$\mathbb{Z}_2$ anomaly and boundaries of topological insulators

Zohar Ringel and Ady Stern

Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel

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We study the edge and surface theories of topological insulators from the perspective of anomalies and identify a $\mathbb{Z}_2$ anomaly associated with charge conservation. The anomaly is manifested through a two-point correlation function involving creation and annihilation operators on two decoupled boundaries. Although charge conservation on each boundary requires this quantity to vanish, we find that it diverges. A corollary result is that under an insertion of a flux quantum, the ground state evolves to an exactly orthogonal state independent of the rate at which the flux is inserted. The anomaly persists in the presence of disorder and imposes sharp restrictions on possible low-energy theories. Being formulated in a many-body, field-theoretical language, the anomaly allows one to test the robustness of topological insulators to interactions in a concise way.

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I. INTRODUCTION

Topological insulators have attracted much attention in recent years due to their novel bulk and surface properties. The bulk of these materials is insulating and characterized by topological indices which measure certain twists in the band structure. The topological properties of the bulk imply, via the bulk-edge correspondence, that the surfaces of these materials are necessarily metallic and support gapless excitations. The theories that emerge on the $(d-1)$-dimensional surfaces of a d-dimensional topological insulator may be understood as "fractions" of the theories of "stand-alone" $(d-1)$-dimensional systems.[1,2] For example, the edge of an integer quantum Hall effect (IQHE) at filling factor one, arguably the simplest and most striking topological state of matter, may be thought of as half a spinless one-dimensional wire.

The surface theories of topological phases have several important properties, which are clearly demonstrated for the case of the IQHE. Although IQHE edges border between two insulating gapped phases, they appear to violate a conservation law of the entire system, namely, charge conservation. Indeed, the low-energy theory of each edge is invariant under the U(1) charge symmetry of the bulk. However, the application of an electric field parallel to the edges of a Hall bar leads to current flow across the bar. Consequently, charges are exchanged between the edges and charge conservation, per edge, is violated. This property is known as an anomaly and will occupy an important part of our discussion. In this case, it is sometimes referred to as the Schwinger anomaly[3] or a one-dimensional (1D) chiral anomaly.[4]

While, technically, anomalies appear as a subtle cutoff scale effect, they are in fact present at all energy scales and have quite direct consequences.[5] For example, the above Schwinger anomaly fixes the commutation relations between the density operators on the edge[6] to a quantized nonzero value, while naively one may think that density operators at different points on the edge commute. Ignoring the anomaly in this case, one may arrive at the wrong impression that the edge conductance vanishes, as if the edge was insulating, instead of being e^2/h.

The identification of the anomaly associated with an edge or surface of a topological state is particularly easy for the case of the IQHE, where charge conservation is being violated, or for the spin quantum Hall effect (QSHE), where spin conservation is violated. Generalizations to other topological phases have been explored;[7,8] however, no anomaly was found to be associated with $\mathbb{Z}_2$ topological insulators (TI) in 2D or 3D. In this respect, it is important to note an anomalous behavior, mathematically similar to the SU(2) global gauge anomaly,[9] associated with the $O(2N)/O(N) \times O(N)$ symmetry of the disorder-averaged action in the replica formalism.[10] However, since the symmetry is not a gauge symmetry, this anomalous behavior does not imply a symmetry violation or an inconsistency in the theory. Thus, it remained unclear whether any symmetry becomes anomalous in $\mathbb{Z}_2$ topological insulators.

In this work, we study topological insulators from the perspective of anomalies and identify a $\mathbb{Z}_2$ anomaly that is associated with charge conservation on the boundary. The anomaly unifies the various aspects of topological insulators in a concise way through a field-theoretical language and allows us to calculate further topological properties of TIs. To formulate the problem, we consider the quantum action of two decoupled boundary theories, corresponding to two distinct edges or surfaces of a TI, and consider the cutoff scale as a bulk gap. We then dynamically insert a flux quantum so that an electric field along the boundaries is generated. Although the two boundary theories remain formally decoupled, we find that certain quantities are exchanged via the anomaly. This is manifested in two ways. First, regardless of how adiabatically the flux insertion is carried out, the final state is always an excited states which is exactly orthogonal to the ground state. Second, even without any direct coupling between the edges, certain two-point correlation functions involving creation and annihilation operators on different edges diverge rather than vanish.

II. THE INTEGER QUANTUM HALL EFFECT AND THE CHIRAL ANOMALY

We begin by reformulating some well-known results of the IQHE in the language of chiral anomalies.[5,11] This formulation will be generalized to TIs in Sec. III. Consider an IQHE bar, at filling factor 1, which is periodic in the x direction and finite in the y direction, i.e., an annulus. The topological nature of the bulk is manifested by two chiral edge states with opposite velocity, which appear on the two disconnected edges. The basic Hamiltonian describing each edge is simply

$$H_{\text{edge}} = v_0(i\hbar\partial_x - eA_x) + \mu, \tag{1}$$

where μ is the chemical potential and we allow coupling to a gauge field (A_x). Below we study in detail the case where the edges are identical, except they have opposite velocities. Generalizations follow readily from the topological nature of our results.

The spectrum of H_{edge} performs a spectral flow as a function of flux (A_x) so that the states at $A_x = 0$ are exactly those at $A_x = h/(eL)$ displaced by a single state, where L is the edge circumference. Thus, as one inserts a flux quantum adiabatically, exactly one electron is transferred between the edges even though H_{edge}, on each edge, has a U(1) charge symmetry. The spectral flow therefore implies the aforementioned charge anomaly of the IQHE edge.

We find it convenient to introduce a formalism that treats both edges simultaneously, and express this anomaly as a 2D chiral anomaly. To this end, consider the action of *both* of the edges following a multiplication of $\bar{\psi}$ by $i\sigma_x$,

$$S = \textstyle\int dx d\tau \bar{\psi}_\sigma [\hat{\mathrm{S}}_{ch}]_{\sigma,\sigma'} \psi_{\sigma'}, \tag{2}$$

$$\hat{\mathrm{S}}_{ch} = (\alpha i\hbar\partial_\tau + i\mu)\sigma_x + v_0\sigma_y(i\hbar\partial_x - eA_x) \equiv \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix}, \tag{3}$$

where the σ spinor is associated with the edge index, $\bar{\psi},\psi$ are Grassman variables, and $\alpha = 1(i)$ for Euclidean (real-time) action. For simplicity, we choose A_x to be independent of x. Furthermore, it couples symmetrically to the edges since we do not allow time-dependent magnetic fields in the bulk. The action operator $\hat{\mathrm{S}}_{ch}$ acts in an extended Hilbert space which includes the extra edge index and the time coordinate. Notably, the U(1) symmetry associated with charge difference between the edges is now reflected by the fact that $\{\hat{\mathrm{S}}_{ch},\sigma_z\} = 0$, which from now on we refer to as the chiral symmetry.

Chiral symmetries are often anomalous[3] and the former is no exception. Using Noether's procedure, one finds that the chiral current is

$$\vec{j}_{ch} = ev_0\bar{\psi}\sigma_z\vec{\sigma}\psi, \tag{4}$$

where $\vec{\sigma} = (\sigma_x,\sigma_y)$. A naive application of Noether's procedure will imply that the divergence of this current in space-time $\langle\partial_\mu j_{ch,\mu}\rangle$, with $\mu = x,\tau$, vanishes. This, however, is not necessarily the case in the presence of gauge fields due to changes in the path-integral measure.[5,11] Instead, one finds the following fundamental relation of the chiral anomaly:[5,12]

$$\int d\tau dx \langle\vec{\nabla}\vec{j}_{ch}\rangle = Spectral\ flow = \frac{e}{h}\int d\tau dx F, \tag{5}$$

where $F_{\mu\nu} = \partial_t A_x$, A_x is periodic, up to gauge transformations, between $-\infty$ and ∞ (real time) or $-\beta/2$ to $\beta/2$ (Euclidean time). By $Spectral\ flow$, we denote the integer number characterizing the displacement of the spectrum of $\mathrm{H}_{\text{edge}}[A_x(\tau)]$ between $\tau = -\infty$ and $\tau = \infty$. For example, $Spectral\ flow = 1\,(-1)$ implies that when following an eigenvalue of $\mathrm{H}_{\text{edge}}[A_x(\tau = -\infty)]$ up to $\tau = \infty$, one ends up with the upper (lower) consecutive eigenvalue of the initial one.

Equation (5) clearly reflects the physics of the IQHE: The charge transferred between the edges is related, in a quantized fashion, to the flux inserted into the annulus. This relation is easy to generalize for the spin quantum Hall effect, in which the chiral charge current (4) is replaced by a chiral spin current. However, for a generic 2D TI, there does not seem to be any local conservation law which is violated by an anomaly.[7]

Chiral anomalies are also accompanied by the appearance of action zero modes, which do generalize to TIs and have important physical consequences. The zero modes of the current model can be understood by viewing $\hat{\mathrm{S}}_{ch}$ as a Dirac Hamiltonian on a 2D torus [spanned by (x,τ)] with a uniform magnetic field. In this system, a zero-energy Landau level appears with a degeneracy equal to the number of magnetic monopoles.[13] This relation turns out to generalize to any chiral operator coupled to a gauge field via the Atiyah-Singer (AS) and Atiyah-Patodi-Singer (APS) theorems,[5]

$$\nu = Spectral\ flow = \frac{e}{h}\int d\tau dx F, \tag{6}$$

where the analytic index ν is given by $\nu = \dim\ker[D] - \dim\ker[D^\dagger]$ and $\dim\ker[D]$ ($\dim\ker[D^\dagger]$) is the dimension of the zero-mode space of D ($D^\dagger$). Note that a zero mode of D ($D^\dagger$) is also a zero mode of the action ($\hat{\mathcal{S}}_{ch}$) with $\sigma_z = 1\,(-1)$, as can be verified explicitly from Eq. (3). Generalizations of all of our results to the $\mu \neq 0$ case (non-Hermitian action) are given in Appendix A.

III. TOPOLOGICAL INSULATORS AND A $\mathbb{Z}_2$ CHIRAL ANOMALY

Here we generalize the correspondence between the IQHE and the chiral anomaly to a correspondence between TIs and a particular $\mathbb{Z}_2$ chiral anomaly. We begin with generalizing Eq. (6). Consider a 2D TI cylinder with two distinct edges and again write the action for the two edges after performing the chirality transformation ($\bar{\psi} \to i\sigma_x\bar{\psi}$). This leads to a chiral action of the form

$$\hat{\mathcal{S}}_{ch} = (\alpha i\hbar\partial_\tau + i\mu)\sigma_x + \sigma_y\mathcal{H}_{\text{edge}}[A_x] = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix}, \tag{7}$$

where $\mathcal{H}_{\text{edge}}$ denotes the low-energy Hamiltonian of a single TI edge, which we keep general. Provided that $A_x(\tau) = -A_x(-\tau)$, the action is invariant under time-reversal symmetry (TRS),

$$T\hat{\mathcal{S}}_{ch}T^{-1} = (is_y\sigma_x P_\tau)[\hat{\mathcal{S}}_{ch}]^T(is_y\sigma_x P_\tau)^T, \tag{8}$$

where $s_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ is the Pauli matrix acting on the electron spin and P_τ flips the sign of τ. Since $T^2 = -1$, Kramer's theorem holds and each action eigenvalue is doubly degenerate. This again remains true also for the non-Hermitian case in the sense that each Jordan block has an even dimension (see Appendix A).

The presence of Kramer's theorem for the action allows us to define a $\mathbb{Z}_2$ analytical index similar to ν. To see this, let us first assume $\mathcal{H}_{\text{edge}}$ respects s_z symmetry and decouple the system into spin-up and spin-down components. When there is a single pair of zero modes, the two modes will have opposite s_z and σ_z (see definition of T). If we now break this spin symmetry, the zero modes and every other mode must remain degenerate due to TRS. The combination of the pairwise degeneracy of all modes along with the antisymmetry of the spectrum around zero energy (implied by the chiral symmetry) allows the degeneracy of zero modes to change only in groups of four. This provides for the following topological index:

$$\nu_2 = \dim\ker[D]\mathrm{mod}2 = \dim\ker[D^\dagger]\mathrm{mod}2. \quad (9)$$

The TIs have an analog to the spectral flow of the IQHE edges in the form of "pair switching."[14] The "pair" refers here to the Kramer's pairs of edge states that characterize the spectrum at values of the flux that are time-reversal symmetric, namely, zero and one-half flux quanta. The term "switching" refers to the fact that states change their Kramer partners between zero flux and half a flux quantum. As we show below, this pair-switching behavior is related to the ν_2 index of the action via

$$\nu_2 = Pair\ switching. \quad (10)$$

The right-hand side (rhs) is equal to 1 (0) if the spectrum of $\mathcal{H}_{\text{edge}}[A_x]$ performs (does not perform) pair switching as a function of $A_x(\tau)$. This generalization of the APS theorem relates the spectral motion characteristic of TI boundaries with action zero modes in the presence of electric fields.

To prove the relation in Eq. (10), let us study the evolution of the action zero modes with τ for $A_x(\tau) = \frac{h\tau}{eL\beta}$. If a zero mode, $[\varphi_0(r,\tau),0]^T$, with $\sigma_z = +1$ exists, the following equation must be satisfied:

$$iD\varphi_0 = \left[\alpha\partial_\tau + \mathcal{H}_{\text{edge}}[A_x(\tau)] - \mu\right]\varphi_0 = 0, \quad (11)$$

with the boundary conditions

$$\varphi_0(x,\tau = -\beta/2) = -\varphi_0(x,\tau = \beta/2)e^{-2\pi i x/L}. \quad (12)$$

To be concrete, let us work with $\alpha = 1$ (Euclidean action) and take an adiabatic limit so that $\beta \to \infty$ (the topological protection of the zero modes will later allow us to change these parameters). Provided that β is large enough, the flux insertion can be viewed as the adiabatic evolution of φ_0 in imaginary time. While it is not obvious that one may still use the adiabatic theorem in imaginary time, this turns out to be true under some limitations.[15] The (imaginary) adiabatic theorem then implies

$$\varphi_0(x,\tau) = \sum_n c_n e^{-\int d\tau(E_n(\tau)-\mu)}\psi_n(x;\tau),$$
$$\mathcal{H}_{\text{edge}}[A_x(\tau)]\psi_n(x;\tau) = [E_n(\tau) - \mu]\psi_n(x;\tau). \quad (13)$$

To relate Eq. (13) with Eq. (10), two observations are important. First, pair switching and TRS imply that there are an odd number of time points, denoted by τ_n, for which a state E_m exists such that $E_m(\tau_n) = \mu$ and $\partial_\tau E_m(\tau_n)$ is positive. Second, upon choosing $c_n = \delta_{nm}$, for each such point the adiabatic evolution yields

$$\varphi_0(x,\tau) = e^{-\partial_\tau E_m(\tau_n)(\tau-\tau_n)^2}\psi_m(x;\tau) + O(e^{-\beta\Delta}), \quad (14)$$

where Δ is the level spacing between two consecutive states of $\mathcal{H}_{\text{edge}}$. In the adiabatic limit, such states are localized near τ_n and trivially satisfy the boundary conditions in Eq. (12). Therefore, an odd number of solutions to Eq. (11) exist. Looking for zero modes with $\sigma_z = -1$, one obtains a variant of Eq. (11) with the sign of $\mathcal{H}_{\text{edge}}$ switched. Repeating the above analysis yields another odd set of zero modes. According to Eq. (9), this implies $\nu_2 = 1$.

In contrast, without pair switching, the edge spectrum $[E_m(\tau)]$ crosses μ, with a positive slope, an even number of times. Consequently, in the adiabatic limit, the $e^{-\int d\tau[E_m(\tau)-\mu]}$ factor in Eq. (13) guarantees an even number of zero modes for either D or $D^\dagger$ yielding $\nu_2 = 0$ according to Eq. (9).

IV. PHYSICAL IMPLICATIONS OF THE $\mathbb{Z}_2$ CHIRAL ANOMALY

So far we have established the stability of zero modes and their relation to pair switching. In this section, we wish to point out the physical consequence of these mathematical observations. To this end, consider a TI in its ground state on an annulus, in which the flux threading the hole is adiabatically varied from $-\Phi_0/2$ to $\Phi_0/2$ ($\Phi_0 = h/e$). We will now show that as long as the rate at which the flux is turned on is much smaller than the cutoff, the TI will end up being in a state that is exactly orthogonal to the ground state. Let us denote the ground state with $-\Phi_0/2$ by $|gs\rangle$ and, consequently, the ground state at $\Phi_0/2$ is $G|gs\rangle$, where $G = \exp^{i\frac{2\pi}{L}\int dx x\psi^+(x)\psi(x)}$. The state to which the system evolves after the flux is inserted is $U|gs\rangle$, with U being the time evolution operator. We examine the overlap $\langle gs|G^+U|gs\rangle$ which, as shown in Appendix B, can be written as the following path integral:

$$\langle gs|G^\dagger U|gs\rangle = Z,$$
$$Z = \int D[\bar\psi\psi]e^{-S[\bar\psi,\psi]}, \quad (15)$$
$$S = \int dtdx\bar\psi\{\hat{\mathcal{S}}_{ch}[\alpha(t)]\}\psi + \int dt\alpha(t)E_{gs},$$

where E_{gs} is the many-body ground-state energy, the limit of $\beta \gg \Delta^{-1}$ is assumed, and we take $\alpha(t) = i$ for the period of the flux insertion ($t \in [-\Delta t,\Delta t]$) and $\alpha = 1$ for $t \in [-\beta/2,\Delta t],[\Delta t,\beta/2]$. As shown in Appendix B, the boundary conditions are again those in Eq. (12).

The path-integral expression can be formally evaluated using the action eigenvalues to yield

$$\langle gs|G^\dagger U|gs\rangle = \mathrm{Det}[\hat{\mathcal{S}}_{ch}] = \Pi_n\beta\lambda_n, \quad (16)$$

where λ_n are the eigenvalues (or the Jordan eigenvalues) of $\hat{\mathcal{S}}_{ch}$. Clearly, in the presence of action zero modes, this overlap vanishes. In the absence of zero modes and provided that the flux insertion rate is sufficiently small, the adiabatic theorem implies that this overlap is one, up to phases. Thus the overlap serves as a sharp distinction between a topological and trivial insulator, formulated in a many-body language. Note also that while coincidental level crossings may appear in any system, they will require fine tuning. In contrast, the above relation is topologically robust. It also holds for finite temperatures as long as it is much smaller than the cutoff (bulk gap). A rigorous derivation of this result, which takes into account regularization issues, appears in Appendix C.

For the QSHE, the vanishing overlap follows from the change in spin on each of the edges, following the insertion of flux. For a generic TI, this behavior is much less obvious.

Nonetheless, in an extreme adiabatic limit, in which the flux insertion rate is smaller than Δ, it may be understood by elementary means. Consider the spectrum of a TI edge and how it evolves with flux as it increases from $-\Phi_0/2$ to $\Phi_0/2$. At $\Phi_0/2$, all the states below the chemical potential are occupied. Following the pair-switching motion with flux, one finds that the ground state is degenerate at zero flux. This crossing of many-body states is not avoided due to TRS. Therefore, increasing the flux to $\Phi_0/2$ yields an excited state, orthogonal to the original ground state.

Last, let us examine the effect zero modes have on the chiral symmetry which, at least naively, implies the conservation of charge difference between the edges. Consider the interedge two-point correlation function, $G_{\text{hop}} = \langle \int dx dt \bar{\psi}\sigma_0 s_0 \psi\rangle$, where the average is taken with the chiral action. Since G_{hop} switches sign following a π chiral rotation, it should seemingly vanish. However, a direct calculation yields a different result. To show this, we include a source term $\hat{\mathcal{S}}_{ch}(m) = \hat{\mathcal{S}}_{ch} - im\bar{\psi}\sigma_0 s_0\psi$, which physically amounts to coupling the charges on both edges, and find that

$$G_{\text{hop}}(m) \equiv \partial_m \ln\left[\int D[\bar{\psi}\psi]e^{-\int dt dx \bar{\psi}\hat{\mathcal{S}}_{ch}(m)\psi}\right]$$
$$= \partial_m \ln(\Pi_n \beta[\lambda_n - im]) = \frac{2\nu_2}{m} + O(m^0, m^1, \ldots). \tag{17}$$

Note that the function $G_{\text{hop}}(m)$ is evaluated here in the limit of $m \to 0$ with the size of the system and β held fixed.[16] This is a physically relevant limit, since the interedge coupling decreases exponentially with L, while the level spacing on each edge decreases only algebraically.

Interestingly, in the presence of zero modes, this two-point function diverges, rather than vanishes, as the edges become more and more decoupled. For this reason, we say that chiral symmetry is anomalous here. Furthermore, the appearance of ν_2 as the coefficient of the $1/m$ pole shows that this is indeed a $\mathbb{Z}_2$ anomaly [see Eq. (9)] so that coupling two anomalous systems yields a nonanomalous system.

In quantum-mechanical language, the anomalous correlation function can be expressed as a diverging weak value.[17] To this end, we note that following standard relations between coherent-state path integrals and operators formalism, one finds

$$G_{\text{hop}} = \int_{-\Delta t}^{\Delta t} dt \frac{\langle Ggs(t)|[\psi^{\dagger}_{\text{top}}\psi_{\text{bottom}} + \text{H.c.}]|gs(t)\rangle}{\langle Ggs(t)|gs(t)\rangle}, \tag{18}$$

where $|gs(t)\rangle = U^t_{-\Delta t}|gs\rangle$, $|Ggs(t)\rangle = [U^{\Delta t}_t]^{-1}G|gs\rangle$, and U^b_a is the unitary evolution between $t = a$ and $t = b$. Thus, G_{hop} appears as the time-averaged weak measurement of the interedge charge-tunneling operator. Physically, this means that if one measures the charge-tunneling operator using a weakly coupled detector while selecting only the events in which the ground state came back to itself after the flux insertion, one will obtain a diverging value rather than a zero value.

V. DISCUSSION

In this work, we have established the existence of an anomaly associated with charge conservation and TRS. The anomaly captures the topological nature of a TI edge in a field-theory language. Physically, it amounts to the statement that the ground state of a TI on a cylinder evolves to an orthogonal state following the insertion of a full flux quantum through the cylinder. Interestingly, the flux insertion rate should be adiabatic only with respect to the bulk gap.

Although this work focused mainly on 2D TIs, generalizations of this anomaly to weak and strong TIs readily follow: Weak TIs can be understood as layers of 2D TIs and, consequently, following its $\mathbb{Z}_2$ nature, the anomaly will either be absent or present depending on the number of layers. For strong TIs, pair switching on the surface depends on two fluxes, namely, one flux induces pair switching conditional on the value of the other flux.[18] Consequently, the anomaly, which follows directly from the pair-switching behavior, will be present or absent depending on whether this other flux is 0 or $\Phi_0/2$.

The $\mathbb{Z}_2$ chiral anomaly may facilitate our understanding of the topological insulator in the presence of disorder and interactions. The anomaly, which persists also in the disordered case, guarantees a nonlocalized phase on the boundary of TIs. This is because anomalous theories cannot be massive or have only short-range correlations at low energy.[19] The anomaly should also be present in the nonlinear-σ-model (NLSM) descriptions of the disordered surfaces[10,20,21] and may, through the anomaly matching concept, allow further investigation of these theories. Generalizations of TIs to the interacting regime have so far used the single-particle Green's function.[22,23] Within this approach, the presence of Green's-function zero modes can sometimes blur the physical distinction between topological and trivial noninteracting phases.[23] In contrast, our criterion [Eq. (17)] can be used as a sharp many-body distinction between interacting topological insulators and band insulators.[24] Thus, provided that the $\mathbb{Z}_2$ anomaly survives interactions, it may further establish the robustness of TI boundaries to disorder and interactions.

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APPENDIX A: THE ANOMALY IN THE NON-HERMITIAN CASE

The chiral Euclidean action is a Hermitian operator only for $\mu = 0$ and, more generally, only when edge symmetric terms are absent. Here we show that the results derived in the main text persist also when such terms are included. We first define the analytic index (ν) for a non-Hermitian action and prove its stability. Next, a variant of Kramer's theorem is proven and used for showing the robustness of the $\mathbb{Z}_2$ analytic index (ν_2).

A general chiral action is given by

$$S = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}, \tag{A1}$$

where D_+ and D_- are not necessarily Hermitian conjugates. The action, when viewed as a matrix, might be nondiagonalizable so that no spanning basis of eigenvectors exists, not even a nonorthogonal one. Nonetheless, a weaker statement holds which is that any matrix can be brought to a Jordan form ($\tilde{S}$) following a similarity transformation ($\tilde{S} = PSP^{-1}$). In its Jordan form,[25] the action becomes block diagonal so that each block (J_n) has a single Jordan eigenvalue (λ_n) on its diagonal, and either 1 or 0 on the upper off-diagonal. All other entries of $\tilde{S}$ are zero. Alternatively stated, a set of Jordan vectors exists ($v_{n,i}$) upon which the matrix $(S - I\lambda_n)$ is nilpotent so that $(A - I\lambda_n)^{N_n} v_{n,i} = 0$, where $N_n \leqslant \dim[J_n]$.

We extend the definition of ν to the non-Hermitian as follows:

$$\nu = \dim[J_0(D_-D_+)] - \dim[J_0(D_+D_-)], \tag{A2}$$

where $J_0(A)$ denotes the Jordan block associated with a zero Jordan eigenvalue of the matrix A. Let us show that for the Hermitian case, the above definition is equivalent to $\nu = \dim\ker[D^\dagger] - \dim\ker[D]$, used in the main text. Note that for a Hermitian operator, $\dim\left[J_0(DD^\dagger)\right] = \dim\ker[DD^\dagger]$, and similarly for the Hermitian conjugate operators. Next, note that if a vector v is in $\ker[D]$ ($\ker[D^\dagger]$), then it must also be in $\ker[D^\dagger D]$ ($\ker[DD^\dagger]$). Furthermore, if v is in the kernel of $\ker[D^\dagger D]$ ($\ker[DD^\dagger]$), it must also be in $\ker[D]$ ($\ker[D^\dagger]$), since $D^\dagger Dv = 0$ implies $v^\dagger D^\dagger Dv = |Dv|^2 = 0$ and therefore $Dv = 0$. Consequently, $\ker[D] = \ker[D^\dagger D]$ and $\ker[D^\dagger] = \ker[DD^\dagger]$, and the two definitions of ν, for the Hermitian case, are clearly equivalent.

In the Hermitian case, each vector in $\ker[D]$ ($\ker[D^\dagger]$) can also be used to build a zero mode of S with $\sigma_z = 1$ ($\sigma_z = -1$), and thus the second definition of ν relates better to the zero modes of S. For the non-Hermitian case, the following weaker relation holds between ν and zero modes of S:

$$\dim[J_0(S)] \geqslant |\nu|. \tag{A3}$$

To prove this last statement, note that

$$S^2 = \begin{pmatrix} D_-D_+ & 0 \\ 0 & D_+D_- \end{pmatrix}, \tag{A4}$$

and therefore $\dim\left[J_0(S^2)\right] \geqslant |\nu|$ and, using Jordan form, we have that $\dim\left[J_0(S^2)\right] = \dim[J_0(S)]$.

Next we show that ν, as defined above, is a stable number. Let $v_{n,i}$ and λ_n denote the Jordan vectors and eigenvalues of D_+D_-. Acting with D_- on $v_{n,i}$ gives us Jordan vectors of D_-D_+, as one can directly show. Furthermore, D_- generates a one-to-one map between the Jordan vectors of these two operators for which $\lambda_n \neq 0$. Indeed, if there exists some linear combination v_n, of $v_{n,i}$ such that $D_-v_n = 0$, then also $D_+D_-v_n = 0$, implying $\lambda_n = 0$. Consider now a small perturbation which increases $\dim[J_0(D_+D_-)]$. This implies that a Jordan block with some small λ_n appears for D_+D_- and must be matched, through the mapping, to an equal-size Jordan block appearing in D_-D_+. This later block can only appear from an equal change of $\dim[J_0(D_-D_+)]$ and, consequently, ν is stable to any perturbation.

Next we wish to show that a variant of Kramer's theorem holds also in the non-Hermitian case. Consider an action which obeys a fermionic TRS symmetry,

$$O^T S O = S^T, \tag{A5}$$

$$O = -O^T, \tag{A6}$$

$$OO^T = 1. \tag{A7}$$

Note that for a Hermitian S and $O = is_y$, the above condition coincides with the usual demand of TRS, namely, $TST^{-1} = S$ where $T = Kis_y$ and K is complex conjugation. We claim that for such a matrix S, each Jordan block has an even dimension. To prove this, let us study how this symmetry acts on the Jordan form. Note that

$$S^T = [P^{-1}\tilde{S}P]^T = P^T\tilde{S}^T[P^T]^{-1} = O^T S O, \tag{A8}$$

and by acting with $[P^{-1}]^T$ and P^T on the two sides of this last equality, one finds

$$\tilde{S}^T = [P^T]^{-1}O^T S O P^T = [P^T]^{-1}O^T P^{-1}\tilde{S}POP^T. \tag{A9}$$

Denoting $\tilde{O} = POP^T$, one has that

$$\tilde{S}^T = \tilde{O}^{-1}\tilde{S}\tilde{O}, \tag{A10}$$

$$\tilde{O}^T = -\tilde{O}. \tag{A11}$$

Next we show that $\tilde{O}$ is block diagonal on the basis, $\hat{e}_{n,i}$, on which $\tilde{S}$ and $\tilde{S}^T$ are block diagonal. For each λ_n block of size N of $\hat{S}$, one can use Eq. (A10) to show that

$$\tilde{O}(\tilde{S}^T - \lambda_n)^N = (\tilde{S} - \lambda_n)^N\tilde{O}. \tag{A12}$$

Acting with $\hat{e}^T_{n,i}$ from the left and using $\hat{e}^T_{n,i}(\tilde{S} - \lambda_n)^N = 0$, one finds

$$\hat{e}^T_{n,i}\tilde{O}(\tilde{S}^T - \lambda_n)^N w = 0, \tag{A13}$$

for any vector w. Using the fact that $\tilde{S}^T - \lambda_n$ is invertible when constrained to the subspace of $\hat{e}_{m,j}$ with $m \neq n$, one may take $w = [(\tilde{S}^T - \lambda_n)^N]^{-1}\hat{e}_{m\neq n,j}$ and obtain

$$\hat{e}^T_{n,i}\tilde{O}\hat{e}_{m,j} = 0 \quad \forall m \neq n, \tag{A14}$$

showing that $\tilde{O}$ is indeed block diagonal on this basis. Last, using the fact that $\tilde{O}$ is invertible and antisymmetric, we find that

$$\det[\tilde{O}_n] \neq 0 \Leftrightarrow \dim[\tilde{O}_n] \in N_{\text{even}}, \tag{A15}$$

where $\tilde{O}_n$ is the n block of $\tilde{O}$. Therefore, each block of the Jordan matrix must have an even number of states and a variant Kramer's theorem holds even in the non-Hermitian case.

The robustness of the ν_2 index, for the non-Hermitian case, follows readily from the definition

$$\nu_2 = (\dim[J_0(S)]\,\mathrm{mod}4)/2. \tag{A16}$$

Indeed, using the chiral and TRS symmetry of the action, one can show that this quantity can only change by multiples of 4.

APPENDIX B: OVERLAP AS A PATH INTEGRAL

Here we show that the partition function discussed in the main text is equal to an overlap of two distinct states: the ground state in the presence of a full flux quantum and the ground state to which we dynamically insert a flux quantum at a rate which is smaller than the cutoff scale or, equivalently, the bulk gap.

To establish this result, we rewrite the overlap, up to phases, as

$$\langle gs|G^\dagger U|gs\rangle = \mathrm{Tr}\,[PG^\dagger UP] = \mathrm{Tr}\,[G^\dagger[GPG^\dagger]UP], \tag{B1}$$

$$P = \lim_{\beta\to\infty} e^{-\beta(H[0]-E_{gs})}, \tag{B2}$$

$$U = e^{-i\int_{-\Delta t}^{\Delta t}(H[A(t)]-E_{gs})}, \tag{B3}$$

$$G = \exp^{i\frac{2\pi}{L}\int dx x\psi^\dagger(x)\psi(x)}, \tag{B4}$$

where $H[A]$ is the full system Hamiltonian in second quantization coupled to a gauge field A, Δt is the duration of the flux insertion process, E_{gs} is a c number equal to the ground-state energy, and therefore P projects on the ground state. The gauge field is given by $A(t) = \frac{e}{h}\frac{t}{2L\Delta t}$ for $t \in [-\Delta t,\Delta t]$, corresponding to the insertion of a single flux quantum.

By taking advantage of the fact that P is presented as a time evolution operator, we may unite $GPG^\dagger$,U and P into a single time evolution operator,

$$GPG^\dagger UP = e^{-\int_{-\beta/2}^{\beta/2} dt\alpha^{-1}(t)H[A(t)]}, \tag{B5}$$

where, for $t \in [-\beta/2,-\Delta t]$, we take $A = -e/(2hL)$ and $\alpha^{-1}(t) = 1$. For $t \in [\Delta t,\beta/2]$, we take $A = +e/(2hL)$ and $\alpha^{-1}(t) = 1$. In the remaining interval, $t \in [-\Delta t,\Delta t]$, we take $A(t) = \frac{e}{h}\frac{t}{2L\Delta t}$,$\alpha^{-1}(t) = i$.

Using Grassmann calculus to take the the trace in Eq. (B1), we obtain

$$\langle gs|G^\dagger U|gs\rangle = \int D[\bar{c}c]e^{-\bar{c}c}\langle -c|G^\dagger e^{-\int_{-\beta/2}^{\beta/2} dt\alpha^{-1}(t)H[A(t)]}|c\rangle, \tag{B6}$$

$$|c\rangle = e^{-c_i a_i^\dagger}|0\rangle, \tag{B7}$$

where $|0\rangle$ is the vacuum, $a_i^\dagger$ is a creation operator for some spanning basis of states indexed by i, $\bar{c}c = \sum_i \bar{c}_i c_i$, the limit $\beta \to \infty$ is assumed, and summation over repeated indices is implicit.

Next we follow the usual steps of constructing a path integral from fermionic coherent states (see, for example, Ref. 26). First, the time evolution is partitioned into a product of $N \to \infty$ infinitesimal time evolutions; next, Grassman resolutions of the identity in the form $\int D[\bar{\psi}_n\psi_n]e^{-\bar{\psi}_n\psi_n}|\psi_n\rangle\langle\psi_n|$ are inserted between each of those; and last, using $a_n|\psi_n\rangle = \psi_n|\psi_n\rangle$, the operator $H(a^\dagger,a)$ is replaced by a Grassmanian functional.

The only nonstandard complication in this path integral is the presence of the gauge transformation G. This turns out to be equivalent to a certain choice of boundary conditions, as we now show. First we present the action of G on a Grassman coherent state $|c\rangle$ through its action on the Grassman variables,

$$\begin{aligned} G|c\rangle &= e^{-c_i G a_i^\dagger G^\dagger}|0\rangle = e^{-c_i g_{ij} a_j^\dagger}|0\rangle, \\ G a_i^\dagger G^\dagger &= g_{ij}a_j^\dagger, \\ G a_i G^\dagger &= g_{ij}^* a_j, \end{aligned} \tag{B8}$$

so that if i is the position basis, then $Ga_i^\dagger G^{-1} = e^{i2\pi x_i/L}a_i^\dagger$ and $g_{ij} = \delta_{ij}e^{i2\pi x_i/L}$. Following this, we obtain

$$\begin{aligned} G|c\rangle &= |cg\rangle, \\ \langle cg| &= \langle c|G^\dagger = \langle 0|e^{-a_j\bar{c}_i g_{ij}^*}. \end{aligned} \tag{B9}$$

Boundary conditions for the path integral are determined by the requirement that the discrete time derivatives generated by the time slicing process remain finite. These time derivatives emerge from the $-\bar{\psi}_n\psi_n$ factors coming from the resolution of the identity and the $\bar{\psi}_n\psi_{n+1}$ coming from the overlap of adjacent coherent states. Focusing only on these elements, the path generates a series of the form

$$\begin{aligned} &e^{-\bar{c}c}\langle -cg|\psi_0\rangle e^{-\bar{\psi}_0\psi_0}\langle\psi_0|\psi_1\rangle(\cdots)e^{-\bar{\psi}_N\psi_N}\langle\psi_N|c\rangle \\ &\quad = e^{-\bar{c}c-\bar{c}g^*\psi_0-\bar{\psi}_0\psi_0+\bar{\psi}_0\psi_1+(\cdots)-\bar{\psi}_N\psi_N+\bar{\psi}_N c}. \end{aligned} \tag{B10}$$

This series can be regrouped to form a sum of derivatives either by grouping adjacent elements that share $\bar{\psi}$ or elements that share ψ (this ambiguity is a discrete version of integration by parts). Consequently, we require both of the following expressions to remain finite as we refine the time slicing:

$$\begin{aligned} &e^{-\bar{c}(c+g^*\psi_0)-\bar{\psi}_0(\psi_0-\psi_1)+(\cdots)-\bar{\psi}_N(\psi_N-c)} \\ &e^{(\bar{\psi}_N-\bar{c})c+(-\bar{c}g^*-\bar{\psi}_0)\psi_0+(\bar{\psi}_0-\bar{\psi}_1)\psi_1+(\cdots)'+(\bar{\psi}_{N-1}-\bar{\psi}_N)\psi_N}. \end{aligned} \tag{B11}$$

Identifying $\psi(\beta/2)$ with ψ_0 and $\psi(-\beta/2)$ with ψ_N, the boundary conditions are

$$\psi(-\beta/2) = -g^*\psi(\beta/2) = -\psi(\beta/2)g^\dagger, \tag{B12}$$

$$\bar{\psi}(-\beta/2) = -\bar{\psi}(\beta/2)g^T = -g\bar{\psi}(\beta/2), \tag{B13}$$

which, in position basis, amounts to

$$\psi(-\beta/2,x) = -e^{-i2\pi x/L}\psi(\beta/2,x), \tag{B14}$$

and its complex-conjugate condition. The path-integral expression for the overlap is, therefore,

$$\begin{aligned} \langle gs|G^\dagger U|gs\rangle &= \int D[\bar{\psi}\psi]e^{-S[\bar{\psi},\psi]}, \\ S &= \bar{\psi}\partial_t\psi - \alpha^{-1}(t)(H[\bar{\psi},\psi] - E_{gs}), \end{aligned} \tag{B15}$$

with field configurations obeying Eq. (B14). Taking $\bar{\psi} \to \bar{\psi}\alpha(t)$ yields

$$S = \bar{\psi}\alpha(t)\partial_t\psi - H[\bar{\psi},\psi] + E_{gs}. \tag{B16}$$

Note that by working with a finite β, one finds that this result holds also for finite temperature.

In the main text, we derived the zero modes associated with the Euclidean chiral action $[\alpha(t) = 1]$; however, these also persist with the extra $\alpha(t)$ factor. Indeed, one can start from $\alpha_0(t) = 1$ and gradually deform it, while respecting TRS as defined in Eq. (8), to $\alpha(t)$. Such a continuous deformation cannot remove the topologically protected zero modes even though the operators associated with S becomes non-Hermitian (see Appendix A).

APPENDIX C: REGULATING THE PATH-INTEGRAL EXPRESSION FOR THE OVERLAP

In the main text, we showed that the path-integral expression for the overlap vanishes in the presence of action zero modes. The argument was, however, based on a formal expression containing an infinite product of action eigenvalues. Here we derive this result with properly regulated expressions.

There are various regulation schemes one can use to make the partition function well defined. For example, a sharp cutoff on action eigenvalues will make the partition function a finite product, at least for a finite system. One can also use a Pauli-Villars regularization in the form of an external bosonic particle with an identical action except for a large imaginary mass. Such a regulator will cause large action eigenvalues coming from the fermionic sector to cancel with similar large action eigenvalues coming from the bosonic sector. Both of these regulators render the partition function finite, while retaining the exact zero modes. Consequently, for any finite value of the regulators, the partition function of an anomalous system will vanish exactly. This is an indication that our results are cut off, independent in the limit where the cutoff is taken to infinity.

Still, within both of these regulation schemes, it is not obvious that the partition function converges to the actual overlap [see Eq. (B15)]. Indeed, working with the sharp cutoff, one is throwing away large multiplicative factors, and using a Pauli-Villars one adds many $(iM)^{-1}$ factors where iM is the imaginary mass of the Pauli-Villars boson. Thus both regulators introduce large unphysical multiplicative factors, leaving at best only a proportionality relation between actual overlap and regularized partition function. Of course, for the sake of showing that the overlap vanishes, a proportionality relation is sufficient. But, clearly, it will be more satisfactory to work with a regulator that is consistent with Eq. (B15).

To this end, we consider the entire topological insulator (TI), i.e., edge theory and bulk. We place the TI on a finite cylindrical lattice geometry. This takes care of all sources of divergences except those related to infinite frequencies. To amend this, we also work in discrete time and take the continuum limit in the end of the calculation. The advantage of this regularization procedure is that the path-integral construction now becomes exact[27] and perfectly consistent with Eq. (B15). The minor disadvantage is that the two TI boundaries become weakly coupled. One mechanism of coupling is through bulk tunneling, which induces $im\sigma_0 s_0$ terms in the chiral action. The second mechanism of coupling is through the bulk modes below and above the band gap (Δ).

We thus wish to bound the overlap $\langle gs;m,\Delta|G^\dagger U|gs;m,\Delta\rangle$, for $m\to 0$ and $\Delta\to\infty$. This limit corresponds to a long cylinder and a large insulating gap. Using the fact that such overlaps are always smaller than one, we have

$$|\langle gs;0,\Delta|G^\dagger U|gs;0,\Delta\rangle| \leqslant \frac{|\langle gs;0,\Delta|G^\dagger U|gs;0,\Delta\rangle|}{|\langle gs;m_0,\Delta|G^\dagger U|gs;m_0,\Delta\rangle|},$$

and combining this inequality with Eq. (B15), we find

$$\begin{aligned}\ln\left(\langle gs;0,\Delta|G^\dagger U|gs;0,\Delta\rangle\right) &< \int_{m_0}^{0} dm\,\partial_m \ln(Z),\\ &= \int_{m_0}^{0} dm\left\langle\int dt dx\,\sigma_0 s_0\right\rangle.\end{aligned} \tag{C1}$$

Thus, to bound the overlap, we wish to show that the rhs is a large negative number.

In the main text, it was shown that due to the zero modes at $m=0$, the above correlation function is singular and behaves as

$$\left\langle\int dt dx\,\sigma_x s_0\right\rangle = \frac{2\nu_2}{m} + O(m^0,m^1,\ldots). \tag{C2}$$

The singularity dominates for m smaller than the action-level spacing around zero, which goes as $\min[1/\beta,1/T]$. Following this, we choose $m_0\approx\min[1/\beta,1/T]$ and neglect these higher-order terms. Following the introduction of bulk states, the zero modes need not be exact anymore, even at $m=0$. In principal, processes in which states get scattered to high energy by the flux insertion may lead to corrections of the order of $\delta=(\Delta T^2)^{-1}$, where T is the rate of the flux insertion. Formally, such corrections can be obtained by calculating the electron self-energy or simply by using second-order degenerate perturbation theory in the time-dependent term. As a result, one finds that the correlation function will have an effective $1/m$ pole at least down to $m\approx\delta$, but will not diverge below this value. Combining these results, the rhs of Eq. (C1) can be bounded by

$$\int_{m_0}^{0}\frac{2\nu_2}{m+\delta} = 2\nu_2\ln(\delta/m_0), \tag{C3}$$

yielding, for $\nu_2=1$,

$$|\langle gs;m,\Delta|G^\dagger U|gs;m,\Delta\rangle| < (\delta/m_0)^2. \tag{C4}$$

Thus, in the limit of $\Delta\to\infty$ (or $T\to\infty$), the overlap is indeed zero up to adiabatic corrections, which go as $(\Delta T)^{-1}$.

While this shows that the overlap goes to zero in the limit where the cutoff is taken to infinity, the convergence rate, estimated from perturbation theory, appears rather slow. A similar situation occurs in the adiabatic approximation where, based on naive time-dependent perturbation theory, one obtains a correction which scales as one over the energy gap to the nearest excited state. However, there it is known that corrections to adiabaticity are typically exponentially suppressed by the gap. This can be shown explicitly for two- and three-level systems[15] and, more generally, by the adiabatic renormalization approach[28] or perturbation theory manipulations (see Ref. 29, p.173). It is likely that the same result will hold here as well, namely, that correction to the vanishing overlap scale as $\exp(-\Delta T)$.

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