Field-induced reentrant superconductivity in thin films of nodal superconductors

M. Hachiya,[1] K. Aoyama,[1,2] and R. Ikeda[1]

[1]*Department of Physics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan*

[2]*The Hakubi Center for Advanced Research, Kyoto University, Kyoto 606-8501, Japan*

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Previous work on nodal d-wave superconductors has shown that a Fulde-Ferrell-Larkin-Ovchinnikov- (FFLO-) like superconducting (SC) state, which is modulated along the film plane, can be realized with no magnetic field when quasiparticles acquire an additional linear term in the wave vector in their dispersion. In the present work, the stability of such a modulated SC state in an artificial film against an applied magnetic field is studied. As a reflection of the presence of two FFLO-like states of different origins, one close to zero field and the other at the high-field end, in a single field vs temperature phase diagram of thin films, the conventional SC state, which is uniform along the film plane, generally tends to appear as a reentrant ordered phase bounded by the normal phase in lower fields.

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I. INTRODUCTION

The conventional superconducting (SC) phase is characterized by a spatially uniform SC order parameter as a reflection of the Bose-Einstein condensation. In several situations, however, a SC ground state can have a spatial modulation in the SC order parameter even if quantized vortices are absent. Such a modulated SC state is regarded as one of the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) states[1,2] and is usually stabilized by a population imbalance of up-spin and down-spin electrons in the momentum space, i.e., by a momentum-*independent* splitting between the two species of Fermi surfaces provided by strong Pauli paramagnetic pair breaking. It is believed at present through extensive studies that the high-field SC phase in the heavy-fermion material $CeCoIn_5$ is the above-mentioned FFLO vortex state with a one-dimensional modulation parallel to the applied magnetic field.[3–5]

It has been clarified recently in different contexts that a similar spatially modulated SC state becomes possible in nodal d-wave superconductors if the quasiparticle dispersion $\varepsilon_{\mathbf{k}}$ includes an additional term linear in $\mathbf{k}$ of the type

$$\varepsilon_{\mathbf{k}} = \varepsilon_{\mathbf{k}}^{(0)} + \mathbf{V}\cdot\mathbf{k}, \tag{1}$$

where $\mathbf{k}$ is the wave vector, $\mathbf{V}$ is $\mathbf{k}$ independent, and $\varepsilon_{\mathbf{k}}^{(0)}$ is an even function of $\mathbf{k}$. Such contexts include (1) zero-field cases in an external current,[6] (2) thin films in zero field,[7] and (3) Rashba noncentrosymmetric superconductors in a magnetic field parallel to a gap node direction.[8,9] The vector $\mathbf{V}$ in Eq. (1) is proportional to the strength of the external current in case (1), the inverse of the film thickness in case (2), and the Zeeman energy and the spin-orbit coupling in case (3). In all of these cases, the interplay between the $\mathbf{k}$ dependence in the last term of Eq. (1) and the corresponding dependence of the SC gap function favors a spatial modulation parallel to a gap-node direction in the SC state. We note that the only FFLO-like state to be realized in these situations with no vortices is a state accompanied only by a pure *phase* modulation.

In the present work, we focus on the above-mentioned case (2), i.e., a SC film sample with anisotropic nodal pairing. In Ref. 7, a tendency for formation of a modulated SC state in a SC film with $d_{x^2-y^2}$-wave pairing has been pointed out in the zero-field case.[7] There, a $d_{x^2-y^2}$-wave paired SC film sample is assumed to have a gap node parallel to the film plane and to be obtained by cutting the bulk sample along the plane formed by the nodal [e.g., (1,1,0)] and z directions, and, in addition, the specular condition on the boundary surfaces is assumed. The main point in Ref. 7 is that a combination of this boundary condition and the d-wave pairing symmetry leads to a spatial modulation of the SC order parameter parallel to the film plane *and* a gap-node direction. In contrast, if the boundary surface is not parallel to any gap-node direction but is parallel to an antinode [e.g., (1,0,0)] direction in the $d_{x^2-y^2}$-pairing case, no stable modulated SC state appears, and just the ordinary uniform SC state[10] is realized in zero field.

Then, it is natural to address whether this nontrivial mechanism leading to a modulated SC state is affected by applying a uniform magnetic field. Since, at least, the Zeeman effect on the conduction electrons breaks a spin-singlet pair, the modulated state in zero field would be *suppressed* with increasing field. However, it is unclear whether, by increasing the field, this state gives way to the normal state or other SC states. In fact, the *conventional* FFLO state *induced* by the momentum-independent Zeeman effect should be realized in high fields and at low temperatures. The resulting field-induced competition between these two modulated states of different origins may lead to unusual field dependences of the phase diagram in intermediate fields.

Hereafter, we investigate possible field vs temperature phase diagrams of films of unconventional superconductors in a magnetic field parallel to the film plane by assuming that the material is close to the Pauli limit so that the presence of vortices may be neglected. Even in film samples with thickness of several times the zero-temperature coherence length ξ_0, field-induced vortices would be inevitably present. Nevertheless, if the applied magnetic field, i.e., the straight vortex axis, is parallel to the modulation direction of the phase-modulated state occurring even in zero field, the vortex lattice pattern in the plane perpendicular to the field is unaffected by this phase modulation, implying that conclusions on the (mean-field) transitions to the normal phase and to the phase-modulated SC state obtained in the Pauli limit, i.e., without orbital pair breaking, are applicable to describing real superconductors with large paramagnetic pair-breaking effect. This argument is based on previous studies on the

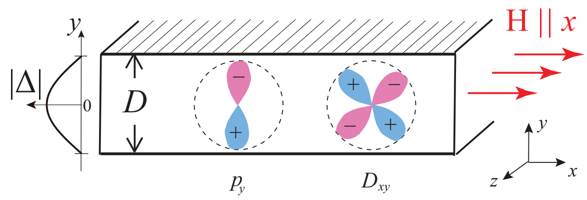

FIG. 1. (Color online) Geometry of superconducting film samples with thickness D and the $d_{x^2-y^2}$ pairing in which the gap nodes are parallel to the film plane (the state in this setup with the $d_{x^2-y^2}$ symmetry is called hereafter the D_{xy} state). The p_y-pairing case is also indicated here for comparison. The flat film surfaces are in the z-x plane. Due to the specular boundary condition on the quasiparticles when the gap nodes are parallel to the film plane, the amplitude of the SC gap function $|\Delta|$ is suppressed to zero on the sample surfaces. The effects of a finite magnetic field applied in the gap-node direction within the two-dimensional (2D) plane (the x direction) will be considered in Sec. III.

corresponding bulk system.[5,11] Of course, the spatially uniform SC phase appearing in our calculation in the Pauli limit would correspond to a vortex lattice in a thin film, which consists of vortices running straight along the field parallel to the film plane.

The geometry of the present system is illustrated in Fig. 1. Although our focus is primarily on the nodal d-wave superconductors, we also discuss a simpler nodal p-wave pairing case in order to clarify that one origin of the present phase-modulated state in zero field is a gap node parallel to the film plane. It is found that, for a film thickness of several times ξ_0, applying a weak parallel magnetic field in the d-wave nodal SC case changes the phase-modulated SC state in low fields to the normal state, while the uniform SC and the conventional FFLO states survive in higher fields. Consequently, reentrant SC phases are expected to occur with increasing field for a film thickness of several times ξ_0. It is pointed out that such a strange field dependence of the SC phases, in part, stems from the nonmonotonic thickness dependence of the SC transition temperature in zero field.[7]

The present paper is organized as follows. In Sec. II, the theoretical formulation is explained, and thickness vs temperature phase diagrams of a d-wave film taken at low enough fields are discussed together with a p-wave case in zero field in Sec. III. In Sec. IV, field vs temperature phase diagrams in the d-wave case, which are our main focus in the present work, are discussed. In Sec. V, this work is summarized.

II. FORMULATION

In the present work, the boundary condition for the SC order parameter Δ on the film surface plays an important role in obtaining a spatial modulation of the SC order parameter. To explain how the boundary condition results in a modulated SC phase, let us start by describing the quasiclassical approach used.

Since we study not only the spin-singlet d-wave pairing in magnetic fields but also a p-wave pairing case in zero field, Eilenberger equations for the quasiclassical Green's functions[12–14] represented in the Nambu space will be considered. Following Refs. 15 and 16, they are expressed by

$$[i\varepsilon_n\hat{\tau}_3 - \hat{\Delta} - \hat{I},\ \hat{g}] + i\mathbf{v}_{\rm F}\cdot\nabla\hat{g} = 0, \tag{2}$$

where the brackets denote the commutator, ε_n is a fermion Matsubara frequency, and $\hat{\tau}_j$ ($j = 1, 2,$ and 3) are the particle-hole Pauli matrices. We largely follow Ref. 16 regarding the notation of the gap function and the Green's functions $\hat{g}$ satisfying $(\hat{g})^2 = -\pi^2$. They will be parametrized in the form

$$\hat{\Delta} = \begin{pmatrix} 0 & i[\Delta(\mathbf{r},\mathbf{p}) + \sum_{j=1,2,3}\boldsymbol{\Delta}_j(\mathbf{r},\mathbf{p})\cdot\sigma_j]\sigma_2 \\ i\sigma_2[\Delta^*(\mathbf{r},\mathbf{p}) + \sum_{j=1,2,3}\boldsymbol{\Delta}_j^*(\mathbf{r},\mathbf{p})\cdot\sigma_j] & 0 \end{pmatrix},$$
$$\hat{g} = \begin{pmatrix} g + \mathbf{g}\cdot\boldsymbol{\sigma} & i(f + \mathbf{f}\cdot\boldsymbol{\sigma})\sigma_2 \\ i\sigma_2(f' + \mathbf{f}'\cdot\boldsymbol{\sigma}) & -g + \mathbf{g}\cdot\boldsymbol{\sigma}* \end{pmatrix}. \tag{3}$$

The matrix $\hat{I}$ expresses the Zeeman energy term. In the parallel field configuration ($\mathbf{H} \perp \hat{y}$) of interest to us in the following sections, $\hat{I}$ is diagonal, and its element gives the Zeeman energy μH in the Nambu space. Further, $\mathbf{r} = (x,y,z)$ is the center-of-mass coordinate of a Cooper pair. As mentioned in Sec. I,[11] the orbital pair-breaking effect inducing the vortices can be safely neglected in the present geometry (see Fig. 1). Thus, the strength of the magnetic field can be identified hereafter with the dimensionless Zeeman energy

$$h \equiv \frac{2}{\pi}e^{\gamma}\frac{\mu H}{T_c(0)}, \tag{4}$$

where γ is the Euler constant. The numerical factor $4e^{\gamma} \simeq 7.1$ in the above expression results from the conventional definition of the Maki parameter $\alpha_{\rm M}$ for an s-wave bulk superconductor.[17]

As is well known, minimization of the free energy with respect to the gap function Δ results in the so-called gap equation, which, in the spin-singlet-pairing case, connects $\hat{\Delta}$ with the scalar anomalous Green's functions $f(\varepsilon_n)$ and $f'(\varepsilon_n) = [f(-\varepsilon_n)]^*$ in the manner

$$\Delta(\mathbf{r})\ln\left(\frac{T_{c0}}{T}\right) = T\sum_{n\geqslant 0}\left\langle \mathcal{Y}^*(\mathbf{p})\left(\frac{2\pi\Delta(\mathbf{r})\mathcal{Y}(\mathbf{p})}{|\varepsilon_n|} - f(\mathbf{r},\mathbf{p};\varepsilon_n) - f'^*(\mathbf{r},\mathbf{p};\varepsilon_n)\right)\right\rangle_{\mathbf{p}}. \tag{5}$$

Here, T_{c0} is the SC transition temperature in zero field, and $\langle\,\rangle_{\mathbf{p}}$ is the angular average over the Fermi surface. Further,

$\mathcal{Y}(\mathbf{p})$ denotes the normalized pairing function satisfying $\langle|\mathcal{Y}(\mathbf{p})|^2\rangle_{\mathbf{p}} = 1$. It appears through the factorization $\Delta(\mathbf{r},\mathbf{p}) = \Delta(\mathbf{r})\mathcal{Y}(\mathbf{p})$ and, in the case with gap nodes parallel to the $\hat{x}$ direction, is proportional to $\hat{p}_y$.

The system geometry is reflected in the boundary condition on $f(\mathbf{r},\mathbf{p};\varepsilon)$ on the outer surface. As in Ref. 7 the specular condition on a flat boundary surface in the z-x plane will be used by assuming the surface roughness to be negligible (see the geometry in Fig. 1). Then we have

$$f(\mathbf{r};\mathbf{p};\varepsilon_n) = f(\mathbf{r};\overline{\mathbf{p}};\varepsilon_n) \tag{6}$$

on the surface, where $\overline{\mathbf{p}} = \mathbf{p} - 2\hat{\mathbf{n}}(\mathbf{p}\cdot\hat{\mathbf{n}})$, and $\hat{\mathbf{n}}$ is the unit vector normal to the surface, which is in the y direction in Fig. 1. It means that, if

$$\mathcal{Y}(\overline{\mathbf{p}}) = -\mathcal{Y}(\mathbf{p}), \tag{7}$$

the SC order parameter $\Delta(\mathbf{r})$ and f vanish on the surface. Among pairing states satisfying Eq. (7), we will focus below on the following:

$$\mathcal{Y}(\mathbf{p}) = \sqrt{2}\hat{p}_x\hat{p}_y, \quad \mathcal{Y}(\mathbf{p}) = \sqrt{2}\hat{p}_y, \tag{8}$$

both of which, in the x-y plane, have a gap node in the direction parallel to the boundary surface. The former case corresponds to a film sample prepared by cutting a $d_{x^2-y^2}$-paired superconductor along, for instance, the (1,1,0) direction. Hereafter, this situation, indicated in Fig. 1, with the $d_{x^2-y^2}$-pairing symmetry will be called the D_{xy} state.

In addition, as a typical $\mathbf{r}$ dependence of $\Delta(\mathbf{r})$ vanishing on $y = \pm D/2$ (see Fig. 1), we choose the form

$$\Delta(\mathbf{r}) = \Delta(x)\sqrt{2}\cos(\pi y/D). \tag{9}$$

For simplicity, the cylindrical Fermi surface with the symmetry axis parallel to $\hat{z}$ will be used throughout this paper.

As will be stressed in Sec. III, a continuous normal-to-superconducting transition point can be found directly from the $O(|\Delta|^2)$ term of the free energy, or equivalently, from Eq. (5) linearized with respect to Δ. To obtain a transition occurring deep in the SC phase such as the onset of a FFLO modulation, the following quasiclassical representation[16] of the SC contribution $\Delta F_{\rm sc}$ to the Luttinger-Ward free energy functional:

$$\Delta F_{\rm sc} = \int dxdy\frac{T}{2}\int_0^1 d\lambda N(0)\sum_{\varepsilon_n}\left\langle {\rm Tr}\hat{\Delta}\left(\hat{g}_\lambda - \frac{1}{2}\hat{g}\right)\right\rangle_{\mathbf{p}} \tag{10}$$

is more useful than the gap equation (5), where $N(0)$ is the density of states (per spin) at the Fermi level in the normal state, and the auxiliary Green's function $\hat{g}_\lambda$ is the solution of the Eilenberger equation with $\hat{\Delta}$ replaced by $\lambda\hat{\Delta}$. The transition from the uniform SC state to a phase-modulated Fulde-Ferrell (FF) SC state

$$\Delta_{\rm FF}(x) = \exp(iq_x x)|\Delta(0)| \tag{11}$$

or to an amplitude-modulated Larkin-Ovchinnikov (LO) SC state

$$\Delta_{\rm LO}(x) = \sqrt{2}\cos(q_x x)|\Delta(0)| \tag{12}$$

is signaled by emergence of a finite equilibrium value of the modulation wave number q_x.

III. THICKNESS VS TEMPERATURE PHASE DIAGRAMS

In this section, we will discuss the zero-field case. By comparing thickness-temperature phase diagrams for the spin-triplet p_y-pairing state and the spin-singlet D_{xy} state[7] cases to each other, we stress that the presence of a gap node parallel to the film plane induces a phase-modulated state at lower temperatures and that a strange thickness dependence of the SC transition temperature is commonly seen in those SC films.

To discuss the character of the normal-to-SC phase transition in the mean-field (MF) approximation which is signaled by vanishing of the SC energy gap, the form of the Ginzburg-Landau (GL) free energy which can be obtained from the gap equation (5) will be explained. By introducing the Fourier-transformation $\Delta(x) = L_x^{-1/2}\sum_{q_x}\Delta_{q_x}\exp(iq_x x)$ to incorporate the possibility of a spatially varying mean-field solution of $\Delta(x)$, the quadratic $[O(|\Delta|^2)]$ term of the GL free energy is recovered in the form

$$\begin{aligned} F_{H=0} &= N(0)\sum_{q_x}\left\{\ln\left(\frac{T}{T_{c0}}\right) + \int_0^\infty d\rho\frac{2\pi T}{\sinh(2\pi T\rho)}\right. \\ &\quad\times\left.\left\langle|\mathcal{Y}(\mathbf{p})|^2\left[1-\cos(\rho v_F\hat{p}_x q_x)\cos\left(\rho\pi v_F\frac{\hat{p}_y}{D}\right)\right]\right\rangle_{\phi_{\mathbf{p}}}\right\}|\Delta_{q_x}|^2 \end{aligned} \tag{13}$$

from the $O(\Delta)$ term of Eq. (5), where $\langle\,\rangle_{\phi_p}$ denotes the average over $\phi_{\mathbf{p}}$ with $p_x + ip_y = p_{\rm F}\exp(i\phi_{\mathbf{p}})$ for the present cylindrical Fermi surface. Here, by first assuming the transition to be continuous, let us consider the situation under which the coefficient of $|\Delta_{q_x}|^2$ in Eq. (13) vanishes. In the cases of interest to us, $\mathcal{Y}(\mathbf{p}) \propto p_y$, i.e., when we have gap nodes in the $\hat{x}$ direction, the sign factor $\cos(\rho\pi v_F\hat{p}_y/D)$ in Eq. (13) may contribute to the negative sign for a broad range of ρ values. Then, inevitably, a state with a finite q_x, i.e., a state with a modulation parallel to a gap node, is favored in order for $\cos(\rho v_F\hat{p}_x q_x)$ to be also accompanied by a negative sign so that a negative sign of the coefficient of the $|\Delta_{q_x}|^2$ term, i.e., SC ordering, may become possible. This is like the mechanism leading to the conventional FFLO state induced by the Zeeman field: the dimensionless film thickness normalized by the zero-temperature coherence length $\xi_0 = v_{\rm F}/(2\pi T_c)$, i.e.,

$$d^{-1} = \frac{v_{\rm F}}{2T_c D} = \pi\frac{\xi_0}{D}, \tag{14}$$

in the present case plays the role of the Zeeman energy in the familiar field-induced FFLO case.

Figure 2 shows the dependences of the normal-to-SC phase boundary on *assumed* q_x values in the d^{-1} vs temperature phase diagram in the p_y-pairing case with zero field. The actual mean-field SC transition line is defined as the envelope of each curve giving the highest T_c and d^{-1} values among those curves. The result of this is presented in Fig. 3(a) as the solid black curve, which implies that, at lower temperatures and/or for larger d^{-1}, a SC phase with a nonzero q_x leads to a higher transition line and thus is favored as the ordered state. The resulting stability region of the modulated FF SC state has been determined by use of Eq. (10). We note that, as pointed out in Ref. 7, the present phase-modulated

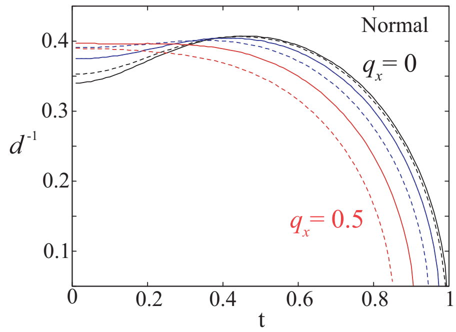

FIG. 2. (Color online) Thickness dependences of the SC transition temperature $T_c(d^{-1})$ of superconducting films with the pairing symmetry $\mathcal{Y}(\mathbf{p}) = \sqrt{2}\hat{p}_y$ in zero field, which have been obtained by assuming various q_x values, where $t = T/T_c(0)$. The envelope of those $T_c(d^{-1})$-curves is nothing but the resulting mean-field SC transition curve between a SC phase and the normal (N) phase and is described as a solid curve in Fig. 3(a). At lower temperatures where the boundary condition (a finite D) is more effective, the $T_c(d^{-1})$-curve with finite q_x tends to be realized. The SC transition is always of second order (see Fig. 3).

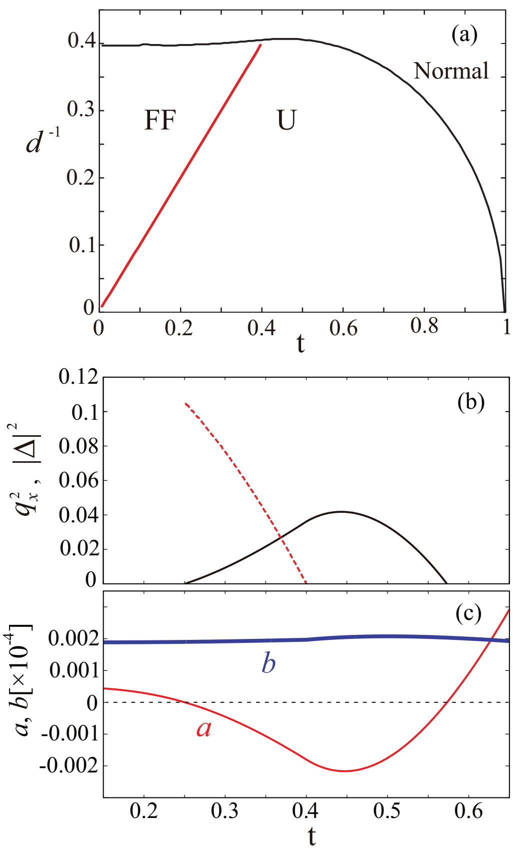

FIG. 3. (Color online) (a) Resulting thickness dependence of the SC transition temperature $T_c(d^{-1})$ (solid black curve) for the p_y pairing obtained from the data in Fig. 2. Although the T_c vs d^{-1} curve at $t < 0.4$ becomes nonmonotonic upon cooling (see the text), the transition at $T_c(d^{-1})$ is of second order. The SC state at lower temperatures and at smaller values of the film thickness is the phase-modulated Fulde-Ferrell state which is separated from the uniform SC (U) state (Ref. 10) via another second-order transition (solid red) curve decreasing upon cooling. (b) Temperature dependences of the squared order parameters $|\Delta|^2$ and q_x^2 at $d^{-1} = 0.399$, and (c) the corresponding coefficient a of the GL quadratic term [thin (red) curve] and the corresponding one b of the quartic term [thick (blue) curve] (see the Appendix for their definition). Here, Δ and q_x are normalized by $T_c(d^{-1} = 0)$ and ξ_0^{-1}, respectively, and hence are dimensionless. The resulting SC and FF transitions are of second order in character, as signaled by the linear vanishing of $|\Delta(t)|^2$ and of $[q_x(t)]^2$, respectively.

FF state is always more stable than the LO state. This phase-modulated state corresponds to the helical state in the context of noncentrosymmetric superconductors with no accompanying field-induced vortices.[8,19]

In addition, to clarify the character of this SC transition, the sign of the quartic $[O(|\Delta|^4)]$ term in the GL free energy needs to be examined. In fact, in the case with a conventional FFLO state induced by the momentum-independent Zeeman effect, the overall coefficient b of the quartic term at lower temperatures is negative near the line on which the quadratic term changes its sign so that the corresponding SC transition is of first order.[5] However, in the present case where the counterpart of the Zeeman energy, i.e., $v_F \hat{p}_y/D$, is linear in $\mathbf{p}$, it is found that the coefficient b of the quartic term remains positive even at lower temperatures [see Fig. 3(c)] and thus the mean-field SC transition is truly of second order. Again, this result is compatible with the closely related result on the low-temperature $H_{c2}(T)$ transition in Rashba noncentrosymmetric superconductors under a field parallel to the basal plane[9] (see the Introduction). On the other hand, in the case of the present Fig. 2, the resulting SC transition curve defined by a sign change of the quadratic term shows a *nonmonotonic* temperature dependence. This feature is in contrast to that in the conventional case with the momentum-independent Zeeman effect, where such a nonmonotonic temperature dependence of the curve on which the quadratic term changes its sign indicates that the mean-field SC transition occurs discontinuously on a different curve.[5]

Figure 4(a) shows the d^{-1} vs temperature phase diagram of a film with the D_{xy} SC state in zero field, which coincides with the result in Ref. 7. Just as in the case of p_y pairing in Fig. 3, the uniform SC phase tends to be destabilized due to formation of the FF state with decreasing film thickness. In the D_{xy} state, there are also gap nodes perpendicular to the film plane which may partially cover the formation of the FF phase modulation stemming from the gap nodes parallel to the plane. In fact, although a nonmonotonic behavior of the second-order transition curve $T_c(d^{-1})$ is present even in the case of the D_{xy} state [see Fig. 4(b)], it is less remarkable than in the p_y-pairing case (Fig. 3). Nevertheless, as the inset of Fig. 4(a) shows, this low-temperature behavior of the $T_c(d^{-1})$ curve is robust against a weak magnetic field. In Sec. III, it will be shown that this nearly flat but nonmonotonic feature close to $t = 0.35$ of the $T_c(d^{-1})$ curve results in a reentry of the FF and normal phases in films with thickness of about $6\xi_0$. In contrast, in sufficiently thick films with large d, the FF phase is destabilized by such a weak magnetic field (see the inset of Fig. 4).

For comparison, we will comment on the $d_{x^2-y^2}$-pairing case in which an antinodal direction is parallel to the film plane. In this case, the boundary condition (7) is not satisfied,

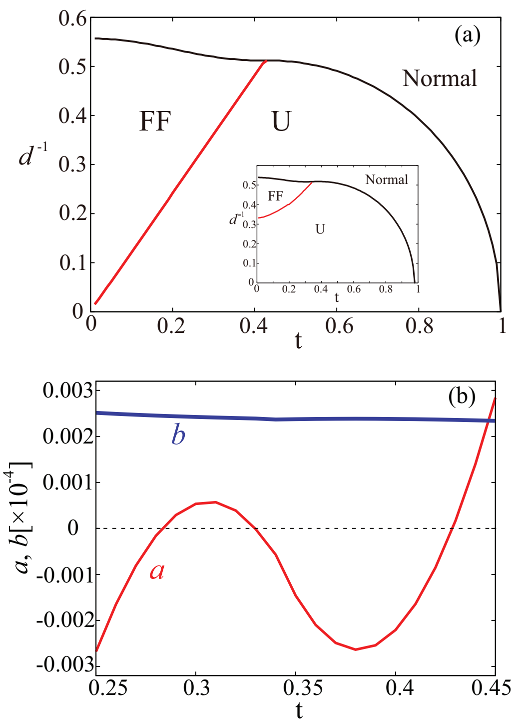

FIG. 4. (Color online) (a) Thickness dependence of the SC transition temperature, $T_c(d^{-1})$, in the case with D_{xy} state. This phase diagram at $h = 0$ quantitatively agrees with that in Ref. 7. The inset shows the corresponding result in the small field $h = 0.32$. The nonmonotonic behavior of the $T_c(d^{-1})$ curve at low temperatures is less remarkable, possibly because of the additional gap nodes perpendicular to the film plane, but nevertheless visible close to $t = 0.35$. (b) Temperature dependences of the GL coefficients a [thin (red) curve] and b [thick (blue) curve] at $d^{-1} = 0.517$ in (a). The positive b values where a changes sign on cooling imply that the SC transition in (a) is of second order.

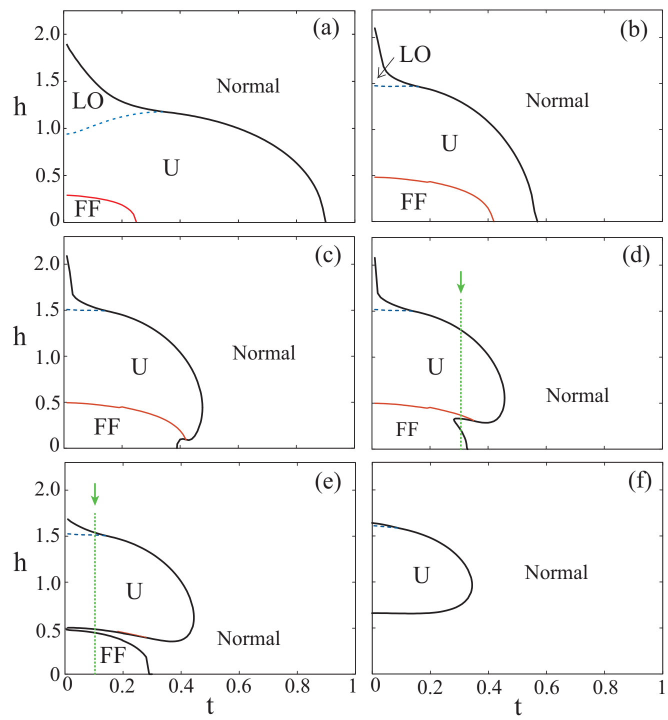

FIG. 5. (Color online) Thickness dependence of the h vs t phase diagram in the case with the D_{xy} state. The used values of d [the normalized thickness defined in Eq. (14)] are 3.33 in (a), 2.0 in (b), 1.95 in (c), 1.93 in (d), 1.92 in (e), and 1.79 in (f). The symbol "LO" indicates the field-induced LO phase. All transitions indicated in the figures are of second order (see Fig. 6). The field dependences of the order parameters on the dotted vertical lines in (d) and (e) are shown in Fig. 6.

and not the FF state but the familiar SC state, which is spatially uniform in any direction, is always favored. Thus, within the approximation used here, disappearance of superconductivity due to the size effect does not occur in this case.

IV. FIELD VS TEMPERATURE PHASE DIAGRAMS

Next, let us examine how the phase diagram in the case with the D_{xy} state is changed by applying a uniform magnetic field. For this purpose, the orbital pair-breaking effect of the magnetic field will be neglected. This assumption is reasonable for describing thin films with thickness of several times ξ_0. Further, as argued in Sec. I, it is at least qualitatively valid even with vortices as long as $\mathbf{H} \parallel \hat{x}$.[11] Then a nonvanishing magnetic field appears only through the Zeeman energy μH, and its effect can be incorporated simply by replacing $\cos(\rho\pi v_F \hat{p}_y/D)$ with $\cos(\rho\pi v_F \hat{p}_y/D)\cos(\rho 2\mu H)$ in Eq. (13) (see Ref. 5 and the Appendix for details). Then, the signs of the two cosine factors compete in the ρ integral in Eq. (13). Physically, this implies that the applied magnetic field frustrates and thus weakens the size effect enhanced by the gap nodes. Since, on the other hand, superconductivity in the present systems is suppressed with decreasing film thickness, the applied magnetic field competing with the size effect may enhance superconductivity. Below, it will be explained that, in thin SC films with gap nodes parallel to the film plane and with thickness of the order of several times ξ_0, rich behaviors reflecting the competition between the magnetic field and the gap-node-induced size effect can occur in the field vs temperature phase diagram, such as coexistence of two spatially modulated FFLO states of different origins and a field-induced reentry of the spatially uniform superconductivity.

The thickness dependence of the h vs t phase diagrams we have obtained is presented in Fig. 5. Although the transition curves in the figures have been determined from the quasiclassical formulation in Sec. II with Eq. (10), all the SC transitions appearing in the present phase diagram can be alternatively obtained from the GL free energy shown in the Appendix because they are of second order in character. As seen in Fig. 3 of Ref. 7 (see also Fig. 4 in the present work), as long as the same boundary condition is used, the FF phase of SC films with a gap node parallel to the film plane under zero field inevitably appears even in sufficiently thick films with $d \gg 1$ at low enough temperatures.[7,18] Reflecting this fact, Fig. 5(a) includes this FF phase at low enough temperatures if the applied field is sufficiently low. With decreasing film thickness, however, the temperature regions of the uniform[10] and LO SC phases are narrower, while the FF phase grows and begins to occupy a broader field and

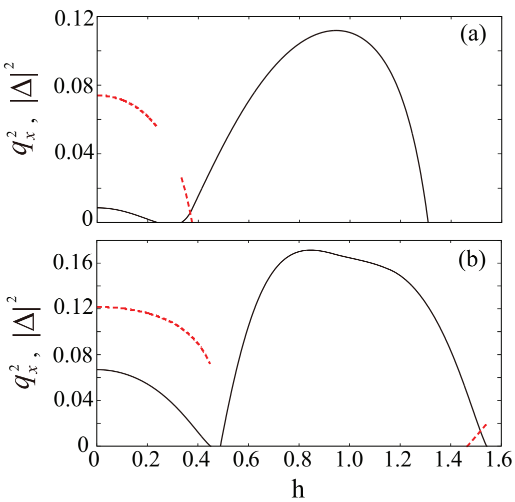

FIG. 6. (Color online) Squared order parameters $|\Delta|^2$ (solid curves) and q_x^2 (dashed curves) taken on the dotted vertical lines in Figs. 5(d) and 5(e) are presented in (a) and (b), respectively. The linear vanishing of $|\Delta|^2$ and of q_x^2 at a finite $|\Delta|$ value ensure that these transitions are of second order in character.

temperature region [see Fig. 5(b)]. In particular, it is notable in Fig. 5(b) that, as the FF phase grows, the field-induced LO phase shrinks and, together with the normal-to-SC transition curve, is pushed up to higher fields. This competition between these two modulated SC phases seems to result in extension of the intermediate *spatially uniform* SC phase. For instance, the point ($t = 0.2$, $h = 1.4$) in Fig. 5(b) is included in the uniform SC phase, although the same point in Fig. 5(a) is in the normal phase, suggestive of an extension of the uniform SC phase in thinner films. More interestingly, for even thinner films with thickness $D < 6.13\xi_0$, the normal phase begins to intervene between the FF and the uniform SC phases at low temperatures, pushing the FF phase down to lower fields with decreasing thickness [see Figs. 5(c) and 5(d)]. It is a remarkable feature that the normal phase begins to enter along the boundary between the FF and uniform phases with decreasing film thickness so that the uniform SC phase at higher fields can remain stable against the normal state [see Figs. 5(e) and 5(f)]. The above-mentioned competition between the size effect and the finite magnetic field and the resulting reentrant survival of the spatially uniform SC phase are nontrivial and the main results in the present work.

As already mentioned, all the transitions appearing in these phase diagrams are of second order. In Fig. 6, the field dependences of the amplitudes of the SC order parameter Δ and the FFLO order parameter q_x taken on the dotted vertical lines in Figs. 5(d) and 5(e) are shown. For instance, in Fig. 6(a), the linear vanishing of $|\Delta|^2$ at three h values and that of q_x^2 in the vicinity of $h = 0.39$ imply the second-order character of these transitions. The q_x^2 curve has a physical meaning when $|\Delta|$ is nonvanishing. For this reason, q_x data are not shown in Fig. 6 in the field regions of the normal state.

In contrast to the case of the D_{xy} state, the FF phase does not occur if the film plane is parallel to an antinodal direction of the $d_{x^2-y^2}$-pairing function. Then the resulting field vs temperature SC phase diagram consists only of the uniform and LO SC phases with no reentry of a low-field normal phase.

V. CONCLUSION

In this paper, we have studied the phase structure of a SC film with an unconventional nodal pairing and, in particular, have focused on its change due to an applied magnetic field parallel to the film plane. It has been found that, when the pairing state has a gap node parallel to the film plane, and consequently the SC energy gap or the SC order parameter is deformed along the film's surface normal, a kind of Fulde-Ferrell SC phase with a spatial modulation parallel to the film plane of the *phase* of the SC order parameter is inevitably induced upon cooling in zero field. When a magnetic field parallel to the gap-node direction is applied, and the system is close to the Pauli limit, this unconventional FF state and the conventional Larkin-Ovchinnikov state in high fields coexist in the same field vs temperature phase diagram. With decreasing film thickness, these modulated SC states seem to repel each other in the phase diagram while keeping intact the uniform SC phase intervening between them. Consequently, the FF phase in lower fields is replaced by the normal phase in thinner films so that a field vs temperature phase diagram with a reentrant uniform SC phase results.

As mentioned in Sec. I, vortices can be neglected in discussing the thermodynamics and the static phase diagram of superconductors close to the Pauli limit. Since, as seen in Fig. 5, the main point in the present work is the reentry of the normal phase in lower fields in thinner films, inclusion of the field-induced vortices in the present theory is not expected to lead to essential changes of our main results.

An intriguing point is an apparent competition between the Fulde-Ferrell state in lower fields and the conventional Larkin-Ovchinnikov state at the high-field end of the SC phase. In fact, the FF phase arises from the anisotropic response to an external field in momentum space [see Eq. (1)] and thus is incompatible with the LO state which is realized even in an isotropic system. Therefore, it may be expected that the field-induced reentry of the uniform SC phase will also occur in other systems with a zero-field FFLO phase, e.g., the cases with a FFLO state induced by the multiband effect or by an electric current.[6] These issues will be considered elsewhere.

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APPENDIX

1. Derivation of the GL quadratic term

The Ginzburg-Landau quadratic term essentially corresponds to the linearized gap equation, so that it can be obtained by expanding Eq. (5) with respect to Δ and picking up the leading-order contributions. In this appendix, we sketch an alternative derivation[5] of the GL free energy functional based on the Gor'kov formalism which is equivalent to the Δ expansion of Eq. (5). The GL quadratic term is formally given by

$$a|\Delta(0)|^2 = \frac{1}{N(0)}\int \frac{d\mathbf{r}}{V}\,\Delta^*(\mathbf{r}) \times \left(\frac{1}{|\lambda|} - \frac{T}{2}\sum_{\varepsilon_n,\sigma}\sum_{\mathbf{p}} \hat{K}_2(-i\nabla_{\mathbf{r}})\right)\Delta(\mathbf{r}),$$
$$\hat{K}_2(-i\nabla_{\mathbf{r}}) = |\mathcal{Y}(\mathbf{p})|^2 \mathcal{G}_{\varepsilon_n,\sigma}(\mathbf{p})\,\mathcal{G}_{-\varepsilon_n,-\sigma}(-\mathbf{p}-i\nabla_{\mathbf{r}}), \tag{A1}$$

where λ denotes a coupling constant of a pairing interaction, $\mathcal{G}_{\varepsilon_n,\sigma}(\mathbf{p}) = [i\varepsilon_n - \xi(\mathbf{p}) + \sigma\mu H]^{-1}$ is the Gor'kov Green's function expressing a quasiparticle in the normal state, ε_n is a fermionic Matsubara frequency, and $\nabla_{\mathbf{r}}$ denotes the gradient with respect to $\mathbf{r}$. By using the replacement $\sum_{\mathbf{p}} \to N(0)\int_{-\infty}^{\infty} d\xi(\mathbf{p})\langle\rangle_{\phi_p}$, we can carry out the summation over $\mathbf{p}$, and then we find

$$\sum_{\mathbf{p}} \hat{K}_2(-i\nabla_{\mathbf{r}}) = N(0)\left\langle \frac{2\pi i\,\mathrm{sgn}(\varepsilon_n)|\mathcal{Y}(\mathbf{p})|^2}{2i\,\varepsilon_n + 2\sigma\mu H + i\mathbf{v}_F\cdot\nabla_{\mathbf{r}}}\right\rangle_{\phi_p}$$
$$= N(0)\int_0^{\infty} d\rho\, 2\pi e^{-2|\varepsilon_n|\rho}\langle|\mathcal{Y}(\mathbf{p})|^2 \times \exp[i\,\mathrm{sgn}(\varepsilon_n)(2\sigma\mu H + i\mathbf{v}_F\cdot\nabla_{\mathbf{r}})\rho]\rangle_{\phi_p}, \tag{A2}$$

where the equation $1/\alpha = \int_0^{\infty} d\rho\exp[-\alpha\rho]$ $(\mathrm{Re}[\alpha] > 0)$ has been used. The summation over ε_n and σ yields

$$\frac{T}{2}\sum_{\varepsilon_n,\sigma,\mathbf{p}} \hat{K}_2(-i\nabla_{\mathbf{r}}) = N(0)\int_0^{\infty} d\rho\,\frac{2\pi T\cos(2\mu H\rho)}{\sinh[2\pi T\rho]} \times \langle|\mathcal{Y}(\mathbf{p})|^2\cos(-i\mathbf{v}_F\cdot\nabla_{\mathbf{r}}\,\rho)\rangle_{\phi_p}. \tag{A3}$$

Since T_{c0} is defined as the SC transition temperature at $D = \infty (d^{-1} = 0)$ and $H = 0$, i.e.,

$$\frac{1}{|\lambda|} - N(0)T_{c0}\sum_{\varepsilon_n>0}^{\omega_c/T_{c0}}\frac{2\pi}{|\varepsilon_n|} = 0, \tag{A4}$$

we have

$$\frac{1}{|\lambda|} \simeq N(0)\left[\ln\left(\frac{T}{T_{c0}}\right) + \int_0^{\infty} d\rho\,\frac{2\pi T}{\sinh[2\pi T\rho]}\right]. \tag{A5}$$

Eventually, we obtain the GL quadratic term as

$$a = \sum_{\mathbf{q}}|C_{\mathbf{q}}|^2\left[\ln\left(\frac{T}{T_{c0}}\right) + \int_0^{\infty} d\rho\,\frac{2\pi T}{\sinh[2\pi T\rho]} \times \langle|\mathcal{Y}(\mathbf{p})|^2[1-\cos(2\mu H\rho)\cos(\mathbf{v}_F\cdot\mathbf{q}\,\rho)]\rangle_{\phi_p}\right], \tag{A6}$$

where $C_{\mathbf{q}}$ is the coefficient of the Fourier expansion $\Delta(\mathbf{r}) = |\Delta(0)|\sum_{\mathbf{q}} C_{\mathbf{q}}\exp[i\,\mathbf{q}\cdot\mathbf{r}]$. For the film with thickness D, the gap function takes the form of Eq. (9), so that it follows that

$$|\Delta(0)|C_{\mathbf{q}} = \Delta_{q_x}[\delta_{q_y,\pi/D} + \delta_{q_y,-\pi/D}]/\sqrt{2}. \tag{A7}$$

Substituting Eq. (A7) into Eq. (A6), we obtain Eq. (13). Note that the coefficient a is given by the same expression for both the FF and LO states.

2. Expression of the GL quartic term

The GL quartic term is given by

$$b|\Delta(0)|^4 = \frac{(2\pi T_c)^2}{2N(0)} \times \int\frac{d\mathbf{r}}{V}\hat{K}_4(\nabla_i)\Delta^*(\mathbf{s}_1)\Delta(\mathbf{s}_2)\Delta^*(\mathbf{s}_3)\Delta(\mathbf{s}_4)\big|_{\mathbf{s}_i\to\mathbf{r}},$$
$$\hat{K}_4(\nabla_i) = \frac{T}{2}\sum_{\varepsilon_n,\sigma}\sum_{\mathbf{p}}|\mathcal{Y}(\mathbf{p})|^4\,\mathcal{G}_{\varepsilon_n,\sigma}(\mathbf{p})\,\mathcal{G}_{-\varepsilon_n,-\sigma}(-\mathbf{p}+i\nabla_1) \times \mathcal{G}_{-\varepsilon_n,-\sigma}(-\mathbf{p}-i\nabla_2)\,\mathcal{G}_{\varepsilon_n,\sigma}[\mathbf{p}+i(\nabla_3+\nabla_2)], \tag{A8}$$

where ∇_i denoting the gradient with respect to $\mathbf{s}_i$ acts on $\Delta(\mathbf{s}_i)$.[5] The summation over $\mathbf{p}$ can be performed in the same manner as that used in obtaining the quadratic term, and then we obtain

$$b = 2\pi^2 T_c^2\sum_{\mathbf{Q}_1+\mathbf{Q}_3=\mathbf{Q}_2+\mathbf{Q}_4} C^*_{\mathbf{Q}_1}C_{\mathbf{Q}_2}C^*_{\mathbf{Q}_3}C_{\mathbf{Q}_4} \times \prod_{j=1}^{3}\int_0^{\infty} d\rho_j\,\frac{2\pi T\,\cos\left(2\mu H\left[\sum_{i=1}^3\rho_i\right]\right)}{\sinh\left[2\pi T\left(\sum_{i=1}^3\rho_i\right)\right]}\langle|\mathcal{Y}(\mathbf{p})|^4$$
$$\times\{\cos[\mathbf{v}_F\cdot\mathbf{Q}_1(\rho_1+\rho_2) - \mathbf{v}_F\cdot\mathbf{Q}_2\rho_2 + \mathbf{v}_F\cdot\mathbf{Q}_3(\rho_2+\rho_3)] + \cos(\mathbf{v}_F\cdot\mathbf{Q}_1\rho_1 + \mathbf{v}_F\cdot\mathbf{Q}_2\rho_2 + \mathbf{v}_F\cdot\mathbf{Q}_3\rho_3)\}\rangle_{\phi_p}, \tag{A9}$$

where $\mathbf{Q}_i = (Q_{x,i}, Q_{y,i})$ and $\mathbf{v}_F = 2\pi T_c\xi_0(\cos(\phi_p), \sin(\phi_p))$. In Eq. (A9), the expression

$$C_{\mathbf{Q}_i} = \frac{\delta_{Q_{y,i},\pi/D} + \delta_{Q_{y,i},-\pi/D}}{\sqrt{2}}\,\frac{\delta_{Q_{x,i},q_x} + \delta_{Q_{x,i},-q_x}}{\sqrt{2}} \tag{A10}$$

is valid for the LO state with $\Delta(\mathbf{r}) = 2\,|\Delta(0)|\cos(y\pi/D)\cos(xq_x)$, while

$$C_{\mathbf{Q}_i} = \frac{\delta_{Q_{y,i},\pi/D} + \delta_{Q_{y,i},-\pi/D}}{\sqrt{2}}\,\delta_{Q_{x,i},q_x} \tag{A11}$$

should be used for the FF state, in which $\Delta(\mathbf{r}) = \sqrt{2}\,|\Delta(0)|\cos(y\pi/D)e^{i\,xq_x}$.

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