}}(i_1,\ldots,i_n) - E_{\rm GS}]$ among all excitations involving the flipping of exactly n spins with respect to the ground state [Eq. (21)],

$$E_{\mathcal{J}}^{\min}(n) \equiv \min_{1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant N} \left[E_{\mathcal{J}}(i_1,\ldots,i_n) - E_{\mathcal{J}}^{\rm GS}\right]. \tag{75}$$

We show in Figs. 1, 2, and 3 that our data for the averaged value over the samples $\mathcal{J}$ of size N,

$$E_N^{\min}(n) \equiv \overline{E_{\mathcal{J}}^{\min}(n)}, \tag{76}$$

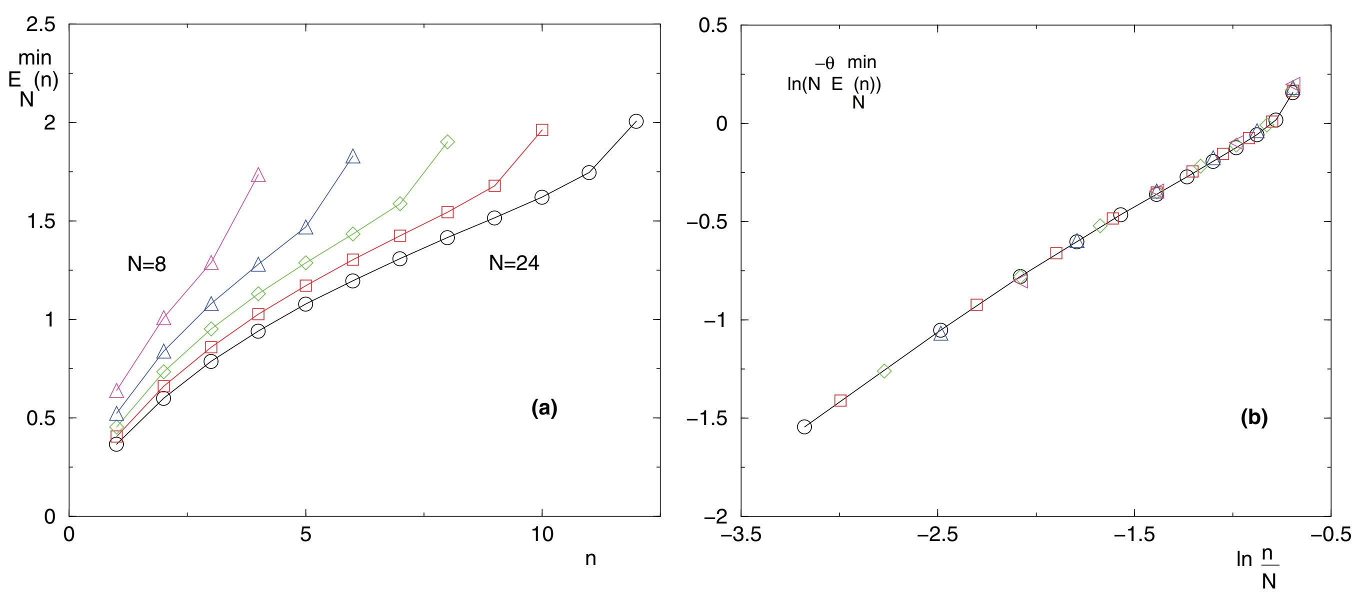

FIG. 3. (Color online) Statistics of the minimal excitation energy involving n spins [Eq. (75)] in the one-dimensional long-ranged model of power $\sigma = 0.75$: (a) Average over samples $E_N^{\min}(n) \equiv \overline{E_{\mathcal{J}}^{\min}(n)}$ as a function of n for sizes $N = 8$, 12, 16, 20, and 24. (b) Data collapse obtained by testing the scaling form of Eq. (77): $\ln[N^{-\theta} E_N^{\min}(n)]$ as a function of $\ln \frac{n}{N}$ with $\theta \simeq 0.17$.

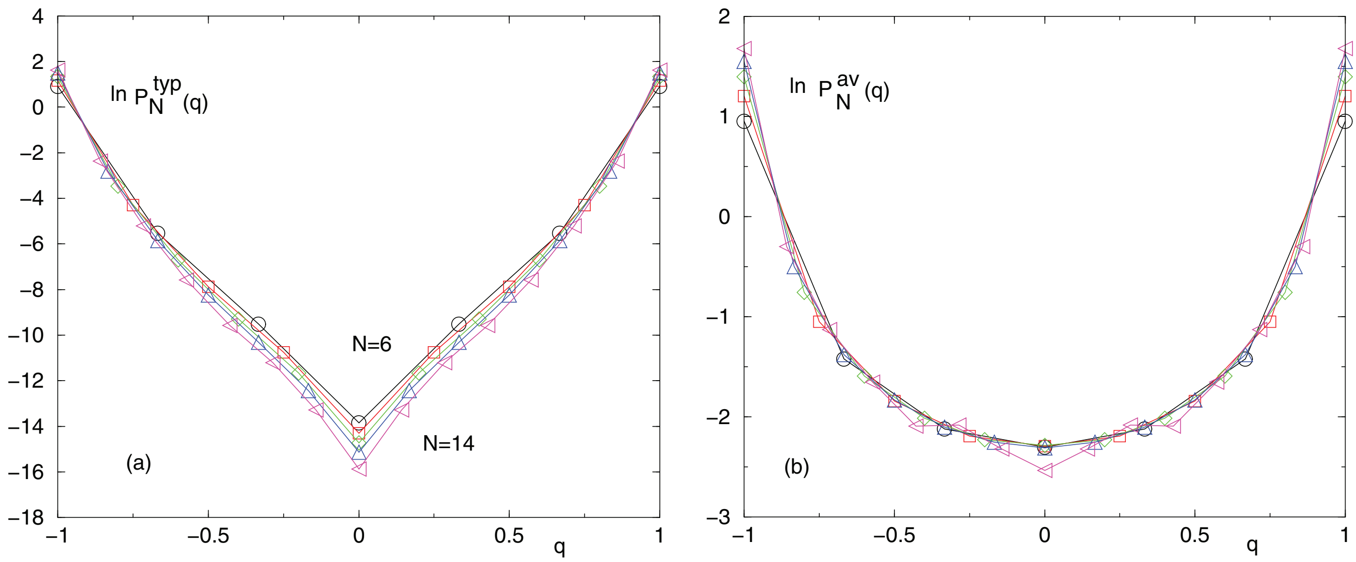

FIG. 4. (Color online) Typical vs averaged overlap distribution for the one-dimensional long-ranged model of power law $\sigma = 0.1$ at temperature $T = 0.1$: (a) $\ln P_N^{\text{typ}}(q)$ as a function of q for the sizes $N = 6, 8, 10, 12$, and 14. (b) $\ln P_N^{\text{av}}(q)$ as a function of q for the sizes $N = 6$, 8, 10, 12, and 14.

can be rescaled in the following form:

$$E_N^{\min}(n) \simeq N^{\theta(\sigma)} g\left(\frac{n}{N}\right), \tag{77}$$

where $\theta(\sigma)$ is the droplet exponent measured previously in Eq. (71).

Since we expect that this averaged value governs the low-temperature behavior of the typical overlap [Eq. (22)], the scaling form of Eq. (77) corresponds to the expectation of Eq. (24).

C. Typical versus averaged overlap distributions

We have also computed directly the overlap distribution $P_{\mathcal{J}}(q)$ via exact enumeration of the 4^N configurations of the two copies of spins for the sizes $6 \leqslant N \leqslant 15$ at the temperature $T = 0.1$ with the following statistics for the number $n_S(L)$ of samples:

$$\begin{aligned} n_s(L=6) &= 2 \times 10^8; \\ n_s(L=8) &= 2 \times 10^7; \\ n_s(L=10) &= 10^6; \\ n_s(L=12) &= 5 \times 10^4; \\ n_s(L=14) &= 1750. \end{aligned} \tag{78}$$

In Figs. 4, 5, and 6, we compare the typical and the averaged overlap distribution for three values of the power σ: in all cases, we find that they are completely different in order of magnitude (see the differences in log scales) and in dependence with the system size N: whereas the averaged value does not change rapidly with N (as found also on bigger sizes[47]), the typical overlap distribution decays with N in the central region around $q = 0$. This effect should be even clearer with the large system sizes used in Ref. 47.

V. FULLY CONNECTED SHERRINGTON-KIRKPATRICK MODEL

The fully connected Sherrington-Kirkpatrick Ising spin-glass model[17]

$$H_{\mathcal{J}} = -\sum_{1 \leqslant i < j \leqslant N} J_{ij} S_i S_j, \tag{79}$$

where the couplings J_{ij} are random quenched variables of zero mean $\overline{J} = 0$ and of variance $\overline{J^2} = 1/N$, can be seen as the limit $\sigma = 0$ of the one-dimensional long-ranged modem described in the preceding section.

A. Statistics of the ground state

The statistics over samples of the ground-state energy has been much studied in the SK model.[26,43,50,57–63] There seems to be a consensus on the shift exponent governing the correction to extensively of the averaged value [Eq. (A6)],

$$\theta_{\text{shift}} \simeq 0.33, \tag{80}$$

which is thus close to the value of the long-ranged one-dimensional model for $\sigma = 0.1$ discussed above. With our exact enumeration on small sizes $6 \leqslant N \leqslant 24$, we see the compatible value

$$\theta \simeq 0.31. \tag{81}$$

On the contrary, the "fluctuation exponent" μ is controversial, with the two main proposals $\mu = 1/4$ and 1/6 (see the discussions in Refs. 43 and 26,50,57–63).

B. Minimal energy of fixed-size excitations in a given sample

We show in Fig. 7 that our data for the averaged value over the samples $\mathcal{J}$ of size N of the minimal energy cost $[E_{\mathcal{J}}(i_1,\ldots,i_n) - E_{\text{GS}}]$ among all excitations involving the flipping of exactly n spins with respect to the ground state

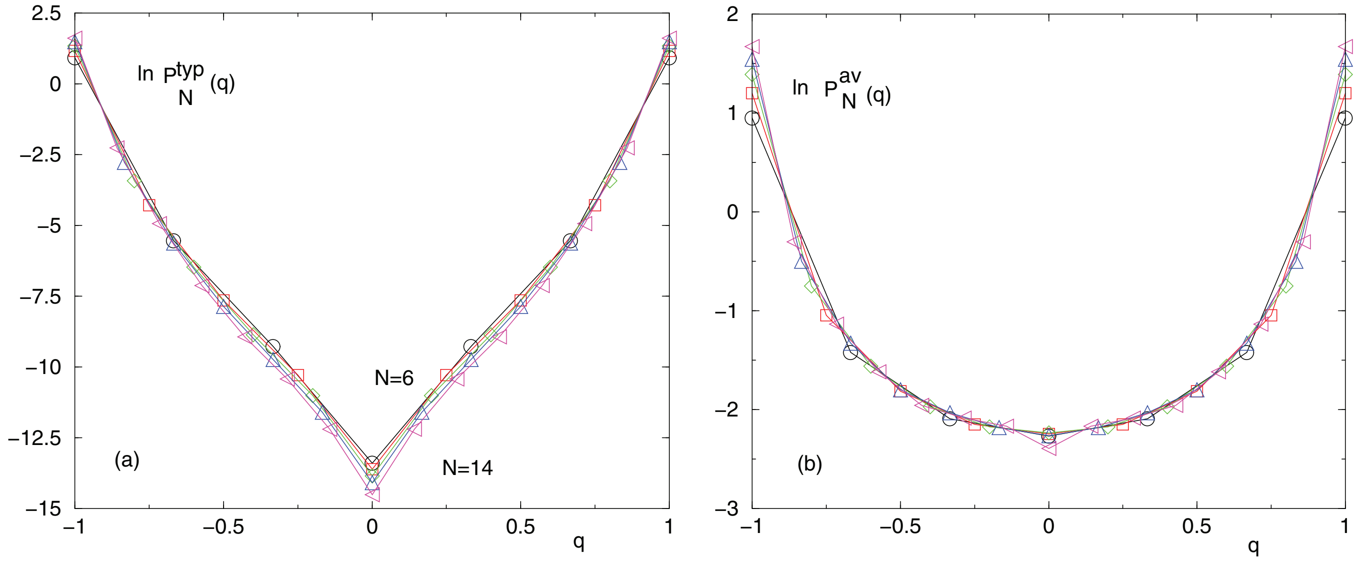

FIG. 5. (Color online) Typical vs averaged overlap distribution for the one-dimensional long-ranged model of power law $\sigma = 0.62$ at temperature $T = 0.1$: (a) $\ln P_N^{\mathrm{typ}}(q)$ as a function of q for the sizes $N = 6, 8, 10, 12$, and 14. (b) $\ln P_N^{\mathrm{av}}(q)$ as a function of q for the sizes $N = 6$, 8, 10, 12, and 14.

[Eq. (21)],

$$E_N^{\min}(n) \equiv \overline{E_{\mathcal{J}}^{\min}(n)}, \tag{82}$$

can be rescaled in the following form:

$$E_N^{\min}(n) \simeq N^{\theta} g\left(\frac{n}{N}\right), \tag{83}$$

where $\theta \simeq 0.31$ is the droplet exponent measured previously in Eq. (81).

Since we expect that this averaged value governs the low-temperature behavior of the typical overlap [Eq. (22)], the scaling form of Eq. (77) corresponds to the expectation of Eq. (24).

C. Typical versus averaged overlap distribution

We have also computed the overlap distribution $P_{\mathcal{J}}(q)$ via exact enumeration of the 4^N configurations of the two copies of spins for the sizes $6 \leqslant N \leqslant 14$ at the temperature $T = 0.1$, with the same statistics as in Eq. (78). As shown in Fig. 8, we find again that the typical and the averaged overlap distribution are completely different in order of magnitude (see the differences in log scales) and in dependence with the

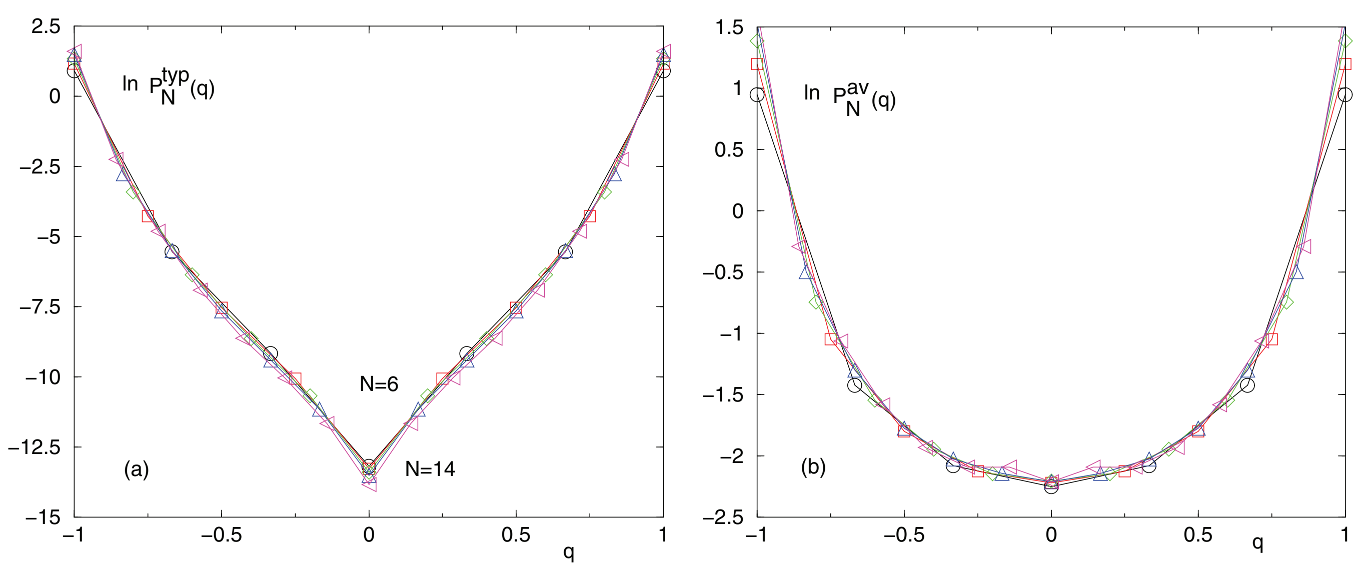

FIG. 6. (Color online) Typical vs averaged overlap distribution for the one-dimensional long-ranged model of power law $\sigma = 0.75$ at temperature $T = 0.1$: (a) $\ln P_N^{\mathrm{typ}}(q)$ as a function of q for the sizes $N = 6, 8, 10, 12$, and 14. (b) $\ln P_N^{\mathrm{av}}(q)$ as a function of q for the sizes $N = 6$, 8, 10, 12, and 14.

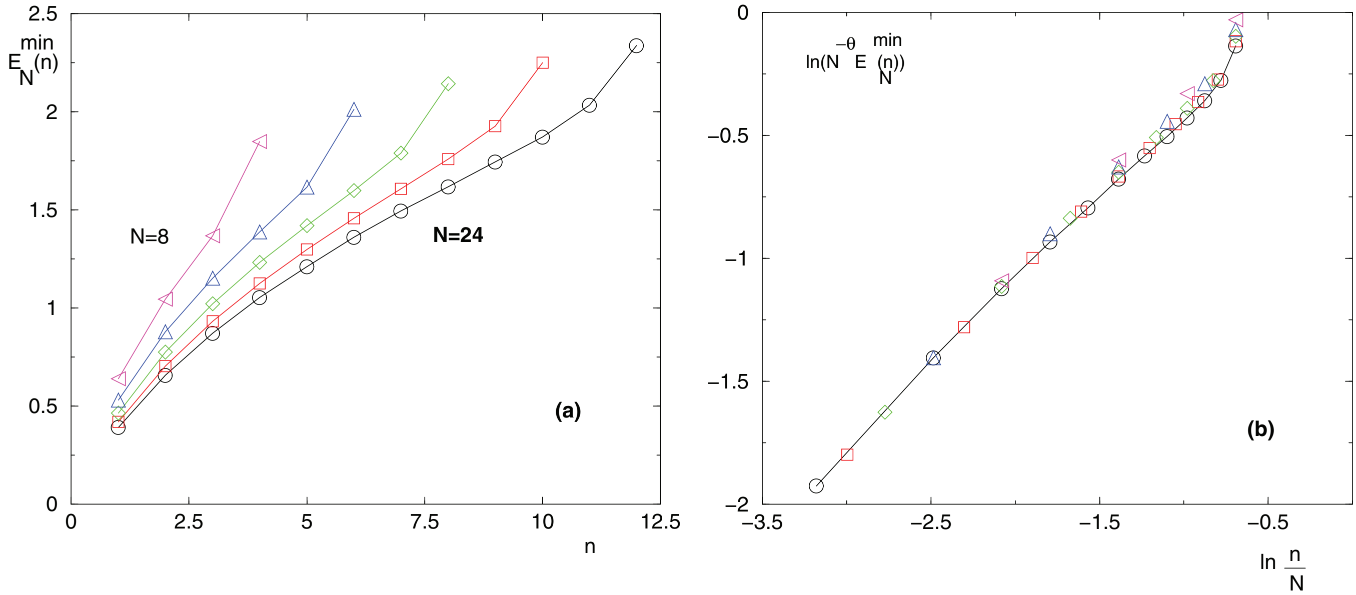

FIG. 7. (Color online) Statistics of the minimal excitation energy involving n spins [Eq. (76)] in the mean-field SK model: (a) Average over samples $E_N^{\min}(n) \equiv \overline{E_{\mathcal{J}}^{\min}(n)}$ as a function of n for sizes $N = 8, 12, 16, 20$, and 24. (b) Data collapse obtained by testing the scaling form of Eq. (77): $\ln[N^{-\theta} E_N^{\min}(n)]$ as a function of $\ln \frac{n}{N}$ with $\theta \simeq 0.31$.

system size N: whereas the averaged value does not change much with N, the typical overlap distribution clearly decays with N in the central region around $q = 0$.

VI. CONCLUSION

In this paper, we have studied the statistical properties over disordered samples ($\mathcal{J}$) of the overlap distribution $P_{\mathcal{J}}(q)$ which plays the role of an order parameter in spin glasses. We have obtained that near zero temperature, the following holds true:

(i) The *typical* overlap distribution is exponentially small in the central region of $-1 < q < 1$:

$$\ln P^{\text{typ}}(q) \equiv \overline{\ln P_{\mathcal{J}}(q)} \sim -\beta N^{\theta} \phi(q), \tag{84}$$

where θ is the droplet exponent defined here with respect to the total number N of spins (in order to consider also fully connected models in which the notion of length does not exist).

(ii) The appropriate rescaled variable to describe sample-to sample fluctuations is

$$v = -\frac{\ln P_{\mathcal{J}}(q)}{\beta N^{\theta}}, \tag{85}$$

which remains an $O(1)$ random positive variable.

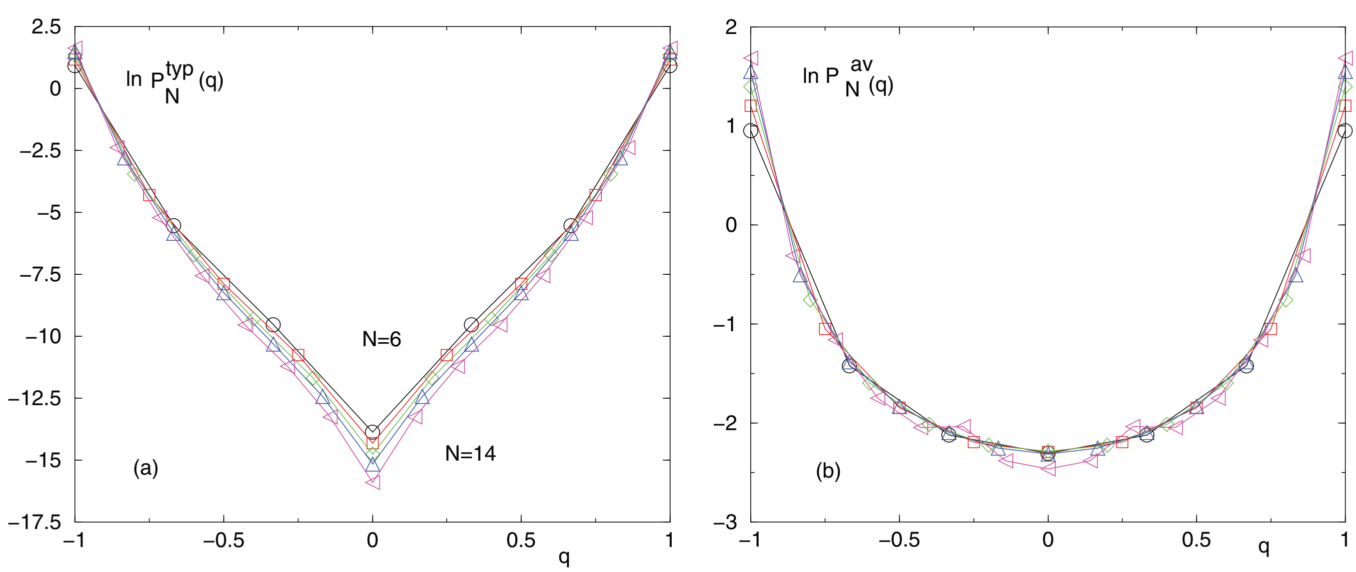

FIG. 8. (Color online) Typical vs averaged overlap distribution for the SK model at temperature $T = 0.1$: (a) $\ln P_N^{\text{typ}}(q)$ as a function of q for the sizes $N = 6, 8, 10, 12$, and 14. (b) $\ln P_N^{\text{av}}(q)$ as a function of q for the sizes $N = 6, 8, 10, 12$, and 14.

(iii) The averaged distribution $\overline{P_{\mathcal{J}}(q)}$ is nontypical, dominated by rare anomalous samples, and it can thus be very misleading.

We first derived these results for the spherical mean-field model with $\theta = 1/3$, $\phi(q) = 1 - q^2$, and the random variable v corresponds to the rescaled difference between the two largest eigenvalues of GOE random matrices.

We then presented numerical results for the long-ranged one-dimensional spin glass with random couplings decaying as $J(r) \propto r^{-\sigma}$ for various values of the exponent σ, and for the SK-mean-field model (corresponding formally to the $\sigma = 0$ limit of the previous model). In all cases, we have obtained that the typical and averaged overlap distributions are completely different, in order of magnitude and in scaling. We have also found that in each case, the same droplet exponent governs the four properties we have measured:

(a) The change in the ground-state energy between different boundary conditions in a given sample.

(b) The correction to extensivity of the averaged ground-state energy.

(c) The minimal energy of excitations of a fixed extensive size $n \propto N$.

(d) The decay of the typical overlap distribution.

Our results are thus in full agreement with the droplet scaling theory.

We hope that future studies on spin glasses will also measure the *typical* values of the overlap distribution, $P^{\text{typ}}(q) = e^{\overline{\ln P_{\mathcal{J}}(q)}}$, or of the cumulative overlap distribution, $I^{\text{typ}}(q_0) = e^{\overline{\ln I_{\mathcal{J}}(q_0)}}$, instead of the nontypical averaged overlap distribution, in order obtain clearer conclusions on the nature of the spin-glass phase.

APPENDIX A: BRIEF REMINDER ON SOME PROPERTIES OF THE DROPLET EXPONENT θ

In the droplet scaling theory,[12–14] the most important notion is the droplet exponent θ, with the following physical meanings.

1. Scaling of renormalized couplings

The initial meaning of the droplet exponent θ_l is the scaling of renormalized couplings J on a length scale L,[12,13]

$$J_L \simeq L^{\theta_l} u, \tag{A1}$$

where u is an $O(1)$ random variable of zero mean (with a probability distribution symmetric in $u \to -u$). This definition is directly used in real-space renormalization studies based on the Migdal-Kadanoff approximation.[15] The definition of Eq. (A1) means that there is no spin-glass phase when $\theta_l < 0$, and that there exists a spin-glass phase when $\theta_l > 0$, which is then governed by a zero-temperature fixed point.

2. Scaling of "optimized excitations" around a given point

The above definition can also be interpreted as the energy scale of "optimized" excitations of linear size L around a given point,[14]

$$E_L^{\text{exc}} \simeq L^{\theta_l} v, \tag{A2}$$

where v is a positive random variable. So the energy scale is L^{θ_l} with a probability $O(1)$, but can also be $O(1)$ with the small probability $P(v < L^{-\theta_l})$, so that these rare events will lead to power laws in various observables.[14]

In the present paper, in order to compare with the fully connected model where the notion of length does not exist, we have chosen to define the droplet exponent with respect to the number of spins N involved, so that Eq. (A2) becomes

$$E_N^{\text{exc}} \simeq N^{\theta} v. \tag{A3}$$

(For short-ranged models in dimension d, where $N = L^d$, the correspondence reads $\theta = \theta_l/d$.)

3. Difference between different boundary conditions in a given sample

The standard procedure to measure the droplet exponent is to compute, in each given sample $\mathcal{J}$, the difference between the ground-state energies corresponding to different boundary conditions,[12–14] for instance periodic-antiperiodic,

$$E_{\mathcal{J}}^{\text{GS(P)}} - E_{\mathcal{J}}^{\text{GS(AP)}} = N^{\theta} u, \tag{A4}$$

where u is an $O(1)$ random variable of zero mean (with a probability distribution symmetric in $u \to -u$). Here the droplet exponent has thus the meaning of a domain-wall exponent, or a stiffness exponent. The link with Eq. (A1) is that the energy difference between different boundary conditions somewhat measures the renormalized coupling between the boundaries. The link with Eq. (A3) is that the change of boundary condition will select an optimized system-size excitation given the new constraints.

For the short-ranged model on hypercubic lattices of dimension d, the values measured for the stiffness exponent θ_l (see Ref. 64 and references therein) read for the exponent $\theta = \theta_l/d$ defined with respect to the number $N = L^d$ of spins

$$\begin{aligned} \theta(d=2) &\simeq -\frac{0.28}{2} \simeq -0.14, \\ \theta(d=3) &\simeq \frac{0.24}{3} \simeq 0.08, \\ \theta(d=4) &\simeq \frac{0.61}{4} \simeq 0.15, \\ \theta(d=5) &\simeq \frac{0.88}{5} \simeq 0.176, \\ \theta(d=6) &\simeq \frac{1.1}{6} \simeq 0.183. \end{aligned} \tag{A5}$$

4. Role of the droplet exponent in the statistics of the ground-state energy

The statistics over samples of the ground-state energy in spin glasses has been much studied recently (see Refs. 43 and 26,50,57–63 and references therein) with the following conclusions:

(i) The averaged value over samples of the ground-state energy reads

$$\overline{E^{GS}_{\mathcal{J}}(N)} \simeq N e_0 + N^{\theta_{\text{shift}}} e_1 + \cdots . \quad \text{(A6)}$$

The first term $N e_0$ is the extensive contribution, whereas the second term $N^{\theta_{\text{shift}}} e_1$ represents the leading correction to extensivity.

(ii) The fluctuations around this averaged value are governed by some fluctuation exponent μ,

$$E^{GS}_{\mathcal{J}}(N) - \overline{E^{GS}_{\mathcal{J}}(N)} \simeq N^{\mu} u + \cdots , \quad \text{(A7)}$$

where u is an $O(1)$ random variable of zero mean $\overline{u} = 0$ by definition.

For spin glasses in finite dimension d, it has been proven that $\mu = 1/2$ and that the distribution of u is simply Gaussian[65] suggesting some central limit theorem coming from the random couplings. But the shift exponent of Eq. (A6) is nontrivial and coincides with the droplet exponent,[57]

$$\theta_{\text{shift}} = \theta . \quad \text{(A8)}$$

The link with Eq. (A3) is that the boundary conditions always induce some system-size frustration, and thus some system-size excitations. This contribution of order N^{θ} is distributed, but the corresponding fluctuations are subleading with respect to the bigger fluctuations corresponding to $\mu = 1/2$.

Besides short-ranged models, the fluctuation exponent μ and the scaling distribution of u have also been studied for long-ranged models,[47,49,50] and the fully connected SK model.[26,43,50,57–63]

APPENDIX B: BRIEF REMINDER ON THE OVERLAP WITHIN THE REPLICA THEORY

Within the replica theory,[16] the probability distribution of the cumulative overlap distribution

$$Y_{\mathcal{J}}(q_0) \equiv \int_{|q| \geqslant q_0} dq\, P_{\mathcal{J}}(q) \quad \text{(B1)}$$

has a nontrivial limit in the thermodynamic limit $N = +\infty$:[66,67] the translation for the cumulative overlap distribution over the central region

$$I_{\mathcal{J}}(q_0) \equiv \int_{-q_0}^{q_0} dq\, P_{\mathcal{J}}(q) = 1 - Y_{\mathcal{J}}(q_0) \quad \text{(B2)}$$

yields that the probability distribution $\Pi_{\mu}(I)$ is indexed by the parameter

$$\mu = \mu(\beta, q_0) = 1 - \overline{Y_{\mathcal{J}}(q_0)} = \overline{I_{\mathcal{J}}(q_0)}. \quad \text{(B3)}$$

Near the origin $I \to 0^+$, there is the power-law divergence,

$$\Pi_{\mu}(I) \underset{I \to 0}{\propto} I^{-(1-\mu)}, \quad \text{(B4)}$$

whereas near the other boundary $I \to 1^-$, there is an essential singularity,

$$\Pi_{\mu}(I) \underset{I \to 1}{\propto} e^{-\frac{1}{4 z_0(\mu)(1-I)}}, \quad \text{(B5)}$$

where $z_0(\mu)$ is given in Refs. 66 and 67. Other singularities appear at $(1 - I) = 1/n$, where n is an integer.[68]

From the point of view of the typical value I^{typ} discussed in the text, the important point is that it remains finite for $N = +\infty$,

$$\ln I^{\text{typ}} \equiv \overline{\ln I} = \int_0^1 dI (\ln I) \Pi_{\mu}(I). \quad \text{(B6)}$$

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