

Topological classification with additional symmetries from Clifford algebras

Takahiro Morimoto¹ and Akira Furusaki^{1,2}¹*Condensed Matter Theory Laboratory, RIKEN, Saitama 351-0198, Japan*²*RIKEN Center for Emergent Matter Science (CEMS), Saitama 351-0198, Japan*

(Received 11 June 2013; revised manuscript received 13 August 2013; published 20 September 2013)

We classify topological insulators and superconductors in the presence of additional symmetries such as reflection or mirror symmetries. For each member of the 10 Altland-Zirnbauer symmetry classes, we have a Clifford algebra defined by operators of the generic (time-reversal, particle-hole, or chiral) symmetries and additional symmetries, together with gamma matrices in Dirac Hamiltonians representing topological insulators and superconductors. Following Kitaev's approach, we classify gapped phases of noninteracting fermions under additional symmetries by examining all possible distinct Dirac mass terms which can be added to the set of generators of the Clifford algebra. We find that imposing additional symmetries in effect changes symmetry classes and causes shifts in the periodic table of topological insulators and superconductors. Our results are in agreement with the classification under reflection symmetry recently reported by Chiu, Yao, and Ryu [*Phys. Rev. B* **88**, 075142 (2013)]. Several examples are discussed including a topological crystalline insulator with mirror Chern numbers and mirror superconductors.

DOI: [10.1103/PhysRevB.88.125129](https://doi.org/10.1103/PhysRevB.88.125129)

PACS number(s): 72.10.-d, 73.20.-r, 73.43.Cd

I. INTRODUCTION

Since the theoretical proposals and experimental confirmations of two- and three-dimensional topological insulators with \mathbb{Z}_2 topological numbers,^{1–12} topological characterization of gapped phases has been intensively studied as a new way to classify states of matter, beyond the conventional way in terms of broken symmetries. The concept of topological insulators can be extended to any system of noninteracting fermions with excitation gap including superconductors where fermionic quasiparticles are well described by the BCS mean-field theory. More generally, the topological insulators and superconductors can be defined as systems of noninteracting (or weakly interacting) fermions which have gapped excitation spectra in the bulk and topologically stable gapless boundary excitations. There exist a large variety of topological insulators and superconductors in the broad sense defined above. A prominent example of topological insulators is integer quantum Hall states.¹³ Examples of topological superconductors include a one-dimensional p -wave superconductor¹⁴ and a chiral p -wave superconductor in two dimensions.¹⁵

The zoo of topological insulators and superconductors has been classified theoretically.^{16–22} Systems of noninteracting fermions are known to be divided into ten Altland-Zirnbauer (AZ) symmetry classes,²³ in terms of the presence or absence of time-reversal symmetry (TRS), particle-hole symmetry (PHS), and chiral symmetry (or sublattice symmetry). According to the classification,^{16–19} in every spatial dimension, there exist five distinct classes of topological insulators and superconductors out of the ten AZ symmetry classes. Among the five classes of topological insulators and superconductors, three are characterized by an integer (\mathbb{Z}) topological index, and two characterized by a binary (\mathbb{Z}_2) topological index. Topological indices are defined from Bloch wave functions or Hamiltonians. The general classification of topological insulators and superconductors has been achieved in various ways: stability analysis of gapless boundary states against (random) perturbations,^{16,17} dimensional reduction in representative massive Dirac Hamiltonians,¹⁸ and application of

K theory and Clifford algebras.^{19,20} The last approach is the most elegant and mathematically powerful.

The so-called periodic table of topological insulators and superconductors was obtained by Kitaev¹⁹ using K theory and Clifford algebras. In this approach, a Clifford algebra is formed from generic symmetry transformations such as TRS, PHS, or chiral symmetry. One can then ask how many different types of generators (related to Hamiltonians after spectral flattening) can be added to the set of generators of the Clifford algebra. The answer to this question is provided by “classifying space.” This formulation naturally leads to the periodic table of topological insulators and superconductors for the 10 AZ symmetry classes in any spatial dimension, which has the periodic structure of period 2 or 8 coming from the Bott periodicity.

As an attempt to find a novel class of topological insulators in noninteracting systems, Fu introduced the idea of topological crystalline insulators,²⁴ i.e., band insulators which become topologically nontrivial only when some crystalline symmetry is present in addition to TRS.^{24,25} Furthermore, Fu and his collaborators made a theoretical proposal²⁶ that SnTe should be a topological crystalline insulator whose topological stability is guaranteed by a mirror Chern number²⁷ defined on a mirror-invariant plane in the Brillouin zone. Subsequent experimental studies have confirmed that SnTe and its alloys $\text{Sn}_{1-x}\text{Pb}_x\text{Te}$ are topological crystalline insulators.^{28–30} The successful discovery of topological crystalline insulators urges us to generalize the classification theory of topological insulators and superconductors to include crystalline symmetries. Indeed, Chiu *et al.* have recently developed such classification by explicitly constructing possible topological invariants for representative Dirac Hamiltonians with a mirror symmetry for each symmetry class.³¹

In this paper we follow the approach pioneered by Kitaev¹⁹ to classify topological insulators and superconductors with additional symmetries such as reflection symmetries. Our approach is complementary to the one taken by Chiu *et al.* and takes advantage of simple and powerful mathematics of

representation theory of Clifford algebras and K theory.³² A drawback of our approach is that it does not give us an explicit formula of topological invariants.

While we focus on noninteracting systems in this paper, we note that new topological phases can appear in interacting systems, since interactions can modify the above-mentioned classifications of noninteracting fermions. For one-dimensional systems of interacting particles, modifications of the classification are explicitly shown and the full classification is obtained, e.g., in terms of matrix product states.^{33–36} For bosonic systems, classification of symmetry-protected topological (SPT) phases in higher dimensions is recently proposed using group cohomology³⁷ and $(2+1)$ -dimensional Chern-Simons theory.³⁸

This paper is organized as follows. In Secs. II and III we briefly review Clifford algebras and their application to the classification of topological insulators and superconductors in zero dimensions for the 10 AZ symmetry classes. These sections give a summary of the theoretical formalism which is used in the following sections. In Sec. IV we apply the formalism to classify zero-dimensional topological insulators and superconductors in the presence of an additional symmetry constraint. We find that the additional symmetry in effect shifts symmetry classes in the periodic table. In Sec. V we take gamma matrices in Dirac Hamiltonians as additional symmetry generators, to obtain classification of d -dimensional topological insulators and superconductors from classification at $d = 0$. In Sec. VI we derive the periodic table of topological insulators/superconductors with a reflection symmetry. Our table is in agreement with the one obtained earlier by Chiu *et al.* In Sec. VII we study cases when multiple additional symmetries are imposed. We find more complicated shuffling of symmetry classes in the periodic table. In Sec. VIII we discuss several examples of topological insulators and superconductors which are protected by reflection symmetries, including topological crystalline insulators with mirror Chern numbers or \mathbb{Z}_2 indices. Some mathematical details are summarized in appendices.

II. FORMALISM

We briefly introduce AZ symmetry classes and Clifford algebras and summarize our program for classifying topological insulators and superconductors with additional symmetries in terms of real and complex Clifford algebras.

A. Ten symmetry classes

The AZ symmetry classes give classification of Hamiltonians of free fermion systems. Hamiltonians can be block diagonalized when they commute with a unitary matrix representing (continuous) symmetry transformation such as (spin) rotation or translation. In the following discussions we assume that Hamiltonians are already block diagonalized. The Hamiltonians may still have discrete symmetries represented by antiunitary operators, and are classified into the 10 AZ symmetry classes according to the presence or absence of time-reversal and particle-hole symmetries (Table I).²³

When Hamiltonian H has neither time-reversal nor particle-hole symmetry, H is in class A or AIII of the complex classes;

TABLE I. Altland-Zirnbauer symmetry classes for (a) complex and (b) real cases. The presence of time-reversal symmetry (TRS) and particle-hole symmetry (PHS) is denoted $+1$ or -1 , depending on whether they square to $+1$ or -1 . The absence of the TRS or PHS symmetry is denoted by “0”. The next to last columns show the classifying spaces (a) C_q and (b) R_q that characterize zero-dimensional Hamiltonian in each symmetry class. The last columns denote type of possible topological numbers, i.e., the number of disconnected parts of each classifying space.

(a) Complex classes				
Class	Chiral	C_q	$\pi_0(C_q)$	
A	0	C_0	\mathbb{Z}	
AIII	1	C_1	0	
(b) Real classes				
Class	TRS	PHS	R_q	$\pi_0(R_q)$
AI	+1	0	R_0	\mathbb{Z}
BDI	+1	+1	R_1	\mathbb{Z}_2
D	0	+1	R_2	\mathbb{Z}_2
DIII	-1	+1	R_3	0
AII	-1	0	R_4	\mathbb{Z}
CII	-1	-1	R_5	0
C	0	-1	R_6	0
CI	+1	-1	R_7	0

see Table I(a).^{18,23} When there exists a unitary transformation Γ that changes the sign of H (i.e., $\Gamma^{-1}H\Gamma = -H$), the Hamiltonian H has so-called chiral symmetry and is a member of class AIII.

Both time-reversal operator T and particle-hole operator C are an antiunitary operator whose square equals either plus or minus identity operator. When H has either time-reversal or particle-hole symmetry (or both), H belongs to real classes which are further divided into eight symmetry classes listed in Table I(b), in terms of the sign of T^2 and C^2 .^{18,23} When H has both T and C symmetries, the product of the two symmetry operations yields a chiral symmetry transformation, a unitary operator that anticommutes with H .

B. Clifford algebras and their extensions

We are going to classify topological insulators and superconductors with additional reflection symmetries using Clifford algebras, which are algebras with generators which anticommute with each other.

A complex Clifford algebra Cl_n has n generators e_i satisfying

$$\{e_i, e_j\} = 2\delta_{i,j}. \quad (1)$$

Linear combination of their products $e_1^{p_1} e_2^{p_2} \cdots e_n^{p_n}$ ($p_i = 0, 1$) multiplied by complex numbers form a 2^n -dimensional complex vector space, accompanied with the multiplication law (1).

A real Clifford algebra $Cl_{p,q}$ has $p+q$ generators satisfying

$$\begin{aligned} \{e_i, e_j\} &= 0 \quad (i \neq j), \\ e_i^2 &= \begin{cases} -1, & 1 \leq i \leq p, \\ +1, & p+1 \leq i \leq p+q. \end{cases} \end{aligned} \quad (2)$$

Their products with real coefficients form a 2^{p+q} -dimensional real vector space, accompanied with the multiplication law (2).

Having introduced real and complex Clifford algebras, we now explain our strategy for classifying topological insulators and superconductors with additional symmetries, which is a natural extension of Kitaev's approach.¹⁹ Namely, we reduce the classification problem of gapped free fermion systems to that of possible extensions of Clifford algebras which are generated by discrete symmetry operators and a Hamiltonian with flattened spectra.

We begin with topological classification of zero-dimensional systems (i.e., systems confined in a finite region) with some symmetries. We first perform "spectral flattening" of Hamiltonian H with an energy gap; i.e., we continuously deform eigenenergies above the gap to $+1$ and those below the gap to -1 while preserving wave functions. This is a continuous deformation of the Hamiltonian and does not change its topological properties. Next we express symmetry constraints as generators $\{e_i\}$ of a Clifford algebra. The relevant Clifford algebras for the complex and real AZ symmetry classes (Table I) are complex and real Clifford algebras, respectively. We consider a matrix representation (of sufficiently large dimension) of the Clifford algebra. We then consider extending the algebra by adding a generator e_0 which is obtained from the flattened Hamiltonian. That is, for a fixed representation of symmetry constraints $\{e_i\}$, we look for possible representations of a new additional generator e_0 . The set of these representations for e_0 forms a "classifying space," denoted as C_q and R_q for complex and real symmetry classes, respectively (see Table VIII in Appendix A). Now, topologically distinct states correspond to topologically distinct extensions of the algebra, and classification of them can be found from a zeroth homotopy group of a classifying space $\pi_0(C_q)$ or $\pi_0(R_q)$, i.e., the number of disconnected parts of C_q or R_q . The resulting classification for the 10 AZ classes is summarized in Table I, whose explicit constructions are given in Sec. III. We will apply this program to complete topological classification in the presence of additional reflection symmetries in the following sections. Once the zero-dimensional systems are classified, the classification of d -dimensional systems can be achieved by considering $\pi_0(C_{q-d})$ and $\pi_0(R_{q-d})$, as shown by Kitaev using K theory.¹⁹ In Sec. V we will give an alternative explanation of the dimensional shift ($q \rightarrow q-d$) for massive Dirac Hamiltonians in d dimensions.

C. Examples of classifying spaces

In order to gain intuitions about classifying spaces and topological invariants, let us look at a couple of examples of massive Dirac Hamiltonians and discuss their classifying spaces by considering what kind of Dirac mass term γ_0 is allowed in specific models. The classifying space corresponds to a set of allowed Dirac mass terms.

First we consider a two-dimensional system in class A, which is described by a $2N \times 2N$ Hamiltonian

$$H_{2D} = k_x \sigma_x \otimes 1_N + k_y \sigma_y \otimes 1_N + \gamma_0, \quad (3)$$

with momenta k_i , 2×2 Pauli matrices σ_i , an $N \times N$ identity matrix 1_N , and a mass term γ_0 . The classifying space corresponds to a set of allowed Dirac mass term γ_0 . Since γ_0 anticommutes with the kinetic terms, γ_0 should have the form

$$\gamma_0 = \sigma_z \otimes A, \quad (4)$$

where A is an $N \times N$ Hermitian matrix and is normalized such that it squares to 1_N . We can diagonalize A with a unitary matrix U as

$$A = U I_{n,m} U^\dagger, I_{n,m} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_m \end{pmatrix}, \quad (5)$$

with $N = n + m$. $I_{n,m}$ is a diagonal matrix whose diagonal entries consist of $+1$ and -1 , appearing n and m times for each. For a given n , A is determined by a choice of U from a unitary group $U(N)$, but there is a redundancy in the choice of bases in each eigenspace of ± 1 . Thus the set of A with fixed n and m corresponds to a complex Grassmanian $U(n+m)/U(n) \times U(m)$. The total classifying space is a union of $U(n+m)/U(n) \times U(m)$ of different values of n . If we assume N to be sufficiently large, complex Grassmanians each labeled with n become almost the same, and we can write the total classifying space as

$$[U(n+m)/U(n) \times U(m)] \times \mathbb{Z}, \quad (6)$$

which is exactly C_0 in Table VIII. Since each complex Grassmanian is a connected manifold, each disconnected part of the classifying space (topologically distinct states) is specified by n , the number of eigenvalues $+1$ of A . Actually, n coincides with the Chern number defined for the 2-dimensional system with a proper regularization. We regularize the Dirac Hamiltonian by adding a k^2 term as $H_{2D} - Ck^2 \sigma_z \otimes 1_N$ with a small positive coefficient C . If we assume $A = I_{n,m}$ for simplicity, then the Hamiltonian decouples into N copies as $H_{2D} = \bigoplus_{i=1}^N H_i$, where

$$H_i = k_x \sigma_x + k_y \sigma_y + (\epsilon_i - Ck^2) \sigma_z, \quad (7)$$

with $\epsilon_i = +1$ for $1 \leq i \leq n$, $\epsilon_i = -1$ for $n+1 \leq i \leq N$. Now the Chern number for H_i is $+1$ for $\epsilon_i = +1$ and 0 for $\epsilon_i = -1$. Therefore the total Chern number is n .

Next let us consider a one-dimensional system in class A, with a $2N \times 2N$ Hamiltonian

$$H_{1D} = k_x \sigma_z \otimes 1_N + \gamma_0. \quad (8)$$

The Dirac mass term γ_0 should be Hermitian, anticommute with σ_z , and square to 1_{2N} , which requires γ_0 to be written as

$$\gamma_0 = \begin{pmatrix} 0 & U \\ U^\dagger & 0 \end{pmatrix}, \quad (9)$$

where U is an $N \times N$ unitary matrix. Thus each choice of U specifies the mass term γ_0 and the classifying space is given by $U(N)$, which is C_1 in Table VIII. Since the unitary group $U(N)$ is connected, we can deform one state into another without closing the energy gap; there exists only one phase which is topologically trivial. This trivial classification is a consequence of the existence of two mass terms that anticommute with each other in the minimal Dirac model. Let us consider a minimal

2×2 Hamiltonian,

$$H_{1D} = k_x \sigma_z + m_1 \sigma_x + m_2 \sigma_y. \quad (10)$$

When there is only one allowed mass term ($m_0 \sigma_z$) as in the previous example [Eq. (7)], two states with different signs of the unique mass term are topologically distinct, and changing the sign of m_0 is only possible via a point $m_0 = 0$ where the bulk gap closes. On the other hand, if we have two masses (m_1 and m_2), the two states with masses $(m_1, m_2) = (\pm 1, 0)$ are connected through a rotation in the plane of (m_1, m_2) ; i.e., we can deform one to the other without closing the bulk gap. This deformation is regarded as a rotation in $U(N)$ with $N = 1$.

III. CLASSIFICATION FOR AZ CLASSES

In this section, we give a concise review of the topological classification of the ten AZ symmetry classes in zero spatial dimensions in terms of an extension problem of the Clifford algebra, in a way complementary to the original Kitaev paper.¹⁹ The two complex classes are classified with complex Clifford algebras while the eight real classes are classified with real Clifford algebras. This section will serve as a starting point for the topological classification in the presence of additional reflection symmetries in the following sections.

A. Complex classes

We start with classification of the complex AZ classes (A and AIII) in terms of complex Clifford algebras. The extension problem for complex Clifford algebra $Cl_n \rightarrow Cl_{n+1}$ is characterized by a classifying space C_n .^{19,32}

When a zero-dimensional system is a member of class AIII, its Hamiltonian H satisfies the chiral symmetry relation

$$\{H, \Gamma\} = 0, \quad (11)$$

where Γ is a unitary operator. After spectral flattening, the zero-dimensional Hamiltonian H has eigenvalues ± 1 , and we set $e_0 := H$. We note that relation (11) is not affected by the spectral flattening. Without loss of generality we may assume $\Gamma^2 = 1$ and regard $e_1 := \Gamma$ as a generator of complex Clifford algebra Cl_1 . We then consider extending the complex Clifford algebra Cl_1 to Cl_2 by adding the generator e_0 to the algebra Cl_1 (Table II). A set of the possible representations of e_0 in the extended algebras forms the classifying space C_1 . Since $\pi_0(C_1) = 0$, zero-dimensional systems in class AIII are topologically trivial (Table I).

For Hamiltonians in class A, we begin with complex Clifford algebra Cl_0 . We consider the extension of Cl_0 to Cl_1 with $e_0 = H$, whose possible representations form the classifying space C_0 (Table II). We then find from $\pi_0(C_0) = \mathbb{Z}$

TABLE II. Classification of the complex AZ classes (A and AIII) in zero dimensions from the extension of complex Clifford algebras. The last column shows classifying spaces C_q .

Class	Generators	Extension	Classifying space
A	$e_0 = H$	$Cl_0 \rightarrow Cl_1$	C_0
AIII	$e_0 = H, e_1 = \Gamma$	$Cl_1 \rightarrow Cl_2$	C_1

that zero-dimensional systems in class A are characterized by an integer topological index (Table I).

B. Real classes

Next we review classification of the real AZ classes in zero spatial dimensions in terms of real Clifford algebras. Time-reversal symmetry (TRS) and particle-hole symmetry (PHS) of Hamiltonian H are written as

$$T^{-1}HT = H, \quad C^{-1}HC = -H, \quad (12)$$

with antiunitary operators T and C , respectively. These relations are not affected by spectral flattening of H ,

$$H^2 = 1. \quad (13)$$

Without loss of generality we can assume

$$[T, C] = 0, \quad (14)$$

and we have

$$T^2 = \epsilon_T, \quad C^2 = \epsilon_C, \quad (15)$$

where ϵ_T and ϵ_C are either $+1$ or -1 ; see Table III. Since both T and C involve complex conjugation \mathcal{K} , we introduce an operator J representing the imaginary unit “ i ” so that we can treat complex structure algebraically in real Clifford algebras. We thus impose the operator J to satisfy the relations

$$J^2 = -1, \quad \{T, J\} = \{C, J\} = [H, J] = 0, \quad (16)$$

as expected for “ i ”.

Equations (12)–(16) are used to define real Clifford algebras $Cl_{p,q}$. According to the absence or presence of the TRS and PHS, we have a different set of generators for real Clifford algebra in each class:

(i) T only (AI and AII): $\{e_1, e_2\} \rightarrow \{e_0, e_1, e_2\}$, where

$$e_0 = JH, \quad e_1 = T, \quad e_2 = TJ, \quad (17a)$$

with

$$e_0^2 = -1, \quad e_1^2 = \epsilon_T, \quad e_2^2 = \epsilon_T. \quad (17b)$$

(ii) C only (C and D): $\{e_1, e_2\} \rightarrow \{e_0, e_1, e_2\}$, where

$$e_0 = H, \quad e_1 = C, \quad e_2 = CJ, \quad (18a)$$

TABLE III. Classification for the real AZ classes in zero dimensions from the extension of real Clifford algebras. The second column (ϵ_T, ϵ_C) shows the sign of squared symmetry operators (T^2, C^2), where the absence of the symmetry is denoted by “0”.

Class	(ϵ_T, ϵ_C)	Extension	Classifying space
AI	(+, 0)	$Cl_{0,2} \rightarrow Cl_{1,2}$	R_0
AII	(-, 0)	$Cl_{2,0} \rightarrow Cl_{3,0}$	R_4
D	(0, +)	$Cl_{0,2} \rightarrow Cl_{0,3}$	R_2
C	(0, -)	$Cl_{2,0} \rightarrow Cl_{2,1}$	$R_{-2} \simeq R_6$
BDI	(+, +)	$Cl_{1,2} \rightarrow Cl_{1,3}$	R_1
DIII	(-, +)	$Cl_{0,3} \rightarrow Cl_{0,4}$	R_3
CII	(-, -)	$Cl_{3,0} \rightarrow Cl_{3,1}$	$R_{-3} \simeq R_5$
CI	(+, -)	$Cl_{2,1} \rightarrow Cl_{2,2}$	$R_{-1} \simeq R_7$

with

$$e_0^2 = 1, \quad e_1^2 = \epsilon_C, \quad e_2^2 = \epsilon_C. \quad (18b)$$

(iii) Both T and C (BDI, DIII, CII, and CI): $\{e_1, e_2, e_3\} \rightarrow \{e_0, e_1, e_2, e_3\}$, where

$$e_0 = H, \quad e_1 = C, \quad e_2 = CJ, \quad e_3 = TCJ, \quad (19a)$$

with

$$e_0^2 = 1, \quad e_1^2 = \epsilon_C, \quad e_2^2 = \epsilon_C, \quad e_3^2 = -\epsilon_T \epsilon_C. \quad (19b)$$

Before adding the generator e_0 , we have real Clifford algebras $Cl_{2,0}$ or $Cl_{0,2}$ for the cases (i) and (ii), and $Cl_{1,2}$, $Cl_{2,1}$, $Cl_{0,3}$, or $Cl_{3,0}$ for the case (iii), depending on the sign of ϵ_T and ϵ_C (Table III). We then consider extension of the real Clifford algebras by adding e_0 . We shall distinguish two cases, $e_0^2 = +1$ and $e_0^2 = -1$. The classifying space for the extension $Cl_{p,q} \rightarrow Cl_{p,q+1}$ ($e_0^2 = +1$) is known to be given by R_{q-p} , while that for $Cl_{p,q} \rightarrow Cl_{p+1,q}$ ($e_0^2 = -1$) is given by R_{p+2-q} .^{19,32} The latter can be understood by noting that we have an isomorphism $Cl_{p,q} \otimes \mathbb{R}(2) \simeq Cl_{q,p+2}$, where $\mathbb{R}(2)$ is an algebra of 2×2 real matrices (Appendix A). By taking the tensor product with $\mathbb{R}(2)$ (which does not affect the extension problem), the extension $Cl_{p,q} \rightarrow Cl_{p+1,q}$ is mapped to the extension $Cl_{q,p+2} \rightarrow Cl_{q,p+3}$, whose classifying space is R_{p+2-q} . From the Bott periodicity,^{19,32} the classifying space has a periodic structure $R_q \simeq R_{q+8}$, and the eight real symmetry classes fall into eight distinct classifying spaces, as listed in Table III (see Appendix A). Finally we find topological classification of each AZ class from zeroth homotopy group of the classifying spaces $\pi_0(R_q)$ (Table I).

IV. ADDITIONAL SYMMETRY

In this section, we study how the topological properties change when an additional symmetry is imposed on each symmetry class. Here we concentrate on an additional symmetry denoted by a unitary operator M that anticommutes with Hamiltonian H ,

$$\{H, M\} = 0. \quad (20)$$

As we will see later in the following sections, this situation has several interesting applications. The additional symmetry is represented as a new generator in Clifford algebras, leading to modification of the Clifford algebras and topological classification. While we consider zero-dimensional Hamiltonian H in this section, we note that, once we find the classifying space for zero dimensions as C_q or R_q , the topological classification for d dimensions is given by $\pi_0(C_{q-d})$ or $\pi_0(R_{q-d})$, according to K theory.¹⁹

A. Complex classes

We first consider zero-dimensional systems in complex classes (see Table IV). For systems originally in class A, M serves as a chiral symmetry operator. Hence the relevant classifying space for class A changes from C_0 to C_1 upon addition of the symmetry M .

For systems in class AIII with chiral symmetry Γ , we impose the condition

$$\Gamma M = \eta_\Gamma M \Gamma, \quad (21)$$

TABLE IV. Shifts of classifying spaces due to an additional symmetry for complex classes in zero spatial dimensions.

Class	η_Γ	Relations	Shift of C_n
A		$\{e_0, M\} = 0$	+1
AIII	+	$[e_0, \Gamma M] = [e_1, \Gamma M] = 0$	0
AIII	-	$\{e_0, M\} = \{e_1, M\} = 0$	+1

where the signature η_Γ designates a commutation (+) or anticommutation (-) relation between M and Γ . If $\eta_\Gamma = -1$, then M serves as an additional generator to the complex Clifford algebra, and we need to consider an extension problem $Cl_2 \rightarrow Cl_3$, instead of the original $Cl_1 \rightarrow Cl_2$. Hence the classifying space is shifted by 1, from C_1 to $C_2 \simeq C_0$. Here we have used the Bott periodicity,^{19,32} $C_n \simeq C_{n+2}$.

On the other hand, when $\eta_\Gamma = +1$, the product ΓM commutes with the original generators Γ and H . Then, in each eigenspace of ΓM , we have the same extension problem of a complex Clifford algebra as before ($Cl_1 \rightarrow Cl_2$), and the topological classification is not changed from Table I.

B. Real classes

Next we consider zero-dimensional systems in the eight real classes with an additional symmetry M . We require M to satisfy

$$[J, M] = 0, \quad M^2 = 1. \quad (22)$$

Furthermore, for systems which are invariant under time-reversal or particle-hole transformation, we assume

$$TM = \eta_T MT, \quad CM = \eta_C MC, \quad (23)$$

where η_T and η_C designate commutation (+) or anticommutation (-) relations with T and C .

If the system has either one of TRS and PHS, or if it has both TRS and PHS with $(\eta_T, \eta_C) = (+, -)$ or $(-, +)$, then we can construct an additional generator \tilde{e} from the symmetry M given in Table V(a), which should be added to the set of generators listed in Eqs. (17)–(19). By considering the extension problem of Clifford algebras, $\{e_1, \dots, \tilde{e}\} \rightarrow \{e_0, e_1, \dots, \tilde{e}\}$, we find that the index q of the classifying space R_q should be shifted by ± 1 , according to the sign of \tilde{e}^2 , as listed in Table V(a).

On the other hand, if the system has both TRS and PHS with $(\eta_T, \eta_C) = (+, +)$ or $(-, -)$, then we can construct, not an additional generator for Clifford algebras, but an operator \tilde{M} which commutes with all generators of the relevant Clifford algebras; see Table V(b). We have the following two cases according to the sign of \tilde{M}^2 .

(1) $\tilde{M}^2 = +1$: The Hilbert space splits into two eigenspaces of \tilde{M} . For each eigenspace of \tilde{M} , we have the same Clifford algebra and classifying space as in Table III, and the topological classification is not changed from Table I.

(2) $\tilde{M}^2 = -1$: \tilde{M} effectively behaves like “ i ” and introduces a complex structure into the real algebra. This situation is formally written as $Cl_{p,q} \otimes_{\mathbb{R}} Cl_{1,0} \simeq Cl_{p,q} \otimes_{\mathbb{R}} \mathbb{C} \simeq Cl_{p+q}$. Then the classification is given by an extension problem of a complex Clifford algebra $Cl_3 \rightarrow Cl_4$, which falls into a classifying space $C_3 \simeq C_1$. This means that the additional symmetry has changed real symmetry classes into class AIII.

TABLE V. Classification of real symmetry classes in the presence of an additional symmetry M in zero dimension. (η_T, η_C) dictates commutation (+) or anticommutation (−) relation with T and C operators [Eq. (23)], or absence of symmetry denoted by “0”. We distinguish the following two cases: (a) The additional symmetry operator M provides a new generator (\tilde{e}) and causes a shift in the classifying space. (b) The additional symmetry operator M provides an operator (\tilde{M}) commuting with all preexisting generators.

Class	(η_T, η_C)	\tilde{e}	\tilde{e}^2	Shift of R_q
AI, AII	(+,0)	JM	−1	+1
	(−,0)	M	+1	−1
D, C	(0,+)	JM	−1	−1
	(0,−)	M	+1	+1
BDI, DIII, CII, CI	(+,−)	M	+1	+1
	(−,+)	JM	−1	−1

Class	(η_T, η_C)	\tilde{M}	\tilde{M}^2	Classifying space
BDI, CII	(+,+)	TCM	+1	same as in Table III
	(−,−)	$TCJM$	−1	C_1
DIII, CI	(+,+)	TCM	−1	C_1
	(−,−)	$TCJM$	+1	same as in Table III

Changes in topological classification of real classes induced by an additional symmetry in zero spatial dimensions are summarized in Table V.

C. Commuting symmetries $[H, M] = 0$

So far we have studied additional symmetries that anticommute with Hamiltonian. Here we briefly discuss additional symmetries that satisfy

$$[H, M] = 0, \quad M^2 = 1. \quad (24)$$

For example, this situation happens in 1D systems with mirror lines, or in 2D systems with mirror planes as discussed in Refs. 39,40. We show below that, upon block diagonalization of H with respect to M , symmetry classes can change if actions of the generic symmetries are not closed in eigenspaces of M .

We start with complex classes. In class A, the block diagonalization does not change the symmetry class nor the classification. In class AIII, when the chiral symmetry also commutes as $[\Gamma, M] = 0$, the classification does not change. On the other hand, when the chiral symmetry anticommutes as $\{\Gamma, M\} = 0$, the chiral symmetry does not hold in subblocks and the symmetry class changes into class A.

Let us move on to real classes. (i) T only (AI and AII) or C only (C and D): When the additional symmetry M commutes with the generic symmetry T or C , the block diagonalization does not change the symmetry class and, hence, the classification. The situation changes for the additional symmetry M that anticommutes with the generic symmetry. Since TRS or PHS does not hold in each eigenspace of M , the symmetry class changes into class A. In terms of Clifford algebras, this situation is written as an extension problem $\{e_1, e_2\} \otimes JM \rightarrow \{e_0, e_1, e_2\} \otimes JM$, where we adopt a new generator JM that commutes with T or C but squares to -1 . Thus the extension $Cl_{p,q} \otimes \mathbb{C} \rightarrow Cl_{p,q+1} \otimes \mathbb{C}$ turns into that of complex algebras $Cl_{p+q} \rightarrow Cl_{p+q+1}$. Then the relevant classifying space is a complex one with $C_{p+q} \simeq C_0$

from Table III, which indicates that the symmetry class changes into class A.

(ii) Both T and C (BDI, DIII, CII, and CI): Since we have a chiral symmetry Γ in this case, we can construct an operator that anticommutes with the Hamiltonian and squares to $+1$ (either $M' = \Gamma M$ or $M' = J\Gamma M$), with which we can use the result of Sec. IV B. We will later discuss an example for this type of symmetry in Sec. VIII B.

V. DIRAC MODEL AND DIMENSIONAL SHIFT

As Kitaev has shown using K theory,^{19,32} once classifying spaces, C_q or R_q , is understood for each AZ symmetry class in zero dimensions, topological classification of d -dimensional systems ($d \geq 1$) of that class is directly found from $\pi_0(C_{q-d})$ or $\pi_0(R_{q-d})$. In this section we consider massive Dirac Hamiltonians to see this dimensional shift explicitly in terms of Clifford algebras with additional symmetries; similar formulations are given in Refs. 20–22.

Let us consider a massive Dirac Hamiltonian in d dimensions,

$$H_d = \sum_{i=1}^d \gamma_i k_i + H, \quad (25)$$

where k_i is a momentum in the i th direction and H is a Dirac mass term which should satisfy appropriate symmetry relations (11) or (12) for each symmetry class. We can impose the constraint $H^2 = 1$ (by taking the Dirac mass as a unit of energy). The gamma matrices obey the anticommutation relations

$$\{\gamma_i, \gamma_j\} = 2\delta_{i,j}, \quad \{H, \gamma_i\} = 0. \quad (26)$$

Furthermore, the TRS and PHS (if present) of the Hamiltonian H_d imply

$$\{T, \gamma_i\} = 0, \quad [C, \gamma_i] = 0, \quad (27)$$

because complex conjugation (\mathcal{K}) changes k_i to $-k_i$. The chiral symmetry (if present) imposes

$$\{\Gamma, \gamma_i\} = 0. \quad (28)$$

Now we show that topological classification of the Dirac Hamiltonian H_d is obtained from $\pi_0(C_{q-d})$ or $\pi_0(R_{q-d})$ when we study the extension (by adding $e_0 := H$) of Clifford algebras generated by generic symmetry operators and the γ_i 's. We proceed by induction. Suppose that the classifying space for the Dirac mass term H in the $(d-1)$ -dimensional system H_{d-1} is found to be C_{q-d+1} or R_{q-d+1} . The d -dimensional Hamiltonian H_d has an additional operator $M = \gamma_d$, which can be considered as an additional symmetry constraint discussed in Sec. IV. For real classes, when the Dirac mass term H in H_d is invariant under T and/or C , the additional operator M anticommuting with H has the signature $\eta_T = -1$ and/or $\eta_C = +1$ from Eq. (27). We find from Table V(a) that the addition of M induces a shift of the relevant classifying space by -1 ; we have R_{q-d} . For complex classes A and AIII ($\eta_\Gamma = -1$), we find from Table IV that the addition of M induces a shift by $+1 = -1 \pmod{2}$ in complex classifying spaces; we have C_{q-d} .

VI. REFLECTION SYMMETRY

Let us discuss topological classification in the presence of reflection symmetry using the Dirac model studied in the preceding section. Our approach is similar but complementary to the one in Ref. 31, where topological invariants are explicitly constructed for Dirac models. We apply the classification theory in terms of Clifford algebras, considering a reflection symmetry as a special case of additional symmetries discussed in Sec. IV. For the sake of simplicity, we first assume translation symmetry and exclude terms which depend on spatial coordinates (such as mass terms of CDW type) in the Dirac Hamiltonian. We will comment on effects of relaxing this condition at the end of this section and in Appendixes B and C.

Let us assume that the Dirac Hamiltonian H_d is invariant under reflection R in the l th direction, where momenta k_j 's are changed as $k_j \rightarrow (-1)^{\delta_{j,l}} k_j$. It follows from $[R, H_d] = 0$ that $\{\gamma_l, R\} = 0$ and $[\gamma_j, R] = 0$ for $j \neq l$. We can always set $R^2 = 1$. Then, as an additional symmetry operator M , we can take

$$M = J\gamma_l R, \quad (29)$$

which satisfy $M^2 = 1$, $\{H, M\} = 0$, and $\{\gamma_j, M\} = 0$ ($1 \leq j \leq d$), where again H is a Dirac mass term satisfying symmetry relation (11) or (12) appropriate for each symmetry class.

Note that the commutation relations of R with C and Γ are different from those of M . Namely, for real classes, when R obeys the relations $RT = \tilde{\eta}_T TR$ and/or $RC = \tilde{\eta}_C CR$ with $\tilde{\eta}_{T,C} = +$ or $-$, M has the signatures $(\eta_T, \eta_C) = (\tilde{\eta}_T, -\tilde{\eta}_C)$.

Similarly, for class AIII, we have $\eta_\Gamma = -\tilde{\eta}_\Gamma$, where $\tilde{\eta}_\Gamma = +$ or $-$ is specified by the relation $R\Gamma = \tilde{\eta}_\Gamma \Gamma R$. Now that we have defined M with the signatures $(\eta_T, \eta_C, \eta_\Gamma)$, we can use Tables IV and V and the dimensional shift discussed in Sec. V, to obtain topological classification in the presence of the reflection symmetry R .

Let us discuss in more detail the consequence of reflection symmetry for each class. We begin with complex classes. First, class A turns into AIII with an effective chiral symmetry M . For class AIII, if we denote R with its commutation relation with Γ as $R^{\tilde{\eta}_\Gamma}$, we find that R^+ changes the classifying space to C_0 , while R^- does not change the classification. Now let us move on to real symmetry classes. We write the R operator with a superscript showing its commutation relations with T or C as $R^{\tilde{\eta}_T}$, $R^{\tilde{\eta}_C}$, $R^{\tilde{\eta}_T \tilde{\eta}_C}$ for symmetry classes with T only, C only, and both T and C , respectively. Suppose that the original classifying space for a given real symmetry class is R_{q-d} for d dimensions. We find from Table V that R^+ and R^{++} shift the classifying space by $+1$ to R_{q-d+1} . Thus we may say that R^+ and R^{++} have the effect of decreasing the spatial dimension by 1. In a similar way, R^- and R^{--} shift the classifying space by -1 to R_{q-d-1} and effectively increase the spatial dimension by 1. As for R^{+-} , topological classifications for BDI and CII remain the same, while DIII and CI change into complex class AIII. On the contrary, with R^{-+} , DIII and CI remain the same, while BDI and CII change into complex class AIII.

Table VI summarizes classification in the presence of various types of reflection symmetries for each spatial dimension. The periodic structures are evident.

A brief comment is in order on the relation between the topological classification of Hamiltonian in the bulk and the presence of gapless boundary states. In topological mirror insulators/superconductors where the bulk Hamiltonian has a nontrivial topological index only in the presence of a reflection symmetry, the existence of gapless states on a boundary depends on whether or not the boundary preserves the reflection symmetry.²⁴ When the presence of a boundary is compatible with reflection symmetry (e.g., when the boundary is normal to a mirror plane), gapless boundary states are stable and protected by the above classification. On the other hand, if a boundary breaks reflection symmetry, the boundary states are generally gapped. This is the case for one-dimensional systems where the presence of an edge breaks the reflection symmetry (unless the mirror plane is parallel to the one-dimensional system itself).

So far we have assumed the translation symmetry. However, as pointed out in Ref. 31, if the condition of translation symmetry is removed, the ‘‘second descendant’’¹⁸ \mathbb{Z}_2 states under the reflection symmetry R^- or R^{--} can be adiabatically deformed into a topologically trivial insulator by introducing an extra mass term with a finite wave number. These unstable \mathbb{Z}_2 phases are denoted by ‘‘ \mathbb{Z}_2 ’’ in Table VI. Once we replace ‘‘ \mathbb{Z}_2 ’’ with 0, Table VI becomes identical to Table I of Ref. 31. These ‘‘ \mathbb{Z}_2 ’’ phases are similar to three-dimensional *weak* \mathbb{Z}_2 topological insulators in class AII with (say) two Dirac cones on a two-dimensional surface, in that the two surface Dirac cones can be gapped out by a perturbation (of CDW type) that couples them. From this analogy we expect that the ‘‘ \mathbb{Z}_2 ’’ states should be stable against disorder, if the disorder average of any mass term is assumed to be spatially uniform. In Appendix B,

TABLE VI. Classification in the presence of a reflection symmetry. The first column shows types of reflection symmetry, where superscripts of R show its commutation relations with basic symmetry operators such as Γ , T , and C , i.e., $R^{\bar{n}r}$ for the complex classes, and $R^{\bar{n}r}$, $R^{\bar{n}c}$, $R^{\bar{n}r\bar{n}c}$ for real classes with T only, C only, and both T and C , respectively. The second column shows symmetry classes in which a given Hamiltonian is classified without reflection symmetry taken into account. The third column shows classifying spaces for zero-dimensional Hamiltonian in the presence of reflection symmetry, and the following columns show topological classifications for spatial dimensions $d = 0, 1, 2, \dots, 7$ (mod 8). Note that the classifying spaces in the third column are shifted from those listed in Table I. The “ \mathbb{Z}_2 ” phases appearing under the reflection R^- or R^{--} turn into topologically trivial (0), when spatially nonuniform perturbations are applied (see the discussion at the end of Sec. VI and Appendix B).

Reflection	Class	C_q or R_q	$d = 0$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$
R	A	C_1	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
R^+	AIII	C_0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0
R^-	AIII	C_1	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
R^+, R^{++}	AI	R_1	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2
	BDI	R_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0
	D	R_3	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}
	DIII	R_4	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0
	AII	R_5	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0
	CII	R_6	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0
	C	R_7	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
	CI	R_0	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2
R^-, R^{--}	AI	R_7	0	0	0	\mathbb{Z}	0	“ \mathbb{Z}_2 ”	\mathbb{Z}_2	\mathbb{Z}
	BDI	R_0	\mathbb{Z}	0	0	0	\mathbb{Z}	0	“ \mathbb{Z}_2 ”	\mathbb{Z}_2
	D	R_1	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	“ \mathbb{Z}_2 ”
	DIII	R_2	“ \mathbb{Z}_2 ”	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0
	AII	R_3	0	“ \mathbb{Z}_2 ”	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}
	CII	R_4	\mathbb{Z}	0	“ \mathbb{Z}_2 ”	\mathbb{Z}_2	\mathbb{Z}	0	0	0
	C	R_5	0	\mathbb{Z}	0	“ \mathbb{Z}_2 ”	\mathbb{Z}_2	\mathbb{Z}	0	0
CI	R_6	0	0	\mathbb{Z}	0	“ \mathbb{Z}_2 ”	\mathbb{Z}_2	\mathbb{Z}	0	
R^{+-}	BDI	R_1	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2
R^{-+}	DIII	R_3	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0	0	\mathbb{Z}
R^{+-}	CII	R_5	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0	0
R^{-+}	CI	R_7	0	0	0	\mathbb{Z}	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
R^{-+}	BDI, CII	C_1	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
R^{+-}	DIII, CI	C_1	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}

we discuss the deformation of “ \mathbb{Z}_2 ” to 0 in more detail in terms of Clifford algebras. We will also show in Appendix C that the deformation of nontrivial states with CDW-type perturbation takes place only for these second descendants “ \mathbb{Z}_2 ” of the R^- and R^{--} cases and no other such deformations are possible in Table VI.

VII. MULTIPLE ADDITIONAL SYMMETRIES

We generalize the analysis of Sec. IV to systems with multiple additional symmetries $\{M_i\}$. Here, we only consider the situation where additional symmetries anticommute with each other,

$$\{M_i, M_j\} = 2\delta_{i,j}, \quad (30)$$

as well as with zero-dimensional Hamiltonian (Dirac mass) H , $\{M_i, H\} = 0$. The dimensional shift discussed in Sec. V is a special case of choosing $M_i = \gamma_i$. Other interesting applications can be found in systems with several independent reflection symmetries (R_i) along different directions. Since independent R_i 's should commute with each other, the additional symmetries $M_i = i\gamma_i R_i$ constructed from R_i anticommute with each other, and the results of this section are applicable.

A. Complex classes

The consequences of imposing multiple additional symmetries on systems in complex classes are as follows.

Class A: The index q of the classifying space C_q is shifted by +1 each time an additional symmetry is imposed, so that the relevant classifying space becomes C_N when the number of M_i 's imposed is N .

Class AIII: Suppose that the M_i 's have the following algebraic relations with the chiral symmetry operator Γ :

$$M_i \Gamma = \begin{cases} -\Gamma M_i, & 1 \leq i \leq m, \\ +\Gamma M_i, & m+1 \leq i \leq m+n. \end{cases} \quad (31)$$

We then define new generators as $e_0 = H$, $e_1 = \Gamma$, $e_i^- = M_i$ ($i = 1, \dots, m$), and $e_i^+ = \Gamma M_{i+m}$ ($i = 1, \dots, n$), such that $\{e_0, e_1, e_1^-, \dots, e_m^-\}$ and $\{e_1^+, \dots, e_n^+\}$ form two Clifford algebras which commute with each other. Thus we have an extension problem $Cl_{m+1} \otimes Cl_n \rightarrow Cl_{m+2} \otimes Cl_n$, for which the classifying space is C_{m+1} .

B. Real classes

We separately discuss the cases where either of TRS or PHS is present and the cases where both are present.

(i) T only (AI and AII) or C only (C and D): Let us write the symmetry operators M_i as $M_i^{\eta_T}$ or $M_i^{\eta_C}$, with a superscript indicating its signature η_T or η_C defined in Eq. (23). We denote the numbers of M_i^+ 's and M_i^- 's by n^+ and n^- , respectively. Now we can construct a new Clifford algebra with the generators

$$\{e_0, e_1, e_2, \tilde{e}_1^+, \dots, \tilde{e}_{n^+}^+, \tilde{e}_1^-, \dots, \tilde{e}_{n^-}^-\}, \quad (32)$$

where e_0, e_1 , and e_2 are defined in Eqs. (17) and (18), and \tilde{e}_i^\pm are defined by

$$\tilde{e}_i^+ = JM_i^+, \quad \tilde{e}_i^- = M_i^-, \quad (33)$$

as in Table V. We thus find that, upon imposing additional symmetries M_i^\pm , the relevant classifying space is changed from R_q to $R_{\tilde{q}}$ with

$$\tilde{q} = \begin{cases} q + n^+ - n^- & (T \text{ only: AI and AII}), \\ q + n^- - n^+ & (C \text{ only: C and D}). \end{cases} \quad (34)$$

(iii) Both T and C (BDI, DIII, CII, and CI): We use the notation $M_i^{\eta_T \eta_C}$ for the additional symmetries with the signatures $\eta_T, \eta_C = +$ or $-$ specifying commutation or anticommutation relations with T and C . We denote the numbers of $M_i^{\eta_T \eta_C}$'s by $n^{\eta_T \eta_C}$. We then define generators as

$$\tilde{e}_i^{+-} = M_i^{+-} \quad (i = 1, \dots, n^{+-}), \quad (35a)$$

$$\tilde{e}_i^{-+} = JM_i^{-+} \quad (i = 1, \dots, n^{-+}), \quad (35b)$$

$$\tilde{e}_i^{++} = TCM_i^{++} \quad (i = 1, \dots, n^{++}), \quad (35c)$$

$$\tilde{e}_i^{--} = TCJM_i^{--} \quad (i = 1, \dots, n^{--}), \quad (35d)$$

with which we have two decoupled Clifford algebras,

$$\{e_0, e_1, e_2, e_3, \tilde{e}_1^{+-}, \dots, \tilde{e}_{n^{+-}}^{+-}, \tilde{e}_1^{-+}, \dots, \tilde{e}_{n^{-+}}^{-+}\} \quad (36a)$$

and

$$\{\tilde{e}_1^{++}, \dots, \tilde{e}_{n^{++}}^{++}, \tilde{e}_1^{--}, \dots, \tilde{e}_{n^{--}}^{--}\}. \quad (36b)$$

For each symmetry class, the generators $\{e_0, e_1, e_2, e_3\}$ are given in Eq. (19), and the original extension problem $Cl_{p,q} \rightarrow Cl_{p,q+1}$ (in zero dimensions) is listed in Table III. With the additional generators, a new extension problem is dictated as

$$Cl_{p+n^+, q+n^+} \otimes Cl_{m_1, m_2} \rightarrow Cl_{p+n^+, q+n^++1} \otimes Cl_{m_1, m_2} \quad (37)$$

with $(m_1, m_2) = (n^{++}, n^{--})$ for DIII and CI, and (n^{--}, n^{++}) for BDI and CII. We can show by using Eq. (A2e) and dropping $\mathbb{R}(2)$ that the extension (37) is equivalent to

$$Cl_{0, \tilde{q}} \otimes Cl_{0, m} \rightarrow Cl_{0, \tilde{q}+1} \otimes Cl_{0, m} \quad (38)$$

with

$$\tilde{q} = q + n^{+-} - p - n^{-+}, \quad (39a)$$

$$m = \begin{cases} n^{--} - n^{++} & (\text{DIII and CI}), \\ n^{++} - n^{--} & (\text{BDI and CII}). \end{cases} \quad (39b)$$

TABLE VII. Classification with multiple additional symmetries for classes with both TRS and PHS (BDI, DIII, CII, and CI). When the numbers of additional symmetries $M^{\eta_T \eta_C}$ are given by $n^{\eta_T \eta_C}$, the relevant Clifford algebras and classifying spaces can be read off from the indices m and \tilde{q} defined in Eqs. (39). The second column shows Clifford algebras to be extended as $\tilde{q} \rightarrow \tilde{q} + 1$, Eq. (38). The last column shows the classifying space, whose zeroth homotopy group gives topological classification.

$m \pmod{8}$	Clifford algebra	Classifying space
0	$Cl_{0, \tilde{q}}$	$R_{\tilde{q}}$
1	$Cl_{0, \tilde{q}} \oplus Cl_{0, \tilde{q}}$	$R_{\tilde{q}} \times R_{\tilde{q}}$
2	$Cl_{0, \tilde{q}}$	$R_{\tilde{q}}$
3	$Cl_{\tilde{q}}$	$C_{\tilde{q}}$
4	$Cl_{0, \tilde{q}+4}$	$R_{\tilde{q}+4}$
5	$Cl_{0, \tilde{q}+4} \oplus Cl_{0, \tilde{q}+4}$	$R_{\tilde{q}+4} \times R_{\tilde{q}+4}$
6	$Cl_{0, \tilde{q}+4}$	$R_{\tilde{q}+4}$
7	$Cl_{\tilde{q}}$	$C_{\tilde{q}}$

We see that n^{+-} and n^{-+} cause a shift in \tilde{q} of a relevant classifying space, which can be understood from successive applications of the procedure described in Sec. IV. On the other hand, n^{--} and n^{++} may effectively alter the base field of the algebra from real to complex or quaternion, according to the value of m ; see Table VIII(b). This change cannot be fully expected from successive applications of the procedure in Sec. IV. Changing the base field into the complex numbers brings real symmetry classes to complex ones ($m = 3, 7 \pmod{8}$), while changing the base field into the quaternion shifts the relevant classifying space by 4 in the periodic table, Table I(b) ($m = 4, 5, 6 \pmod{8}$).

The new classification arising from the extension problem (38) is summarized in Table VII (some mathematical details needed in the derivation can be found in Appendix D). With Table VII, we can find topological classification in the presence of additional symmetries as follows. First, without additional symmetries, we identify the relevant Clifford algebra $Cl_{p,q}$ for a given symmetry class in Table III. Next from the (anti)commutation relations of additional symmetries M_i 's with T and C , we find the numbers $n^{\eta_T \eta_C}$, which determine \tilde{q} and m in Eqs. (39). Table VII then tells us associated extension problem of the relevant Clifford algebra. Once the relevant classifying space is found, the topological classification of zero-dimensional systems is obtained from the zeroth homotopy group, which is listed in Table I. As for d -dimensional systems, we only have to replace \tilde{q} with $\tilde{q} - d$, as we have discussed in Sec. V (each gamma matrix in the kinetic term of Dirac Hamiltonian gives M^{-+}). For a classifying space of a direct product such as $R_q \times R_q$, the d -dimensional topological invariant is also a direct product as $\pi_0(R_{q-d}) \times \pi_0(R_{q-d})$.

Finally, let us point out an interesting consequence from Table VII. As we decrease m from 0 to -2 by imposing some additional symmetries, we find that the classification changes into a complex class at $m = -1$, and at $m = -2$ it comes back to a real symmetry class with a shift of 4 in the Bott periodicity. We will discuss an example of this type in Sec. VIII, where two types of reflection symmetries are imposed to cause such changes in a symmetry class.

VIII. EXAMPLES

We discuss several examples of insulators or superconductors which exhibit a change in topological properties due to addition of reflection or mirror symmetries. We will use “ i ” in real algebras instead of J in this section.

A. Mirror Chern number

In relation to mirror Chern numbers²⁷ which characterize a recently discovered topological crystalline insulator SnTe,^{26,28–30} let us consider Dirac Hamiltonian defined by

$$H = m\sigma_z + v(k_x s_y - k_y s_x)\sigma_x + v_z k_z \sigma_y, \quad (40)$$

where σ_i and s_i are Pauli matrices for orbital and spin degrees of freedom. This Hamiltonian²⁶ describes low-energy excitations near an L point of SnTe and belongs to class AII, since it is invariant under time-reversal operation, $T = i s_y \mathcal{K}$. We can define a topological index which is determined by the sign of m . If we double the degrees of freedom to account for another L point as $H_2 = H \otimes \tau_0$, where τ_0 is a unit 2×2 matrix in the valley sector (L_1 and L_2), then the Hamiltonian can have another T -invariant mass term, $m' s_z \sigma_x \tau_y$, where τ_y is a Pauli matrix and couples the two valleys. With the additional mass m' , insulators with different signs of m are no longer topologically distinguished, because rotation in the m - m' plane can adiabatically connect the two. Thus H is characterized by a \mathbb{Z}_2 topological index.

Let us now take into consideration reflection symmetry in the x direction ($k_x \rightarrow -k_x$). The Hamiltonian H is indeed invariant under the reflection transformation, which can be written as $R^{-1}H(-k_x, k_y, k_z)R = H(k_x, k_y, k_z)$ with $R = s_x$. Following the analysis in Sec. IV B, we define $M = i s_y \sigma_x s_x = \sigma_x s_z$, which anticommutes with T ; i.e., $\eta_T = -1$. We then find from Table VI that class AII with the reflection R^- effectively behaves like class DIII, which is characterized by an integer topological number \mathbb{Z} in $d = 3$. As we discuss below, the integer topological number corresponds to the mirror Chern number.^{26,27} Incidentally, the mirror symmetry does not allow the doubled Hamiltonian $H_2 = H \otimes \tau_0$ to have the additional mass term $m' s_z \sigma_x \tau_y$, so that the topological numbers from the two valley sectors can add up.

The mirror Chern number is a Chern number defined from Bloch states on the mirror plane ($k_x = 0$) for each eigenspace of the reflection R (i.e., $s_x = \pm 1$). In each subspace the Hamiltonian (40) reduces to

$$H^\pm = m\sigma_z \mp v k_y \sigma_x + v_z k_z \sigma_y, \quad (41)$$

which falls into class D (with $C = \sigma_x \mathcal{K}$) and apparently possesses a Chern number \mathbb{Z} . Now let us relate this mirror Chern number to our classification. The original extension problem for class AII with mirror symmetry in $d = 3$ is $Cl_{2,4} \rightarrow Cl_{3,4}$, which we find from Eq. (A2) is equivalent to $Cl_{2,2} \otimes Cl_{0,2} \rightarrow Cl_{2,3} \otimes Cl_{0,2}$. The latter can be regarded as an extension problem for class D in $d = 2$, if we drop the trivial part $Cl_{0,2} \simeq \mathbb{R}(2)$. The original generators of the Clifford algebra for class AII with a mirror forming $Cl_{3,4}$ are $\{e_1, e_2, \gamma_x, \gamma_y, \gamma_z, M, e_0\}$, where $(e_1, e_2, e_0) = (i s_y \mathcal{K}, s_y \mathcal{K}, i \sigma_z)$ are defined in Eq. (17), and $(\gamma_x, \gamma_y, \gamma_z) = (s_y \sigma_x, -s_x \sigma_x, \sigma_y)$ come from kinetic terms in H . Using these generators, we can construct generators for

the new algebra $Cl_{2,3} \otimes Cl_{0,2}$ explicitly as

$$\begin{aligned} & \{-e_2 \gamma_x, e_1 \gamma_x, \gamma_x \gamma_y M, -e_1 e_2 \gamma_z, e_1 e_2 e_0\} \\ & \otimes \{e_1 e_2 \gamma_x M, e_1 e_2 \gamma_x \gamma_y\} \\ & = \{\sigma_x \mathcal{K}, -i \sigma_x \mathcal{K}, i \sigma_x, i \sigma_y, \sigma_z\} \otimes \{s_x, s_z\}. \end{aligned} \quad (42)$$

The latter half of the right-hand side spans the spin degrees of freedom s_i , while the former half corresponds to a Clifford algebra $Cl_{2,3}$ for Dirac Hamiltonians of class D in $d = 2$. We can read from Eqs. (18) and (42) and Table V(a) that the particle-hole transformation is $C = \sigma_x \mathcal{K}$, the gamma matrices in the kinetic terms are (σ_x, σ_y) , and the mass term $\propto \sigma_z$. Indeed this construction reproduces the Hamiltonian in Eq. (41), for which a mirror Chern number is defined. Thus the topological number \mathbb{Z} for class AII with a mirror in $d = 3$ is equivalent to a topological number \mathbb{Z} for class D in $d = 2$, i.e., a mirror Chern number.

B. Topological mirror superconductor ($d = 1$)

Next we consider a one-dimensional model of a time-reversal invariant topological mirror superconductor discussed in Ref. 39, in which an integer number of Majorana Kramers pairs live at an end of a mirror invariant wire. We consider the Hamiltonian of a Rashba quantum wire with proximity coupling to a nodeless s_\pm -wave superconductor,

$$H = (-2t \cos k_x + 2\lambda \sin k_x \sigma_z - \mu)\tau_z + 2\Delta_1 \tau_x \cos k_x, \quad (43)$$

where t is hopping, λ Rashba coupling, σ_i and τ_i are Pauli matrices in the spin and particle-hole spaces, and the condition $|\mu| < 2\lambda$ is assumed.³⁹ This Hamiltonian is in class DIII with $T = i \sigma_y \mathcal{K}$, $C = \sigma_y \tau_y \mathcal{K}$, characterized by a \mathbb{Z}_2 topological number and the existence of a Kramers pair of Majorana zero modes at the edge. If we consider a doubled system $H_2 = H \otimes \rho_0$ where ρ_0 is a 2×2 unit matrix, we have another mass term $\sigma_x \tau_x \rho_y$ that can deform a \mathbb{Z}_2 nontrivial state into a trivial state, indicating that two pairs of Majorana zero modes are unstable.

Now let us impose a “mirror line” symmetry $R = \sigma_z$, which commutes with H . As we discussed in Sec. IV C, combining R with the chiral symmetry TC , we can introduce an additional symmetry $M = \sigma_z \tau_y$, which anticommutes with H . Since the commutation relations of M with symmetry operators T and C are characterized by $(\eta_T, \eta_C) = (+1, +1)$, we have a change in the symmetry class from DIII to AIII [Table V(b)], so that we can define a topological number \mathbb{Z} in each eigenspace of R , which is a mirror winding number.³⁹ The change in the classification from \mathbb{Z}_2 to \mathbb{Z} is reflected in the disappearance of the additional mass term $(\sigma_x \tau_x \rho_y)$ in the doubled system upon imposing R , which guarantees the stability of Kramers pairs of Majorana zero modes.³⁹

C. CI \rightarrow AIII \rightarrow DIII ($d = 2$)

In this section we construct a toy model in $d = 2$ whose symmetry class changes (a) from CI to AIII when we impose an additional symmetry M_1 , and (b) from CI to DIII when we impose two additional symmetries M_1 and M_2 . We assume that both symmetry operators M_1 and M_2 commute with T and C ; i.e., $(\eta_T, \eta_C) = (+, +)$. These symmetries can be thought of as coming from reflection symmetries in the x and y directions,

for example. In the context of Sec. VII, our toy model is an example where a real symmetry class turns into (a) a complex class ($m = -1$ and $\tilde{q} = 7$ in Table VII) and (b) a real class shifted by 4 in the Bott period ($m = -2$ and $\tilde{q} = 7$ in Table VII). The $d = 2$ topological number of our model changes as (a) $0 \rightarrow 0$ and (b) $0 \rightarrow \mathbb{Z}_2$.

Let us consider a two-dimensional 8×8 Dirac Hamiltonian,

$$H = (k_x \sigma_x + k_y \sigma_y) \tau_x + \sum_i m_i g_i. \quad (44)$$

The Dirac mass operators g_i are 8×8 matrices made from 3 sets of Pauli matrices σ , τ , and ρ . We require the system to be in class CI with the symmetry operators,

$$T = \tau_y \rho_y \mathcal{K}, \quad C = \tau_x \rho_y \mathcal{K}, \quad \Gamma = \tau_z, \quad (45)$$

which satisfy $T^2 = +1$, $C^2 = -1$, and $[T, C] = 0$. We find the following set of possible g_i 's which anticommute with the kinetic terms and which are compatible with the symmetries (45):

$$\tau_y \otimes \{\rho_x, \rho_y, \rho_z\}, \quad \sigma_y \tau_x. \quad (46)$$

The two additional symmetries which we are going to impose are

$$M_1^{++} = \tau_z \rho_z, \quad M_2^{++} = \tau_z \rho_y, \quad (47)$$

both of which anticommute with the kinetic terms and commute with T and C . As discussed in Sec. IV and VII, we can define new operators $\tilde{M}_i = T C M_i^{++}$ which commute with H , T , and C : $\tilde{M}_1 = i \rho_z$, $\tilde{M}_2 = i \rho_y$.

First we impose only the M_1 symmetry. The allowed g_i 's which anticommute with M_1 are reduced to

$$\tau_y \rho_z, \quad \sigma_y \tau_x. \quad (48)$$

Since $\tilde{M}_1 = i \rho_z$ commute with H , we can concentrate on the subspace of $\rho_z = +1$. We then find that the relevant symmetry class is AIII with the chiral symmetry $\Gamma = \tau_z$. This is consistent with Table V(b), where the classifying space for class CI with $(\eta_T, \eta_C) = (+, +)$ and $\tilde{M}^2 = -1$ is shown to be C_1 , i.e., class AIII [Table I(a)]. We observe that, when H has a mass term taken from Eq. (48), we can always add a second mass term which anticommutes with the first mass term. This indicates that the system is topologically trivial, in agreement with topological triviality of class AIII in $d = 2$.

When both symmetries M_1 and M_2 are imposed, among those in Eq. (46), the only allowed Dirac mass operator which anticommutes with both M_1 and M_2 is $\sigma_y \tau_x$. Then the Dirac Hamiltonian reads

$$H = (k_1 \sigma_x + k_2 \sigma_z) \tau_x + m_1 \sigma_y \tau_x, \quad (49)$$

which does not contain Pauli matrices ρ 's, since H has to commute with both $\tilde{M}_1 = i \rho_z$ and $\tilde{M}_2 = i \rho_y$. The Hamiltonian (49) belongs to class DIII with new symmetry generators $T = i \tau_y \mathcal{K}$ and $C = \tau_x \mathcal{K}$. It has a unique mass term $m_1 \sigma_y \tau_x$ and is characterized by a \mathbb{Z}_2 topological index.

D. BDI \rightarrow AIII \rightarrow CII ($d = 3$)

Next we construct a toy model in $d = 3$ whose symmetry class changes (a) from BDI to AIII when we impose an addi-

tional symmetry M_1 and (b) from BDI to CII when we impose two additional symmetries M_1 and M_2 . Accordingly, the $d = 3$ topological number changes as (a) $0 \rightarrow \mathbb{Z}$ and (b) $0 \rightarrow \mathbb{Z}_2$.

We consider a Dirac Hamiltonian of the form

$$H = (k_x \sigma_x s_z + k_y s_x + k_z \sigma_z s_z) \tau_x + \sum_i m_i g_i, \quad (50)$$

where the Dirac mass operators g_i 's are 16×16 matrices written as products of Pauli matrices σ , s , τ , and ρ . We choose basic symmetry operators as

$$T = s_y \tau_0 \rho_y \mathcal{K}, \quad C = s_y \tau_z \rho_y \mathcal{K}, \quad \Gamma = \tau_z, \quad (51)$$

which makes the model to be in class BDI ($T^2 = +1$, $C^2 = +1$). The Dirac mass operators g_i 's, which anticommute with the kinetic terms in Eq. (50) and are compatible with the symmetries (51), are taken from the following set:

$$\sigma_y s_z \tau_x, \quad \{s_y \tau_x, \tau_y, \sigma_y s_x \tau_y\} \otimes \{\rho_x, \rho_y, \rho_z\}. \quad (52)$$

The additional symmetries to be imposed are again given by

$$M_1^{--} = \tau_z \rho_z, \quad M_2^{--} = \tau_z \rho_y, \quad (53)$$

which anticommute with T and C ; i.e., $(\eta_T, \eta_C) = (-, -)$. We then define $\tilde{M}_i = i T C M_i^{--}$, which commute with H , T , and C : $\tilde{M}_1 = i \rho_z$, $\tilde{M}_2 = i \rho_y$. We may think of M_1^{--} and M_2^{--} as coming from reflection symmetries along the x and y directions, respectively.

Let us impose only the M_1^{--} symmetry. The allowed g_i 's anticommute with M_1^{--} are reduced to

$$\sigma_y s_z \tau_x, \quad \{s_y \tau_x, \tau_y, \sigma_y s_x \tau_y\} \otimes \rho_z. \quad (54)$$

Since $[\rho_z, H] = 0$, we can concentrate on the eigenspace $\rho_z = +1$ (or -1) and find that the system in this subspace is in class AIII with the chiral symmetry $\Gamma = \tau_z$. This is in agreement with Table V(b), where the classifying space for class BDI with $(\eta_T, \eta_C) = (-, -)$ is shown to be C_1 , as well as with Table VII ($m = -1$ and $\tilde{q} = 1$). We observe that, among the mass operators in Eq. (54), $\sigma_y s_x \tau_y \rho_z$ is special in that it commutes with the other three operators, while these three anticommute among themselves. This means that Hamiltonians with different signs of the mass term $\sigma_y s_x \tau_y \rho_z$ are topologically distinct, which is consistent with the fact that systems in class AIII are characterized by an integer topological index.

When both symmetries M_1 and M_2 are imposed, we have only a single allowed mass operator, $\sigma_y s_z \tau_x$. In this case the Hamiltonian has the form

$$H = (k_x \sigma_x s_z + k_y s_x + k_z \sigma_z s_z) \tau_x + m \sigma_y s_z \tau_x, \quad (55)$$

which turns out to be in class CII with $T = s_y \mathcal{K}$ and $C = s_y \tau_z \mathcal{K}$, and is characterized by a \mathbb{Z}_2 topological index. The change in the symmetry class from BDI to CII is indeed expected from Table VII with $m = -2$ and $\tilde{q} = 1$ indicating the classifying space to be R_5 , which corresponds to class CII (Table I).

IX. SUMMARY

We have studied changes in classification of topological insulators and superconductors due to additional symmetries,

TABLE VIII. Clifford algebras and classifying spaces for (a) complex and (b) real classes (after Table 2 of Ref. 19). The last columns denote the zeroth homotopy group of each classifying space.

(a) Complex classes			
q	Cl_q	C_q	$\pi_0(C_q)$
0	\mathbb{C}	$[U(n+m)/U(n) \times U(m)] \times \mathbb{Z}$	\mathbb{Z}
1	$\mathbb{C} \oplus \mathbb{C}$	$U(n)$	0
(b) Real classes			
q	$Cl_{0,q}$	R_q	$\pi_0(R_q)$
0	\mathbb{R}	$[O(n+m)/O(n) \times O(m)] \times \mathbb{Z}$	\mathbb{Z}
1	$\mathbb{R} \oplus \mathbb{R}$	$O(n)$	\mathbb{Z}_2
2	$\mathbb{R}(2)$	$O(2n)/U(n)$	\mathbb{Z}_2
3	$\mathbb{C}(2)$	$U(2n)/Sp(n)$	0
4	$\mathbb{H}(2)$	$[Sp(n+m)/Sp(n) \times Sp(m)] \times \mathbb{Z}$	\mathbb{Z}
5	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$Sp(n)$	0
6	$\mathbb{H}(4)$	$Sp(n)/U(n)$	0
7	$\mathbb{C}(8)$	$U(n)/O(n)$	0

by considering extension problems of Clifford algebras generated from operators representing symmetry constraints. Our theory provides a simple and clear derivation of topological classification which agrees with the periodic table obtained by Chiu *et al.*,³¹ who studied topological invariants for topological insulators and superconductors with a reflection symmetry. We have also discussed several examples including a topological crystalline insulator characterized by mirror Chern numbers and mirror topological superconductors.

ACKNOWLEDGMENTS

This work was supported by Grants-in-Aid from the Japan Society for Promotion of Science (Grants No. 24840047 and No. 24540338) and by the RIKEN iTHES Project.

APPENDIX A: PROPERTIES OF THE CLIFFORD ALGEBRAS

We summarize some useful formulas of Clifford algebras used in this paper. We begin with complex Clifford algebras:

$$Cl_1 \simeq \mathbb{C} \oplus \mathbb{C}, \quad (\text{A1a})$$

$$Cl_2 \simeq \mathbb{C}(2), \quad (\text{A1b})$$

$$Cl_{n+2} \simeq Cl_n \otimes \mathbb{C}(2). \quad (\text{A1c})$$

The classifying space for the extension problem $Cl_n \rightarrow Cl_{n+1}$ is denoted by C_n . Since a fixed representation for 2×2 complex matrices $\mathbb{C}(2)$ does not affect the extension problem, Eq. (A1) leads to a periodic structure of the classifying space $C_{n+2} \simeq C_n$.

The real Clifford algebras have the following properties:³²

$$Cl_{0,1} \simeq \mathbb{R} \oplus \mathbb{R}, \quad (\text{A2a})$$

$$Cl_{0,2} \simeq \mathbb{R}(2), \quad (\text{A2b})$$

$$Cl_{1,0} \simeq \mathbb{C}, \quad (\text{A2c})$$

$$Cl_{2,0} \simeq \mathbb{H}, \quad (\text{A2d})$$

$$Cl_{p+1,q+1} \simeq Cl_{p,q} \otimes \mathbb{R}(2), \quad (\text{A2e})$$

$$Cl_{p,q} \otimes Cl_{0,2} \simeq Cl_{q,p+2}, \quad (\text{A2f})$$

$$Cl_{p,q} \otimes Cl_{2,0} \simeq Cl_{q+2,p}, \quad (\text{A2g})$$

$$Cl_{p,q} \otimes Cl_{0,4} \simeq Cl_{p,q+4}, \quad (\text{A2h})$$

where \mathbb{H} denotes the set of quaternions. The classifying space for the extension problem $Cl_{p,q} \rightarrow Cl_{p,q+1}$ is given by R_{q-p} . The extension problems $Cl_{p,q} \otimes \mathbb{R}(2) \rightarrow Cl_{p,q+1} \otimes \mathbb{R}(2)$ and $Cl_{p,q} \rightarrow Cl_{p,q+1}$ are equivalent, since the sector of 2×2 real matrices $\mathbb{R}(2)$ with fixed representation does not change the degrees of freedom of the extension. Furthermore, we have

$$Cl_{p+8,q} \simeq Cl_{p,q+8} \simeq Cl_{p,q} \otimes \mathbb{R}(16). \quad (\text{A3})$$

Since $\mathbb{R}(16)$ does not affect extension problems, the classifying space R_q has the periodic structure $R_{q+8} \simeq R_q$.

Clifford algebras and classifying spaces for complex and real classes are summarized in Table VIII.

APPENDIX B: DEFORMATION OF SECOND DESCENDANT \mathbb{Z}_2 INTO 0 UNDER R^- OR R^{--} SYMMETRY

In this appendix we show, first with an explicit example and second by applying Clifford algebras for general cases, that insulating states characterized with a nontrivial second descendant \mathbb{Z}_2 index [$=\pi_0(R_2)$] in the presence of a reflection symmetry R^{--} (or R^-) can be deformed into a topologically trivial state with a mass term that mixes Dirac cones at different momenta, when the translation symmetry is not assumed.

Let us begin with an example of a two-dimensional 8×8 Hamiltonian written as

$$H = \sigma_x \tau_z k_x + \sigma_y \tau_z k_y + \sum_i m_i g_i, \quad (\text{B1})$$

which we assume to be in class CII with the symmetry operators

$$T = i\sigma_y \rho_z \mathcal{K}, \quad C = i\sigma_y \tau_x \rho_z \mathcal{K}, \quad \Gamma = \tau_x. \quad (\text{B2})$$

The Dirac mass terms g_i 's, which anticommute with kinetic terms and are compatible with the generic symmetries in Eq. (B2), are given by

$$g_1 = \sigma_z \tau_z \rho_x, \quad g_2 = \tau_y \rho_x. \quad (\text{B3})$$

The presence of these two mutually anticommuting mass terms implies that the system is topologically trivial "0". Now we impose a reflection symmetry along the x direction,

$$R^{--} = \sigma_y \rho_z, \quad (\text{B4})$$

which anticommutes with T and C defined in Eq. (B2). Then the possible Dirac mass term is g_1 only, and the system is classified by a \mathbb{Z}_2 index.

So far we implicitly assumed translation symmetry. However, if we admit terms breaking the translation symmetry, we can find an extra mass term that anticommutes with g_1 in Eq. (B3) and turns a \mathbb{Z}_2 topological state into a trivial state, as we explain below. First, we add to the Hamiltonian a term $\delta\Delta = \delta\sigma_x \tau_z \rho_x$. Since it commutes with the kinetic term in the x direction, it causes splitting of the Dirac points as $k_x \rightarrow k_x \pm \delta$. We then add an extra mass term $m_2(x)\tau_y \rho_x$, which is reflection symmetric if $m_2(x)$ is an odd function of x .

Now the Hamiltonian reads

$$H = \sigma_x \tau_z (k_x + \delta \rho_x) + \sigma_y \tau_z k_y + m_1 \sigma_z \tau_z \rho_x + m_2(x) \tau_y \rho_x. \quad (\text{B5})$$

Let the Fourier component of $m_2(x)$ at $k_x = 2\delta$ be m_2^δ . Then rotations in the masses (m_1, m_2^δ) will connect any insulating state to trivial insulators. In particular, gapless edge states, formed along the boundary parallel to the x axis, will be gapped out by the m_2 term with a finite m_2^δ . Hence we have only a single (trivial) insulating phase.

The disappearance of the second descendant \mathbb{Z}_2 topological states can be also understood in terms of Clifford algebras. The deformation to a trivial state is possible when we have the two kinds of operators Δ and g_2 with the following properties:

(1) Δ commutes with γ_x , anticommutes with the other kinetic terms and the mass term (g_1), and is compatible with TRS, PHS, and R^{--} .

(2) g_2 anticommutes with g_1 , Δ , and the kinetic terms, and is compatible with TRS and PHS, but *not* with R^{--} .

When these two operators are available, we can split Dirac points along the k_x direction with Δ , and couple the split Dirac points by adding an extra mass term $m_2(x)g_2$ with the mass modulation $m_2(x)$ which is odd in x .

The existence condition for a Δ -type operator can be formulated as an extension problem of Clifford algebra. The Clifford algebra characterizing class CII in $d = 2$ with R^{--} is given by

$$\{e_0, e_1, e_2, e_3, J\gamma_x, J\gamma_y, \gamma_x R^{--}\}, \quad (\text{B6})$$

where e_i 's are defined in Eq. (19), and the extension $Cl_{6,0} \rightarrow Cl_{6,1}$ yields a classifying space R_2 . If we have an operator $\tilde{\Delta}$ that squares to -1 and anticommutes with all the generators in Eq. (B6), then we can define Δ as

$$\Delta = J\tilde{\Delta}R^{--}. \quad (\text{B7})$$

Thus the problem which we need to consider is whether the Clifford algebra $Cl_{5,1}$, generated by $\{e_0, e_1, e_2, e_3, J\gamma_y, \gamma_x R^{--}\}$, can be extended by introducing another generator squaring to -1 (i.e., $J\gamma_x$ or $\tilde{\Delta}$). The extension is $Cl_{5,1} \rightarrow Cl_{6,1}$ with a classifying space R_6 , whose topological index is "0". This ensures simultaneous existence of $J\gamma_x$ and $\tilde{\Delta}$ that anticommute with each other, hence, the existence of Δ . This argument can be applied to other spatial dimensions as well. When the original classification of a given second descendant " \mathbb{Z}_2 " state is characterized by $Cl_{p,q} \rightarrow Cl_{p,q+1}$ (corresponding classifying space, R_{q-p}), the extension problem regarding $\tilde{\Delta}$ is $Cl_{p-1,q+1} \rightarrow Cl_{p,q+1}$, whose classifying space is R_{p-q} . Since we have $q - p = 2 \pmod{8}$ for the second descendant \mathbb{Z}_2 states, we can always find a Δ operator because $\pi_0(R_{p-q}) = \pi_0(R_6) = 0$.

Next, we repeat a similar discussion to show the existence of a g_2 -type operator. If we have an operator \tilde{g}_2 that squares to -1 and anticommutes with all the generators in the Clifford algebra

$$\{e_0, e_1, e_2, e_3, J\gamma_x, J\gamma_y, \gamma_x R^{--}, -J\Delta R^{--}\}, \quad (\text{B8})$$

we can construct g_2 as

$$g_2 = J\Delta\gamma_x\tilde{g}_2. \quad (\text{B9})$$

The existence of \tilde{g}_2 is established by considering the extension problem $Cl_{p,q+1} \rightarrow Cl_{p+1,q+1}$, where we try to add

$\tilde{\Delta}$ to (B6). The topology of the associated classifying space $\pi_0(R_{p-q+1}) = \pi_0(R_7) = 0$ tells that we always have another generator anticommute with $\tilde{\Delta}$, namely, \tilde{g}_2 . Thus we find that a g_2 -type operator also exists. The existence of the two types of operators (g_2 and Δ) reduces the second descendant \mathbb{Z}_2 states to trivial "0" states.

A similar argument can be applied to the second descendant \mathbb{Z}_2 states under a R^- symmetry, and we conclude that " \mathbb{Z}_2 " in Table VI are changed into 0, when we include nonuniform terms such as $m_2(x)g_2$.

Finally, we discuss stability of the surface states of a " \mathbb{Z}_2 " nontrivial insulator with R^{--} or R^- symmetry. We have shown above that the surface states are gapless if $m_2^\delta = 0$. Now we ask what happens if we take $m_2(x)$ to be a random odd function of x . We argue below that the surface states remain critical, as long as random perturbation is spatially uniform on average (disorder average of m_2^δ vanishes). To this end, we consider classification of the g_2 -type mass term. Let us examine the extension problem of (B8) with \tilde{g}_2 . The extension is characterized with $Cl_{p+1,q+1} \rightarrow Cl_{p+2,q+1}$, whose classifying space turns out to be R_{p-q+2} . This means that the g_2 -type operator is classified with $\pi_0(R_0) = \mathbb{Z}$. For slow modulation of $m_2(x)$ we can imagine that the surface is divided into domains possessing various values of \mathbb{Z} , with gapless edge states running along domain boundaries and percolating through the surface. Thus we can expect that the gapless surface states of a " \mathbb{Z}_2 " topological insulator remain critical, when random perturbations are spatially uniform on average. This is similar to criticality of surface states of 3D weak topological insulators in the presence of random potential with zero mean.⁴¹⁻⁴⁴

APPENDIX C: STABILITY OF TOPOLOGICAL STATES WITH REFLECTION SYMMETRY

In Appendix B, we have shown that the second descendant \mathbb{Z}_2 states with R^{--} or R^- reflection symmetry (labeled with " \mathbb{Z}_2 " in Table VI) can be deformed into trivial states with a CDW-type mass term. Here we prove that, besides the second descendant " \mathbb{Z}_2 " states, there are no such deformations of topological states to trivial ones in Table VI.

For simplicity we concentrate on systems with both TRS and PHS, for which we have the Clifford algebra

$$\{e_0, e_1, e_2, e_3, J\gamma_1, J\gamma_2, \dots, J\gamma_d\}, \quad (\text{C1})$$

supplemented by either \tilde{e} or \tilde{M} in Table V related to the reflection symmetry $M = J\gamma_1 R$.

To deform a topologically nontrivial state into a trivial state by introducing a CDW-type mass perturbation, we need to have two terms Δ and g_2 satisfying the following conditions:

(1) Δ commutes with γ_1 , anticommutes with the other γ_i 's and the mass term (e_0), and is compatible with generic symmetries (TRS, PHS, and chiral) and the reflection symmetry.

(2) g_2 anticommutes with e_0 , Δ , and all the γ_i 's, and is compatible with generic symmetries, but *not* with the reflection symmetry.

In general, the existence of an operator e_+ (e_-) which squares to $+1$ (-1) and which can be used as a new generator to extend a Clifford algebra $Cl_{p,q}$ is judged by considering

an extension problem of an algebra with one generator fewer. Namely, we look at an extension problem $Cl_{p,q-1} \rightarrow Cl_{p,q}$ for e_+ ($Cl_{p-1,q} \rightarrow Cl_{p,q}$ for e_-). If topological classification of the associated classifying space is “0”, the existence of e_{\pm} is guaranteed. On the other hand, if the classification is \mathbb{Z} or \mathbb{Z}_2 , we cannot have e_{\pm} . We note that this last statement is valid under the assumption that we are dealing with minimal models where the Hamiltonian is already block diagonalized with respect to possible commuting unitary operators.⁴⁵

1. R^{--} case

We assume that the original classification for the mass term e_0 is given by the extension $Cl_{p,q} \rightarrow Cl_{p,q+1}$, where $Cl_{p,q+1}$ has generators

$$\{e_0, e_1, e_2, e_3, J\gamma_1, \dots, J\gamma_d, \gamma_1 R^{--}\}, \quad (C2)$$

whose classifying space is R_{q-p} . The question as to whether the Δ -type operator can be added to the algebra as

$$\{e_0, e_1, e_2, e_3, J\gamma_1, \dots, J\gamma_d, \gamma_1 R^{--}, J\Delta R^{--}\} \quad (C3)$$

is answered by examining the extension problem $Cl_{p-1,q+1} \rightarrow Cl_{p,q+1}$, because $(J\Delta R^{--})^2 = -1$. We find that the classifying space is R_{p-q} . As for a g_2 -type operator, the extended algebra with g_2 is written as

$$\{e_0, e_1, e_2, e_3, J\gamma_1, \dots, J\gamma_d, \gamma_1 R^{--}, J\Delta R^{--}, J\Delta\gamma_1 g_2\}, \quad (C4)$$

for which the extension is $Cl_{p,q+1} \rightarrow Cl_{p+1,q+1}$ because $(J\Delta\gamma_1 g_2)^2 = -1$. We then find that the associated classifying space is R_{p-q+1} . Thus the conditions for the existence of both Δ and g_2 are given by $\pi_0(R_{p-q}) = \pi_0(R_{p-q+1}) = 0$, which is met when $q - p = 2, 3$. At $q - p = 3$, the original classification is trivial. The case of $q - p = 2$ is exactly the second descendant \mathbb{Z}_2 marked in Table VI as “ \mathbb{Z}_2 ”.

So far we have discussed classes with both TRS and PHS. Symmetry classes with either TRS or PHS can be discussed in the same way. For classes with PHS and R^- , the above discussion is applicable if we drop e_3 from the algebras. For classes with TRS and R^- , while constructions of algebras are slightly different, the extension problems turn out to have the same structures and the resulting existence conditions become the same as the R^{--} case. Thus, we can conclude that the deformation only occurs for the second descendant “ \mathbb{Z}_2 ”, when we have reflection symmetries R^- or R^{--} .

2. R^{++} case

The original classification for the mass term e_0 is obtained from the extension $Cl_{p,q} \rightarrow Cl_{p,q+1}$ with the extended algebra $Cl_{p,q+1}$ generated by

$$\{e_0, e_1, e_2, e_3, J\gamma_1, \dots, J\gamma_d, J\gamma_1 R^{++}\}, \quad (C5)$$

whose classifying space is R_{q-p} . Let us examine whether the Δ -type operator can be added to the algebra as

$$\{e_0, e_1, e_2, e_3, J\gamma_1, \dots, J\gamma_d, J\gamma_1 R^{++}, \Delta R^{++}\}. \quad (C6)$$

The existence condition for Δ is found from the extension $Cl_{p,q} \rightarrow Cl_{p,q+1}$, for $(\Delta R^{++})^2 = +1$. We note that this is the same extension problem as that for the classification of e_0 . This implies that, when we have a topologically nontrivial

classification for e_0 , which is the case of our interest, we cannot have a Δ -type operator. Thus deformation of a topologically nontrivial state to a trivial one is impossible for the R^{++} reflection symmetry.

For classes with either TRS or PHS and with a reflection symmetry R^+ , we can repeat similar discussions to show that the deformation does not occur.

3. R^{-+} and R^{+-} case

The Clifford algebra with the reflection symmetry R^{-+} or R^{+-} is written as

$$\{e_0, e_1, e_2, e_3, J\gamma_1, \dots, J\gamma_d\} \otimes \{\tilde{M}\}, \quad (C7)$$

with \tilde{M} that squares to either $+1$ or -1 . \tilde{M} is given in Table V(b) with $M = J\gamma_1 R$, according to which \tilde{M} is either of $TC\gamma_1 R$ or $TCJ\gamma_1 R$ depending on the symmetry class and the type of R .

First let us consider the case where $\tilde{M}^2 = +1$. We look into the existence condition for Δ with the algebra

$$\{e_0, e_1, e_2, e_3, J\gamma_1, \dots, J\gamma_d\} \otimes \{\tilde{M}, J\gamma_1 \Delta \tilde{M}\}. \quad (C8)$$

Since $\tilde{M}^2 = (J\gamma_1 \Delta \tilde{M})^2 = +1$, the existence of Δ is determined from the extension $Cl_{p,q+1} \otimes Cl_{0,0} \rightarrow Cl_{p,q+1} \otimes Cl_{0,1}$. This corresponds to setting $\tilde{q} = 0$ in Table VII, and we find that the classification is always \mathbb{Z} (or $\mathbb{Z} \times \mathbb{Z}$). Thus, nontrivial states cannot be deformed to a trivial one due to the lack of a Δ -type operator.

When $\tilde{M}^2 = -1$, we consider extending an algebra with Δ and g_2 to

$$\{e_0, e_1, e_2, e_3, J\gamma_1, \dots, J\gamma_d, J\gamma_1 \Delta g_2\} \otimes \{\tilde{M}, J\gamma_1 \Delta\}. \quad (C9)$$

Since $\tilde{M}^2 = (J\gamma_1 \Delta)^2 = -1$, the existence of Δ is associated with the extension problem $Cl_{p,q+1} \otimes Cl_{0,0} \rightarrow Cl_{p,q+1} \otimes Cl_{1,0}$, or equivalently, $Cl_{p,q+1} \otimes Cl_{0,2} \rightarrow Cl_{p,q+1} \otimes Cl_{0,3}$. By setting $\tilde{q} = 2, m = q + 1 - p$ in Eq. (38) and Table VII, we find that Δ can exist when $q - p = 3, 4, 5$. Among these cases, only when $q - p = 4$, the original classification of the Dirac mass e_0 is nontrivial. We should then look at the existence condition for g_2 at $q - p = 4$. Since $(J\gamma_1 \Delta g_2)^2 = -1$, the existence of g_2 is associated with $Cl_{p-1,q+1} \otimes Cl_{2,0} \rightarrow Cl_{p,q+1} \otimes Cl_{2,0}$, or equivalently, $Cl_{q+3,p-1} \rightarrow Cl_{q+3,p}$, classification of which is $\pi_0(R_0) = \mathbb{Z}$ for $q - p = 4$, and g_2 does not exist.

Thus we conclude that no deformation of nontrivial states into trivial states can take place in the cases with R^{+-} and R^{-+} .

4. Complex cases

Finally we briefly discuss the absence of the deformation in the complex classes. Here we concentrate on the cases when addition of reflection symmetry changes the topological classification: class AIII with R^+ and class A. Let us consider the existence condition for a Δ -type operator which is added to the algebra as

$$\{e_0, (e_1,)\gamma_1, \dots, \gamma_d, \gamma_1 R, \Delta R\}, \quad (C10)$$

where $e_1 = \Gamma$ is present for class AIII but is absent for class A. Since there is no distinction between generators squaring

to +1 and those to -1 in the complex Clifford algebra, we see that the extension problems for classifying e_0 and for the existence of Δ are the same. Therefore, if the classification of e_0 is nontrivial, which is the case of our interest, then Δ -type operators do not exist, and no deformation into a trivial state occurs.

APPENDIX D: CLASSIFICATION FOR MULTIPLE SYMMETRIES

We briefly explain how to derive classification of time-reversal invariant topological superconductors in the presence of multiple additional symmetries. For systems with both time-reversal and particle-hole symmetries, the extension problem is given as Eq. (37), which is equivalent to

$$Cl_{0,\bar{q}} \otimes Cl_{0,m} \rightarrow Cl_{0,\bar{q}+1} \otimes Cl_{0,m}. \quad (\text{D1})$$

This equivalence can be understood by using Eq. (A2e) several times.

For each value of m , we make use of the following relations:

$$Cl_{0,\bar{q}} \otimes Cl_{0,1} \simeq Cl_{0,\bar{q}} \oplus Cl_{0,\bar{q}}, \quad (\text{D2a})$$

$$Cl_{0,\bar{q}} \otimes Cl_{0,2} \simeq Cl_{0,\bar{q}} \otimes \mathbb{R}(2), \quad (\text{D2b})$$

$$Cl_{0,\bar{q}} \otimes Cl_{0,3} \simeq Cl_{0,\bar{q}} \otimes Cl_{1,0} \otimes Cl_{0,2} \simeq Cl_{\bar{q}} \otimes \mathbb{R}(2), \quad (\text{D2c})$$

$$Cl_{0,\bar{q}} \otimes Cl_{0,4} \simeq Cl_{0,\bar{q}+4}, \quad (\text{D2d})$$

$$Cl_{0,\bar{q}} \otimes Cl_{0,5} \simeq Cl_{0,\bar{q}+4} \otimes Cl_{0,1} \simeq Cl_{0,\bar{q}+4} \oplus Cl_{0,\bar{q}+4}, \quad (\text{D2e})$$

$$Cl_{0,\bar{q}} \otimes Cl_{0,6} \simeq Cl_{0,\bar{q}+4} \otimes Cl_{0,2} \simeq Cl_{0,\bar{q}+4} \otimes \mathbb{R}(2), \quad (\text{D2f})$$

$$Cl_{0,\bar{q}} \otimes Cl_{0,7} \simeq Cl_{0,\bar{q}+4} \otimes Cl_{0,3} \simeq Cl_{\bar{q}+4} \otimes \mathbb{R}(2), \quad (\text{D2g})$$

which can be derived using Eqs. (A2). Having in mind that $Cl_{0,q} \rightarrow Cl_{0,q+1}$ is classified with R_q and that $Cl_q \rightarrow Cl_{q+1}$ is classified with C_q , we obtain the classifying space listed in the last column in Table VII.

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⁴⁵For nonminimal (or reducible) models, we have the following counterexample. Let us assume that operators e_1, \dots, e_{p+q-1} in a given nonminimal Dirac model form a Clifford algebra $Cl_{p,q-1}$. We further assume that the model has three additional operators g_1, g_2, g_3 which anticommute with all e_i 's and obey the following conditions: $[g_1, g_2] = [g_1, g_3] = \{g_2, g_3\} = 0$ and $g_1^2 = g_2^2 = g_3^2 =$

$+1$. In this case the product $g_1 g_2$ is a constant of motion, and the Hamiltonian can be block diagonalized and therefore is nonminimal. We can extend the Clifford algebra $Cl_{p,q-1}$ to $Cl_{p,q}$ by adding g_1 as a unique allowed generator, and topology of the associated classifying space is either \mathbb{Z} or \mathbb{Z}_2 . However, it is also possible to extend $Cl_{p,q}$ generated by $\{e_1, \dots, e_{p+q-1}, g_2\}$, to $Cl_{p,q+1}$ generated by $\{e_1, \dots, e_{p+q-1}, g_2, g_3\}$. This situation does not happen for a (fully diagonalized) minimal model, since we cannot have operators such as g_2 or g_3 which leads to a block diagonalization with respect to $g_1 g_2$ or $g_1 g_3$.