

**Un-Fermi liquids: Unparticles in strongly correlated electron matter**

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(Received 7 May 2013; revised manuscript received 16 August 2013; published 17 September 2013)

Since any nontrivial infrared dynamics in strongly correlated electron matter must be controlled by a critical fixed point, we argue that the form of the single-particle propagator can be deduced simply by imposing scale invariance. As a consequence, the unparticle picture proposed by Georgi<sup>14</sup> is the natural candidate to describe such dynamics. Unparticle stuff is scale-invariant matter with no particular mass. Scale invariance dictates that the propagator has an algebraic form which can admit zeros and hence is a candidate to explain the ubiquitous pseudogap state of the cuprates. We refer to the nonperturbative electronic state formed out of unparticles as an un-Fermi liquid. We show that the underlying action of the continuous mass formulation of unparticles can be recast as an action in anti-de Sitter space which serves as the generating functional for the propagator. We find that this mapping fixes the scaling dimension of the unparticle to be  $d_U = d/2 + \sqrt{d^2 + 4}/2$  and ensures that the corresponding propagator has zeros with  $d$  the space-time dimension of the unparticle field. Should  $d = 2 + 1$ , unparticles acquire the nontrivial phase  $2\pi d_U$  upon interchange. Because  $d_U$  is noninteger and in general not half integer, clockwise and counterclockwise interchange of unparticles do not lead to the same phase and time-reversal symmetry is broken spontaneously as reported in numerous experiments in the pseudogap phase of the cuprates. The possible relevance of this mechanism to such experiments is discussed. We then formulate the analogous BCS gap using unparticles and find that in contrast to the Fermi-liquid case, the transition temperature increases as the attractive interaction strength decreases, indicating that unparticles are highly susceptible to a superconducting instability.

DOI: [10.1103/PhysRevB.88.115129](https://doi.org/10.1103/PhysRevB.88.115129)

PACS number(s): 71.10.Hf

**I. INTRODUCTION**

The key problem that arises from the strong correlations in the normal state of the copper-oxide superconductors is identifying the weakly interacting entities that make a particle interpretation of the current possible. Superconductivity would then be reduced to a pairing instability of such objects. However, there is good reason to believe that the construction of such entities may not be possible. The reason lies in the fact that both the parent and pseudogap phases are characterized by a vanishing of the single-particle propagator  $G(E, \mathbf{p})$ , evaluated at zero energy.<sup>1-5</sup> While  $\text{Det}[G(E = 0, \mathbf{p})] = 0$ <sup>6</sup> for certain momenta in the pseudogap state, the single-particle propagator vanishes for all momenta in the parent Mott insulating state<sup>4,5,7</sup> as would be expected for a hard gap in the absence of symmetry breaking. If we write the propagator simply as  $1/[E - \epsilon_{\mathbf{p}} - \Sigma(E, \mathbf{p})]$ , a vanishing propagator (where both the real and imaginary parts vanish) is possible only if the self-energy diverges. Because the imaginary part of the self-energy defines the decay rate of a state, a divergent self-energy implies that no stable particlelike excitation is present. In fact, to date the only known kind of excitations that emerge from zeros of a single-particle propagator are bound or composite states, for example Cooper pairs in the context of superconductivity, which do not admit a particle interpretation in terms of a quadratic action<sup>8</sup> with canonical fields. Nonetheless, a highly influential result, the Luttinger count,<sup>9</sup> which has been applied widely in the field of strong correlations,<sup>4,10,11</sup> equates the number of excitations which can be given a particle interpretation not only with the number of poles, or quasiparticles, but also with the number of zeros

of the single-particle Green function. If this were true, this would be truly remarkable as it would imply that even in the limit diametrically opposed to the Fermi-liquid or quasiparticle regime, the particle concept still applies. Hence, on physical grounds, equating the Luttinger count with the particle density is difficult to fathom and inconsistent with the work of 't Hooft<sup>12</sup> who has shown that the analogous problem in QCD in  $d = 1 + 1$  implies that there are “no physical quark states” at low energies.

Since we believe that strongly correlated electron matter is no different, one of us<sup>13</sup> analyzed the precise mathematical statement underlying Luttinger's claim,

$$n_L = 2 \sum_{\mathbf{p}} \Theta[\text{Re}G(\omega = 0, \mathbf{p})], \quad (1)$$

with  $G(\omega, \mathbf{p})$  the single-particle propagator,  $\Theta$  the Heaviside step function and  $\mathbf{p}$  the momentum, and has shown that it is, understandably, false in the sense that when zeros are present,  $n_L$  is not necessarily the particle density. The failure of Eq. (1) to yield the particle density in the normal state of the cuprates where zeros are present has a profound consequence. The right-hand side of Eq. (1) is evaluated strictly at zero energy or at the chemical potential. Hence, if Eq. (1) fails to yield the charge density, there are charged degrees of freedom left over which have no interpretation in terms of low-energy physics alone, that is particles.

Consequently, the key question that arises is what is the stuff in the normal state of the cuprates (or more generally strongly correlated electron matter) that couples to the current but has no particle interpretation? We put forth here that

the unparticle stuff proposed by Georgi<sup>14</sup> several years ago provides a reasonable answer to this problem. We stick with the characterization used by Georgi<sup>14</sup> that unparticles should be called stuff because all other characterizations imply, mistakenly, a particle correspondence. There is strictly none to be had for unparticle matter. Unparticle stuff is scale invariant matter with no particular mass but with nontrivial (non-Gaussian in terms of canonical fields) infrared (IR) dynamics. This construction is natural in strongly correlated systems because the interactions exist on all length scales. A key prediction of this correspondence is that strongly correlated electron matter that has a vanishing propagator exhibits fractional statistics in  $d = 2 + 1$ , which is experimentally falsifiable.

## II. PRECURSORS

We motivate the relevance of unparticle stuff to strongly correlated electron matter by recalling what happens when the Wilsonian procedure is carried out on the basic model for a doped Mott insulator, namely, the Hubbard model. In this model, electrons hop among neighboring lattice sites with amplitude  $-t$  and encounter an on-site repulsion of magnitude  $U$  when two electrons with opposite spin occupy the same site. As  $U$  is the high-energy scale, the goal of a Wilsonian procedure is to integrate out the  $U$ -scale physics. However, this is not simple to do because the physics at the  $U$ -scale involves a four-fermion term. However, we have shown how this can be done exactly. The procedure is well documented<sup>15-17</sup> so we will just recount the essential parts.

The key idea<sup>15-17</sup> is to extend the Hilbert space by introducing a new fermionic operator,  $D_i^\dagger$ , which permits a clean identification of the  $U$ -scale physics. The operator  $D_i$  enters the  $UV$ -complete Lagrangian with a mass of  $U$ . However, it only corresponds to the creation of double occupancy through a constraint,  $\delta(D_i - \theta c_{i\uparrow} c_{i\downarrow})$ , where  $\theta$  is a Grassmann variable and  $c_{i\sigma}$  is the electron annihilation operator for site  $i$  with spin  $\sigma$ . In essence, we have fermionized double occupancy by the introduction of the Grassmann field and hence  $D_i$  to some extent represents a superfield. The constraint is imposed with a Lagrange multiplier  $\varphi_i$ , which must have charge  $2e$  because  $D_i$  has charge  $2e$ . The  $UV$ -complete theory written in terms of the  $D_i$  field is formally equivalent to the Hubbard model: integration over  $\varphi_i$  and then integration over  $D_i$  results in the action for the Hubbard model. However, equivalence at the  $UV$  scale is not the key point here. The reason for adopting this new language is to be able to integrate out the  $U$ -scale physics exactly. Because  $D_i$  is a canonical fermionic field and it enters the action in a quadratic fashion, it can be integrated out exactly. The Lagrangian that results will describe the IR physics and will depend on the constraint field  $\varphi_i$ .

We note that in the integration over the  $U$ -scale physics,  $D_i$  represents whatever the physics is on the  $U$  scale. Only in the atomic limit is this strictly double occupancy. Similarly, in the IR theory, the emergent field  $\varphi_i$  also will not represent double occupancy but whatever physics the upper Hubbard band produces in the lower band by virtue of dynamical spectral weight transfer. When all of this is done, what is most important here is that one can identify what the charge degrees of freedom look like in the IR by adding a minimally coupled source term to the  $UV$ -complete theory. The source

term acts in the extended space and is carefully chosen so that when the constraint is solved, the bare electron operator is then minimally coupled to the source term.

However, integrating out the  $D_i$  field tells another story. The new IR charge which is now minimally coupled to the source, let us call it  $\psi_{i\sigma}$ , depends explicitly on  $\varphi_i$ . The square of this field, which defines the number density of such excitations, can be computed explicitly.<sup>18</sup> For a lower Hubbard band with  $x$  holes, the conserved charge is  $1 - x$ . Nonetheless,  $\langle |\psi_{i\sigma}|^2 \rangle < 1 - x$ . The deficit corresponds to all the stuff that couples to  $\varphi_i$ , the Lagrange field that tethers  $D_i$  to the  $UV$  scale. Hence, all of the charge degrees of freedom which depend on  $\varphi_i$  contribute to the current but *not* to the particle density. In fact, they create zeros of the single-particle Green function<sup>18</sup> and hence cannot be given a particle interpretation, thus their vanishing contribution to the particle density.

This implies that the Wilsonian procedure on the Hubbard model provides a clear example of an emergent IR theory with charged degrees of freedom that have no particle (electron) interpretation. In fact, the culprit emerges from the effective interpolating<sup>19</sup> field  $\varphi_i$ , which is made manifest in the IR theory entirely by eliminating the  $UV$ -scale physics. It couples to particle stuff and leads to zeros of the single-particle Green function.<sup>18</sup> In the context of physics beyond the standard model, Georgi<sup>14</sup> has proposed that such fields can arise and lead to nontrivial IR dynamics by interacting with the particle sector, precisely in the manner found here. Such fields generate<sup>19</sup> unparticle physics.

That such physics should enter the Hubbard model can be seen by comparing the number of electron states at low energy with the total spectral weight. In a Fermi liquid, the two are necessarily equal, but it is not so in an expansion around the atomic limit of the Hubbard model. The number of electron states is delineated from a pure stoichiometric argument, namely counting the number of sites. The Mott state corresponds to one electron per site. Hence, there are  $N$ -electron addition and  $N$ -electron removal states in an  $N$ -site system. Per site, this simply means that there is a single removal and addition state. Let  $x = n_h/N$ , where  $x$  is the number of empty sites. The number of electron removal states when  $x$  holes are present is  $1 - x$ . Each hole can be filled with either a spin-up or a spin-down electron. Hence,  $x$  holes produce  $2x$  electron states, which lie at low energy. Therefore, the number of low-energy electron states is just a sum of the electron removal plus the number of hole states,  $1 - x + 2x = 1 + x$ . This number is not affected by the dynamics. However, the spectral weight does change as this is determined by the true propagating degrees of freedom in the IR and hence the dynamics. As shown many years ago<sup>20</sup> and observed experimentally,<sup>21,22</sup> the spectral weight in the lower band is given by  $1 + x + f(x, t/U)$  where  $f(x, t/U) > 0$ . Consequently, counting electrons cannot exhaust the number of degrees of freedom in the lower band. The missing degrees of freedom do not have a particle interpretation. Furthermore, such degrees of freedom play a crucial role in the doped system as they provide a mechanism for zeros of the single-particle propagator, the key mechanism for the breakdown of Luttinger's theorem. Further, it has been suggested<sup>23</sup> (without proof in the last sentence of the paper) that dynamical spectral weight transfer can only be captured by a low-energy theory if

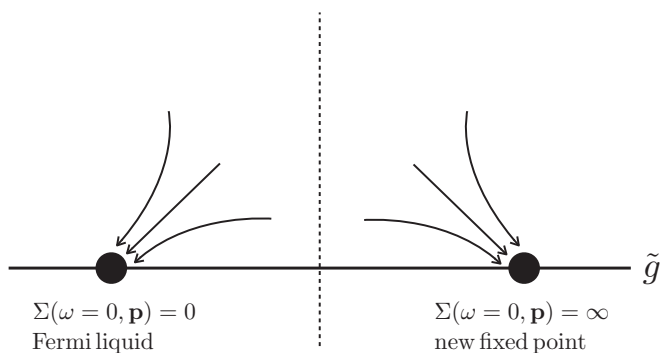


FIG. 1. Heuristic renormalization group flow for the Fermi-liquid fixed point in which the self-energy is zero or negligible in the IR and one in which zeros of the single-particle propagator appear. Since the self-energy diverges in the latter, the excitations which appear here are not adiabatically connected to the Fermi-liquid fixed point. The breakdown of the particle concept at the new fixed point suggests that an unparticle picture is valid. The horizontal axis represents the strength of the coupling constant.

the fundamental excitations in the IR have fractional statistics. As will be seen, this feature appears naturally in our low-energy construction.

### III. UNPARTICLES

The breakdown of the Fermi-liquid picture in a doped Mott insulator stems generically from a divergent self-energy at zero energy (see Fig. 1). Any excitations that arise from such a divergence are clearly not adiabatically connected to the noninteracting or Fermi-liquid fixed point. Consequently, if any new excitations emerge, they must arise fundamentally from a new fixed point as illustrated in Fig. 1. While it is difficult to establish the existence of a nontrivial (non-Gaussian in the UV variables) IR fixed point of the Hubbard model (or any strongly coupled model for that matter), the cuprates, Mott systems in general which display quantum critical scaling,<sup>24,25</sup> and the Wilsonian procedure for the Hubbard model suggest that such a nontrivial fixed point emerges out of the strong relevant on-site interactions, evidenced in part by the numerous experimental signatures<sup>26–28</sup> of scale invariance in the normal state as well as highly successful phenomenology<sup>28</sup> that relies on such invariance. What permits immediate quantitative progress here is that all critical fixed points exhibit scale invariance and this principle anchors fundamentally the kind of single-particle propagators that arise as pointed out by Georgi.<sup>14</sup> Since mass and scale invariance are incompatible, the excitations which emerge have no particular mass and are called unparticles.<sup>14</sup> The propagator<sup>14</sup> in the scalar unparticle sector can be written down strictly from scale invariance,

$$G_U(\mathbf{k}) = \frac{A_{d_U}}{2 \sin(d_U \pi)} \frac{i}{(\mathbf{k}^2 - i\epsilon)^{d/2 - d_U}}, \quad (2)$$

$$A_{d_U} = \frac{16\pi^{5/2}}{(2\pi)^{2d_U}} \frac{\Gamma(d_U + 1/2)}{\Gamma(d_U - 1)\Gamma(2d_U)},$$

where  $\mathbf{k}$  is the  $d$  momentum,  $d$  is the dimension of the space-time the unparticle lives in, and  $d_U$  is the scaling dimension of the unparticle operator, typically not an integer. Here, we

adopt the notation where the diagonal entries of the metric are mostly positive.

Consequently, the failure of the particle concept is manifest by the branch cut in  $G_U$ . As a result,  $G_U$  bears some resemblance to the propagator for Luttinger liquids.<sup>29</sup> However,  $G_U$  lacks the topological term in the denominator that preserves the singularity at the Fermi momentum  $k_F$ , ultimately the key to the successful implementation of the bosonization scheme for Luttinger liquids.

Our precise claim is that the low-energy physics of strongly correlated electron systems, at least the sector governed by the interpolating field  $\varphi_i$  in the context of the Hubbard model, is described by a spectral function that scales with an anomalous exponent. Since  $\varphi_i$  is only present when the hopping matrix element is nonzero, our analysis does not apply to the atomic limit in which the Green function can be obtained exactly. Within the unparticle proposal, the spectral function should have the form

$$A(\Lambda\omega, \Lambda^{\alpha_A} \mathbf{k}) = \Lambda^{\alpha_A} A(\omega, \mathbf{k}), \quad (3)$$

$$A(\omega, \mathbf{k}) = \omega^{\alpha_A} f_A\left(\frac{\mathbf{k}}{\omega^{\alpha_A}}\right).$$

We take  $\alpha_A = 2d_U - d$ . The scaling for a Fermi liquid corresponds to  $d_U = (d - 1)/2$ . Because of the constraints on unparticles,  $d_U$  always exceeds  $d/2 - 1$  for scalar unparticles and  $(d - 1)/2$  for unfermions, where the rule to obtain unfermions is to set  $d \rightarrow d + 1$ . We term a correlated system with such scaling an un-Fermi liquid as the basic excitations are unparticles. Un-Fermi liquids are non-Fermi liquids composed of unparticles, whose propagator is given by the unfermionic analog of Eq. (2),

$$S_U(\mathbf{k}) \sim (\mathbf{k}^2 - i\epsilon)^{d_U - (d+1)/2} (\mathbf{k} + \cot(d_U \pi) \sqrt{\mathbf{k}^2 - i\epsilon}), \quad (4)$$

which contains a nonlocal mass term. Un-Fermi liquids should not be construed as Fermi liquids with poles at the unparticle energies. Because the unparticle fields cannot be written in terms of canonical ones, there is no sense in which a Gaussian theory can be written down from which polelike excitations can be deduced. Note that the scaling form, if it were to satisfy any kind of sum rule, can only be a valid approximation over a finite-energy range.

As pointed out by Georgi,<sup>14</sup> the phase space for  $N$  massless particles is identical to the unparticle propagator but with  $d_U$  replaced by  $N$  in four dimensions. In general the relationship is  $d_U = N(d/2 - 1)$ . Since  $d_U$  is in general not an integer, interpreted loosely as an anomalous dimension, the dynamics of unparticles with scaling dimension  $d_U$  are equivalent to those of a fractional number of massless particles. Indeed, Luttinger-liquid Green functions have been proposed previously<sup>30,31</sup> as the source of non-Fermi liquid behavior in  $(2 + 1)$ -dimensional systems but without justification because such Green functions have a rigorous basis only in  $(1 + 1)$  dimension. The unparticle construction applies regardless of  $d$  and hence offers a way around this conundrum. In addition, when  $d_U > d/2$  [ $d_U > (d + 1)/2$  for fermions], the propagator can vanish, giving rise to zeros and an explicit violation of the Luttinger count. One of our key results is that  $d_U$  has a natural lower bound that guarantees that  $G_U(0) = 0$ , leading to a realistic model of zeros, exploited recently in

the context of overlap fermions.<sup>32</sup> Consequently, we propose here that unparticles are in part responsible for the dynamics at nontrivial IR fixed points in strongly correlated electron matter. While in general the breakdown of the particle concept is well accepted to obtain at critical points,<sup>33,34</sup> our specific proposal is that the propagator can be deduced immediately from scale invariance<sup>14</sup> and that such scaling is valid in an entire phase (namely the pseudogap), not just at a single point, as long as zeros of the single-particle propagator persist.

### A. Action on AdS

We show in this section that there is an intimate link between unparticles and the gauge-gravity duality.<sup>35</sup> In fact, the latter helps fix  $d_U$  such that the unparticle propagator will always have zeros. We start with the action

$$S_\phi = \frac{1}{2} \int d^4 \mathbf{x} (\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2) \quad (5)$$

for a massive free theory, where  $\phi$  is a scalar field and  $m$  is the mass, and make it scale-invariant<sup>36,37</sup> by simply integrating over mass,

$$S = \int_0^\infty dm^2 B(m^2) S_\phi, \quad (6)$$

where  $\phi \equiv \phi(x, m^2)$  now depends explicitly on the mass at each scale. Here, we have included a mass-distribution function of the form  $B(m^2) = a_\delta (m^2)^\delta$ . Even though we are explicitly considering the case of scalar field here, our conclusions apply equally to formulation with Dirac fields.

While it is common<sup>38</sup> to relate the emergent unparticle field directly to the scalar field  $\phi$  using a different mass-distribution function  $f(m^2)$  through a relationship of the form

$$\phi_U(\mathbf{x}) = \int_0^\infty dm^2 \phi(\mathbf{x}, m^2) f(m^2), \quad (7)$$

this is not correct as it would imply that the unparticle field  $\phi_U$  has a particle interpretation in terms of the scalar field  $\phi$ . For example,  $\phi_U$  would then obey a canonical commutator and the resultant unparticle propagator could be interpreted as that of a Gaussian theory. We demonstrate this explicitly in the Appendix. In actuality, the unparticle field should not be a sum of  $\phi(x, m^2)$ s, which are independent functions of mass, but rather should involve some unknown product of the particle fields.<sup>14</sup> Consequently, unparticle physics cannot be accounted for in a Gaussian theory.

Although  $\phi_U$  and  $\phi$  are not related in any straightforward way, we will show that the action, Eq. (6), gives rise to a generating functional for the unparticle strictly in the gauge/gravity duality sense. Indeed, some link has been noted previously between the unparticle idea and an action on anti-de Sitter (AdS) space. However, such a connection<sup>39,40</sup> remains heuristic as there has been no explicit mapping between the mass integration in the unparticle action and the AdS metric. It is this missing link that we provide here. The key idea is that we transform the mass into a length scale through  $m = z^{-1}$  and  $z$  will appear as the radial direction in AdS. We then deduce the generating functional for the unparticle simply by constructing the AdS on-shell action. This will effectively remove the degree of freedom that determines the

scaling dimension of the unparticle field in the underlying AdS formulation, namely the mass of the scalar field living in the AdS space.<sup>39,40</sup>

The substitution of  $m = z^{-1}$  introduces an extra factor of  $z^{5+2\delta}$  into the action as can be seen from

$$\mathcal{L} = a_\delta \int_0^\infty dz \frac{2R^2}{z^{5+2\delta}} \left[ \frac{1}{2} \frac{z^2}{R^2} \eta^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) + \frac{\phi^2}{2R^2} \right]. \quad (8)$$

We propose that the correct starting point for the unparticle construction is thus the action on  $\text{AdS}_{5+2\delta}$ ,

$$S = \frac{1}{2} \int d^{4+2\delta} \mathbf{x} dz \sqrt{-g} \left( \partial_a \Phi \partial^a \Phi + \frac{\Phi^2}{R^2} \right), \quad (9)$$

where

$$ds^2 = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2), \quad (10)$$

is the corresponding metric,  $R$  is the AdS radius,  $\sqrt{-g} = (R/z)^{5+2\delta}$ , and  $\phi \rightarrow (a_\delta 2)^{-1/2} R^{3/2} \Phi$ . All of the factors of  $z^{5+2\delta}$  and the  $z^2$  in the gradient terms appear naturally with this metric. According to the gauge-gravity duality, the on-shell action then becomes the generating functional for unparticle stuff which lives in an effective dimension of  $d = 4 + 2\delta$ . Therefore, we would like to have  $\delta \leq 0$ . Furthermore, even though it may seem that unwanted dynamics in the  $z$  direction has been introduced, the solution to the equation of motion prescribed by gauge/gravity duality can be thought of as compensating for this, *i.e.*, the solution we are substituting into the action is the one that is non-normalizable at the boundary.

That the unparticle propagator falls out of this construction can be seen as follows. The equation of motion is given by

$$z^{d+1} \partial_z \left( \frac{\partial_z \Phi}{z^{d-1}} \right) + z^2 \partial_\mu \partial^\mu \Phi - \Phi = 0. \quad (11)$$

For spacelike momenta  $k^2 > 0$ , the solutions are identical to those of Euclidean AdS. The solution that is smooth in the interior is given by

$$\Phi(z, \mathbf{x}) = \int \frac{d^d \mathbf{k}}{(2\pi)^d} e^{ik \cdot \mathbf{x}} \frac{z^{\frac{d}{2}} K_\nu(kz)}{\epsilon^{\frac{d}{2}} K_\nu(k\epsilon)} \tilde{\Phi}(\mathbf{k}), \quad (12)$$

where  $\mathbf{k}$  is a  $d$ -momentum transverse to the radial  $z$  direction and

$$\nu = \frac{\sqrt{d^2 + 4}}{2}. \quad (13)$$

We note that this solution decays exponentially in the interior and thus, even though it is a  $z$ -dependent solution, one can think of  $\Phi$  as localized at the boundary  $z = \epsilon \rightarrow 0$ . Here, we have explicitly cut off the AdS geometry to regularize the on-shell action,

$$\begin{aligned} S &= \frac{1}{2} \int d^d \mathbf{x} g^{zz} \sqrt{-g} \Phi(z, \mathbf{x}) \partial_z \Phi(z, \mathbf{x}) \Big|_{z=\epsilon} \\ &= \frac{1}{2} \frac{R^{d-1}}{\epsilon^{d-1}} \int \frac{d^d \mathbf{p}}{(2\pi)^d} \frac{d^d \mathbf{q}}{(2\pi)^d} (2\pi)^d \delta^{(d)}(\mathbf{p} + \mathbf{q}) \\ &\quad \times \tilde{\Phi}(\mathbf{p}) \frac{d}{d\epsilon} (\ln[\epsilon^{\frac{d}{2}} K_\nu(p\epsilon)]) \tilde{\Phi}(\mathbf{q}). \end{aligned} \quad (14)$$

Interpreting this as a generating functional for the unparticle field  $\Phi_U$  living in a  $d$ -dimensional space-time, we can then



read the (regulated) two-point function, which scales like  $p^{2\nu}$ . We can then analytically continue to the case of timelike momenta, which corresponds to choosing the non-normalizable solution to the bulk equation of motion. This analytically continued solution will also be localized at the boundary.

The two-point function of the unparticle in real space is then given by

$$\langle \Phi_U(\mathbf{x}) \Phi_U(\mathbf{x}') \rangle = \frac{1}{|\mathbf{x} - \mathbf{x}'|^{2d_U}}, \quad (15)$$

where

$$d_U = \frac{d}{2} + \frac{\sqrt{d^2 + 4}}{2} > \frac{d}{2}, \quad (16)$$

and  $d$  is the dimension of the spacetime the unparticle lives in. We note that in this construction, there is only one possible scaling dimension for the unparticle instead of two, due to the fact that the square of the mass of the AdS scalar field  $\Phi$  is positive.<sup>41</sup> As a result, the unparticle propagator has zeros defined by  $G_U(0) = 0$ , not infinities. This is the principal result of this construction.

The AdS interpretation of an unparticle that we propose here is different than those in Refs. 39 and 40 and the differences are as follows. In Refs. 39 and 40, the spacetime dimension of the AdS space is 5, while here, the spacetime dimension is  $d + 1 = 5 + 2\delta$ . Second, the mass  $m_{AdS}$  of the particle living in the AdS interpretation of Refs. 39 and 40 is a free parameter which is related to the scaling dimension of the unparticle field. In particular,

$$m_{AdS}^2 = \frac{d_U(d_U - 4)}{R^2}. \quad (17)$$

In our interpretation, however,  $m_{AdS}^2 = 1/R^2$ . Third, since  $m_{AdS}$  is a free parameter, the scaling behavior of the two-point function of the unparticle field is undetermined in Refs. 39 and 40, while in our case, it depends solely on the dimension of the space-time the unparticle lives in.

### B. Statistics

Our proposal that unparticles are the fundamental excitations in strongly correlated systems (at least ones that possess zeros) has a key experimental prediction. As pointed out by Georgi,<sup>14</sup> unparticles have a propagator equivalent to that of  $N$  massless particles where  $N = d_U/(d/2 - 1)$ . Consider the case of  $d = 2 + 1$ . Extending the arguments used in the context of anyons,<sup>42-44</sup> we find that because of the branch-cut structure of the propagator, there should be a nontrivial phase upon unparticle exchange given by  $e^{i2\pi d_U} \neq \pm 1$  that is directly related to the fact that interchange of unparticles amounts to an interchange of  $d_U/(d/2 - 1)$  massless particles. Since  $d_U$  is noninteger [see Eq. (16)], any statistics are possible. Consequently, clockwise and counterclockwise rotations of unparticles do not yield the same phase, thereby indicating a spontaneous time-reversal symmetry breaking (TRSB). The TRSB found here for unparticles arises fundamentally from the interactions that lead to the nontrivial excitations in the IR and hence avoids the argument against TRSB based on quasiparticles.<sup>45</sup> Our arguments apply strictly to  $d = 2 + 1$  and hence if applicable to doped Mott insulators

are relevant only to a single copper-oxide layer. Certainly subtleties will arise in applying them to bulk three-dimensional materials. Nonetheless, it is interesting to note that numerous experiments<sup>46-48</sup> have reported observations consistent with the breaking of time-reversal symmetry in the pseudogap phase (a phenomenon distinct from the surface-induced breaking of time-reversal symmetry in the superconducting state<sup>49</sup>). However, recent work<sup>50,51</sup> has suggested that Kerr effect<sup>48</sup> measurements are more consistent with the breaking of inversion symmetry rather than time-reversal because the signal fails to change sign when a measurement is made on the opposite surface. Certainly all of the anyon constructions<sup>42-44</sup> require both inversion and time-reversal symmetries to be broken. However, since scale-invariant matter does not require inversion symmetry breaking, it seems unlikely that the nontrivial statistics associated with unparticles in  $d = 2 + 1$  would result in a sign change of the Kerr signal for measurements on opposing surfaces. Consequently, there exists no *a priori* contradiction between the unparticle construction and the Kerr effect<sup>48</sup> observations. Hence, experiments designed to search for nontrivial statistics in the pseudogap regime are most relevant here. Since unparticles yield zeros, they can be localized and hence could be interchanged thereby making a direct measurement of their statistics possible. Experiments along these lines would certainly be sufficient to falsify the relevance of unparticles to pseudogap matter. Nonetheless, the zero feature of the unparticle propagator is noteworthy because it can explain the dip in the density of states (that is the pseudogap), a feature which is absent in other work,<sup>52</sup> which can explain just the presence of potential TRSB. Regardless of the applicability of these results to the experiments,<sup>46-48</sup> this work indicates that TRSB can quite generally arise from the strong correlations that remain from the Mott state, that is, Mottness, thereby offering a realization of the general principle underlying interaction-induced fractional statistics advocated by Jones-Marshall and Wilczek.<sup>43</sup>

### IV. SUPERCONDUCTING INSTABILITY

Because unparticles do not have any particular energy, they should be useful in describing physics in which no coherent quasiparticles appear, as in the normal state of the cuprates. Since all formulations of superconductivity start with well-defined quasiparticles, we explore what happens when we use a quasiparticle spectral function with a scaling form. Such an approach is warranted given that the cuprates exhibit a color change<sup>53,54</sup> upon a transition to the superconducting state as evidenced most strikingly by the violation of the Ferrell-Glover-Tinkham sum rule.<sup>53</sup> Some initial work along these lines has been proposed previously,<sup>30,31,55</sup> which have all been based on the Luttinger-liquid Green function. As remarked earlier, the advantage of the unparticle approach is that it is completely general regardless of the spatial dimension unlike the Luttinger-liquid one which must be restricted to  $d = 1 + 1$ . To obtain the general result, we work first with the scaling form of the spectral function as in Eq. (3), generalizing the procedure in Ref. 31.

Since the machinery to deal with pairing instabilities with nontrivial statistics does not exist for arbitrary anomalous dimension  $d_U$ , our goal in this section is to see if something

new arises in the superconducting instability when fermionic particles are described by a scale-invariant spectral function of the form of Eq. (3). We take a system of fermions that have a separable two-body interaction  $V(\mathbf{k} - \mathbf{k}') = \lambda w_{\mathbf{k}}^* w_{\mathbf{k}'}$ , but are described by a spectral function that has an effective scaling form up to some energy scale  $W$  and some lower bound close to zero. The equation for the existence of an instability in terms of the Green function is

$$1 = i\lambda \sum_{\mathbf{k}} |w_{\mathbf{k}}|^2 G(\mathbf{k} + \mathbf{q}) G(-\mathbf{k}). \quad (18)$$

This gives the zero-temperature result. We take the interaction strength  $\lambda$  as a constant of mass dimension  $2 - d$ , and  $w_{\mathbf{k}}$  as a filling factor. We work in the center-of-mass frame, such that  $\mathbf{q} = (q_0, 0)$ . The critical temperature for a second-order transition corresponds to  $q_0 = 0$ . We then switch to imaginary time to work at finite temperature, so that the new equation reads

$$1 = \lambda T \sum_{n, \mathbf{k}} |w_{\mathbf{k}}|^2 G(\omega_n, \mathbf{k}) G(-\omega_n, -\mathbf{k}). \quad (19)$$

The Green function is related to the spectral function via

$$G(\omega_n, \mathbf{k}) = \int_{-\infty}^{\infty} dx \frac{A(x, \mathbf{k})}{x - i\omega_n}. \quad (20)$$

Then we obtain, using  $\omega_n = \pi T(2n + 1)$ ,

$$1 = \frac{\lambda}{2} \int dx dy \sum_{\mathbf{k}} |w_{\mathbf{k}}|^2 A(x, \mathbf{k}) A(y, -\mathbf{k}) \times \frac{\tanh(x/2T) + \tanh(y/2T)}{x + y}. \quad (21)$$

The  $\mathbf{k}$  dependence is recast in terms of  $\xi(\mathbf{k})$ , which is in general some function of  $\mathbf{k}$  with units of energy, which for instance can always be done for an isotropic system. In BCS,  $\xi(\mathbf{k})$  would correspond to kinetic energy. Take  $\sum_{\mathbf{k}} |w_{\mathbf{k}}|^2 \rightarrow (\text{volume})^{-1} \times N(0) \int d\xi$  so we pull out a constant density of states. The integral is now rewritten as

$$1 = \frac{g}{2} \int dx dy \int_0^{\omega_c} d\xi A(x, \xi) A(y, \xi) \times \frac{\tanh(x/2T) + \tanh(y/2T)}{x + y}, \quad (22)$$

where  $\lambda N(0) \times (\text{volume})^{-1} = g$  such that  $g$  is a dimensionless measure of interaction strength. Evaluating the integral at any temperature gives the minimal coupling to cause a pairing instability, and this equation traces out a phase diagram for  $g$  and  $T$ .

We can extract some qualitative features knowing the spectral function is attenuated at low and high energy. In the limit of extremely low critical temperature, the entire range of nonzero  $A$  falls in the region where  $\tanh(x/2T) \approx 1$ , and therefore the integral becomes independent of critical temperature to this order, so  $g \approx \text{const}$  (in the case of BCS, this would just be 0 as  $T \rightarrow 0$  due to the logarithmic divergence from the integral). Oppositely in the limit of very high critical temperature, over the entire region it is valid to take  $\tanh(x/2T) \approx x/2T$ , thus  $g \approx 4T/\omega_c$ . Thus regardless of the specific form of the spectral function,  $g$  will increase with critical temperature for high temperatures.

Now we look in the intermittent region of energy where we take advantage of the scaling form. The “ $\beta$  function” looks like

$$\frac{dg}{d \ln T} = \frac{g^2}{4T} \int dx dy \int_0^{\omega_c} d\xi A(x, \xi) A(y, \xi) \times \frac{x \text{sech}^2(x/2T) + y \text{sech}^2(y/2T)}{x + y}. \quad (23)$$

We can recover the BCS result using  $A(\omega, \xi) = \delta(\omega - \xi)$ ,

$$\frac{dg}{d \ln T} = g^2 \tanh \frac{\omega_c}{2T} \approx g^2. \quad (24)$$

The  $\text{sech}^2$  terms in Eq. (23) exponentially suppress high-energy contributions. With an appropriate attenuation on the low-energy side from the spectral function, it is possible for the  $\beta$  function to be negative seeing how  $(x \text{sech}^2 x + y \text{sech}^2 y)/(x + y)$  dips into negative values, which are most pronounced along the line  $x + y = 0$ . In addition these values can outweigh ones for the corresponding positive entries along  $x - y = 0$ . For instance, the smallest minimum of the function is at  $(x, y) = (1.35, -1.35)$  and we obtain a value of  $-0.321$  but for  $(x, y) = (1.35, 1.35)$  we find  $0.235$ . With appropriate weight and suppression, the negative values can dominate the integral and confer a total negative sign. This suppression seems natural in the realm of  $\alpha_A > 0$ , where the scaling form naturally takes on smaller values at lower energies. This would mean that coupling strength increases with decreasing temperature, and therefore there is a bottoming out. That is, a minimum coupling strength that can confer superconductivity exists.

Let us more closely examine what happens when we impose the scaling form at the outset. Then approximately

$$1 = \frac{g}{2} \tilde{T}^{2(1+\alpha_A)} \int dx dy \int_0^{\omega_c/\tilde{T}} d\xi A(x, \xi) A(y, \xi) \times \frac{\tanh(x/2W) + \tanh(y/2W)}{x + y}, \quad (25)$$

where the tilde denotes the ratio of that energy to  $W$ , e.g.,  $\tilde{T} \equiv \frac{T}{W}$ . The scaling form of the spectral function confers a scaling form for  $g$  like

$$g(\tilde{T}, \tilde{\omega}_c) = \tilde{T}^{-2(1+\alpha_A)} f_g \left( \frac{\tilde{T}}{\tilde{\omega}_c} \right). \quad (26)$$

Now sequentially we take a logarithm, derivative, and finally rescale the remaining integral back to obtain

$$\frac{dg}{d \ln \tilde{T}} = -2(1 + \alpha_A) g + \frac{g^2}{2} \omega_c \int dx dy A(x, \omega_c) A(y, \omega_c) \times \frac{\tanh(x/2T) + \tanh(y/2T)}{x + y}. \quad (27)$$

The second term is positive-definite. This term can conceivably be small if there is relatively little spectral weight near  $\omega_c$  within the scaling form or, equivalently, that  $g$  is not very susceptible to changes in  $\omega_c$ . In this event, then we have

$$\frac{dg}{d \ln \tilde{T}} = -2(1 + \alpha_A) g + O(g^2). \quad (28)$$

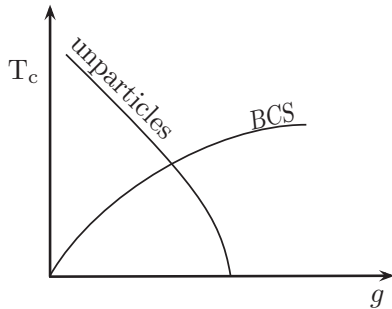


FIG. 2. Plot of the  $\beta$  function for the superconducting transition in the ladder approximation for unfermions, Eq. (28). We have considered the case  $d_U > d/4$  for the scaling dimension of the unparticle field, a condition naturally satisfied in the AdS construction of unparticles. The contrast with the standard prediction of BCS theory is shown.

The right-hand side of this expression is strictly negative for our region of interest where  $\alpha_A > 0$ . Hence, we find quite generally that the critical temperature increases as the coupling constant decreases!

This stands in stark contrast to the Fermi-liquid case in which just the opposite state of affairs obtains. This is illustrated clearly in Fig. 2. In the context of the cuprate superconductor problem, the opposing trends for  $T_c$  versus the pairing interaction suggests that perhaps a two-fluid model underlies the shape of the superconducting dome assuming, of course, that a similar behavior for  $T_c$  as a function of doping persists.<sup>56</sup> Since the transition to the superconducting state breaks scale invariance, the particle picture should be reinstated. Consequently, we expect the broad spectral features dictated by the branch cut of the unfermion propagator to vanish and sharp quasiparticle features to appear upon the transition to the superconducting state as is seen experimentally.<sup>57</sup>

## V. CLOSING

We have proposed that using scale invariance as an organizing principle aids in computing the properties of strongly correlated electron systems at low energy. The necessity for a non-Gaussian fixed point arises any time the single-particle Green function vanishes. Such a vanishing obtains at either a single point or in an entire phase, such as in the pseudogap phase of the cuprates. Our key proposal here is that scale invariance persists as long as the single-particle Green function vanishes over a locus of points in momentum space. The key signature of this behavior which is testable experimentally is the critical scaling of the spectral function in Eq. (3) in the entire region.

We have also proposed an interpretation of unparticles using the AdS construction, which permits us to fix the scaling dimension of the unparticle field and ensures that its propagator will have zeros. Since this propagator possesses zeros, it is a candidate to explain the breakdown of Fermi-liquid theory from strong interactions.

There is a simple way of understanding why a divergent self-energy results in excitations which have fractional statistics. A divergent self-energy represents an orthogonality catastrophe, implying that the underlying excitations

have no overlap with the starting particle fields. Hence, some new fundamental objects which have no canonical particle interpretation carry the charge. The excitations that emerge are composites. In  $d = 2 + 1$ , the new excitations can acquire nontrivial exchange statistics. While experiments to detect fractional statistics are notoriously difficult, we hope this work provides the impetus to search for them in the pseudogap phase of the cuprates, where the zeros of the propagator exist. Note that the zeros give rise to localized excitations and hence unparticle interchange is certainly feasible experimentally. Two other predictions that are falsifiable experimentally are (i) a spectral function exhibiting the scaling of Eq. (3) and (ii) a deviation from the Luttinger count, the latter having already received experimental confirmation.<sup>13</sup> The latter prediction puts this theory in direct contrast with the leading phenomenological theory of the pseudogap regime in which the Luttinger count is strictly maintained.<sup>10</sup>

Finally, we have shown that, in contrast to the Fermi-liquid case, the branch cut singularity in the unparticle propagator gives rise to a superconducting instability in which the critical temperature increases as the coupling constant decreases. This suggests that perhaps a two-fluid model underlies the shape of the superconducting dome of the cuprate superconductors.

## ACKNOWLEDGMENTS

P.P. thanks M. Zubkov for the email communication pointing out the connection between unparticles and zeros, S. Balatsky for a clarifying exchange on Ref. 26, T. Hughes for comments on the statistics section of the paper which led to our emphasis on time-reversal symmetry breaking, and N. P. Armitage for an exchange on time-reversal symmetry breaking. The work of B. Langley and P. Phillips was supported by NSF Grant No. DMR-1104909 which grew out of earlier work funded by the Center for Emergent Superconductivity, a DOE Energy Frontier Research Center, Grant No. DE-AC0298CH1088. J. Hutasoit was supported by NSF Grants No. DMR-1005536 and No. DMR-0820404 (Penn State MRSEC).

## APPENDIX

We show here that only if a linear relationship between the unparticle and particle fields is maintained [as in Eq. (7)], then a Gaussian action for the unparticles is obtained. Let us turn the action in terms of the massive fields into an action in terms of unparticle fields. The original partition function is given by

$$\mathcal{Z} = \int \mathcal{D}\phi_n e^{i \int d^d p \mathcal{L}[\{\phi_n\}]},$$

$$\mathcal{L} = \frac{1}{2} \sum_n B_n \phi_n(p) (p^2 - M_n^2) \phi_n(-p),$$

where  $n$  indicates a sum over the mass  $M_n$ . This sum can remain a general sum over various free fields, but we will ultimately take the limit where the sum is a continuous sum over all masses. The factor  $B_n$  is a weight factor that, in the continuous mass limit, will change the mass dimension of  $\phi_n$ . We introduce a Lagrange multiplier through a factor of unity

and simply integrate over the all fields that are not  $\phi_U$  to obtain

$$\begin{aligned} \mathcal{Z} &= \int \mathcal{D}\phi_n \mathcal{D}\phi_U \mathcal{D}\lambda \exp \left\{ i \int d^d p \left[ \frac{1}{2} \sum_n B_n (p^2 - M_n^2) \phi_n^2 + \lambda \left( \phi_U - \sum_n F_n \phi_n \right) \right] \right\} \\ &= \int \mathcal{D}\phi_U \mathcal{D}\lambda \exp \left\{ i \int d^d p \left( \lambda \phi_U - \frac{1}{2} \lambda^2 \sum_n \frac{F_n^2}{B_n (p^2 - M_n^2)} \right) \right\} \\ &= \int \mathcal{D}\phi_U \exp \left\{ \frac{i}{2} \int d^d p \phi_U(p) \left( \sum_n \frac{F_n^2}{B_n (p^2 - M_n^2)} \right)^{-1} \phi_U(-p) \right\} \end{aligned}$$

with repeated absorptions of normalization constants into the measure. The factor  $F_n$  is another weight factor, this time chosen to determine the scaling dimension of the unparticle field  $\phi_U$ . Because  $F_n$  is chosen to give  $\phi_U(x)$  a scaling dimension  $d_U$ , in the continuous mass limit the ratio  $F_n^2/B_n \sim (M_n^2)^{d_U-d/2}$ . This is necessary because of how  $F_n$  imposes the scaling dimension. Hence we identify the propagator of the

unparticle field as

$$G_U(p) = \sum_n \frac{F_n^2}{B_n (p^2 - M_n^2)} \sim (p^2)^{d_U-d/2}.$$

This argument can also be run in reverse. Namely, if we assume a Gaussian action for the unparticles then the Lagrange multiplier constraint in the form of Eq. (7) is implied.

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