Classification of topological insulators and superconductors in the presence of reflection symmetry

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We discuss a topological classification of insulators and superconductors in the presence of both (nonspatial) discrete symmetries in the Altland-Zirnbauer classification and spatial reflection symmetry in any spatial dimensions. By using the structure of bulk Dirac Hamiltonians of minimal matrix dimensions and explicit constructions of topological invariants, we provide the complete classification, which still has the same dimensional periodicities with the original Altland-Zirnbauer classification. The classification of reflection-symmetry-protected topological insulators and superconductors depends crucially on the way reflection symmetry operation is realized. When a boundary is introduced, which is reflected into itself, these nontrivial topological insulators and superconductors support gapless modes localized at the boundary.

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I. INTRODUCTION

Topological insulators and superconductors are symmetry-preserving fermionic systems with a bulk energy gap.[1,2] Relevant symmetry conditions that are necessary to define these symmetry-protected topological states can be divided into two categories: *nonspatial* symmetries and *spatial* symmetries. The Hamiltonians of nonspatial symmetric systems may possess time-reversal symmetry (TRS), particle-hole symmetry (PHS), or chiral symmetry. They have gapless boundary states that are topologically protected and are related to physical quantities, such as the Hall conductivity. The subject started with the recognition by Kane and Mele[3,4] that by incorporating a spin-orbit coupling in the tight-binding model for graphene, the system will become what is now known as a two-dimensional (2D) $\mathbb{Z}_2$ topological insulator with TRS, as well as the theoretical prediction[5] and experimental observation[6] of such $\mathbb{Z}_2$ topological states in the HgTe/CdTe quantum well. After that, three-dimensional (3D) $\mathbb{Z}_2$ topological insulators were predicted[7–9] and observed;[10–13] the identification of ^{3}He-B as a topological superconductor was realized.[14–17] It turns out that those topological insulators and superconductors are just a part of a larger scheme; a complete classification of topological insulators and superconductors has been developed with a unified periodic table.[18–20]

Topological phases protected by these nonspatial discrete symmetries are stable against spatially homogeneous as well as inhomogeneous deformations. In addition, the protected boundary modes (edge, surface, etc.) that appear at the boundary of topological phases in the periodic table are completely immune to disorder; for arbitrary strong disorder, as long as the bulk topological character is not altered in the bulk through a phase transition, the boundary can never be Anderson localized.[18]

With a set of discrete spatial symmetries, a topological distinction among gapped phases (i.e., "symmetry-protected topological phase") can arise as well. One example is inversion-symmetry-protected topological insulators,[21–23] where inversion symmetry is defined as the invariance of the system under the sign flip of the spatial coordinate $r \to -r$, where $r = (r_1, r_2, \dots, r_d)$ is the spatial coordinates in d spatial dimensions, $r \in \mathbb{R}^d$ (for lattice systems, r labels a site on a d-dimensional lattice). Unlike the case of nonspatial discrete symmetries, for topological phases protected by a set of spatial symmetries, nontrivial bulk topology is not necessarily accompanied by a gapless boundary mode, as the boundary might break the spatial symmetries in question. The implications of certain specific point group symmetries on the topological distinction of ground states have been also discussed.[24–27] More importantly, Hsieh *et al.*[28] predicted that one such spatial symmetric insulator can possess gapless surface states protected by reflection symmetry. The observation of these predicted surface states on $Pb_{1-x}Sn_xTe$ (Ref. 29), SnTe (Ref. 30), and $Pb_{1-x}Sn_xSe$ (Ref. 31) reveals a new generation of topological insulators in nature.

In this paper, we discuss the implication of a reflection (or mirror) symmetry in one spatial direction: It is an invariance of the system under the sign flip of, say, the first component of Cartesian coordinates, $r \to \tilde{r} \equiv (-r_1, r_2, \dots, r_d)$. (For an earlier study of this subject, see Ref. 32.) While an inversion symmetry singles out a special point, reflection symmetry singles out a special $(d-1)$-dimensional plane ($r_1 = 0$ in the this case). As a consequence, when we terminate the system with a $(d-1)$-dimensional boundary (plane) which is orthogonal to the reflection plane ($r_1 = 0$), the boundary with constant r_i $(i \neq 1)$ is reflection symmetric under $(r_1, r_2, \dots, r_d) \to (-r_1, r_2, \dots, r_d)$. Reflection symmetry is arguably the simplest spatial symmetry of a system for which certain boundaries can respect the spatial symmetry in question. This boundary property is an important distinction from the inversion-symmetric topological phases, for which a plane boundary to the system alone does not inherit the spatial symmetry (inversion symmetry) in the bulk. (A pair of boundaries can be inversion symmetric to each other, though). With the special choice of the boundary above, we argue that for topologically nontrivial phases protected by reflection there is a stable boundary mode, in the manner similar to topologically phases protected by nonspatial discrete symmetries. The correspondence still holds between the nontrivial bulk topology and the gapless boundary modes when the boundary that reflects to itself is chosen as shown in Fig. 1.

Although topological insulators and superconductors protected by nonspatial or spatial symmetries have been studied

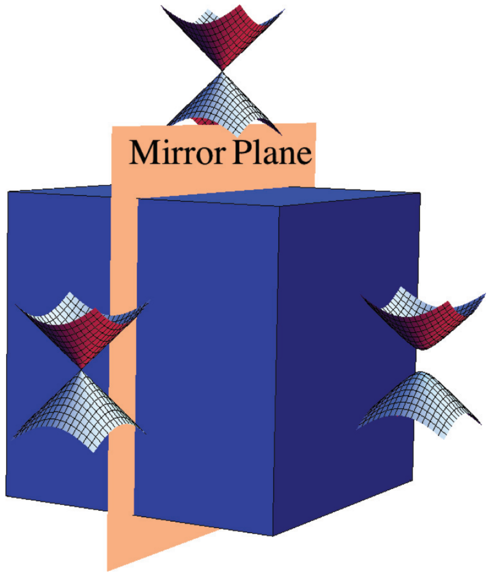


FIG. 1. (Color online) For topological insulators and superconductors that are protected by reflection symmetry, the correspondence between gapless surface states and bulk topology holds only when the surface that reflects to itself is chosen. The figure shows that nontrivial bulk topology guarantees gapless states in the *self-reflected* surfaces. Furthermore, gapped states in the *non-self-reflected* surfaces does not imply trivial bulk topology.

separately, their recognition does not directly provide a complete classification of topological systems in the presence of both nonspatial and spatial discrete symmetries. Therefore, we consider the topological classification of reflection-symmetric systems with a subset of the three nonspatial symmetries: TRS, PHS, and chiral symmetry. We found the topological classification depends not only on the set of symmetries which are respected but also on the way reflection symmetry is realized, i.e., algebraic relations satisfied among reflection and other nonspatial discrete symmetries when they exist. Our result of the classification of reflection-symmetric systems is summarized in Table I. The algebraic relations among reflection and nonspatial symmetry operations are denoted by $R_{\pm}$ and $R_{\pm\pm}$ in Table I.

Nontrivial topological states displayed in Table I are characterized by a topological invariant of integer ($\mathbb{Z}_2$) or $\mathbb{Z}_2$ type. For example, the entries in Table I marked by $M\mathbb{Z}$ indicate the presence of topologically protected states by a topological invariant defined on mirror invariant planes in the Brillouin zone ("mirror topological invariant"). These topological states include 3D topological insulators protected by the "mirror Chern number" discussed in Ref. 32 and 2D topological superconductors with TRS and reflection symmetry (class DIII + R) discussed in Ref. 33, which are characterized by reflection winding numbers in the 1D mirror lines. We generalize those mirror numbers to any spatial dimensions and relevant symmetry classes. Furthermore, we show that some systems are protected by the $\mathbb{Z}$ topological number and the mirror Chern number simultaneously; the larger one of these two numbers gives a new integral topological invariant (denoted by $\mathbb{Z}^1$ in Table I). In other cases, the topological insulators and superconductors in the original periodic table turn out to be invariant under reflection. If this is the case, the same topological invariant also characterizes the nontrivial topology of reflection-symmetric topological insulators and superconductors. These cases are indicated by "0", $\mathbb{Z}_2$, and $\mathbb{Z}$ in Table I. In short, we claim that the reflection-symmetric topological states are characterized by one of the topological invariants, 0, $\mathbb{Z}_2$, $\mathbb{Z}$, $M\mathbb{Z}$, and $\mathbb{Z}^1$.

In this paper, we use "the minimal Dirac Hamiltonian method" to characterize the AZ-symmetry classes with reflection symmetry. Without reflection symmetry the topological classification (Table II) of the AZ symmetry classes can be studied by Anderson localization,[18] K theory,[19] and minimal Dirac Hamiltonians.[34,35] The minimal Dirac Hamiltonian method provides a direct way to produce the original classification (Table II). In this method, we first write a bulk Dirac Hamiltonian preserving system's symmetries in the minimal matrix dimension. The topological class of the system is determined by the existence of a symmetry-preserving extra mass term (SPEMT), which keeps the system in the same topological phase during the continuous deformation. If this term exists in the minimal Dirac Hamiltonian, this phase is characterized by 0 topological invariant. If not, we consider a bigger system including two minimal Dirac Hamiltonians. The presence of a SPEMT in this system of the two copies implies $\mathbb{Z}_2$ topological invariant character. Otherwise, the absence of a SPEMT implies $\mathbb{Z}$ character. When classifying reflection-symmetry topological insulators and superconductors, we study the existence of a SPEMT in Dirac Hamiltonians to determine topological characters. Complementary to the minimal Dirac Hamiltonian method, we also look for topological invariants (0, $\mathbb{Z}_2$, $\mathbb{Z}$, $M\mathbb{Z}$, and $\mathbb{Z}^1$) in the presence of reflection symmetry to determine bulk topology. The classification of bulk topology in terms of topological invariants is fully consistent with the minimal Dirac Hamiltonian method.

Topological insulators might have topological invariants of strong and weak indices.[36] In the original classification table[18,19] for a d-dimensional system, the strong index is the topological invariant in d dimensions and the weak indices are captured by the strong indices in the dimensions less than d. However, the complications arise when the weak indices are considered in the classification of reflection-symmetric systems. The reason is that the weak indices may not be described by the strong indices in the dimensions less than d in the reflection classification table. Moreover, the weak indices might depend on spatial directions. That is, in different directions the weak indices are different because reflection-symmetry operation only flips one direction. In this paper, we focus on only the strong indices of the classification.

To name a few physically interesting topological systems in Table I, in symmetry class AII in 3D, with reflection symmetry specified by R_{--}, there are topological insulators protected by the $M\mathbb{Z}$ invariant. These are nothing but the topological insulator that was proposed by Hsieh *et al.*[28] and observed by several groups.[29–31] Their observation is the first experimental realization of crystalline topological insulators. This particular reflection-symmetric topological insulator continues to be topologically nontrivial even in the absence of TRS, as indicated by "$M\mathbb{Z}$" in symmetry class A in 3D in Table I. Table I also includes topological

TABLE I. The complete classification table of reflection-symmetric topological insulators and superconductors: For class AIII, $R_{\pm}$ indicates that the reflection-symmetry operator (R) commutes/anticommutes with S. For four real symmetry classes ($R_{\pm\pm}$, $R_{\pm\mp}$) that have TRS and PHS, the first sign $\pm$ of R indicates that R commutes/anticommutes with T and the second sign $\pm$ indicates that R commutes/anticommutes with C. For the four other real symmetry classes ($R_{\pm}$) that preserve only one nonspatial symmetry, the sign $\pm$ indicates that R commutes/anticommutes with system's nonspatial symmetry operator. The Hamiltonian in the mirror-symmetry plane can be block-diagonalized to two blocks in the eigenspace $R = \pm 1$. The superscript of 2 in the mirror-symmetry classes (MSC) (see Appendix B) indicates that these two blocks of $R = \pm 1$ are independent.

AZ class	T	C	S	R operator	MSC	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d = 8$
AIII	0	0	1	R_{+}	AIII^2	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$
				R_{-}	A	$\mathbb{Z}^1$	0	$\mathbb{Z}^1$	0	$\mathbb{Z}^1$	0	$\mathbb{Z}^1$	0
A	0	0	0	R	A^2	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0
AI	+	0	0	R_{+}[a]	AI^2	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
				R_{-}	A	0	0	$2M\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$M\mathbb{Z}$	0
BDI	+	+	1	R_{++}[a]	BDI^2	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0	$\mathbb{Z}_2$
				R_{--}	AIII	0	0	0	$2M\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$M\mathbb{Z}$
				R_{+-}	AI	$2\mathbb{Z}^1$	0	0	0	$\mathbb{Z}^1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
				R_{-+}	D	$2\mathbb{Z}$	0	$2M\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2M\mathbb{Z}$	0
D	0	+	0	R_{+}[a]	D^2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0
				R_{-}[b]	A	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0	0	$\mathbb{Z}_2$
DIII	−	+	1	R_{++}	DIII^2	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$
				R_{--}[b]	AIII	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0	0
				R_{+-}	AII	$2M\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2M\mathbb{Z}$	0	0	$2\mathbb{Z}$
				R_{-+}	D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^1$	0	0	0	$2\mathbb{Z}^1$	0
AII	−	0	0	R_{+}	AII^2	$2M\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0
				R_{-}[b]	A	0	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0
CII	−	−	1	R_{++}	CII^2	0	$2M\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0
				R_{--}	AIII	0	0	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$
				R_{+-}	AII	$2\mathbb{Z}^1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^1$	0	0	0
				R_{-+}	C	$2\mathbb{Z}$	0	$2M\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2M\mathbb{Z}$	0
C	0	−	0	R_{+}[c]	C^2	0	0	$2M\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0
				R_{-}	A	$2M\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0
CI	+	−	1	R_{++}[d]	CI^2	0	0	0	$2M\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$
				R_{--}	AIII	0	$2M\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0
				R_{+-}	AI	$2M\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2M\mathbb{Z}$	0	0	$2\mathbb{Z}$
				R_{-+}	C	0	0	$2\mathbb{Z}^1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^1$	0

[a]Spinless systems.
[b]Spin-$\frac{1}{2}$ systems.
[c]Spin-$\frac{1}{2}$, $C^2 = 1$, SU(2) symmetry for the spin.
[d]Spin-$\frac{1}{2}$, $C^2 = 1$, and $T^2 = -1$, SU(2) symmetry for the spin.

TABLE II. The original classification table of topological insulators and superconductors without reflection symmetry (Refs. 18 and 19). The first column represents the names of the ten symmetry classes associated with the presence or absence of TR, PH, and chiral symmetries in the last three columns. The number 0 in the last three columns denotes the absence of the symmetry. The numbers +1 and −1 denote the presence of the symmetry and indicate the signs of the square TR operator and the square PH operator.

AZ class\\d	0	1	2	3	4	5	6	7	T	C	S
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	0
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	1
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	+	0	0
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	+	+	1
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	0	+	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	−	+	1
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	−	0	0
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	−	−	1
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	−	0
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	+	−	1

superconductors protected by reflection symmetry, such as with TRS and reflection-symmetry 2D topological superconductors (class DIII + R_{--}),[33] which are classified by an integral-valued topological invariant. For symmetry class D in 2D, which hosts T-breaking topological superconductors in the absence of reflection symmetry, there is a reflection-symmetric topological superconductor characterized by a $\mathbb{Z}_2$ topological invariant. Other examples are also discussed in Sec. VI.

The paper is organized as follows. In Sec. II, we provide the background knowledge of reflection symmetry in band theory and, in particular, describe how we distinguish different realizations of reflection-symmetry operation in the presence of other nonspatial discrete symmetries. In Sec. III, we review the connection between the minimal bulk Dirac Hamiltonians and the topology possessing 0, $\mathbb{Z}_2$, and $\mathbb{Z}$ topological invariants in the Altland-Zirnbauer (AZ) symmetry classes without reflection symmetry. Moreover, by considering reflection-symmetric Dirac Hamiltonians we show the correspondence between gapless boundary states and bulk topology. In Sec. IV, we consider the two kinds of the classifications: The reflection-symmetry operator commutes with all of the nonspatial discrete symmetries and anticommutes with TRS and PHS operators. In Sec. V, we classify the remaining AZ-symmetry classes possessing TRS and PHS under the condition that one of these symmetry operators commutes with the reflection-symmetry operator and the the other anticommutes with the reflection-symmetry operator. In Sec. VI, the concrete examples for topological insulators/superconductors protected by reflection symmetry are provided.

II. REFLECTION SYMMETRY IN BAND INSULATORS

To describe our schemes for reflection-symmetric topological insulators and superconductors, in the following we start from a tight-binding Hamiltonian. We focus on electronic insulators, i.e., systems with a conserved U(1) charge, but a similar tight-binding formalism can be developed for BdG Hamiltonians of topological superconductors, since both kinds of Hamiltonians can be treated as noninteracting systems.[18] Let us consider a tight-binding (noninteracting) Hamiltonian,

$$H = \sum_{r,r'} \psi^\dagger(r)\,\mathcal{H}(r,r')\,\psi(r'), \tag{1}$$

where $\psi(r)$ is a N_f component fermion annihilation operator, and index $r = (r_1, r_2, \ldots, r_d)$ labels a site on a d-dimensional lattice (the internal indices are suppressed). Each block in the single-particle Hamiltonian $\mathcal{H}(r,r')$ is an $N_f \times N_f$ matrix, satisfying the Hermiticity condition $\mathcal{H}^\dagger(r',r) = \mathcal{H}(r,r')$, and we assume the total size of the single-particle Hamiltonian is $N_f V \times N_f V$, where V is the total number of lattice sites. The components in $\psi(r)$ can describe, e.g., orbitals or spin degrees of freedom, as well as different sites within a crystal unit cell centered at r.

Provided the system has translational symmetry, $\mathcal{H}(r,r') = \mathcal{H}(r - r')$, with periodic boundary conditions in each spatial direction (i.e., the system is defined on a torus T^d), we can perform the Fourier transformation and obtain in momentum space

$$H = \sum_{k\in \mathrm{BZ}} \psi^\dagger(k)\,\mathcal{H}(k)\,\psi(k), \tag{2}$$

where the crystal momentum k runs over the first Brillouin zone (BZ), and the Fourier component of the fermion operator and the Hamiltonian are given by $\psi(r) = V^{-1/2}\sum_{k\in\mathrm{BZ}} e^{ik\cdot r}\psi(k)$ and $\mathcal{H}(k) = \sum_r e^{-ik\cdot r}\mathcal{H}(r)$, respectively. The Bloch Hamiltonian $\mathcal{H}(k)$ is diagonalized by

$$\mathcal{H}(k)|u^a(k)\rangle = \varepsilon^a(k)|u^a(k)\rangle, \quad a = 1,\ldots,N_f, \tag{3}$$

where $|u^a(k)\rangle$ is the ath Bloch wave function with energy $\varepsilon^a(k)$. We assume that there is a finite gap at the Fermi level, and therefore we obtain the unique ground state by filling all states below the Fermi level. [In this paper, we always adjust $\varepsilon^a(k)$ in such a way that the Fermi level is at zero energy.] We assume there are N_- (N_+) occupied (unoccupied) Bloch wave functions with $N_+ + N_- = N_f$. We call the set of filled/unfilled Bloch wave functions as $\{|u^-_{\hat{a}}(k)\rangle\} \equiv \{|v_{\hat{a}}(k)\rangle\}$, $\{|u^+_a(k)\rangle\} \equiv \{|w_a(k)\rangle\}$, respectively, where hatted indices $\hat{a} = 1,\ldots,N_-$ label the occupied bands only.

In discussing symmetry-protected topological phases, we consider a set of (discrete) symmetry conditions imposed on the tight-binding Hamiltonians. Altland-Zirnbauer discrete symmetries, i.e., TRS, PHS, and chiral symmetry, act on the Bloch Hamiltonian as

$$\begin{aligned} T^{-1}\mathcal{H}(-k)T &= \mathcal{H}(k),\\ C^{-1}\mathcal{H}(-k)C &= -\mathcal{H}(k),\\ S^{-1}\mathcal{H}(k)S &= -\mathcal{H}(k), \end{aligned} \tag{4}$$

respectively, where T and C are antilinear operators, and S is a unitary operator. These are on-site (purely local) symmetries. While PHS can most naturally be introduced in the context of BdG Hamiltonians, one can still impose a PHS for electronic systems with conserved particle number.

On the other hand, reflection ($\mathcal{R}$) is a nonlocal operation; by definition, a reflection $\mathcal{R}$ in the x direction (=r_1 direction), say, connects fermion operators at $r = (r_1, r_2, \ldots, r_d)$ and at $\tilde{r} \equiv (-r_1, r_2, \ldots, r_d)$, as

$$\mathcal{R}\,\psi(r)\,\mathcal{R}^{-1} = R\psi(\tilde{r}), \tag{5}$$

where R is an $N_f \times N_f$ unitary matrix implementing reflection. The invariance of H under $\mathcal{R}$ implies, in momentum space,

$$R^{-1}\mathcal{H}(k)R = \mathcal{H}(\tilde{k}), \tag{6}$$

where $\tilde{k} = (-k_1, k_2, \ldots) = (-k_1, k_\perp)$.

For example, for a spineless system, possible realizations of these symmetries are $R = \mathbb{I}$, $T = \mathbb{I}\Theta$, and $C = \tau_x\Theta$, where τ_x is the first Pauli matrix acting on the particle-hole grading (in the BdG Hamiltonian), and Θ is the complex conjugate operator. For a spin-$\frac{1}{2}$ system, $R = i\sigma_x$ (Ref. 37), $T = i\sigma_y\Theta$, and $C = \tau_x\Theta$, where σ_i indicates Pauli matrices acting on spin degrees of freedom. We consider more realizations of these symmetries later.

Before discussing different realizations, we here note that, when there is a conserved U(1) charge, there is a phase ambiguity in the definition of the reflection operator;[37] when a Hamiltonian is invariant under a reflection, $\mathcal{R}: \psi(r) \to R\psi(\tilde{r})$, the system is also invariant under the reflection followed by a U(1) gauge transformation, $\psi(r) \to e^{i\phi} R \psi(\tilde{r})$, where $e^{i\phi}$ is an arbitrary phase factor. The combined transformation, $\mathcal{R}' : \psi(r) \to R' \psi(\tilde{r})$, with $R' = e^{i\phi} R$, is also qualified to be called reflection operation. This redefinition changes, e.g., the eigenvalues of the reflection transformations.

In this paper, we require R to be *Hermitian*. For example, in the spin-$\frac{1}{2}$ case, we add an extra $3\pi/2$ phase factor in R so that $R = \sigma_x$ anticommutes with T and C. With this convention, we construct the classification tables in terms of possible commutation and anticommutation relations of R with the three nonspatial symmetry operations. In this regard we can display the classification tables in a well-organized manner. While this is a matter of convention in classifying electrical insulators, this may not be so in classifying superconductors (BdG systems). (It should be noted that for a AZ-symmetry class that can be interpreted as a BdG systems, it can also be realized as an electrical system with some fine tuning.)

One reason for this convention is that for different choices for the phase of reflection operator (e.g., $R' = e^{i\phi} R$), its algebraic relation with T and C is different.

Note that while $\mathcal{R},\mathcal{C},\mathcal{T}$, when acting on a fermion operator, may not commute due to the phase factor $e^{i\phi} R$, they always commute when acting on any fermion bilinears. Physically, all of the point group symmetry operators are expected to commute with three nonspatial symmetry operators: TR, PH, and chiral symmetry operators. Although the requirement of the Hermiticity of R may not correspond to the real system, it simplifies the classification tables.

We consider the topological classification when *Hermitian* R commutes or anticommutes with T, C, and S. For simplicity, we define R_S, R_T, and R_C obeying

$$SRS^{-1} = R_S R, \quad TRT^{-1} = R_T R, \quad CRC^{-1} = R_C R. \tag{7}$$

Hence, $R_X = \pm 1$ indicates the commutation or anticommutation relation between R and the nonspatial symmetry operator X. Furthermore, for the complex symmetry classes, we define the symbol of the reflection-symmetry operator R_{R_S} to display the algebraic relation between R and S. For four real symmetry classes that preserve TRS and PHS, the symbol R_{R_T,R_C} shows the similar property for T and C and provides the relation between R and S, which is the combination of T and C. For the four other real symmetry classes that have only one nonspatial symmetry (TRS or PHS), the symbol of the reflection-symmetry operator $R_{R_{T/C}}$ is defined to show the algebraic relation between R and the nonspatial symmetry operator. In short, classes AIII, AI, D, AII, and C, which preserve only one nonspatial symmetry, have two possible reflection-symmetry operators R_- and R_+. On the other hand, classes BDI, DIII, CII, and CI, which preserve TRS and PHS, possess four possible reflection-symmetry operators: R_{--}, R_{++}, R_{-+}, and R_{+-}. To further simplify our notations, we define in a *real* symmetry class the reflection-symmetry operator (R_+ and R_{++}) that commutes all nonspatial symmetry operators as $\mathcal{R}^+$. Similarly, $\mathcal{R}^-$ indicates the reflection-symmetry operator (R_- and R_{--}) anticommuting with T and/or C in a *real* symmetry class.

III. TOPOLOGICAL CLASSIFICATION OF DIRAC HAMILTONIANS WITHOUT REFLECTION SYMMETRY

To capture the essential topological features in an efficient manner, we use the minimal Dirac Hamiltonian method.[35,38] This method simply considers the minimal matrix form of Dirac Hamiltonians in each AZ-symmetry class and spatial dimension. (See below for more details.) When applied to topological insulators and superconductors without spatial symmetries, this method reproduces the periodic table of topological insulators and superconductors in the AZ-symmetry classes (see Table II). Such a Dirac Hamiltonian represents a generic Hamiltonian with the same topological features when the spectra of the two systems can be continuously deformed from one to the other without closing the bulk band gap. Therefore, we still use the method of minimal Dirac Hamiltonians to classify topological phases of reflection-symmetric topological insulators and superconductors.

First, let us review the method of *minimal Dirac Hamiltonians*. The Dirac Hamiltonian in the minimal matrix dimension in d spatial dimensions, which respects the set of symmetries under consideration, is written as

$$\mathcal{H} = m\gamma_0 + k_1\gamma_1 + \sum_{i\neq 1}^{d} k_i\gamma_i, \tag{8}$$

where m is a constant and γ matrices γ_i obey the anticommutation relations

$$\{\gamma_i,\gamma_j\} = 2\delta_{ij}\mathbb{I}, \quad i = 0,1,\ldots,d. \tag{9}$$

In this paper, we consider the classification of reflection-symmetric systems with three local symmetries: TRS, PHS, and chiral symmetry. That is, we seek the topological features of the AZ symmetry classes[39] with reflection symmetry. When the Dirac Hamiltonian obeys TRS, PHS, and chiral symmetry equations as shown in Eq. (4), the γ matrices satisfy

$$[\gamma_0,T] = 0, \quad \{\gamma_{i\neq 0},T\} = 0, \tag{10}$$

$$\{\gamma_0,C\} = 0, \quad [\gamma_{i\neq 0},C] = 0, \tag{11}$$

$$\{\gamma_i,S\} = 0, \tag{12}$$

respectively. In addition, when the system preserves reflection symmetry, each term in the Dirac Hamiltonian obeys

$$\{\gamma_1,R\} = 0, \quad [\gamma_{i\neq 1},R] = 0. \tag{13}$$

Consider a system in a real symmetry class, which possesses TRS and/or PHS. If a term has the same anticommutation or commutation relation of γ_0 in Eq. (10) and/or Eq. (11) and anticommutes with the other γ matrices in the Dirac Hamiltonian, we call this term as a *mass* term. Similarly, if a term behaves like $\gamma_{i\neq 0}$ in Eq. (10) and/or Eq. (11), this term is identified as a *kinetic* term. Furthermore, if a mass term anticommutes with γ_0, this is called as an *extra* mass term. For a complex symmetry class, we also can define an extra mass term, which anticommutes with each γ matrix in the Dirac Hamiltonian.

The presence of an extra mass term in the Dirac Hamiltonian plays an essential role in distinguishing topological phases in a system. We consider a system that preserves nonspatial symmetries above with/without other symmetries. If any extra mass term does not preserve the system's symmetries, this term cannot be added to the Hamiltonian. When $m = 0$, the bulk spectrum becomes gapless. With negative energies filled, we identify $m = 0$ as a quantum phase transition point, where the gap between the empty and occupied band closes at $k = 0$. That is, the phases with positive and negative m are topologically different. On the contrary, an extra mass term preserving the system's symmetries can be added to the Hamiltonian as a perturbation. We define this term as a SPEMT. It is worth mentioning that by definition in a real symmetry class an extra mass term is always a SPEMT when only TRS and/or PHS are considered. When m varies from $-\infty$ to ∞, there are no gap closing points. Therefore, the system for any m is always in the same phase, which is topologically *trivial*.

A. Topological invariant 0

For the topological classification of the AZ-symmetry classes there are three different kinds of topological invariants: 0, $\mathbb{Z}_2$, $\mathbb{Z}$. For a given set of symmetries and spatial dimension, we write down a Dirac Hamiltonian of the minimal matrix dimension, which is in the form of Eq. (8). If a SPEMT ($\mathfrak{M}$) is allowed to be added to the Hamiltonian, the system is always in the trivial phase; we can classify this phase as topological invariant 0.

For example, consider a 1D Dirac Hamiltonian in class AII. We write a Dirac Hamiltonian in the form of the minimal matrix dimension

$$h = M\tau_z + k_x\sigma_z\tau_x, \tag{14}$$

where σ_i describes spin degree freedom and τ_i describes orbital degree freedom. TRS is preserved with TRS operator $T = i\sigma_y\Theta$. An extra mass term $\sigma_z\tau_y$, which preserves TRS, plays a SPEMT role ($\mathfrak{M}$). When M varies, the system is always in the same trivial phase.

For the other two cases ($\mathbb{Z}_2$ and $\mathbb{Z}$) any SPEMT does not exist in the minimal model.[35,38] Therefore, the system has at least two different phases by varying m in Eq. (8). To distinguish $\mathbb{Z}_2$ and $\mathbb{Z}$, we need to enlarge the Hamiltonian and then check the presence of a SPEMT.

B. Topological invariant "$\mathbb{Z}_2$"

While enlarging the Dirac Hamiltonian, we consider in the new system two minimal Dirac Hamiltonians, which may have the same or opposite orientations. That is, one is given by Eq. (8) and the other is in the form of Eq. (8) with some $\gamma_i \to -\gamma_i$. Moreover, each new γ matrix in the enlarged Hamiltonian must anticommute with each other and keep the original symmetries. The expression of the enlarged Hamiltonian of the two minimal Dirac Hamiltonians can be written as

$$\mathcal{H}_2 = \sum_i k_{n_i}\gamma_{n_i}\otimes\sigma_z + \sum_{\text{remain}} k_{n_j}\gamma_{n_j}\otimes\mathbb{I}. \tag{15}$$

The orientation of the second minimal Dirac Hamiltonian is determined by σ_z and to simplify the expression of the equation, let $m = k_0$. The first summation is over an arbitrary set of γ_{n_i} ($n_i = 0,1,2,\dots,d$) and the second summation is over γ_{n_i}'s that are not picked up by the first summation. For the system with a $\mathbb{Z}_2$ topological invariant, a SPEMT can always be added to the enlarged Hamiltonian in Eq. (15) so the system is in the trivial phase. Therefore, the corresponding symmetries and spatial dimension restrict that the system can be in the only two different phases when the system is characterized by the sign of m in the minimal Dirac Hamiltonian.

We provide an example to explain $\mathbb{Z}_2$ properties for Dirac Hamiltonians. Consider the 2D low-energy Hamiltonian[5] (h_{AII}) in HgTe quantum wells, which preserves TRS in class AII and is one of the minimal models,

$$h_{\text{AII}} = M\tau_z + k_1\sigma_z\tau_x + k_2\tau_y. \tag{16}$$

Each matrix in the Hamiltonian satisfies the TRS conditions in Eq. (10) with TRS operator $T = i\sigma_y\Theta$. In this case, all possible extra mass terms, which anticommute with $\sigma_z\tau_z$, $\sigma_z\tau_x$, and τ_y, are $\sigma_x\tau_x$ and $\sigma_y\tau_x$. However, these two terms, which do not preserve TRS, are not allowed to be added to the Hamiltonian so SPEMTs are absent. Therefore, positive and negative M represent two different topological phases.

Since this system is classified as $\mathbb{Z}_2$, a new system that is constructed by the two Hamiltonians in HgTe quantum wells is always in the same topological phase. The Hamiltonian for the new system is in form of

$$H_{\text{AII}} = \begin{pmatrix} h_{\text{AII}} & 0 \\ 0 & h'_{\text{AII}} \end{pmatrix}, \tag{17}$$

where $h'_{\text{AII}\pm\pm\pm} = \pm M\tau_z \pm k_1\sigma_z\tau_x \pm k_2\tau_y$. The signs determine that the new Hamiltonian might be in eight possible forms. It is not difficult to show for each form that at least one SPEMT can be present in the new Hamiltonian. For example, for $+++$ the SPEMTs can be $\sigma_y\tau_x\otimes\sigma_x$ and $\sigma_x\tau_y\otimes\sigma_y$. Thus, the new system is always in the same topological phase.

C. Topological invariant "$\mathbb{Z}(2\mathbb{Z})$"

For the system with a $\mathbb{Z}$ (or $2\mathbb{Z}$) topological invariant, when the first summation in Eq. (15) includes an *odd* number of γ_{n_i}'s, a SPEMT can be treated as a perturbation added to the Hamiltonian. However, when there are *even* number of γ_{n_i}'s in the first summation, a SPEMT does not exist. Therefore, the system can go through a quantum phase transition as m varies from positive to negative.

To explain a system with the $\mathbb{Z}(2\mathbb{Z})$ invariant, we consider n copies of the minimal Dirac Hamiltonian but with different m's:

$$\mathcal{H}_m = \gamma_{n_i}\otimes\begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & m_n \end{pmatrix} + \sum_{i=1}^{d} k_{n_j}\gamma_i\otimes\mathbb{I}. \tag{18}$$

Assume all m_i's are positive. When one of the m_i's varies from positive to negative, the system goes through a quantum phase transition. This phase transition cannot be avoided in the absence of SPEMTs due to the $\mathbb{Z}(2\mathbb{Z})$ invariant.[35,38] By adjusting m_i's the system passes through n times of different quantum phase transitions. Hence, in different m_i's,

$n+1$ different quantum phases, which are labeled by the $\mathbb{Z}$ topological invariant, describe the system.

To explain $\mathbb{Z}$ invariant, we consider the low-energy Hamiltonian of quantum anomalous Hall effect[1] as an example. This 2D system, which does not preserve any symmetry, belongs to class A. One of the simplest Hamiltonians, which is also a minimal Dirac Hamiltonian, can be written as

$$h_{\mathrm{A}} = M\sigma_z + k_x\sigma_x + k_y\sigma_y. \tag{19}$$

It is impossible to find an extra mass term because only three γ matrices can be present in the 2×2 matrix dimension. Therefore, the system can be changed to the different topological phase by varying M. This is similar with a system with $\mathbb{Z}_2$ invariant. To distinguish $\mathbb{Z}$ and $\mathbb{Z}_2$, introducing two copies of h_{A} is necessary.

First, the twice-as-big Hamiltonian with two identical h_{A}'s is given by

$$H_{rmA} = M\sigma_z\otimes\mathbb{I}_{2\times2} + k_x\sigma_x\otimes\mathbb{I}_{2\times2} + k_y\sigma_y\otimes\mathbb{I}_{2\times2}, \tag{20}$$

for which there is no extra mass term so the system can be in several different topological phases. Second, we change the sign of one of the Pauli matrices in the second h_{A}. The Hamiltonian is written as

$$H'_{rmA} = M\sigma_z\otimes\mathbb{I}_{2\times2} + k_x\sigma_x\otimes\sigma_z + k_y\sigma_y\otimes\mathbb{I}_{2\times2}. \tag{21}$$

Two extra mass terms ($\sigma_x\otimes\sigma_x$ and $\sigma_x\otimes\sigma_y$), which are also SPEMTs, can be found. Hence, the system is always in the same phase.

In a system possessing a $\mathbb{Z}$ (or $2\mathbb{Z}$) topological invariant, we compute the topological number of the Dirac Hamiltonian in the form of Eq. (18). Let $m_i = M - k^2$. By calculating the winding number and the Chern number (see Appendix A), the topological number in this system is $n(2n)$ if M is positive and 0 if M is negative. As M is positive, one sign flipping ($\gamma_i \to -\gamma_i$) causes the topological number to change its sign. When the two systems with and without the sign switching are coupled, a SPEMT can be present in the mixed Hamiltonian. The entire system in the trivial phase is consistent with the zero value of the topological number ($n - n = 0$).

D. The correspondence between gapless edge states and bulk topology

The topologically protected gapless edge states are present on the boundary between the trivial and nontrivial phases. The presence and absence of such gapless edge states determines the topological bulk phases: (i) the *presence* of *intact* gapless edge states implies the nontriviality in the bulk. (ii) the *absence* of such states guarantees the system in the trivial phase. However, the latter statement is not always true[23] when spatial symmetries are introduced. In the following, we show that this statement holds when a system only preserves TRS, PHS, or chiral symmetry. Furthermore, for reflection symmetry this correspondence between bulk and boundaries is also true when we choose the boundaries reflected to themselves and the translation symmetry in the reflection direction is preserved.

First, consider the minimal Dirac Hamiltonian in Eq. (8) in a symmetry class with $m = M - k^2$. Moreover, we consider a domain wall in the r_d direction: Let $M = M_0$ be a positive constant in the region $r_d > 0$, which is in the nontrivial phase. For the trivial phase region $r_d < 0$, let $M = -M_0$ be a negative constant. Therefore, k_d is not a good quantum number. We replace k_d by $-i\partial/\partial r_d$. The Dirac Hamiltonian can be rewritten as

$$\mathcal{H} = \gamma_0\left(m\mathbb{I} - i\gamma_0\gamma_d\frac{\partial}{\partial r_d}\right) + \sum_{i=1}^{d-1} k_i\gamma_i, \tag{22}$$

where $m = M + (\frac{\partial}{\partial r_d})^2 - \sum_{i=1}^{d-1} k_i^2$. To have the gapless energy states, we expect to find the wave functions so that the terms in the parentheses vanish. To satisfy this vanishing condition, there are two possible solutions $i\gamma_0\gamma_d\vec{\phi} = \pm\vec{\phi}$. We choose the minus sign to have the normalizable wave functions

$$\begin{aligned}\Phi(r_d > 0) &= \left(c_1 e^{-\frac{1}{2}(1-m_-)r_d} + c_2 e^{-\frac{1}{2}(1+m_-)r_d}\right)\vec{\phi},\\ \Phi(r_d < 0) &= e^{-\frac{1}{2}(1-m_+)r_d}\vec{\phi},\end{aligned} \tag{23}$$

where $m_\pm = \sqrt{1 \pm 4M_0 - 4\sum_{i=1}^{d-1} k_i^2}$. Our focus is on the low-energy spectrum near $k = 0$ so $M_0 > \sum_{i=1}^{d-1} k_i^2$. Therefore, $m_+ > 1$ and $\mathrm{Re}(m_-) < 1$ show that the wave function is normalizable.

Because $i\gamma_0\gamma_d$ commutes with $\gamma_{i\neq 0,\,d}$, by using the projection operator $\mathbf{P} = (\mathbb{I} - i\gamma_0\gamma_d)/2$ we can discuss the projective Hamiltonian for the edge states in the $(i\gamma_0\gamma_d = -1)$ eigenspace

$$\mathcal{H}_{\mathrm{eff}} = \sum_{i=1}^{d-1} k_i\gamma_{\mathbf{p}i}, \tag{24}$$

where $\gamma_{\mathbf{p}i} = \mathbf{P}\gamma_i\mathbf{P}$. The energy spectrum $(\pm\sqrt{\sum_{i=1}^{d-1} k_i^2})$ shows the gapless behavior of the edge states. Using the projective Hamiltonian, we can prove statement (i) by considering a domain wall. When both sides of the domain wall are in the trivial phase, a SPEMT(Γ) can be added into the Hamiltonian. Because Γ anticommutes with all of the other γ matrix, $\gamma = \mathbf{P}\Gamma\mathbf{P}$ does not vanish and anticommutes with all of $\gamma_{\mathbf{p}i}$'s. Therefore, the gapless edge states become gapped. Furthermore, this statement implies that when the gapless edge states are intact, at least one side of the domain wall must be in a nontrivial phase.

To investigate statement (ii), we focus on the behavior of the bulk Hamiltonian when the edge states are gapped without breaking any symmetry. The only one way to gap the *edge states* in $\mathcal{H}_{\mathrm{eff}}$ is to add a symmetry-preserving term that anticommutes with H_{eff}, say $\tilde{\gamma}$. In the bulk Hamiltonian there exists corresponding symmetry preserving $\tilde{\Gamma}$ so that $\tilde{\gamma} = \mathbf{P}\tilde{\Gamma}\mathbf{P}$. Therefore, $\tilde{\Gamma}$ must commute with $i\gamma_0\gamma_d$ (Ref. 40). There are two possibilities of commutation and anticommutation relations of $\tilde{\Gamma}$ with each γ_i. First, $\tilde{\Gamma}$ anticommutes with each γ_i (Ref. 41). Second, $\tilde{\Gamma}$ anticommutes with $\gamma_{i\neq 0,d}$ but commutes with γ_0 and γ_d (Ref. 42). In the first case $\tilde{\Gamma}$ plays a role of SPEMT keeping the system in the trivial phase. The second case needs to be investigated scrupulously. Although $\tilde{\Gamma}$ is not a SPEMT, in the case of some specific symmetries, there exist a SPEMT, which is a Hermitian matrix $i\tilde{\Gamma}\gamma_0\gamma_d$. If the system preserves TRS, PHS, and chiral symmetry, then by Eqs. (12) to (11) $i\tilde{\Gamma}\gamma_0\gamma_d$ also preserves those symmetries. The correspondence between edge states and bulk topology can be applied for these three symmetries.

Let reflection symmetry reflect only in the k_d direction, then $\{R,\gamma_d\} = 0$ and $[R,\gamma_{i\neq d}] = 0$. Therefore, $i\tilde{\Gamma}\gamma_0\gamma_d$ breaks reflection symmetry and then the absence of the gapless edge states does not imply the trivial bulk topology. However, if reflection symmetry is not in the k_d direction, then $i\tilde{\Gamma}\gamma_0\gamma_d$ preserves the symmetry and can be present in the Hamiltonian as a SPEMT. The gapped edge states possessing reflection symmetry guarantee the triviality in bulk.

When the translational symmetry in the reflection direction is broken, the correspondence between gapless edge states and bulk topology does not hold. However, we still can use the midgap states in the entanglement spectrum to distinguish topological trivial and nontrivial phase.[21,23] We leave this issue in the future discussion. In the paper, we use the existence of SPEMTs in the minimal Dirac Hamiltonians to determine possible topological phases.

IV. THE CLASSIFICATION OF R_+-, $\mathcal{R}^+$-, AND $\mathcal{R}^-$-SHIFTED PERIODIC TABLE

We consider the *real* AZ-symmetry classes with the reflection-symmetry commuting ($\mathcal{R}^+$) and anticommuting ($\mathcal{R}^-$) with T and C. For $\mathcal{R}^+$, the classification table is obtained from the original table without reflection symmetry by "upward shift" in spatial dimensions as shown in Table III. The topological invariant $\mathbb{Z}$ is replaced by a new topological invariant $M\mathbb{Z}$, which is explained later ("mirror" topological invariant). Similarly, for $\mathcal{R}^-$, in d dimensions the topological invariants in the new table is the ones in $d+1$ dimensions in the original table ("downward shift"), except for the absence of the second descendant $\mathbb{Z}_2$[43] of $\mathbb{Z}$ as shown in Table IV. In the following, for the real symmetry classes, we construct these two tables for $\mathcal{R}^-$ and $\mathcal{R}^+$ and define the topological invariant $M\mathbb{Z}$. We leave the discussion of the *complex* symmetry classes (R_+) for interested readers.

A. Classification of Dirac Hamiltonians

Let us start by giving a brief description of the mechanism behind these dimensional shifts. By knowing that γ_1 anticommutes with all of the other γ matrices, we construct a Hermitian matrix $i\gamma_1 R$ satisfying the anticommutation relation

$$\{i\gamma_1 R, \mathcal{H}\} = 0. \tag{25}$$

First, for the case of $\mathcal{R}^-$, from Eqs. (13), (10), and (11) $i\gamma_1 R$ can be used as a γ matrix to construct another Dirac kinetic term in one higher dimensions $(d+1)$, say as γ_{d+1}. Because $[i\gamma_1 R, C] = 0$ and $\{i\gamma_1 R, T\} = 0$, the $(d+1)$-dimensional Hamiltonian preserves the same set of local AZ symmetries. Alternatively, to construct a Hamiltonian in d dimensions with $\mathcal{R}^-$, we can start from a system in $d+1$ spatial dimensions preserving the same local symmetries, but not reflection. By removing one γ matrix γ_{d+1} (and momentum component) from the kinetic, we obtain the d-dimensional Hamiltonian with reflection symmetry. We later make use of the topological classification of the $d+1$-dimensional Hamiltonians without reflection to discuss the topological classification of our d-dimensional target Hamiltonians with reflection symmetry.

On the other hand, for the case of $\mathcal{R}^+$, $i\gamma_1 R$ can be used as an extra mass term: It can be added to the Hamiltonian without changing its AZ-symmetry class since $\{i\gamma_1 R, C\} = 0$ and $[i\gamma_1 R, T] = 0$, while $i\gamma_1 R$ breaks the reflection symmetry. Because of the algebraic structure of the Clifford algebra, adding a mass term effectively acts as removing one kinetic γ matrix and therefore effectively decreases the spatial dimension by one.[35,38] Therefore, the topological classification of the d-dimensional Hamiltonians with $\mathcal{R}^+$ is related to the classification of $(d-1)$-dimensional Hamiltonians in the corresponding AZ-symmetry class without reflection symmetry. This "upward" shift is also supported by considering the Hamiltonian in the $(d-1)$-dimensional mirror plane in the BZ. The topological invariant defined for the $(d-1)$-dimensional Hamiltonian directly determines the topological class of the original d-dimensional Hamiltonian with reflection symmetry. (See below for more details.)

For those two cases ($\mathcal{R}^-$ and $\mathcal{R}^+$), when a SPEMT in $d \pm 1$ dimensions in the AZ-symmetry class without reflection symmetry is still a SPEMT in d dimensions with reflection symmetry, both of the systems share the same topological invariant. In the following we show that when systems in $d \pm 1$ dimensions possesses 0 and $\mathbb{Z}_2$ topological invariants, the corresponding reflectional systems have the same topological invariants. Likewise, for a $\mathbb{Z}$ invariant in $d \pm 1$ dimensions, the corresponding reflection-symmetric system in d dimension has a $\mathbb{Z}$-like topological invariant.

TABLE III. The classification table for R(class A), R_+(class AIII), and $\mathcal{R}^+$(real symmetry classes). Each nonspatial symmetry operator commutes with R. For class A, a system has only reflection symmetry so no commutation issue is in this class. If we treat $M\mathbb{Z}(2M\mathbb{Z})$ as $\mathbb{Z}(2\mathbb{Z})$, this table is obtained from the original table just by "upward shift" in spatial dimensions.

							Commutation relations of R		
AZ class + $R\backslash d$	1	2	3	4	5	6	T	C	S
A	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	0	0	0
AIII	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	$M\mathbb{Z}$	0	0	1
AI	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0	+	0	0
BDI	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	+	+	1
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	0	+	0
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	−	+	1
AII	$2M\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	−	0	0
CII	0	$2M\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$M\mathbb{Z}$	−	−	1
C	0	0	$2M\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	−	0
CI	0	0	0	$2M\mathbb{Z}$	0	$\mathbb{Z}_2$	+	−	1

TABLE IV. The classification table for $\mathcal{R}^-$. For the eight real symmetry classes, R anticommutes with T and C but commutes with S. For class AIII, the only nonspatial symmetry operator S anticommutes with R. We treat $M\mathbb{Z}(2M\mathbb{Z})$ as $\mathbb{Z}(2\mathbb{Z})$, then the topological invariants in this table in d dimensions are the ones in the original table in $d+1$ dimensions, except for the second descendant $\mathbb{Z}_2$ of $\mathbb{Z}$.

							Anticommutation relations of R		
AZ class + $R\backslash d$	1	2	3	4	5	6	T	C	S
AI	0	0	$2M\mathbb{Z}$	0	0	$\mathbb{Z}_2$	+	0	0
BDI	0	0	0	$2M\mathbb{Z}$	0	0	+	+	1
D	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	0	0	+	0
DIII	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	$2M\mathbb{Z}$	−	+	1
AII	0	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	0	−	0	0
CII	0	0	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	−	−	1
C	$2M\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$M\mathbb{Z}$	0	0	−	0
CI	0	$2M\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$M\mathbb{Z}$	+	−	1

(a) Topological invariant 0. Consider a system in an AZ-symmetry class in $d \pm 1$ that has a 0 topological invariant. Therefore, the presence of a SPEMT ($\mathfrak{M}$) in the minimal Dirac Hamiltonian in Eq. (8) keeps the system in the trivial phase. We define the reflection operator $R = i\Lambda\gamma_1$, where $\Lambda = \tilde{\gamma}_1(\gamma_{d+1})$ corresponds to $\mathcal{R}^+(\mathcal{R}^-)$. Because $\mathfrak{M}$ anticommutes with Λ and γ_1, the system in d spatial dimensions with $\mathfrak{M}$ preserves reflection symmetry. The same AZ-symmetry class with reflection symmetry in d dimensions has a 0 topological invariant.

(b) Topological invariant "$\mathbb{Z}_2$." For a $\mathbb{Z}_2$ topological invariant, in $d \pm 1$ dimensions the minimal Dirac Hamiltonian in Eq. (8) in one of the AZ-symmetry classes has no SPEMTs. Therefore, SPEMTs do not exist for the minimal Dirac Hamiltonian in d dimensions with reflection symmetry $R = i\Lambda\gamma_1$. Again, to find the topological property for the reflection symmetry, the minimal Dirac Hamiltonian in d dimensions can be enlarged in several ways,

$$\mathcal{H}_2^d = k_1\gamma_1 \otimes \mathbb{I} + \sum_{n_i \neq 1} k_{n_i}\gamma_{n_i} \otimes \sigma_z + \sum_{\text{remain}} k_{n_j}\gamma_{n_j} \otimes \mathbb{I}, \quad (26)$$

$$\mathcal{H}_2^{d'} = k_1\gamma_1 \otimes \sigma_z + \sum_{n_i \neq 1} k_{n_i}\gamma_{n_i} \otimes \sigma_z + \sum_{\text{remain}} k_{n_j}\gamma_{n_j} \otimes \mathbb{I}, \quad (27)$$

with the unchanged reflection-symmetry operator $R = i\Lambda\gamma_1 \otimes \mathbb{I}$. Because of the $\mathbb{Z}_2$ topological invariant, for $\mathcal{H}_2^d$ we can construct the Hamiltonian in $d \pm 1$ dimensions in the same AZ-symmetry class with a SPEMT $\mathfrak{M}$ without the reflection symmetry as

$$\mathcal{H}_2^{d\pm1} = \mathcal{H}_2^d + \lambda\Lambda \otimes \mathbb{I} + \mathfrak{M}, \quad (28)$$

where $\lambda = k_{d+1}(\tilde{m})$ in $d+1(d-1)$ dimensions. To check whether $\mathfrak{M}$ preserves the reflection symmetry, the commutation relation between $R = i\Lambda\gamma_1 \otimes \mathbb{I}$ and $\mathfrak{M}$ should be considered. By the definition of a SPEMT, $\mathfrak{M}$ anticommutes with $\Lambda \otimes \mathbb{I}$ and $\gamma_1 \otimes \mathbb{I}$. Therefore, $\mathfrak{M}$ preserves the reflection symmetry. The Hamiltonian can be gapped out without breaking any symmetry.

For the second Hamiltonian $\mathcal{H}_2^{d'}$, the corresponding gapped Hamiltonian in $d \pm 1$ dimensions is written as

$$\mathcal{H}_2^{d\pm1'} = \mathcal{H}_2^{d'} + \lambda\Lambda \otimes \sigma_z + \mathfrak{M}. \quad (29)$$

Because the SPEMT $\mathfrak{M}$ commutes with $\gamma_1 \otimes \mathbb{I}$ and $\Lambda \otimes \mathbb{I}$, $\mathfrak{M}$ preserves the reflection symmetry. This is so since if one of $\gamma_1 \otimes \mathbb{I}$ and $\Lambda \otimes \mathbb{I}$ anticommutes with $\mathfrak{M}$, $\mathbb{I} \otimes \sigma_z$ must commute with $\mathfrak{M}$ so both $\gamma_i \otimes \mathbb{I}$ and $\Lambda \otimes \mathbb{I}$ anticommute with $\mathfrak{M}$. This contradicts with the assumption that there are no SPEMTs in the minimal Hamiltonian. In short, the d-dimensional system with reflection symmetry has a $\mathbb{Z}_2$ invariant inherited from the $d \pm 1$-dimensional system without reflection symmetry. However, there is an *exception*. For the anticommutation case a reflection system from $d+1$ dimensions corresponding to the second descendants[43] $\mathbb{Z}_2$ of $\mathbb{Z}$ has 0 topological invariant instead of $\mathbb{Z}$, which is discussed later.

(c) Topological invariant "$\mathbb{Z}$." Consider a system in $d \pm 1$ dimensions in an AZ-symmetry class that has a $\mathbb{Z}$ topological invariant. Therefore, in the corresponding d-dimensional system with reflection symmetry any SPEMT does not exist for the minimal Dirac Hamiltonian, which is similar with the $\mathbb{Z}_2$ case. To distinguish the topological invariant from $\mathbb{Z}_2$, we need to enlarge the minimal Dirac Hamiltonian in the forms of Eqs. (26) and (27). The corresponding Hamiltonians without reflection symmetry are written in the forms of Eqs. (28) to (29), respectively, so that a SPEMT $\mathfrak{M}$ preserves the reflection symmetry. However, by the definition of the $\mathbb{Z}$ topological invariant, the SPEMT $\mathfrak{M}$ are present in Eqs. (28) and (29) only when the first summation is over *odd* number of the γ matrices. Thus, for the system with reflection symmetry the presence/absence of a SPEMT is determined by the first summation odd/even number of γ matrices in Eqs. (26) and (27) but does not depend on the way to enlarge γ_1. In the next paragraph we prove that such a system possesses a topological invariant. We label this invariant by $M\mathbb{Z}$. The reason is that the $M\mathbb{Z}$ system behaves the same with the $\mathbb{Z}$ one in the nonreflectional symmetry direction but topological property of $M\mathbb{Z}$ is insensitive in the reflectional symmetry direction. Moreover, we show that $M\mathbb{Z}$ number is defined in the reflection (mirror) symmetry planes ($k_1 = 0$ or π) so M means mirror.

B. Topological numbers

1. Topological numbers of $M\mathbb{Z}$

For the case where the reflection operator R commutes with the nonspatial discrete symmetries, the topological numbers $M\mathbb{Z}$ can be defined from the bulk Hamiltonian to characterize bulk topology and protected gapless edge states. The

Hamiltonian without k_1 commutes with R; therefore, the Hamiltonian can be block diagonalized in the two eigenspaces $R = \pm 1$ because R is Hermitian. Each individual block Hamiltonian is (not) invariant under the original symmetries if such symmetry operators (anti)commute with R. However, the Hamiltonian still belongs to one of the AZ-symmetry classes, which corresponds to the nonspatial symmetry operators commuting with R. We name this symmetry class in the mirror planes of $k_1 = 0$ or π as a *mirror-symmetry class*. The details of mirror-symmetry classes are discussed in Appendix B.

We focus on one of the blocks with a definite eigenvalue, $R = 1$, say, because the topological numbers of these two blocks differ by signs when the weak topological index vanishes. The reason is that a d-dimensional system with a nonzero weak index can be understood by a stacking limit of $d-1$-dimensional topologically nontrivial layers.[44] Furthermore, for any k_1 the sum of these two topological numbers is invariant; hence, if this total number does not vanish, by definition the weak index is nonzero. In the following, we always consider the case that the weak index vanishes to define the $M\mathbb{Z}$ number.

(a) $M\mathbb{Z}$ for $\mathcal{R}^+$. First, suppose the nonspatial symmetry operators commute with R. For a d-dimensional system possessing a $M\mathbb{Z}$ topological invariant, the topological number does not depend on the k_1 direction (direction of the reflection symmetry). Furthermore, in $d-1$ dimensions the mirror-symmetry class, which is the same with the original AZ-symmetry class, has a $\mathbb{Z}$ topological invariant. Hence, to obtain the $M\mathbb{Z}$ number, we can calculate the $\mathbb{Z}$ number in one of the blocks with a definite reflection eigenvalue in $d-1$ dimensions by using Eq. (A3) or Eq. (A6) because the Dirac Hamiltonian without γ_1 commutes with R. This $\mathbb{Z}$ property is protected by the corresponding block diagonal nonspatial symmetry operators since R commutes with these operators.

In the continuum model the $M\mathbb{Z}$ number can be properly defined for the block Hamiltonians at $k_1 = 0$. However, in the lattice model, which can be obtained by the replacement $k_1 \to \sin nk_1$ $(n \in \mathbb{Z})$, $\sin nk_1$ vanishes in the Hamiltonian only when $k_1 = 0, \pm\pi/n, \pm 2\pi/n, \ldots, \pm\pi$. These points are the possible positions to have a $(d-1)$-dimensional $\mathbb{Z}$ number. However, the $\mathbb{Z}$ numbers which are not at the symmetry points $(k_1 = 0, \pi)$, are fragile, or not protected: They can vanish by coupling the opposite $\mathbb{Z}$ numbers in the other block of $R = -1$ without breaking reflection symmetry. Furthermore, no SPEMTs are allowed in the bulk Dirac Hamiltonian around the symmetry points so the $\mathbb{Z}$ numbers at the symmetry points are invariant. Therefore, we can calculate the two numbers ν_0^{d-1} and ν_π^{d-1} at $k_1 = 0, \pi$, respectively, in the block of $R = 1$.

To have the topological number of the strong index, we need to consider translational symmetry breaking. The presence of translational symmetry breaking along the r_1 direction connects the $\mathbb{Z}$ numbers at the two symmetric points so the total topological invariant number is $\nu_0^{d-1} + \nu_\pi^{d-1}$. However, this number is not the strong index. The strong index $N_{M\mathbb{Z}}$ is given by

$$N_{M\mathbb{Z}} = \nu_0^{d-1} + \nu_\pi^{d-1} - 2N_{\text{weak}}, \tag{30}$$

where N_{weak} is the *mirror* weak index, which is the weak index in one of the blocks of $R = \pm 1$ and invariant for any k_1. Such a mirror weak index is determined by $d-1$-dimensional nontrivial layers,[44] which are stacked to a nontrivial weak system. To have the strong index, we determine the mirror weak index first by considering two possible situations: $\nu_0^{d-1}\nu_\pi^{d-1} > 0$ and $\nu_0^{d-1}\nu_\pi^{d-1} < 0$. On the one hand ($\nu_0^{d-1}\nu_\pi^{d-1} > 0$), the mirror weak index is

$$N_{\text{weak}} = \text{sgn}\left(\nu_0^{d-1}\right)\min\left(\left|\nu_0^{d-1}\right|,\left|\nu_\pi^{d-1}\right|\right). \tag{31}$$

Because the total invariant number is the sum of $2N_{\text{weak}}$ and the strong index, we can write the strong index topological number as

$$\text{sgn}\left(\nu_0^{d-1}\right)\left|\nu_0^{d-1} - \nu_\pi^{d-1}\right|. \tag{32}$$

On the other hand, when $\nu_0^{d-1}\nu_\pi^{d-1} < 0$ the mirror weak index is absent. The strong index is the total invariant number $\nu_0^{d-1} + \nu_\pi^{d-1}$. From these two cases, the $M\mathbb{Z}$ number is defined as

$$N_{M\mathbb{Z}} = \text{sgn}\left(\nu_0^{d-1} - \nu_\pi^{d-1}\right)\left(\left|\nu_0^{d-1}\right| - \left|\nu_\pi^{d-1}\right|\right). \tag{33}$$

The signs determine the orientation; however, the $N_{M\mathbb{Z}}$ does not have the summation property like $\mathbb{Z}$ topological invariant ($N_{\mathbb{Z}}$). Consider that a system is the collection of several subsystems. Each subsystem has its own $\mathbb{Z}$ number $N_{\mathbb{Z}}^i$. Therefore, the $\mathbb{Z}$ number of the entire system is given by $N_{\mathbb{Z}} = \sum_i N_{\mathbb{Z}}^i$. This relation does not hold for $M\mathbb{Z}$. To obtain $N_{M\mathbb{Z}}$, we have to compute the two $\mathbb{Z}$ numbers (ν_0^{d-1} and ν_π^{d-1}) for the entire system in the reflection-symmetric planes and then use Eq. (33).

(b) $M\mathbb{Z}$ for $\mathcal{R}^-$. Similarly, we consider the case that a d-dimensional system with reflection-symmetry operator $\mathcal{R}^-$, which anticommutes with the TRS operator or PHS operator. If d is even, the system possessing $M\mathbb{Z}$ preserves chiral symmetry. This chiral symmetry operator commutes with $\mathcal{R}^-$ so in the Hamiltonian block diagonalized by $\mathcal{R}^-$ each block has the corresponding chiral symmetry operator. Neither the TRS nor the PHS is block diagonalized at the same time because of the anticommutation relations. Therefore, each block in the Hamiltonian belongs to class AIII in even dimensions and then the topological number can be evaluated as the winding number by Eq. (A3). On the other band, if d is odd, the off-diagonal Hamiltonians, which do not preserve any symmetry, belong to class A in odd dimensions. Hence, in class A the Chern number in Eq. (A6) can characterize the topological number at the symmetry points. In short, the topological number in the anticommutation case still can be described by Eq. (33).

2. Topological numbers of $\mathbb{Z}_2$

(a) $\mathbb{Z}_2$ for $\mathcal{R}^+$. Consider a d-dimensional $\mathbb{Z}_2$ system with $\mathcal{R}^+$ commuting with the local symmetries. In such a system, the topological invariant for a block in the R-block-diagonal Hamiltonian is $\mathbb{Z}_2$ in $d-1$ dimensions in the mirror-symmetry class, which is the same with the system's symmetry class. Such a $\mathbb{Z}_2$ topological invariant was already evaluated in several ways.[3,17,36,45] Therefore, at the two symmetry points $k_1 = 0, \pi$, the $\mathbb{Z}_2$ numbers are defined as ν_0 and ν_π, respectively, in the block of the Hamiltonian. The $\mathbb{Z}_2$ number for the entire system is

$$N_{\mathbb{Z}_2} = \nu_0 + \nu_\pi \bmod 2. \tag{34}$$

The reason is that the reflection symmetry does not prevent a translational symmetry breaking density wave from coupling and gapping out a pair of bulk band-gap closing (quantum phase transitions) at these two symmetry points. Only one bulk band-gap closing survives under arbitrary symmetry preserving perturbations when a system possesses an odd number of closing.

(b) $\mathbb{Z}_2$ for $\mathcal{R}^-$. Consider the case where the reflection-symmetry operator anticommutes with T and/or C, which is more complicated. We discuss two possible cases, respectively: the first and second descendants $\mathbb{Z}_2$ of $\mathbb{Z}$ in $d+1$ dimensions.

First, consider a system with reflection symmetry corresponding to the first descendant $\mathbb{Z}_2$ of $\mathbb{Z}$ in $d+1$ dimensions. We note that in d dimensions the original topological classification gives a $\mathbb{Z}_2$ topological invariant. The topological number in this case can be defined by the original $\mathbb{Z}_2$ number. That is, a system in such a symmetry class with and without the reflection symmetry has the same topological invariant.

Second, in a system with reflection symmetry corresponding to the second descendant $\mathbb{Z}_2$ of $\mathbb{Z}$ in $d+1$ dimensions, the topological number cannot be properly defined. The reason is that T and/or C anticommute with R so the mirror-symmetry class is class A. Therefore, no $\mathbb{Z}_2$ topological numbers can be defined at $k_1 = 0, \pi$. Furthermore, in d dimensions the corresponding topological invariant is 0 in the original classification. It turns out that a SPEMT can be present in the Hamiltonian to prevent the bulk gap closing, so this case is classified as 0. The further discussion is in the following. Without enlarging the minimal Dirac Hamiltonian in d dimensions in the corresponding symmetry class, $d+3$ kinetic γ matrices $(\gamma_1, \gamma_2, \ldots, \gamma_{d+3})$ and one mass γ matrix (γ_0) can be present.[35,46] The reflection-symmetry operator is defined as $R = i\gamma_0\gamma_{d+1}$ to satisfy the anticommutation relations with TRS and PHS operators. Due to the presence of γ_{d+1}, γ_{d+2}, γ_{d+3}, we have more choices to add some symmetry-preserving terms in the minimal Dirac Hamiltonian in Eq. (8),

$$\mathcal{H}_\delta = m\gamma_0 + k_1\gamma_1 + \sum_{i\neq 1}^{d} k_i\gamma_i + \delta\Delta, \tag{35}$$

where δ is a positive constant and $\Delta = i\gamma_1\gamma_{d+1}\gamma_{d+2}$, which is invariant under all system's symmetries. Since Δ commutes with only γ_1 in $\mathcal{H}_\delta$, the eigenvalues of $\gamma_1(k_1\mathbb{I} + \delta i\gamma_{d+1}\gamma_{d+2})$ are $k_1 \pm \delta$ and $-k_1 \pm \delta$ due to the eigenvalues ± 1 of $i\gamma_{d+1}\gamma_{d+2}$. Therefore, when the quantum phase transition $(m = 0)$ occurs, the bulk gap closing points shift $k_1 = \pm\delta$ and $k_\perp = 0$. Now we can add another symmetry preserving term to prevent the bulk gap closing at the new transition points by breaking translational symmetry. This gap opening term is written in the form of the second quantization,

$$\hat{\mathfrak{N}} = \sum_{-\eta\leqslant k_1<\eta} (ic^\dagger_{k_1+\eta+\delta}\mathfrak{N}c_{k_1-\eta+\delta} + \text{H.c.}) + \sum_{-\eta<k_1\leqslant\eta} (ic^\dagger_{k_1+\eta-\delta}\mathfrak{N}c_{k_1-\eta-\delta} + \text{H.c.}), \tag{36}$$

where η is a positive constant less than δ and $\mathfrak{N} = i\gamma_{d+1}\gamma_{d+2}\gamma_{d+3}$, which anticommutes all of the terms in $\mathcal{H}_\delta$. Also, $\hat{\mathfrak{N}}$ preserves TSR and PHS by Eqs. (10) and (11). By the definition of the reflection-symmetry operator $\hat{R} = \sum_{k_1} c^\dagger_{k_1}(i\gamma_0\gamma_{d+1})c_{-k_1}$ (Ref. 47), it is easy to check that $\hat{\mathfrak{N}}$ preserves the reflection symmetry. The last thing we need to verify is that $\hat{\mathfrak{N}}$ prevents the bulk gap closing. To have the low-energy spectrum, consider that case $m = 0$ and $k_\perp = 0$ so $\mathcal{H}_\delta$ is a function of k_1. The Hamiltonian $\mathcal{H}_\delta(k_1)$ with $c\hat{\mathfrak{N}}$ is in the form of the second quantization is written as

$$\hat{\mathcal{H}}_{\delta,c} = \sum_{-\eta\leqslant k_1<\eta} (\Psi^\dagger_{k_1+\delta}\mathcal{H}_\eta(k_1+\delta)\Psi_{k_1+\delta} + \Psi^\dagger_{-k_1-\delta}\mathcal{H}_\eta(-k_1-\delta)\Psi_{-k_1-\delta}) + \text{high energy terms}, \tag{37}$$

where $\Psi_{p_1} = (c_{p_1+\eta}\ c_{p_1-\eta})^T$ and

$$\mathcal{H}_\eta(p_1) = \begin{pmatrix} \mathcal{H}_\delta(p_1+\eta) & ic\mathfrak{N} \\ -ic\mathfrak{N} & \mathcal{H}_\delta{}'(p_1-\eta) \end{pmatrix}. \tag{38}$$

We compute the eigenvalues of the two blocks $[\mathcal{H}_\eta(k_1+\delta)$ and $\mathcal{H}_\eta(-k_1-\delta)]$ to capture the low-energy spectrum, because those two blocks are the reflection-symmetry partners sharing the same energy spectrum. Therefore, consider the energy spectrum of one of the blocks, say

$$\mathcal{H}_\eta(k_1+\delta) = (k_1+\delta)\mathbb{I}\otimes\gamma_1 + \eta\sigma_z\otimes\gamma_1 + \delta\mathbb{I}\otimes\Delta + c\sigma_y\otimes\mathfrak{N}. \tag{39}$$

We note that $\mathfrak{N}$ anticommutes with Δ and γ_1 and Δ commutes with γ_1. Therefore, the expression of the energy square is

$$E^2 = \left(\eta \pm \sqrt{k_1^2 + c^2}\right)^2, \quad [\eta \pm \sqrt{(k_1+2\delta)^2 + c^2}]^2. \tag{40}$$

Hence, when c is larger than η, the energy never becomes zero. The bulk gap closing has been blocked by the symmetry-preserving terms as SPEMTs. Therefore, the topological invariant in this case is 0. In Sec. VI C, one example is provided to show that the translational symmetry-breaking $\hat{\mathfrak{N}}$ gaps the edge states and destroys the midgap states in the entanglement spectrum.

V. THE CLASSIFICATION OF R_{-+} AND R_{+-}

In this section, we consider the topological classification of insulators and superconductors for the cases of R_{-+} and R_{+-}. To have these anticommutation and commutation relations we have to consider the AZ-symmetry classes that preserve both TRS and PHS: classes DIII, CII, CI, and BDI. The results are summarized in Tables V and VI.

A. Classification of Dirac Hamiltonians with $R = i\gamma_1 S$

As a start, let us consider, as a possible definition of reflection operator, $R = i\gamma_1 S$. The commutation/anticommutation relations of $R = i\gamma_1 S$ with TRS and PHS operators are summarized in in Table V. They can be verified as follows. Let us go back to the expressions of TRS and PHS operators

$$T = U_T\Theta, \quad C = U_C\Theta, \tag{41}$$

where U_T and U_C are complex matrices. To simplify our problem, we assume that U_T and U_C are Hermitian and unitary.

TABLE V. The classification table for the case of reflection-symmetry operator given by $i\gamma_1 S$. By using the similar discussion of $R = i\gamma_1 S$, a reflection-symmetric system in class AIII + R_- possesses a $\mathbb{Z}^1$ invariant in odd dimensions and 0 invariant in even dimensions. The AZ-symmetry classes with the reflection symmetry have $\mathbb{Z}_2$ and $\mathbb{Z}^1$ corresponding to $\mathbb{Z}_2$ and $\mathbb{Z}$ without the reflection symmetry, respectively.

Class	TR	PH	Ch	R	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
AIII	0	0	1	R_-	0	$\mathbb{Z}^1$	0	$\mathbb{Z}^1$	0
BDI	+1	+1	1	R_{+-}	0	0	0	$2\mathbb{Z}^1$	0
DIII	−1	+1	1	R_{-+}	$\mathbb{Z}_2$	$\mathbb{Z}^1$	0	0	0
CII	−1	−1	1	R_{+-}	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^1$	0
CI	+1	−1	1	R_{-+}	0	$2\mathbb{Z}^1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

To define chiral operator S, which is Hermitian, we let $S = TC$ if $[U_C^*, U_T] = 0$ or $S = iTC$ if $\{U_C^*, U_T\} = 0$. Therefore, $R = i\gamma_1 S$ is Hermitian. To determine the commutation and anticommutation relations of R with T and C, we have to check the relations of S with T and C. In the both cases ($[U_C^*, U_T] = 0$ and $\{U_C^*, U_T\} = 0$), we have the same relations,

$$TST^{-1} = \pm S, \quad CSC^{-1} = \pm S, \tag{42}$$

where we pick up the plus sign in front of S when $T^2 = \pm 1$ and $C^2 = \pm 1$, whereas we pick up the minus sign when $T^2 = \pm 1$ and $C^2 = \mp 1$. The reason is that

$$\pm 1 = T^2 = U_T U_T^*, \quad \pm 1 = C^2 = U_C U_C^*. \tag{43}$$

By using Hermitian and unitary properties of U_T and U_C,

$$U_T = \pm U_T^*, \quad U_C = \pm U_C^*. \tag{44}$$

By Eqs. (10) and (11), we obtain the relations exactly shown in Table V: $[T,R] = 0$ and $\{C,R\} = 0$ when $T^2 = \pm 1$ and $C^2 = \pm 1$ and $\{T,R\} = 0$ and $[C,R] = 0$ when $T^2 = \pm 1$ and $C^2 = \mp 1$.

This construction of R does not require any new γ matrix. Hence, the absence of a SPEMT in the original classification is unchanged when the reflection symmetry is considered. When a SPEMT exists in the original Hamiltonian, we have to check whether this SPEMT preserves the reflection symmetry. If so, the system is in the trivial phase. If not, the reflection symmetry provides new topological phases. We consider the following three cases separately.

(1) Let us first suppose that a system in an AZ-symmetry class without reflection symmetry has 0 topological invariant. A SPEMT ($\mathfrak{M}$) exists in the minimal Hamiltonian. That is, $\mathfrak{M}$ anticommutes with S and γ_i. Therefore, when the reflection symmetry is imposed on the system, $R = i\gamma_1 S$ commutes with $\mathfrak{M}$ so the reflection symmetry is preserved. The system is still in the trivial phase and classified by the 0 topological invariant.

(2) For the AZ-symmetry class that has a $\mathbb{Z}_2$ topological invariant, no SPEMTs are allowed in the minimal Hamiltonian. However, when the size of the Hamiltonian is doubled in the form of Eq. (15), without considering reflection symmetry a SPEMT ($\mathfrak{M}$) does exist. We found that $\gamma_{n_i} \otimes \mathbb{I}$ in Eq. (15) must commute with $\mathfrak{M}$ because if not, the minimal Hamiltonian can have a SPEMT. Also, we know $\{\gamma_{n_j} \otimes \mathbb{I}, \mathfrak{M}\} = 0$. Therefore, if $n_i \neq 1$, $[R = i\gamma_1 S, \mathfrak{M}] = 0$ shows that the reflection symmetry is preserved. Otherwise, the reflection symmetry is broken. Hence, any SPEMT seems to be absent in this case; however, later we show that a SPEMT can be present, which is similar to the counterfeit $\mathbb{Z}_2$ case in Sec. IV B2. Hence, the reflection symmetry does not provide any extra topological phase. Such a system with the reflection symmetry is still classified as $\mathbb{Z}_2$.

(3) Finally, a system with a $\mathbb{Z}$ topological invariant guarantees no SPEMTs in the minimal Hamiltonian. After doubling the size of the minimal Hamiltonian, the original $\mathbb{Z}$ topological invariant forbids any SPEMT in Eq. (15) when there are even terms in the first summation. When reflection symmetry is considered, by reasoning similar to that in the previous discussion, $\{R = i\gamma_1 S, \mathfrak{M}\} = 0$ breaks reflection symmetry as $n_i = 1$ in Eq. (15). In short, the system can be gapped out by a SPEMT only when the first summation in Eq. (15) is over an odd number of the γ matrices, excluding γ_1. Let us go back to the discussion of the $M\mathbb{Z}$ topological invariant. Equations (26) and (27) provide all of the possible twice-as-big minimal Dirac Hamiltonian. Only when the first summation of Eq. (26) is over an odd number of the γ matrices, a SPEMT can be present in the Hamiltonian. Therefore, for an even number of the γ matrices in the first summation of Eq. (26) a SPMET is forbidden by the $\mathbb{Z}$ topological invariant. In Eq. (27) since $k_1\gamma_1 \otimes \sigma_z$ in the Hamiltonian with an even number of γ matrices in the first summation, a mass term preserving the nonspatial symmetries breaks the reflection symmetry. Therefore, the bulk topology is nontrivial for both of the even numbers of γ matrices. For the former, the $\mathbb{Z}$ number is nonzero as well as for the latter, the $M\mathbb{Z}$ number does not vanish. Thus, the bulk topology *might* be protected by $\mathbb{Z}$ and $M\mathbb{Z}$ topological invariants. In the next section, we show that for such a system the topological invariant labeled by $\mathbb{Z}^1$ is determined by $\mathbb{Z}$ and $M\mathbb{Z}$ topological numbers.

TABLE VI. The classification table for the rest of the commutation and anticommutation relations. Nontrivial topology only shows up in odd spatial dimensions.

Class	TR	PH	Ch	R	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
BDI	+1	+1	1	R_{-+}	0	$2M\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2M\mathbb{Z}$
DIII	−1	+1	1	R_{+-}	0	$2\mathbb{Z}$	0	$2M\mathbb{Z}$	0	$2\mathbb{Z}$
CII	−1	−1	1	R_{-+}	0	$2M\mathbb{Z}$	0	$2\mathbb{Z}$	0	$2M\mathbb{Z}$
CI	+1	−1	1	R_{+-}	0	$2\mathbb{Z}$	0	$2M\mathbb{Z}$	0	$2\mathbb{Z}$

The above considerations lead to the classification summarized in Table V. In the following, we present a proper definition of topological invariants.

1. Topological number of $\mathbb{Z}^1$

To define the proper $\mathbb{Z}^1$ number, characterizing bulk topology and intact gapless edge modes, we consider the $\mathbb{Z}$ number ($N_{\mathbb{Z}}$) and the $M\mathbb{Z}$ number ($N_{M\mathbb{Z}}$) as our candidates. In the following, we prove that the $\mathbb{Z}^1$ number is

$$N_{\mathbb{Z}^1} = \text{Max}(|N_{\mathbb{Z}}|,|N_{M\mathbb{Z}}|). \tag{45}$$

To simplify the problem, we consider the presence of nontrivial topology only at the $k_1 = 0$ plane. In other words, $\mathbb{Z}$ and $M\mathbb{Z}$ numbers can be determined by the Hamiltonian around this symmetry plane. We leave the general proof in Appendix C for interested readers. In a d-dimensional system the $\mathbb{Z}$ number $N_{\mathbb{Z}}$ and the $M\mathbb{Z}$ number $N_{M\mathbb{Z}}$, which is the $\mathbb{Z}$ number ν_{d-1} at $k_1 = 0$, are both defined by either Eq. (A3) or Eq. (A6). We define $(\pm,\pm)$ and $(\pm,\mp)$ to describe the signs of the γ matrices in Eq. (8). The first slot of the parentheses indicates the sign of γ_1 and the second slot $+(-)$ shows an even(odd) number of the other γ matrices having the minus sign. According to the $\mathbb{Z}^1$ properties, the system with $(+,+)$ and $(+,-)$ can be gapped out by a SPEMT. Similarly, the pair of $(-,-)$, $(-,+)$ are in the trivial phase, too. However, two minimal Hamiltonians from these two different pairs are protected by the $\mathbb{Z}^1$ topological invariant. We can treat these two pairs be two independent systems of $(+,\pm)$ and $(-,\pm)$. Let the numbers of the minimal Hamiltonians of $(\pm,\pm)$ and $(\pm,\mp)$ be $N_{\pm,\pm}$ and $N_{\pm,\mp}$ respectively. A nontrivial minimal Hamiltonian provides one protected gapless edge mode so the numbers of intact gapless edge modes for these systems are given by, respectively,

$$N_{+,\pm} = N_{+,+} - N_{+,-}, \quad N_{-,\pm} = N_{-,+} - N_{-,-}. \tag{46}$$

We note that a system $(+,\pm)$ has $N_{\mathbb{Z}} = \pm 1$ and $N_{M\mathbb{Z}} = \pm 1$ and with $(-,\pm)$ has $N_{\mathbb{Z}} = \mp 1$ and $N_{M\mathbb{Z}} = \pm 1$ when $m = M - k^2$ in Eq. (8). Therefore, the numbers of the intact gapless modes in the expression of $N_{\mathbb{Z}}$ and $N_{M\mathbb{Z}}$ are written as

$$N_{+,\pm} = \frac{N_{\mathbb{Z}} + N_{M\mathbb{Z}}}{2}, \quad N_{-,\pm} = \frac{N_{\mathbb{Z}} - N_{M\mathbb{Z}}}{2}. \tag{47}$$

The total number of the intact gapless edge modes in the system is the $\mathbb{Z}^1$ number $N_{\mathbb{Z}^1} = |N_{+,\pm}| + |N_{-,\pm}|$, as in shown Eq. (45).

2. Topological number of $\mathbb{Z}_2$

In Table V the $\mathbb{Z}_2$ number from a system with reflection symmetry can be computed in the same way of the $\mathbb{Z}_2$ number without reflection symmetry. The reason is that the reflection symmetry does not give rise to any new topological phase. From the previous discussion, only for the two-copy minimal Dirac Hamiltonian $\mathcal{H}_2^{d'}$ in the form of Eq. (27) the bulk gap closing cannot be prevented by any symmetry preserving *homogeneous* term. However, we show later that this Hamiltonian can be kept gapped by some *inhomogeneous* SPEMTs.

This $\mathbb{Z}_2$ class can be separated into two slightly different cases represented by the original $\mathbb{Z}_2$ invariants in Table II: the second descendant $\mathbb{Z}_2$ of $\mathbb{Z}$ in odd dimensions and the first descendant $\mathbb{Z}_2$ of $\mathbb{Z}$ in even dimensions. In the following, we consider both of the cases together; in the twice-as-big minimal Dirac Hamiltonian there exists a SPEMT preserving reflection symmetry ($R = i\gamma_1 S$). Therefore, this reflection symmetry still keeps the original $\mathbb{Z}_2$ invariants.

For the second descendant, without enlarging the dimension of the minimal Dirac Hamiltonian $d+2$ kinetic γ matrices $\gamma_1,\gamma_2,\ldots,\gamma_{d+2}$ and one mass matrix γ_0 (Ref. 35) can be used for the SPEMT construction.

For the first descendent in even dimensions, we have the same γ matrices, except for missing γ_{d+2}. Therefore, from those γ matrices some symmetry-preserving terms can be constructed and added into the enlarged minimal Dirac Hamiltonian,

$$\mathcal{H}_2^{d'} = k_1\gamma_1 \otimes \sigma_z + \delta\Delta + \sum_{n_i \neq 1} k_{n_i}\gamma_{n_i} \otimes \sigma_z + \sum_{\text{remain}} k_{n_j}\gamma_{n_j} \otimes \mathbb{I}, \tag{48}$$

where $\Delta = (i)\gamma_{d+1}\prod_{n_i\neq 1}^{\text{even}} \gamma_{n_i} \otimes \sigma_u$ when the first summation is over an even number of terms or $\Delta = (i)\prod_{n_i\neq 1}^{\text{odd}} \gamma_{n_i} \otimes \sigma_u$ when it is over an odd number of terms. The presence or absence of i keeps Δ being Hermitian and choosing $u = x,y$ lets Δ preserve TRS and PHS. We can check that Δ preserves all of the system's symmetries, including reflection symmetry $R = i\gamma_1 S \otimes \mathbb{I}$. The situation is similar with Eq. (35): The presence of Δ shifts the bulk gap closing points at $m = 0$, $k_1 = \pm\delta$, $k_\perp = 0$. However, this system of the twice-as-big minimal Dirac Hamiltonian still can be gapped out by a special SPEMT,

$$\hat{\mathfrak{N}} = \sum_{-\eta \leqslant k_1 < \eta} (ic_{k_1+\eta+\delta}\mathfrak{N}c_{k_1-\eta+\delta} + \text{H.c.}) + \sum_{-\eta < k_1 \leqslant \eta} (ic_{k_1+\eta-\delta}\mathfrak{N}c_{k_1-\eta-\delta} + \text{H.c.}), \tag{49}$$

where $\mathfrak{N} = (i)\gamma_1\prod_{n_i\neq 1}^{\text{even}} \gamma_{n_i} \otimes \sigma_u$ when the first summation in Eq. (48) is over an even number of terms, or $\mathfrak{N} = (i)\gamma_{d+1}\gamma_1\prod_{n_i\neq 1}^{\text{odd}} \gamma_{n_i} \otimes \sigma_u$ when the summation is over an odd number of terms. Therefore, $\mathfrak{N}$ anticommutes with all of the terms in Eq. (48). Again the presence or absence of i guarantees the Hermiticity of $\mathfrak{N}$. By properly choosing $u = x,y$, $\hat{\mathfrak{N}}$ preserves TSR and PHS by Eqs. (10) and (11). By the definition of the reflection-symmetry operator $\hat{R} = \sum_{k_1} c_{k_1}^\dagger (i\gamma_1 S \otimes \mathbb{I})c_{-k_1}$, $\hat{\mathfrak{N}}$ preserves the reflection symmetry. Following the similar discussion in Sec. IV B2, the low-energy spectrum is shown in Eq. (40). When $c > \eta$, the original bulk gap closing is blocked: no quantum phase transitions, no topological nontrivial phases. The system of the twice-as-big minimal Dirac Hamiltonian is trivial. Hence, this case is classified as $\mathbb{Z}_2$ from the original topological invariant without the reflection symmetry.

B. The rest of the classification

The classification of the rest of the commutation and anticommutation relations has to be discussed case by case. The result of this classification for the four symmetry classes is shown in Table VI.

1. Even spatial dimensions

We consider systems in even dimensions. In the following we show that the topological invariant in such a system is zero. For such a d-dimensional system, in the Dirac Hamiltonian if only $\gamma_0, \gamma_1, \ldots, \gamma_d$, and S are ingredients to construct reflection-symmetry operator R, it is not possible that R satisfies the commutation and anticommutation relations in Table VI. To have R from the γ matrices, we need to introduce an extra mass term $\tilde{\gamma}$ or an extra kinetic term γ_{d+1} preserving the nonspatial symmetries. At the same time, the Dirac Hamiltonian might be enlarged to have a new γ matrix.

With an extra mass term, R can be constructed and satisfies the required commutation and anticommutation relations:

$$R_{+-} = i\tilde{\gamma} \prod_{i=0, i\neq 1}^{d} \gamma_i, \quad \text{as} \quad d = 4n+2, \tag{50}$$

$$R_{-+} = \tilde{\gamma} \prod_{i=0, i\neq 1}^{d} \gamma_i, \quad \text{as} \quad d = 4n. \tag{51}$$

It is easy to check that $\tilde{\gamma}$ preserves the reflection symmetry. Hence, this γ matrix plays a role in a SPEMT. This shows that as $R_T = \mp 1$, $R_C = \pm 1$, and $d = 4n(+2)$, those four symmetry classes possess the 0 topological invariant, which explains several 0's in even dimensions in Tables V and VI.

To have the opposite commutation and anticommutation relations in the previous case, we use γ_{d+1} to construct R:

$$R_{-+} = i \prod_{i=0, i\neq 1}^{d+1} \gamma_i, \quad \text{as} \quad d = 4n+2, \tag{52}$$

$$R_{+-} = \prod_{i=0, i\neq 1}^{d+1} \gamma_i, \quad \text{as} \quad d = 4n. \tag{53}$$

We note that in $4n+2$ dimensions for class BDI and CII and in $4n$ dimensions for class DIII and CI without enlarging the Dirac Hamiltonian, an extra mass term,

$$\tilde{\gamma}_1 = \begin{cases} S \prod_{i=0}^{d} \gamma_i & \text{as} \quad d = 4n+2, \\ iS \prod_{i=0}^{d} \gamma_i & \text{as} \quad d = 4n, \end{cases} \tag{54}$$

can be found. It is to easy to check that $\tilde{\gamma}_1$ preserves the system's nonspatial symmetries by Eq. (42). Furthermore, $\tilde{\gamma}_1$ anticommutes with $\gamma_{i\neq d+1}$ but commutes with γ_{d+1} so that the reflection symmetry is preserved. This SPEMT $\tilde{\gamma}_1$ implies that this system is classified by a 0 original topological invariant.

2. Odd spatial dimensions

In odd spatial dimensions without $\tilde{\gamma}$ and γ_{d+1}, we can use only the original γ matrices to construct R satisfying the aforementioned commutation and anticommutation relations with T and C,

$$R_{-+} = i \prod_{i=0, i\neq 1}^{d} \gamma_i, \quad \text{as} \quad d = 4n-1, \tag{55}$$

$$R_{+-} = \prod_{i=0, i\neq 1}^{d} \gamma_i, \quad \text{as} \quad d = 4n+1. \tag{56}$$

The corresponding symmetry classes are class BDI and CII as $d = 4n-1$ and class DIII and CI as $d = 4n+1$. The original classification of the AZ-symmetry class shows that those classes have 0 topological invariants so that the nonspatial SPEMT $\tilde{\gamma}$ can be present in the minimal Dirac Hamiltonian without enlarging the dimension of the matrix. However, the anticommutation relation $\{\tilde{\gamma}, \gamma_i\} = 0$ implies that $\tilde{\gamma}$ breaks the reflection symmetry. Such a system can be in a nontrivial phase. To find the topological invariant, we have to investigate the presence of the SPEMT in the enlarged Dirac Hamiltonian in Eq. (15).

All of the possible ways in Eq. (15) to doubling the size of the Dirac Hamiltonian can be separated into two expressions in Eqs. (26) and (27). To construct a SPEMT, all of the ingredients are the γ matrices, a mass term $\tilde{\gamma}$, and the Pauli matrices. We find that from the first summation over only *odd* number of the terms in Eqs. (26) and (27) the SPEMT $\mathfrak{M}$ can be found. That is, $\mathfrak{M} = (i) \prod_{n_i \neq 1}^{\text{odd}} \gamma_{n_i} \otimes \sigma_u$ in Eq. (26) and $\mathfrak{M} = (i) \tilde{\gamma} \gamma_1 \prod_{n_i \neq 1}^{\text{odd}} \gamma_{n_i} \otimes \sigma_u$ in Eq. (27), where the Hermiticity is adjusted by the presence or absence of i and $u = x$ or y to satisfy the nonspatial symmetries. The commutation and anticommutation $[\mathfrak{M}, \gamma_{n_i} \otimes \mathbb{I}] = 0$ and $\{\mathfrak{M}, \gamma_{n_j} \otimes \mathbb{I}\} = 0$ implies that $\mathfrak{M}$ commutes with the enlarged reflection-symmetry operator $R' = (i) \prod_{i=0, i\neq 1}^{d} \gamma_i \otimes \mathbb{I}$. Therefore, the reflection symmetry is preserved. On the one hand, $\sum_{n_i \neq 1} k_{n_i} \gamma_{n_i} \otimes \sigma_z$ over an *odd* number of terms gives 0 topological invariant. On the other hand, an *even* number provides a nontrivial topological phase. In short, this shows the properties of $\mathbb{Z}$ topological invariants when k_1 vanishes. Such a system has the $2M\mathbb{Z}(M\mathbb{Z})$ (Ref. 48) topological invariant.

The cases that we have not discussed are classes DIII and CI in $4n-1$ dimensions and classes BDI and CII in $4n+1$ dimensions. In the original classification without reflection symmetry those symmetry classes possess $\mathbb{Z}$ and $2\mathbb{Z}$ topological invariants. Let us first consider the $\mathbb{Z}$ case. To build the reflection-symmetry operator R satisfying the sign changing of R_T and R_C in Eqs. (55) and (56), only using $\gamma_0, \gamma_1, \ldots, \gamma_d$, and S is not possible. Therefore, because σ_y plays a role in switching signs of R_T and R_C, we construct the reflection-symmetry operator as the direct product of σ_y and the reflection-symmetry operator in the form of Eqs. (55) and (56),

$$R_{+-} = i \prod_{i=0, i\neq 1}^{d} \gamma_i \otimes \sigma_y, \quad \text{as} \quad d = 4n-1, \tag{57}$$

$$R_{-+} = \prod_{i=0, i\neq 1}^{d} \gamma_i \otimes \sigma_y, \quad \text{as} \quad d = 4n+1. \tag{58}$$

At the same time, the Dirac Hamiltonian must be enlarged in this unique way,

$$\mathcal{H}_2^{\mathbb{Z}} = m\gamma_0 \otimes \mathbb{I} + \sum_{i\neq 0}^{d} k_i \gamma_i \otimes \mathbb{I}, \tag{59}$$

to preserve all of the system's symmetries. From the properties of the $\mathbb{Z}$ original topological invariant, a SPEMT is absent from this Hamiltonian. Therefore, to distinguish this nonzero topological invariant, $\mathcal{H}_2^{\mathbb{Z}}$ needs to be enlarged in the two possible forms:

$$\mathcal{H}_4^{\mathbb{Z}} = k_1\gamma_1 \otimes \mathbb{I} + \sum_{n_i} k_{n_i}\gamma_{n_i} \otimes \mathbb{I} \otimes \sigma_z + \sum_{n_j} k_{n_j}\gamma_{n_j} \otimes \mathbb{I}_{4\times 4}, \tag{60}$$

$$\mathcal{H}_4'^{\mathbb{Z}} = k_1\gamma_1 \otimes \sigma_z + \sum_{n_i} k_{n_i}\gamma_{n_i} \otimes \mathbb{I} \otimes \sigma_z + \sum_{n_j} k_{n_j}\gamma_{n_j} \otimes \mathbb{I}_{4\times 4}. \tag{61}$$

Again, the original $\mathbb{Z}$ protects the nontrivial topological phases when an even number of σ_z in the Hamiltonians. For the case of an odd number, although an extra mass term, which preserves the nonspatial symmetries, is present, we have to confirm that the reflection symmetry is also preserved so that the system is in the trivial phase. The expression of the mass term is $\mathfrak{M} = (i)\prod_{n_i\neq 0}^{\text{odd}} \gamma_{n_i} \otimes \sigma_x \otimes \sigma_u$ for Eq. (60) and $\mathfrak{M} = (i)\gamma_0 \prod_{n_i\neq 0}^{\text{odd}} \gamma_{n_i} \otimes \mathbb{I} \otimes \sigma_u$ for Eq. (61), where the presence or absence of i keeps the Hermiticity of $\mathfrak{M}$ and choosing $u = x$ or y makes $\mathfrak{M}$ preserve the nonspatial symmetries. It is easy to check that $\mathfrak{M}$ commutes with $R = (i)\prod_{i=0,i\neq 1}^{d} \gamma_i \otimes \sigma_y \otimes \mathbb{I}$. Hence, the behavior of even and odd numbers of σ_z shows that even with reflection symmetry such a system still possesses $\mathbb{Z}$ properties. However, the reflection symmetry requires the dimension of the Dirac Hamiltonian to be doubled in Eq. (59). The classification of the topological invariant changes from $\mathbb{Z}$ to $2\mathbb{Z}$.

For the $2\mathbb{Z}$ case, we know that in the proper basis the minimal Dirac Hamiltonian of $2\mathbb{Z}$ is the two copies of the minimal Dirac Hamiltonian of $\mathbb{Z}$:

$$\mathcal{H}^{2\mathbb{Z}} = m\gamma_0^{\mathbb{Z}} \otimes \mathbb{I} + \sum_{i\neq 0}^{d} k_i\gamma_i^{\mathbb{Z}} \otimes \mathbb{I}. \tag{62}$$

The nonspatial symmetry operators can be expressed by the symmetry operator of $\mathbb{Z}$

$$T^{2\mathbb{Z}} = T^{\mathbb{Z}} \otimes \sigma_y, \quad C^{2\mathbb{Z}} = C^{\mathbb{Z}} \otimes \sigma_y. \tag{63}$$

Following the similar discussion of the $\mathbb{Z}$ case, a system of the $2\mathbb{Z}$ original topological invariant with the reflection symmetry is still classified as $2\mathbb{Z}$.

VI. EXAMPLES

Let us now discuss several examples of topological phases protected by reflection symmetry: More specifically, we consider the following.

Class $AIII + R_+$ and $BDI + R_{++}$ in $d = 2$. All of the nonspatial symmetry operators commute with R. Therefore, Table III shows that both of the symmetry classes possess $M\mathbb{Z}$ topological invariants.

Class $DIII + R_{--}$ in $d = 2$. In this symmetry class, R anticommutes with T and C. This situation hence falls into Table IV. As described below, gapped phases in this case are characterized an $M\mathbb{Z}$ topological invariant.

Class $D + R_+$ in $d = 2$. With reflection symmetry, gapped phases in this symmetry class are characterized as a $\mathbb{Z}_2$ topological invariant in Table III.

Class $CII + R_{--}$ in $d = 2$. For class CII no nonzero topological invariants are in the original classification or for *mirror*-symmetry class A in the reflection-symmetric plane. The gapped phases are always trivial.

Class $DIII + R_{-+}$ in $d = 3$. R anticommutes with T but commutes with C. Hence, Table V shows that there is a $\mathbb{Z}^1$ in this case.

A. Class AIII + R_+, BDI + R_{++}, and DIII + R_{--} in $d = 2$

Symmetry class AIII has two physical interpretations, one in terms of charged (complex) fermions with conserved fermionic number and the other in terms of BdG Hamiltonians. As an electron system, a way to obtain such a system is to consider lattice fermion systems with bipartite hopping only. In this context, chiral symmetry of class AIII is *sublattice symmetry*. Alternatively, symmetry class AIII can be realized as a time-reversal symmetric BdG Hamiltonian with conserved S_z spin rotation. While it is perhaps fair to say that the BdG interpretation is more experimentally realizable, since achieving an exact sublattice symmetry is challenging, in this section we focus on electronic realizations of symmetry class AIII.

Similar to class AIII, symmetry class BDI has two physical interpretations, one in terms of charged (complex) fermions with conserved fermionic number and the other in real (Majorana) fermions. Below, we first discuss realization in terms of complex fermions; we later discuss a realization in terms of Majorana fermions.

Recall that in $d = 2$, there is no topological insulator in AIII and BDI if we do not impose reflection symmetry. Class DIII is of $\mathbb{Z}_2$ type. With reflection, there are topological insulators in these three classes characterized by $M\mathbb{Z}$ topological invariants.

(a) Bulk Hamiltonian. Let us start by considering the following tight-binding Hamiltonian:

$$\begin{aligned} H = \sum_r \psi^\dagger(r)\begin{pmatrix} t & i\Delta \\ i\Delta & -t\end{pmatrix}\psi(r+\hat{x}) + \text{H.c.} \\ + \psi^\dagger(r)\begin{pmatrix} t & \Delta \\ -\Delta & -t\end{pmatrix}\psi(r+\hat{y}) + \text{H.c.} \\ + \psi^\dagger(r)\begin{pmatrix} \mu & 0 \\ 0 & -\mu\end{pmatrix}\psi(r), \end{aligned} \tag{64}$$

where the two-component fermion annihilation operator at site r, $\psi(r)$, is given in terms of the electron annihilation operators with spin-up and spin-down, $c_{r,1/2}$, as $\psi^T(r) = (c_{r,1}, c_{r,2})$, and we take $t = \Delta = 1$ and $\mu = m + 2$. The chiral p-wave superconductor has been discussed in the context of superconductivity in strontium ruthenate[49] and paired states in the fractional quantum Hall effect.[50] There are four phases separated by three quantum critical points at $\mu = 0, \pm 4$, which are labeled by the Chern number as Ch $= 0$ ($|\mu| > 4$), Ch $= -1$ ($-4 < \mu < 0$), and Ch $= +1$ ($0 < \mu < +4$). The nonzero Chern number implies the IQHE in the spin transport.[51] In momentum space,

$$H = \sum_{k\in \text{BZ}} \psi^\dagger(k)[\vec{n}(k)\cdot\vec{\sigma}]\psi(k),$$

$$\vec{n}(k) = \begin{pmatrix} -2\Delta\sin k_x \\ -2\Delta\sin k_y \\ 2t(\cos k_x + \cos k_y) + \mu \end{pmatrix}. \tag{65}$$

A lattice model topological insulator in symmetry class AIII + R_+, BDI + R_{++}, and DIII + R_{--} can be constructed by taking the two copies of the above two-band Chern insulator with opposite chiralities. Consider the Hamiltonian in momentum space,

$$H = \sum_{k\in \mathrm{BZ}} \sum_{s=\uparrow,\downarrow} \psi_s^\dagger(k)[\vec{n}_s(k)\cdot\vec{\sigma}]\psi_s(k), \tag{66}$$

where $s=\uparrow,\downarrow$ represent "pseudospin" degrees of freedom, and $\vec{n}_s(k)$ is given, in terms of $\vec{n}$, as

$$\vec{n}_\uparrow(k) = \vec{n}(k), \quad \vec{n}_\downarrow(k) = \vec{n}_\uparrow(\tilde{k}) = \vec{n}(\tilde{k}), \tag{67}$$

where $\tilde{k} = (-k_1,k_2,\ldots) = (-k_1,k_\perp)$. That is,

$$\mathcal{H}(k) = n_x(k)\tau_z\sigma_x + n_y(k)\tau_0\sigma_y + n_z(k)\tau_0\sigma_z. \tag{68}$$

The model is chiral symmetric:

$$S^{-1}\mathcal{H}(k)S = -\mathcal{H}(k), \quad S = \tau_x\sigma_x. \tag{69}$$

The Hamiltonian is invariant under the following two TRSs:

$$\begin{aligned} T^{-1}\mathcal{H}(-k)T &= \mathcal{H}(k), \quad T=\tau_x\sigma_0\Theta, \quad T^2=+1\\ T^{-1}\mathcal{H}(-k)T &= \mathcal{H}(k), \quad T=\tau_y\sigma_0\Theta, \quad T^2=-1. \end{aligned} \tag{70}$$

Also, the corresponding particle-hole symmetries with $C = ST$ are

$$\begin{aligned} C^{-1}\mathcal{H}(-k)C &= -\mathcal{H}(k), \quad C=\tau_0\sigma_x\Theta, \quad C^2=+1,\\ C^{-1}\mathcal{H}(-k)C &= -\mathcal{H}(k), \quad C=i\tau_z\sigma_x\Theta, \quad C^2=+1. \end{aligned} \tag{71}$$

Imposing the former form of TRS, the system falls into symmetry class BDI, whereas with the latter form of TRS, the system falls into symmetry class DIII.

We now impose the following reflection symmetry:

$$R^{-1}\mathcal{H}(\tilde{k})R = \mathcal{H}(k), \quad R = \tau_x. \tag{72}$$

Chiral and reflection symmetries commute with each other in the sense that

$$[S,R] = 0. \tag{73}$$

For class BDI, R commutes with T and C (R_{++}). For class DIII, R anticommutes with T and C (R_{--}). From Tables III and IV, both of the cases possess $M\mathbb{Z}$ topological invariants. Similarly, class AIII + R_+ is also classified by a $M\mathbb{Z}$ topological invariant.

(b) $M\mathbb{Z}$ bulk topological invariant. At the reflection-symmetric plane, i.e., only $k_x=0$ in the continuum model, $\mathcal{H}(k)$ commutes with R, and hence, it can be block diagonalized. Furthermore, since reflection R commutes with chiral symmetry in Eq. (73), each block has an off-diagonal structure in a proper basis,

$$\begin{aligned} \mathcal{H}(0,k_y) &= \begin{pmatrix} \mathcal{H}^+(0,k_y) & 0\\ 0 & \mathcal{H}^-(0,k_y)\end{pmatrix},\\ \mathcal{H}^\pm(0,k_y) &= \begin{pmatrix} 0 & D_\pm(k_y)\\ D_\pm^\dagger(k_y) & 0\end{pmatrix}, \end{aligned} \tag{74}$$

where $\pm$ indicates the eigenspace of $R=\pm1$. For each block, the 1D winding number (the topological invariant of symmetry class AIII in $d=1$) is well-defined;[18] by the definition of the $M\mathbb{Z}$ number in Eq. (33), we only need to focus on the winding number in one of the eigenspaces of R, say $+1$, due to the absence of the weak index. The winding number is defined from $q(k)$ in the Q matrix in Eq. (A2). To construct the Q matrix from the (k_y-dependent) occupied wave functions, we have to solve the eigenvalue problem

$$\mathcal{H}^+\Phi^a_\pm = \pm\varepsilon^a\Phi^a_\pm, \tag{75}$$

where a runs over occupied bands, $a=1,2$ in our case and

$$\Phi^a_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} u^a\\ \pm v^a\end{pmatrix}^T. \tag{76}$$

Here we assume u^a and v^a are normalized so $\Phi^a_\pm$ is also normalized. Because $(\mathcal{H}^+)^2\Phi^a_\pm = (\varepsilon^a)^2\Phi^a_\pm$, u^a and v^a are the eigenfunctions of $D_RD_R^\dagger$ and $D_R^\dagger D_R$, respectively, and share the same positive eigenvalue $(\varepsilon^a)^2$:

$$DD^\dagger u^a = (\varepsilon^a)^2u^a, \quad D^\dagger D\, v^a = (\varepsilon^a)^2 v^a. \tag{77}$$

Therefore, we can compute the projector of the occupied bands which have negative energies

$$\begin{aligned} P(k_y) &= \frac{1}{2}\sum_a \begin{pmatrix} u_a\\ -v_a\end{pmatrix}\begin{pmatrix} u_a^\dagger & -v_a^\dagger\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}\mathbb{I} & 0\\ 0 & \mathbb{I}\end{pmatrix} - \frac{1}{2}\sum_a\begin{pmatrix} 0 & u_av_a^\dagger\\ v_au_a^\dagger & 0\end{pmatrix}. \end{aligned} \tag{78}$$

Therefore, due to $Q(k_y) = \mathbb{I} - 2P(k_y)$ and the definition of $q(k_y)$ in Eq. (A2), we have the expression of

$$q(k_y) = \sum_a |u^a(k_y)\rangle\langle v^a(k_y)|, \tag{79}$$

and the topological invariant for $\mathcal{H}^+$ at $k_x=0$ is defined by

$$\nu_0 = \frac{i}{2\pi}\int dk_y\, \mathrm{tr}\,[q^\dagger(k_y)\partial_{k_y}q(k_y)]. \tag{80}$$

The winding number ν_π at the other symmetric point $k_x=\pi$ can be computed in the similar way so we have the $M\mathbb{Z}$ number $N_{M\mathbb{Z}}$ by Eq. (33).

Following the general discussion, we now calculate the topological invariant of the model in Eq. (66). We first look for a basis where (i) R is diagonal, (ii) $\mathcal{H}(k)$ is block diagonal for each reflection eigenvalue $R=\pm1$, and (iii) the chiral symmetry operation looks identical for both sectors of R. This can be achieved by the unitary transformation $U=U_1U_2$, where

$$U_1 = \frac{1}{\sqrt{2}}(\tau_0\sigma_0 + \tau_z\sigma_y), \quad U_2 = \frac{1}{\sqrt{2}}(\tau_0 + i\tau_y). \tag{81}$$

Under the unitary transformation,

$$\begin{aligned} URU^\dagger &= \tau_z\sigma_0, \quad USU^\dagger = \tau_0\sigma_z,\\ U\mathcal{H}(K)U^\dagger &= \begin{pmatrix}\mathcal{H}^+(K) & 0\\ 0 & \mathcal{H}^-(K)\end{pmatrix}, \end{aligned} \tag{82}$$

where $K=0,\pi$, and

$$\mathcal{H}^\pm_K(k_y) = \begin{pmatrix} 0 & -in_y \mp n_z\\ in_y \mp n_z & 0\end{pmatrix}. \tag{83}$$

The topological invariant $\nu_K^\pm$ for the blocks $\mathcal{H}_K^\pm$ satisfy $\nu_K^+ = -\nu_K^-$ due to the absence of the weak index. In this specific model, $\nu_0^+ = 1$ and $\nu_\pi^+ = 0$ as $-4 < \mu < 0$ and $\nu_0^+ = 0$ and $\nu_\pi^+ = 1$ as $0 < \mu < 4$. Therefore, by Eq. (33) $N_{M\mathbb{Z}} = 1$ and -1 as $-4 < \mu < 0$ and $0 < \mu < 4$, respectively.

(c) Edge theory. Let us now introduce a boundary to the system. In the continuum model, one way to do this is to make the mass y-dependent $m \to m(y)$. The edge Hamiltonian, in a suitable basis, is

$$\mathcal{H}(k_x) = k_x\sigma_3, \quad \{\mathcal{H}(k_x),\sigma_1\} = 0. \tag{84}$$

Or, in the second quantized language, the edge mode is described by

$$H = \int dx[\psi_L^\dagger i\partial_x\psi_L - \psi_R^\dagger i\partial_x\psi_R]. \tag{85}$$

Since there are left and right movers, one could give a mass to gap them out. We can write down two such masses,

$$m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L), \quad im_5(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L), \tag{86}$$

if we have not imposed any discrete symmetry. Let us now impose the two discrete symmetries. With chiral symmetry, the first mass will be eliminated: The first mass term can be written as $m\Psi^\dagger\sigma_x\Psi$, where $\Psi := (\psi_L,\psi_R)^T$. Since $\{m\sigma_x,\sigma_x\} \neq 0$, this mass term is not compatible with chiral symmetry. On the other hand, the second mass is $m_5\Psi^\dagger\sigma_y\Psi$. Since $\{m_5\sigma_y,\sigma_x\} = 0$, this mass term is chiral symmetric, and allowed to exist. With reflection symmetry, we should be able to prohibit the second mass term. By definition, reflection should exchange ψ_L and ψ_R, and observing reflection should commute with chiral symmetry, R, given by

$$\mathcal{R}\psi_L(x)\mathcal{R}^{-1} = \psi_R(-x), \quad \mathcal{R}\psi_R(x)\mathcal{R}^{-1} = \psi_L(-x). \tag{87}$$

Observe that reflection and chiral symmetry commute,

$$[R,\sigma_x] = 0. \tag{88}$$

The first mass as well as the kinetic term is invariant under $\mathcal{R}$. However, the second mass is not invariant under $\mathcal{R}$. Therefore, with both chiral and reflection symmetries, the edge state is stable.

We have treated the case with unit topological invariant. By increasing the number of edge channels with the same or different signs in Eq. (85), any mass term is still prohibited. We can consider cases with $M\mathbb{Z}$ topological invariants.

B. Class $\mathbf{D} + R_+$ in $d = 2$

For symmetry class D (i.e., generic BdG systems without any symmetry) in $d = 1$, the system is classified as a $\mathbb{Z}_2$ topological superconductor. Therefore, a $\mathbb{Z}_2$ topological superconductor can be realized in a symmetry class D system with reflection symmetry R_{++} due to the upward shifting in Table III. As follows, we describe this case in more detail with an example.

Observe that the pairing terms in the x direction for the spin-up and spin-down sectors differ by sign. The model is invariant under reflection defined by

$$\mathcal{R}c_{sr}\mathcal{R}^{-1} = c_{-s\tilde{r}} \quad (s = \uparrow,\downarrow). \tag{89}$$

Furthermore, for spin-$\frac{1}{2}$ system, $R^2 = -1$. Without affecting the equation above, we perform a phase shift so that $R = \sigma_x$ is Hermitian. In particular, note that the pairing terms in the x direction transform under reflection $\mathcal{R}$ as $\mathcal{R}$: $\sum_r \Delta(c_{\uparrow r}^\dagger c_{\uparrow r+\hat{x}}^\dagger - c_{\uparrow r+\hat{x}}^\dagger c_{\uparrow r}^\dagger) \to -\sum_r \Delta(c_{\downarrow r}^\dagger c_{\downarrow r+\hat{x}}^\dagger - c_{\downarrow r+\hat{x}}^\dagger c_{\downarrow r}^\dagger)$, and hence $\mathcal{R}$ exchanges $H_{p+ip}^\uparrow$ and $H_{-p+ip}^\downarrow$.

(a) Bulk spectrum. With the periodic boundary condition, we make use of the Fourier transforms which transform the Hamiltonian into

$$\begin{aligned} H &= \sum_{0\leqslant k_x\leqslant\pi}\sum_{k_y}\Psi_{k_x}^\dagger(k_y)\mathcal{H}_{k_x}(k_y)\Psi_{k_x}(k_y), \\ \Psi_{k_x}^\dagger(k_y) &:= (c_{\uparrow,k}^\dagger,\ c_{\downarrow,-k},\ c_{\uparrow,k}^\dagger,\ c_{\downarrow,-k}), \end{aligned} \tag{90}$$

where the kernel $\mathcal{H}_{k_x}(k_y)$ is block diagonal in spin indices and given by

$$\begin{aligned} \mathcal{H}_{k_x}(k_y) &= \begin{pmatrix} \mathcal{H}_{k_x}^\uparrow(k_y) & 0 \\ 0 & \mathcal{H}_{k_x}^\downarrow(k_y) \end{pmatrix}, \\ \mathcal{H}_{k_x}^s(k_y) &= \begin{pmatrix} \xi_k & \Delta_k^s \\ \Delta_k^{s*} & -\xi_k \end{pmatrix}, \end{aligned} \tag{91}$$

where $\xi_k = 2t(\cos k_x + \cos k_y) - \mu$ and $\Delta_k^s = 2\Delta(-s\sin k_x + i\sin k_y)$. The PHS operator $C = \tau_x\Theta$ commutes with $R = \sigma_x$. It corresponds to class $\mathrm{D} + R_{++}$. At the reflection-symmetric points $k_x = 0,\ \pi$, following the general discussion, we take a basis in which R is diagonal: This is achieved by a unitary transformation,

$$R \to URU^{-1} = \sigma_z, \quad U = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{92}$$

Accordingly, the BdG Hamiltonian at $k_x = 0,\pi$, $\mathcal{H}_{k_x=0,\pi}(k_y)$, is transformed as

$$\begin{aligned} \mathcal{H}_{k_x}(k_y) &\to \begin{pmatrix} \mathcal{H}_{k_x}^+(k_y) & 0 \\ 0 & \mathcal{H}_{k_x}^-(k_y) \end{pmatrix}, \\ \mathcal{H}_{k_x}^{R=\pm}(k_y) &= \tfrac{1}{2}[\mathcal{H}_{k_x}^\uparrow(k_y) + \mathcal{H}_{k_x}^\downarrow(k_y)], \end{aligned} \tag{93}$$

where we note that $\mathcal{H}_{k_x=0,\pi}^\uparrow(k_y) = \mathcal{H}_{k_x=0,\pi}^\downarrow(k_y)$ in this example. The Hamiltonian for $R = +1$ and $R = -1$ sectors are identical and given by

$$\begin{aligned} \mathcal{H}_{k_x=0,\pi}^\pm(k_y) &= \begin{pmatrix} \xi_{k_y} & \Delta_{k_y} \\ \Delta_{k_y}^* & -\xi_{k_y} \end{pmatrix}, \\ \xi_{k_y} &= 2t(\pm 1 + \cos k_y) - \mu, \\ \Delta_{k_y} &= 2i\Delta\sin k_y, \end{aligned} \tag{94}$$

where $+$ for $k_x = 0$ and $-$ for $k_x = \pi$.

(b) Bulk topological invariant. Once we decompose the Hamiltonian into reflection-symmetric/antisymmetric ($R = \pm$) sectors, we can define a $d = 1$ topological invariant for each $R = \pm$ sector, following Ref. 45: Let us consider class D systems in $d = 1$ with translation invariance. In momentum space, it can be written as

$$H = \frac{i}{4}\sum_{\alpha,\beta}\sum_{k_y}\tilde{B}_{\alpha\beta}(k_y)\lambda_\alpha(k_y)\lambda_\beta(-k_y), \tag{95}$$

where α,β are a band index, and we are using a Majorana fermions $\lambda_\alpha(k_y)$ to represent a class D system. Then the $\mathbb{Z}_2$ topological invariant is given by

$$(-1)^\nu := \mathrm{sgn}[\mathrm{Pf}\ \tilde{B}(0)]\mathrm{sgn}[\mathrm{Pf}\ \tilde{B}(\pi)] = \pm 1. \tag{96}$$

In our problem, this Kitaev $\mathbb{Z}_2$ invariant can be defined for $k_x = 0, \pi$ and for each $R = \pm$ sector. For a given sector ($R = +$, say), by performing the transformation to real Majorana fermion operators,

$$\lambda_k := c^\dagger_{k_y} + c_{k_y}, \quad \lambda'_k := (c_{-k_k} - c^\dagger_{-k_y})/i, \quad \Lambda_k := \begin{pmatrix} \lambda_{k_y} \\ \lambda'_{k_y} \end{pmatrix}, \tag{97}$$

the Hamiltonian in Eq. (94) is given by

$$H^R_{k_x=0,\pi}(k_y) = \frac{-i}{4}\sum_k \Lambda^T_{-k_y}\begin{pmatrix} 2\Delta\sin k_y & -\xi_{k_y} \\ \xi_{k_y} & 2\Delta\sin k_y \end{pmatrix}\Lambda_{k_y}. \tag{98}$$

Let us focus on the $\mathbb{Z}_2$ topological invariant at $k_x = 0$. Thus, the $\tilde{B}$ matrix at $k_y = 0, \pi$ is given by

$$\tilde{B}_0(0) = \begin{pmatrix} 0 & 4t-\mu \\ -4t+\mu & 0 \end{pmatrix}, \quad \tilde{B}_0(\pi) = \begin{pmatrix} 0 & -\mu \\ +\mu & 0 \end{pmatrix}, \tag{99}$$

and hence the topological invariant is

$$\begin{aligned}(-1)^{\nu_0} &= \mathrm{sgn}[\mathrm{Pf}\,\tilde{B}_0(0)]\mathrm{sgn}[\mathrm{Pf}\,\tilde{B}_0(\pi)] \\ &= \mathrm{sgn}(4t-\mu)\cdot\mathrm{sgn}(-\mu).\end{aligned} \tag{100}$$

Similarly, the topological invariant at $k_x = \pi$ is

$$(-1)^{\nu_\pi} = \mathrm{sgn}(-4t-\mu)\cdot\mathrm{sgn}(-\mu). \tag{101}$$

Therefore, the $\mathbb{Z}_2$ topological invariant in Eq. (34) for the entire reflection system is

$$(-1)^{N_{\mathbb{Z}_2}} = \mathrm{sgn}\,(+4t-\mu)\cdot\mathrm{sgn}\,(-4t-\mu). \tag{102}$$

The presence of the topological nontrivial phase is in the region $-4t < \mu < 4t$.

(c) Edge state. A consequence of nontrivial topology in the bulk is the presence of edge modes. Let us now consider an edge of $d = 2$ class D topological superconductor:

$$S = \int dx d\tau [\psi_\uparrow(-i\partial_x)\psi_\uparrow + \psi_\downarrow(i\partial_x)\psi_\downarrow], \tag{103}$$

where $\psi_\uparrow$ and $\psi_\downarrow$ are a left-moving (right-moving) Majorana fermion. A mass term $im\psi_\uparrow\psi_\downarrow$ is prohibited because of a reflection symmetry:

$$R: \psi_\uparrow \to \psi_\downarrow, \quad \psi_\downarrow \to \psi_\uparrow. \tag{104}$$

In other words, with reflection the edge state is stable, and there is a bulk topological superconductor protected by reflection.

Such protection by reflection symmetry is of $\mathbb{Z}_2$ type as we can readily see by considering two copies of the edge theory:

$$S = \int dx d\tau \sum_{i=1}^{2}[\psi_{\uparrow i}(-i\partial_x)\psi_{\uparrow i} + \psi_{\downarrow i}(i\partial_x)\psi_{\downarrow i}]. \tag{105}$$

The corresponding Hamiltonian is

$$H = \int dx\, \Psi^T\mathcal{H}\Psi, \quad \mathcal{H} = -i\partial_x\tau_0\sigma_z, \tag{106}$$

where $\Psi^T = (\psi_{\uparrow 1},\psi_{\downarrow 1},\psi_{\uparrow 2},\psi_{\downarrow 2})^T$. Mass terms $i\psi_{\uparrow 1}\psi_{\downarrow 1}$ and $i\psi_{\uparrow 2}\psi_{\downarrow 2}$ are again prohibited. However, $-2i(\psi_{\uparrow 1}\psi_{\downarrow 2} + \psi_{\downarrow 1}\psi_{\uparrow 2}) = \Psi^T\tau_y\sigma_x\Psi$ and $-2i(\psi_{\uparrow 1}\psi_{\uparrow 2} + \psi_{\downarrow 1}\psi_{\downarrow 2}) = \Psi^T\tau_y\sigma_0\psi$ are invariant under reflection and hence allowed to be added as a perturbation.

C. Class CII + R_{--} in $d = 2$

In class CII a system is invariant under TS, PH with $T^2 = C^2 = -1$. When we impose reflection symmetry into the system and require R to anticommute with T and C, Table IV shows that class CII + R_{--} in 2D is always in the trivial phase. In this section, we provide one example to show that the gapless edge states and the entanglement midgap states inevitably are gapped by the translational breaking term, which preserves all of the discrete symmetries, in Eq. (37). Consider the Hamiltonian in class CII + R_{--},

$$\mathcal{H} = M\gamma_0 + \sin k_1\gamma_1 + \sin k_2\gamma_2, \tag{107}$$

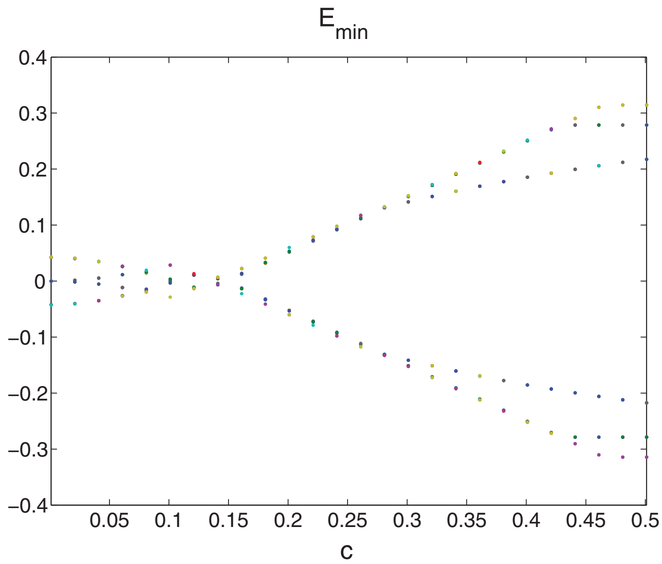
where $M = m + \cos k_1 + \cos k_2$, $\gamma_0 = \tau_z$, $\gamma_1 = \tau_x\sigma_x$, and $\gamma_2 = \tau_x\sigma_y\mu_x$. The corresponding nonspatial symmetry operators are $T = \sigma_y\Theta$, $C = \tau_x\mu_y\Theta$. Physically, τ_i, σ_j, and μ_l can be treated as particle-hole, one-half spin, and pseudospin degrees of freedom, respectively. Without enlarging the matrix dimension, we have three extra kinetic terms $\gamma_3 = \tau_x\sigma_z$, $\gamma_4 = \tau_y$, and $\gamma_5 = \tau_x\sigma_y\mu_z$. In addition, $\sin k_i\gamma_{3,4,5}$ preserve TRS and PHS. Therefore, the reflection-symmetry operator can be defined as $R = i\gamma_1\gamma_3 = \sigma_y$, which anticommutes with T and C. For such a reflection system, as $-2 < m < 2$ the topological phase is nontrivial if the translational symmetry is preserved. That is, the real spectrum gapless edge states still can be observed at the y direction edge and the entanglement midgap states are present when we make a spatial cut exactly in half at $x = 0$. However, those states are unstable when the translational symmetry is broken. Consider the Hamiltonian $\mathcal{H}$ with $\delta\Delta = i\delta\gamma_1\gamma_3\gamma_4 = \tau_y\sigma_y$ and a translational symmetry-breaking term $c\hat{\mathfrak{N}}$ in Eq. (37), where $\mathfrak{N} = i\gamma_{d+1}\gamma_{d+2}\gamma_{d+3} = \tau_y\sigma_x\mu_z$. As shown in Fig. 2(a) the edge states are gapped when $c > \eta$. Furthermore, Fig. 2(b) shows that the midgap in the entanglement spectrum is destroyed; this confirms that in the region $-2 < m < 2$ the system is in the trivial phase.

D. Class DIII + R_{-+} in $d = 3$

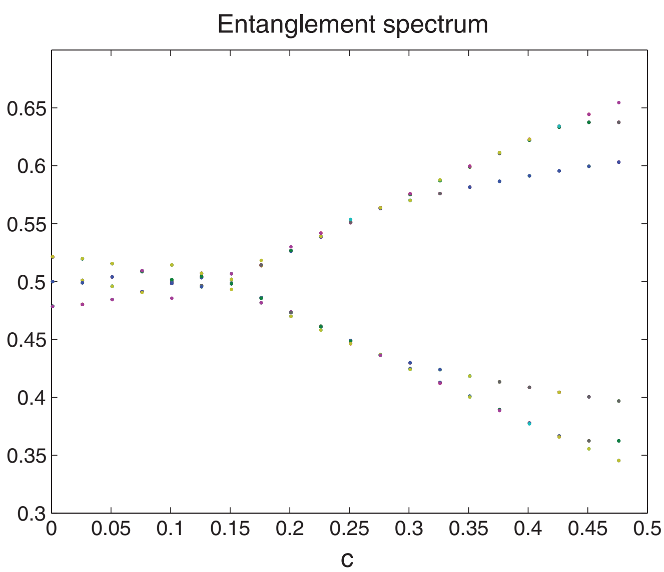
By reading Table V, in three spatial dimensions class DIII + R_{-+} possesses $\mathbb{Z}^1$. In what follows, we demonstrate $\mathbb{Z}^1$ topological properties by considering this reflection-symmetric system in class DIII in 3D. First, without reflection symmetry we introduce a 3D time-reversal invariant topological insulator possessing *artificial* chiral symmetry corresponding to class DIII. We choose the Hamiltonian of such a system to be identical to the Bogoliubov–de Gennes Hamiltonian of the ^{3}He-B phase.[52] The system of ^{3}He-B is recently shown to be a topological superfluid preserving TRS and PHS in 3D.[15–17] Now instead of PHS, we require our system to be invariant under chiral symmetry even if TRS is broken. Therefore, we write the Hamiltonian of the 3D topological insulator possessing TRS and chiral symmetry as

$$H_{\mathrm{DIII}} = \sum_p \Psi^\dagger\begin{pmatrix} \epsilon_p & 0 & tp_+ & -tp_z \\ 0 & \epsilon_p & -tp_z & -tp_- \\ tp_- & -tp_z & -\epsilon_p & 0 \\ -tp_z & -tp_+ & 0 & -\epsilon_p \end{pmatrix}\Psi, \tag{108}$$

where $\Psi(p) = (a_{\uparrow p},a_{\downarrow p},b_{\uparrow p},b_{\downarrow p})$, $p_\pm = p_x \pm ip_y$, and $a_{\sigma p}$ and $b_{\sigma p}$ operators indicate the two different sublattices instead of particle-hole degree freedom in the ^{3}He-B phase. Let $t = 2$



(a) Real spectrum: size $L_x \times L_y = 140 \times 24$



(b) Size $L_x \times L_y = 140 \times 8$

FIG. 2. (Color online) (a) The energy spectrum for the y edge states near $E = 0$ as a function of a translational symmetry breaking term (c) and (b) the entanglement spectrum around the entanglement eigenvalue 1/2 as a function of c. We consider that the open boundary condition is the y discretion. The parameters are set to $m = -1$, $\delta = \arcsin(0.1\pi)$, and $\eta = 0.05\pi = 0.157$. As $c > 0.157$, the edge states start being gapped and the entanglement midgap states begin to move out from 0.5. However, as $c < 0.157$, $E_{\rm min}$ is nonzero and the entanglement eigenvalue is not 0.5 due to the finite size effect in the numerical stimulation.

and in the lattice model the Hamiltonian can be written as

$$\mathcal{H}_{\rm DIII} = \epsilon_p \tau_z + \sin p_x \tau_x \sigma_z + \sin p_y \tau_y + \sin p_z \tau_x \sigma_x, \quad (109)$$

where $\epsilon_p = m - \cos p_x - \cos p_y - \cos p_z$ and τ_i/σ_j describes the sublattice/spin degrees freedom. The expressions of the TRS operator and chiral symmetry operator are $T = \sigma_y \Theta$ and $S = \tau_x \sigma_y \Theta$ so the *pseudo*-PHS operator is $C = \tau_x \Theta$. Since $T^2 = -1$ and $C^2 = 1$, such a system belongs to class DIII. Due to chiral symmetry preserving, by using a method similar to that in in Sec. VI A, the winding number $N_{\mathbb{Z}}$ can be computed:

$$N_{\mathbb{Z}} = \begin{cases} -2, & \text{as } |m| < 1, \\ 1, & \text{as } 1 < |m| < 3, \\ 0, & \text{elsewhere.} \end{cases} \quad (110)$$

In the following, we put our focus on $m = 2$. In the original classification, this winding number corresponds to the number of the gapless surface modes and the entanglement midgap states, which are intact against any symmetry-preserving disorder.

Now we introduce reflection symmetry, which changes $x \to -x$. There are two possible expressions of the reflection-symmetry operators R. First, $a_{\sigma p}$ and $b_{\sigma p}$ exchange under reflection so $R = \tau_x \sigma_x$ (we require R to be Hermitian). This reflection symmetry is broken when the Hamiltonian is in the expression of Eq. (109). Second, the sublattice is invariant under reflection so $R = \sigma_x$. The reflection symmetry is preserved in our system. Therefore, R anticommutes with T and commutes with C. (For ^{3}He-B, such relations change because $R = \tau_z \sigma_x$.) Since R commutes with C, the Hamiltonian at the reflection-symmetry planes $k_x = 0, \pi$ corresponds to class D in 2D, which has a $\mathbb{Z}$ topological invariant. Therefore, the mirror Chern numbers can be computed in those symmetry planes. We perform a unitary transformation on the symmetry operators and the Hamiltonian

$$\sigma_x \to \sigma_z, \quad \sigma_z \to -\sigma_x, \quad \Theta \to \Theta, \quad (111)$$

so that $R = \sigma_z$ and the Hamiltonian is transformed to

$$\mathcal{H}'_{\rm DIII} = \begin{pmatrix} \mathcal{H}_+ & -\sin p_x \tau_x \\ -\sin p_x & \mathcal{H}_- \end{pmatrix}, \quad (112)$$

where $\mathcal{H}_\pm = \epsilon_p \tau_z + \sin p_y \tau_y \pm \sin p_z \tau_x$ in the eigenspace of $R = \pm 1$. As $m = 2$, by Eq. (A6) the Chern numbers for $\mathcal{H}_+$ at $k_x = 0, \pi$ are $\nu_0 = 1, \nu_\pi = 0$, respectively. Therefore, the mirror Chern number defined by Eq. (33) is

$$N_{M\mathbb{Z}} = 1. \quad (113)$$

The topological phase of the system protected by the $\mathbb{Z}$ and $M\mathbb{Z}$ topological invariants is described by the $\mathbb{Z}^1$ number in Eq. (45). In this case, $N_{\mathbb{Z}^1} = 1$, corresponding to the number of the gapless surface states and the entanglement midgap.

Now consider the two copies of the topological insulator Hamiltonian in a system but with a sign changing. The enlarged BdG Hamiltonian can be written as

$$\mathcal{H}^2_{\rm DIII} = \begin{pmatrix} \mathcal{H}_{\rm DIII} & 0 \\ 0 & \mathcal{H}'_{\rm DIII} \end{pmatrix}, \quad (114)$$

where we choose $\mathcal{H}'_{\rm DIII} = \mathcal{H}_{\rm DIII}(p_x \to -p_x)$. The two topological numbers for $\mathcal{H}'_{\rm DIII}$ are $N_{M\mathbb{Z}} = 1$ and $N'_{\mathbb{Z}} = -1$ and then for the whole system are $N_{M\mathbb{Z}} = 2$ and $N_{\mathbb{Z}} = 0$ so $N_{\mathbb{Z}^1} = 2$. Although the $\mathbb{Z}$ topological number vanishes, the topological phase is protected by $N_{M\mathbb{Z}}$.

Let $\mathcal{H}'_{\rm DIII}$ in $\mathcal{H}^2_{\rm DIII}$ become $\mathcal{H}_{\rm DIII}$ with $p_x \to -p_x, p_y \to -p_y$. Although $N_{M\mathbb{Z}} = 0$ for the entire system, $N_{\mathbb{Z}} = 2$ so $N_{\mathbb{Z}^1} = 2$. That is the reason that there are two robust gapless surface modes.

Also we consider $\mathcal{H}'_{\rm DIII} = \mathcal{H}_{\rm DIII}$ with only $p_y \to -p_y$. Both of the two topological numbers for $\mathcal{H}'_{\rm DIII}$ switch the signs. Therefore, the total numbers vanish: $N_{M\mathbb{Z}}$ and $N_{\mathbb{Z}^1}$. This phase is trivial because we can find a SPEMT,

$$\begin{pmatrix} 0 & \tau_y \\ \tau_y & 0 \end{pmatrix}, \tag{115}$$

which anticommutes with $\mathcal{H}^2_{\rm DIII}$.

VII. CONCLUSION

Combining Tables III to VI, we write the complete classification results (27 symmetry classes $+R$) in Table I. The classification of reflection-symmetric topological insulators and superconductors still has the same spatial dimensional periodicities with the original AZ classification in Table II. The complex and real symmetry classes with reflection symmetry have the periods of two and eight, respectively. Although the reflection classification tables seem to be complicated, the two ingredients can slightly simplify the classifications. To define these ingredients, we consider a d-dimensional system in an AZ-symmetry class with reflection symmetry. We denote N^d as a topological invariant of the original strong index without reflection symmetry as shown in Table II. Furthermore, we define N^{d-1} as the $(d-1)$-dimensional topological invariant in the corresponding mirror-symmetry class (Appendix B). By observing the reflection tables, some of the topological invariants of the strong index are determined by N^d and N^{d-1} as shown in Table VII. In other cases, there is an ambiguity to determine the strong topological invariants of reflection-symmetric systems. Therefore, the approach of using a minimal Dirac model provides a systematic way to find topological invariants.

While topological insulators and superconductors protected by a set of spatial discrete symmetries are more fragile, in general, than those protected by nonspatial symmetries, they are still fairly relevant to realistic systems. We list several realistic reflection-symmetric systems in Table I.

For example, we again note that for the $\mathbb{Z}_2$ TRS topological insulator in class AII (without reflection), its Dirac representative has reflection symmetry R_{--} in three spatial dimensions. This is, from the point of view of TRS topological insulators, somewhat accidental. However, as we discussed, this is related to the fact that there is a topological distinction of ground states even without time reversal when reflection symmetry is a good symmetry. The associated topological invariant is integer valued ($M\mathbb{Z}$), as opposed to $\mathbb{Z}_2$.

In fact, many experimentally realized topological insulators such as Bi_2Se_3 "accidentally" have reflection symmetry. From the discussion above, even breaking TRS, a surface Dirac cone will not go away if both the TRS breaking perturbation and the surface geometry respect reflection symmetry. For example, we consider a surface which respects the reflection symmetry. When an in-plane magnetic field, namely parallel to the surface, is applied to the system, the TRS is broken. However, the reflection symmetry is still preserved when the direction of the in-plane magnetic field is tuned to coincide with the reflected direction. If this occurs, the surface Dirac cone is stable since it is protected by the preserved reflection symmetry even though the TRS has been broken. These stable surface Dirac cones may be detected by STM, say. (ARPES may not be ideal if we use a magnetic field to break TRS.) When the direction of applied magnetic field is away from the reflection direction, the system respects neither reflection nor TRS. As a consequence, a gap should open in the surface states. We predict that the surface magnetoresistance with an in-plane magnetic field changes significantly, when the field direction is rotated and when the chemical potential is close to the Dirac point. This is because the surface gap varies with the rotation of the field direction.

We close with discussion on the effects of disorder on topological insulators and superconductors protected by reflection symmetry. In the tenfold classification of topological insulators and superconductors, it has been proved useful to consider the boundary (edge, surface, etc.) Anderson localization problem: For a topological bulk, one should find a boundary mode which is completely immune to disorder. In turn, once one finds such "Anderson delocalization" at the boundary, it means there is a topologically nontrivial bulk. Not only can this bulk-boundary correspondence be used to find and classify bulk topological phases *in the absence of disorder*, it immediately tells us such topological phases are stable against disorder; two phases which are topologically distinct cannot be adiabatically connected by either spatially homogeneous or inhomogeneous deformations. For topological phases protected by a set of spatial symmetries, stability against disorder is, in general, not trivial, since spatial inhomogeneity does not respect the spatial symmetries. One can still consider, however, situations where the spatial symmetries are preserved *on average*.[33,53,54] Below, we consider the stability of reflection-protected topological phases we identified earlier against disorder which is reflection symmetric on average.

Let us consider as an example the reflection-symmetric topological insulator in symmetry class A in 3D (class A + R). (For other examples in 2D, see Ref. 33.) For symmetry class A in 3D, we have a topological insulator protected by reflection symmetry, which is characterized by an integer topological invariant. Let us consider the surface Hamiltonian: $\mathcal{H}(r) = \mathcal{H}_0(r) + \mathcal{V}(r)$ where r denotes the 2D coordinates on the surface, $\mathcal{H}_0(r)$ is a kinetic term (the surface Dirac kinetic term). We have added a random perturbation $\mathcal{V}(r)$. The disorder-free part is reflection symmetric under R, which is reflection inherited from the bulk, $R^{-1}\mathcal{H}_0(\tilde{r})R = \mathcal{H}_0(r)$, while disorder $\mathcal{V}(r)$ is not so. The reflection symmetry can be, however, imposed on average: $R^{-1}\overline{\mathcal{V}(\tilde{r})}R = \overline{\mathcal{V}(r)}$,

TABLE VII. The topological invariants of reflection-symmetric topological insulators and superconductors, and the strong topological index for d-dimensional topological states in the original periodic Table I (N_d), and the $d-1$-dimensional mirror topological invariant (N^{d-1}). In several cases, N^d and N^{d-1} still cannot completely determine the reflection topological invariant. We have to go back to the minimal Dirac Hamiltonian method to determine the topological characters.

$N^{d-1} N^d$	0	$\mathbb{Z}_2$	$\mathbb{Z}$
0	0	0, $\mathbb{Z}_2$	0, $\mathbb{Z}$
$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$,$\mathbb{Z}$
$\mathbb{Z}$	0,$M\mathbb{Z}$	$M\mathbb{Z}$	$\mathbb{Z}^1$

where $\overline{\cdots}$ represents the quenched disorder averaging. We could approach this problem by means of effective field theories of Anderson localization, the nonlinear σ models (NLσMs); they describe slowly varying degrees of freedom in a disordered metal, which are related to a diffusion motion of electrons (called "diffusions" and "Cooperons"). When derived for the disordered surface problem, the action of the NLσM is given by

$$S_{\mathrm{NL}\sigma\mathrm{M}} = \frac{1}{\lambda}\int d^2r\,\mathrm{tr}\,[\partial_\mu Q\partial_\mu Q] + \frac{\Theta}{16\pi i}\int d^2r\,\epsilon^{\mu\nu}\mathrm{tr}\,[Q\partial_\mu Q\partial_\nu Q], \quad (116)$$

where a matrix field $Q(r)$ is the NLσM field $Q\in \mathrm{U}(2N_r)/\mathrm{U}(N_r)\times\mathrm{U}(N_r)$ and N_r is the number of replicas; λ is the coupling constant of the NLσM, which is the strength of interactions among diffusons and Cooperons and is inversely proportional the conductivity. The last term in the action is the topological term (Pruisken term), which counts the nontrivial winding associated to $\pi_2[\mathrm{U}(2N_r)/\mathrm{U}(N_r)\times\mathrm{U}(N_r)]=\mathbb{Z}$. (Here we are considering the real space which is topologically equivalent to a sphere.) In the absence of any discrete symmetry, Θ can take, in principle, any value, in which case one can make electrons to be Anderson localized. While the action for the generic value of Θ breaks reflection symmetry (on the surface), $\Theta=(\text{integer})\times\pi$ turns out to be consistent with reflection symmetry. Moreover, when $\Theta=(\text{odd integer})\times\pi$ there is no Anderson localization. The NLσM (116) can be derived for the Dirac representative of the surface mode, which consists of N flavors of two-component Dirac fermions when the bulk topological invariant is $N\in\mathbb{Z}$. The θ angle is given by $\Theta=N\pi$. Therefore, there is an even-odd effect; when the bulk topological invariant is odd (even), the surface mode is stable (unstable) against disorder. This would mean that, in the presence of spatially inhomogeneity which nevertheless preserves reflection symmetry on average, the topological distinction is not $\mathbb{Z}$, but $\mathbb{Z}_2$.

Recently, two complementary and independent preprints (Refs. 55 and 56) appeared in which symmetry class D with reflection symmetry R_+ and symmetry class DIII with reflection symmetry R_{--} are discussed, respectively.

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APPENDIX A: THE CALCULATION OF $\mathbb{Z}$ NUMBER

We review the calculation of the topological number for the symmetry classes possessing a $\mathbb{Z}$ or $2\mathbb{Z}$ topological invariant. For the classification without reflection symmetry, in odd ($d=2n+1$) spatial dimensions such symmetry classes always have chiral symmetry, which is a key point to define the topological number. In even ($2n+2$) spatial dimensions, the topological number is exactly the same with the $n+1$th Chern number.

To seek the expression of the topological number in $2n+1$ spatial dimensions, first we introduce the spectral projector onto the filled Bloch states and the "Q matrix" (flat band)[43] by

$$P(k)=\sum_{\hat{a}}|u^-_{\hat{a}}(k)\rangle\langle u^-_{\hat{a}}(k)|, \quad Q(k)=\mathbb{I}-2P(k). \quad (A1)$$

We note that $P(k)^2=P(k)$ so $Q(k)^2=\mathbb{I}$. If the system possesses the chiral symmetry, which is described by Eq. (4), with negative energy filled $N_+=N_-$. The Q matrix can be brought into block off-diagonal form,

$$Q(k)=\begin{pmatrix}0 & q(k)\\ q^\dagger(k) & 0\end{pmatrix}, \quad q(k)\in U(N_-), \quad (A2)$$

in some basis. The topological number in $2n+1$ spatial dimensions is characterized by the winding number,

$$\nu_{2n+1}[q]=\frac{(-1)^n n!}{(2n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\int_{\mathrm{BZ}^{d=2n+1}}\mathrm{tr}[(q^{-1}dq)^{2n+1}]. \quad (A3)$$

Here, BZ^d means the integration over d-dimensional k space. That is, the region is the first BZ in the lattice model.

For the topological number in $2n+2$ spatial dimensions, first we define the non-Abelian Berry connection of the occupied bands

$$\mathcal{A}^{\hat{a}\hat{b}}(k)=A^{\hat{a}\hat{b}}_\mu(k)dk_\mu=\langle u^-_{\hat{a}}(k)|du^-_{\hat{b}}(k)\rangle, \quad (A4)$$

where $\mu=1,\ldots,d$, $\hat{a}$, $\hat{b}=1,\ldots,N_-$. The Berry curvature is defined by

$$\mathcal{F}^{\hat{a}\hat{b}}(k)=d\mathcal{A}^{\hat{a}\hat{b}}+(\mathcal{A}^2)^{\hat{a}\hat{b}}. \quad (A5)$$

The topological number is captured by the $n+1$th Chern number

$$\nu_{2n+2}=\mathrm{Ch}_{n+1}[\mathcal{F}]=\frac{1}{(n+1)!}\int_{\mathrm{BZ}^{d=2n+2}}\mathrm{tr}\left(\frac{i\mathcal{F}}{2\pi}\right)^{n+1}. \quad (A6)$$

APPENDIX B: SYMMETRY CLASSES IN MIRROR PLANES

In the mirror planes $k_1=0,\pi$, the Hamiltonian commutes with the reflection operator R,

$$[\mathcal{H}_{k_1=0,\pi},R]=0. \quad (B1)$$

Furthermore, because R is Hermitian and $R^2=1$, the Hamiltonian can be decomposed to two diagonal blocks $\mathcal{H}_\pm$ corresponding to the eigenspaces $R=\pm1$, respectively. Now consider nonspatial symmetries, which determine the symmetry class of the system. If all of the nonspatial symmetry operators commute with R, the two block Hamiltonians $\mathcal{H}_\pm$ belong to the same symmetry class. Furthermore, we name this symmetry class for $\mathcal{H}_\pm$ in the mirror planes as *mirror-symmetry class*. However, the mirror-symmetry class changes when at least one of the nonspatial symmetry operators anticommutes with R. The Hamiltonians $\mathcal{H}_\pm$ are invariant under the R-commuting symmetries but not invariant under the R-anticommuting symmetry. Therefore, the mirror-symmetry class is determined by the R-commuting symmetries. Table I shows the mirror-symmetry classes for each possible algebraic relation between nonspatial symmetry operators and R.

APPENDIX C: THE PROOF OF THE $\mathbb{Z}^1$ NUMBER DEFINITION

We prove that the bulk topology of the $\mathbb{Z}^1$ system is determined by the maximum value of $|N_{\mathbb{Z}}|$ and $|N_{M\mathbb{Z}}|$ in general cases. We simplify the problem by considering the Dirac Hamiltonian with the coefficient of the mass term:

$$m = M - (k_x \pm \delta_i)^2 - \tilde{k}^2. \tag{C1}$$

In the proof of Sec. V A1 we consider only $\delta_i = 0$. In general, δ_i can be any number in BZ. As $\delta_i = 0,\pi$, $N_{\mathbb{Z}}$ and $N_{M\mathbb{Z}}$ can be computed for such Dirac Hamiltonians. As δ_i is not at the mirror-symmetry points, the Dirac Hamiltonians only make a contribution to $N_{\mathbb{Z}}$. To achieve the proof of the $\mathbb{Z}^1$ number definition, we consider the Dirac Hamiltonians in any possible distribution and determine the interplay between the phase of the nontrivial Dirac Hamiltonians, $N_{\mathbb{Z}}$, and $N_{M\mathbb{Z}}$.

In general, the $M\mathbb{Z}$ topological number is computed in the two symmetry planes $k_x = 0,\pi$. To simplify the problem, we consider that the weak mirror index in Eq. (31) vanishes so $\nu_0\nu_\pi \leqslant 0$; hence, the $M\mathbb{Z}$ number is given by $N_{M\mathbb{Z}} = \nu_0 + \nu_\pi$. However, with translational symmetry breaking by folding the BZ, the two symmetry planes collapse to the one symmetry plane $k = 0$. The $M\mathbb{Z}$ number of this $d-1$-dimensional plane is $\nu_0 + \nu_\pi$. Since $N_{M\mathbb{Z}}$ is invariant under density waves that connect these two points, the $d-1$-dimensional $\mathbb{Z}$ number in the new $k_x = \pi$ plane vanishes. Thus, we still can discuss any $M\mathbb{Z}$ number only in the $k_x = 0$ plane without loss of generality.

The $\mathbb{Z}$ topological number ($N_{\mathbb{Z}}$) for the entire system, in general, cannot be defined in the mirror-symmetry planes. The discussion of the $\mathbb{Z}$ number can be separated to two parts: $\delta_i = 0,\pi$ and $\delta_i \neq 0$ or π. First, the part $N^0_{\mathbb{Z}}$ of the $\mathbb{Z}$ number is contributed from the Dirac Hamiltonians with $\delta_i = 0,\pi$. The problem can be simplified by considering only the $\delta_i = 0$ case. The reason is that the contribution of the $\delta_i = \pi$ can be moved to $\delta_i = 0$ by folding the BZ. Second, the other part $N'_{\mathbb{Z}}$ of the $\mathbb{Z}$ number are contributed from $\delta_i \neq 0$ or π. Due to the reflection symmetry, $N'_{\mathbb{Z}}$ must be even. Out of the mirror-symmetry planes, the bulk topology is losing the reflection-symmetry protection. Therefore, if the Dirac Hamiltonians ($\pm\delta_i$) possess $\mathbb{Z}$ numbers that differ by signs, the system combined by these two Hamiltonian is in the trivial phase. Hence, we put our focus on these Dirac Hamiltonians having the same sign of the $\mathbb{Z}$ numbers.

We define $(+)$ as a pair of the Dirac Hamiltonians ($\pm\delta_i$) with an even number of the γ matrices having the minus sign. In other words, the $\mathbb{Z}$ number in this case is 2. Similarly, $(-)$ indicates an odd number of the minus-sign γ matrices and $N'_{\mathbb{Z}} = -2$. Consider a system possessing the Dirac Hamiltonians with $(+,+)$ (see Sec. V A1) and $(-)$. Hence, $N_{\mathbb{Z}} = -1$ and $N_{M\mathbb{Z}} = 1$. The entire Hamiltonian for the system is written as

$$H = \sum_{k_1}[\mathcal{H}_{++}b^\dagger_{k_1}b_{k_1} + \mathcal{H}_-(c^\dagger_{k_1+\delta}c_{k_1+\delta} + c^\dagger_{k_1-\delta}c_{k_1-\delta})], \tag{C2}$$

where $\mathcal{H}_{++} = m\gamma_0 + \sum_{i=1} k_i\gamma_1$ and $\mathcal{H}_- = \sum_{n_i} k_{n_i}\gamma_{n_i} - \sum^{\text{odd}}_{n_j} k_{n_j}\gamma_{n_j}$. Due to Eq. (13) the Hamiltonian is invariant under the reflection symmetry with the reflection operator in the second quantization,

$$\hat{R} = \sum_{k_1}(b^\dagger_{k_1}Rb_{-k_1} + c^\dagger_{k_1}Rc_{-k_1}). \tag{C3}$$

The SPEMT is found to prevent two of the three minimal Dirac Hamiltonians passing through quantum phase transition,

$$\hat{\mathfrak{N}}_\pm = \frac{\mathfrak{N}}{\sqrt{2}}(b^\dagger_{k_1}c_{k_1+\delta} \pm b^\dagger_{k_1}c_{k_1-\delta}) + \text{H.c.}, \tag{C4}$$

where $\mathfrak{N} = (i)\prod^{\text{odd}}_{n_j}\gamma_{n_j}$ and we choose the presence or absence of "i" to preserve TRS and PHS. Moreover, in order to preserve the reflection symmetry $+/-$ in $\hat{\mathfrak{N}}$ corresponds to $n_j \neq 1/n_j = 1$ in the product of $\mathfrak{N}$. The entire Hamiltonian with $\hat{\mathfrak{N}}_\pm$ and a real coupling coefficient g is given by

$$H_\pm = \sum_{k_1}\Psi^\dagger_\pm\begin{pmatrix}\mathcal{H}_{++} & g\mathfrak{N} & 0\\ g\mathfrak{N}^\dagger & \mathcal{H}_- & 0\\ 0 & 0 & \mathcal{H}_-\end{pmatrix}\Psi_\pm \tag{C5}$$

on the basis of

$$\Psi_\pm = \left(c_{k_1}\ \frac{1}{\sqrt{2}}(c_{k_1+\delta} \pm c_{k_1-\delta})\ \frac{1}{\sqrt{2}}(c_{k_1+\delta} \mp c_{k_1-\delta})\right)^T. \tag{C6}$$

Hence, the first two Dirac Hamiltonians become trivial and the last Dirac Hamiltonian survives. The topological number is one, which equals to $N_{\mathbb{Z}^1}$ in Eq. (45).

Now we add an extra $(+,+)$ into the original system. That is, $N_{\mathbb{Z}} = 0$ and $N_{m\mathbb{Z}} = 2$. The entire Hamiltonian with the SPEMTs is written as

$$H_\pm = \sum_{k_1}\Phi^\dagger_\pm\begin{pmatrix}\mathcal{H}_{++} & 0 & g_1\mathfrak{N} & 0\\ 0 & \mathcal{H}_{++} & g_2\mathfrak{N} & 0\\ g_1\mathfrak{N}^\dagger & g_2\mathfrak{N}^\dagger & \mathcal{H}_- & 0\\ 0 & 0 & 0 & \mathcal{H}_-\end{pmatrix}\Phi_\pm \tag{C7}$$

on the basis of

$$\Phi_\pm = \left(a_{k_1}\ b_{k_1}\ \frac{1}{\sqrt{2}}(c_{k_1+\delta} \pm c_{k_1-\delta})\ \frac{1}{\sqrt{2}}(c_{k_1+\delta} \mp c_{k_1-\delta})\right)^T. \tag{C8}$$

The SPEMT coupling from the last Dirac Hamiltonian $\mathcal{H}_{\text{odd}}$ to the others is forbidden by the symmetries. By performing a proper unitary transition, only two Dirac Hamiltonians couple so that these subsystems become trivial. The two other Dirac Hamiltonians keep the original bulk topology structure. Hence, the topological number is two, which is the maximum value of $|N_{\mathbb{Z}}|$ and $|N_{M\mathbb{Z}}|$.

Instead of adding an extra $(+,+)$, we add $(-,-)$

$$\mathcal{H}_{--} = -m\gamma_0 - k_1\gamma_1 + \sum_{i=2}k_i\gamma_i \tag{C9}$$

into the original system. In other words, consider the system with $N_{\mathbb{Z}} = N_{M\mathbb{Z}} = 0$. The entire Hamiltonian with the SPEMT is given by

$$H_{\pm} = \sum_{k_1} \Phi_{\pm}^{\dagger} \begin{pmatrix} \mathcal{H}_{--} & 0 & 0 & g_1\mathfrak{L} \\ 0 & \mathcal{H}_{++} & g_2\mathfrak{N} & 0 \\ 0 & g_2\mathfrak{N}^{\dagger} & \mathcal{H}_{-} & 0 \\ g_1\mathfrak{L}^{\dagger} & 0 & 0 & \mathcal{H}_{-} \end{pmatrix} \Phi_{\pm}, \quad \text{(C10)}$$

where $\mathfrak{L} = i\mathfrak{N}\gamma_0\gamma_1$ preserves all of the symmetries. All of the Dirac Hamiltonians couple so the entire system is in the trivial phase. As expected, the topological number vanishes. Thus, only one $(+,+)$ can couple with a pair of Dirac Hamiltonians $(-)$ and then one Dirac Hamiltonian survives and provides the nontrivial bulk topology. In general, consider $N_{+,+}$ $(+,+)$ Dirac Hamiltonians and $N_-/2$ pairs of $(-)$ Dirac Hamiltonians in a system. The number of the nontrivial Dirac Hamiltonians is $|N_{+,+} - N_-/2|$. Similarly, the number of the protected modes for the other cases separately are $|N_{+,-} - N_+/2|$, $|N_{-,-} - N_-/2|$, and $|N_{-,+} - N_+/2|$, where $N_+/2$ is the number of the pairs of $(+)$ Dirac Hamiltonians. However, the discussion in Sec. V A1 $(+,\pm)$ the number of the protected modes for the entire system is $|N_{+,+} - N_{+,-}| + |N_{-,+} - N_{-,-}|$. In this case, we replace

$$N_{\pm,\mp} \to N_{\pm,\mp} - N_+/2, \qquad N_{\pm,\pm} \to N_{\pm,\mp} - N_-/2. \quad \text{(C11)}$$

Moreover, a system with the same number of $(-)$ and $(+)$ is trivial. Thus, the total number of the protected modes is given by

$$\left|N_{+,+} - N_{+,-} + \frac{N_+ - N_-}{2}\right| + \left|N_{-,+} - N_{-,-} + \frac{N_+ - N_-}{2}\right|, \quad \text{(C12)}$$

which is the topological number for the $\mathbb{Z}^1$ system. By applying those identities,

$$N_{\mathbb{Z}}^0 \pm N_{M\mathbb{Z}} = N_{\pm,+} - N_{\pm,-}, \quad \text{(C13)}$$

$$N'_{\mathbb{Z}} = N_+ - N_-, \quad N_{\mathbb{Z}} = N_{\mathbb{Z}}^0 + N'_{\mathbb{Z}}, \quad \text{(C14)}$$

we can simplify the expression of the $\mathbb{Z}^1$ number:

$$N_{\mathbb{Z}^1} = \left|\frac{N_{\mathbb{Z}}^0 + N_{M\mathbb{Z}} + N'_{\mathbb{Z}}}{2}\right| + \left|\frac{N_{\mathbb{Z}}^0 - N_{M\mathbb{Z}} + N'_{\mathbb{Z}}}{2}\right| = \mathrm{Max}(|\nu_d|,|N_{M\mathbb{Z}}|). \quad \text{(C15)}$$

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[40]In the proper basis, $i\gamma_0\gamma_d$ can be written as $\sigma_z \otimes \mathbb{I}$. Therefore, the decomposition of $\tilde{\Gamma}$ is uniquely in the form of $\tilde{\Gamma}_C + \tilde{\Gamma}_A$, where $[i\gamma_0\gamma_d,\tilde{\Gamma}_C] = 0$ and $\{i\gamma_0\gamma_d,\tilde{\Gamma}_A\} = 0$. Let W be system's symmetry operator. We know $W\tilde{\Gamma}W^{-1} = \tilde{\Gamma}$ and $Wi\gamma_0\gamma_d W^{-1} = \pm i\gamma_0\gamma_d$. Hence, $[W\tilde{\Gamma}_C W^{-1}, i\gamma_0\gamma_d] = 0$, so $W\tilde{\Gamma}_C W^{-1} = \tilde{\Gamma}_C$. That is, $\tilde{\Gamma}_C$ preserves system's symmetry. Now let a new $\tilde{\Gamma}$ be $\tilde{\Gamma}_C$, which commutes with $i\gamma_0\gamma_d$.

[41]We know that $\tilde{\gamma}\gamma_{\mathbf{p}j} = -\gamma_{\mathbf{p}j}\tilde{\gamma}$, where $j \neq 0,d$. Therefore, $\mathbf{P}\tilde{\Gamma}\mathbf{P}^2\gamma_j\mathbf{P} = -\mathbf{P}\gamma_j\mathbf{P}^2\tilde{\Gamma}\mathbf{P}$. Since γ_j and $\tilde{\Gamma}$ commute with $i\gamma_0\gamma_d$, $\mathbf{P}\tilde{\Gamma}\gamma_j = -\mathbf{P}\gamma_j\tilde{\Gamma}$. Similarly, consider the domain wall with $M \to -M$. The projection operator becomes $\mathbf{P}' = (\mathbb{I} + i\gamma_0\gamma_d)/2$. We have $\mathbf{P}'\tilde{\Gamma}\gamma_j = -\mathbf{P}'\gamma_j\tilde{\Gamma}$. Thus, $\tilde{\Gamma}\gamma_j = -\gamma_j\tilde{\Gamma}$.

[42]In the proper basis, $\gamma_0 = \sigma_z \otimes \mathbb{I}$. Therefore, $\tilde{\Gamma}$ can be written in the form of $\tilde{\Gamma}'_C + \tilde{\Gamma}'_A$, where $\tilde{\Gamma}'_C$ commutes with γ_0 and $\tilde{\Gamma}'_A$ anticommutes with γ_0. Following the similar derivation with Ref. 39, $\tilde{\Gamma}'_C$ and $\tilde{\Gamma}'_A$ both preserve system's symmetry.

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[46]We give two examples: For class CII in $d = 2$, the symmetry operators are $C = \tau_x\mu_y$ and $T = \sigma_x\Theta$. The γ matrices are $\gamma_0 = \tau_z, \gamma_1 = \tau_x\sigma_x, \gamma_2 = \tau_x\sigma_y\mu_x, \gamma_3 = \tau_x\sigma_z, \gamma_4 = \tau_y$, and $\gamma_5 = \tau_x\sigma_y\mu_z$. For class C in $d = 3$, only PHS is kept and the γ matrices are the same with class CII, except for the extra kinetic term $\gamma_6 = \tau_x\sigma_y\mu_y$.

[47]The other definition of the reflection symmetry is $\hat{R} = \sum_{k_1} c^\dagger_{k_1}(i\gamma_0\gamma_{d+1})c_{-k_1}e^{-ik_1}$. We redefine $\mathfrak{N} = e^{-i\eta}i\gamma_{d+1}\gamma_{d+2}\gamma_{d+3}$. The topological phase is still trivial.

[48]In the same dimensions the sizes of the minimal Dirac Hamiltonians in this case are always twice as big, having $M\mathbb{Z}$ invariants for $R_{\pm,\pm}$ (Ref. 35). For example, the size of the minimal Hamiltonian in class D + R_{++} is 4 and the size of the minimal Hamiltonian is class BDI + R_{-+} and CII + R_{-+} is 8. The situation is similar with that in the AZ classification the sizes of the minimal Dirac Hamiltonians having $2\mathbb{Z}$ invariants are twice as big, having $\mathbb{Z}$ invariants. Hence, in this reflection-symmetry case the systems possess $2M\mathbb{Z}$ topological invariants.

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