Majorana fermions on the Abrikosov flux lattice in a $p_x + ip_y$ superconductor

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We show that a periodic lattice of Abrikosov vortices in the type-II superconductor can support fermionic states with zero energy. Zero modes appear at the intersection of electronic Bloch bands with the Fermi level. In a chiral $p_x + ip_y$ wave superconductor the spectrum contains Majorana states at the center of an effective Brillouin zone. The Bloch bands formed by the overlapping vortex core states can transmit energy flow across the lattice. The hallmark of zero modes in electronic heat conductivity is discussed.

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I. INTRODUCTION

Exotic properties of fermionic spectrum in chiral $p_x + ip_y$ wave superconductors and Fermi superfluids are determined by nontrivial topology of the ground state.[1] Such pairing symmetry was found in a superfluid A phase of ^{3}He films[2] with chiral $p_x + ip_y$ structure of the superfluid order parameter corresponding to the Cooper pairing with angular momentum $L_z = \pm 1$. The same state is suggested to be realized in layered triplet p wave superconductor Sr_2RuO_4.[3]

In particular, fermionic sectors of ^{3}He-A and Sr_2RuO_4 contain zero energy states localized near domain walls and solitons,[4] boundaries,[5] and quantized vortices.[6] The zero energy fermionic modes can be described in terms of the self-conjugated Majorana fermions which were theoretically predicted to appear in several other two-dimensional systems such as the fractional quantum Hall liquid at filling 5/2,[7] heterostructures of topological insulators and superconductors,[8] and possibly certain Iridates which effectively realize the Kitaev honeycomb model.[9]

An appealing possibility offered by the nontrivial structure of fermionic spectrum in the vortex phase of chiral $p_x + ip_y$ superconductors is the realization of quantum matter with exotic non-Abelian quasiparticle statistics.[10,11] In this case, the non-Abelian anyons are presented by vortex excitations supporting zero-energy Majorana fermions residing inside their cores. That is the spectrum of vortex core fermions is given by

$$\varepsilon = \omega(n + \gamma), \tag{1}$$

where n is integer number, $\gamma = 1/2$ for s wave,[12] and $\gamma = 0$ for $p_x + ip_y$ wave[6] superconductors. Thus in topologically nontrivial superconductors the spectrum of vortex core fermions (1) contains zero-energy modes with $n = 0$ which can be conveniently described in terms of the Majorana self-conjugated fermionic field.[11] Such a possibility provides an extra motivation for the study of vortices in $p_x + ip_y$ superconductors due to their potential application in topological quantum computing.[13]

Vortex core Majorana fermions have an important property of being stable with respect to the impurity scattering[6] and order parameter perturbations.[11] However, the spectrum of vortex core states is extremely sensitive to the intervortex quasiparticle tunneling. The corresponding spectrum modification in finite clusters of vortices was investigated first by Mel'nikov and Silaev.[14,15] It was shown that in a pair of vortices the intervortex quasiparticle tunneling removes the twofold degeneracy of vortex core states. Such splitting of zero energy levels opens the gap in the fermionic spectrum. Later the arguments were advanced that it can break the quantum coherence during the vortex permutation which is important for the fault tolerance of topological quantum computations.[16]

Generalization of the two-vortex problem for M-vortex clusters is straightforward and was discussed in detail.[14,15] Based on these results it is easy to demonstrate that a cluster consisting of an odd number of vortices in $p_x + ip_y$ superconductor always has at least one zero energy state irrespective of the vortex positions. Thus the splitting of Majorana fermions is not a generic effect and depends on the parity of the total number of vortices M. On the other hand, in clean type-II superconductors without disorder and pinning centers vortices in a finite magnetic field form a periodic Abrikosov flux lattice. Therefore, the natural question considered in the present paper is whether the spectrum of fermions on the vortex lattice in chiral $p_x + ip_y$ superconductor is gapped or contains zero energy Majorana states.

Previously, various types of spectral problems for two-dimensional lattices of Majorana fermions were considered.[17–21] These models take into account only the tunneling between lowest energy states in the vortex cores. As shown in Refs. 14 and 15 for the generic problem of two vortices the shift of vortex core energy levels becomes larger than the interlevel energy already in small magnetic fields $H \geqslant 0.1 H_{c2}$. Therefore, the one level approximation of lattice models is not applicable for intermediate vortex densities. Instead in the present paper we consider the eigenvalue problem of genuine Bogoliubov–de Gennes equations with gap potential corresponding to the periodic Abrikosov flux lattice. To treat this problem we generalize the original approach developed earlier.[14,15] This approach allows one to calculate the spectrum when the intervortex distance is larger than the superconducting coherence length.

II. MODEL

In general, the problem of identifying the quasiparticle energies in superconductors is to solve the Bogoliubov–de Gennes (BdG) equations:

$$\hat{H}_0 \Psi + \begin{pmatrix} 0 & \hat{\Delta} \\ \hat{\Delta}^+ & 0 \end{pmatrix} \Psi = \varepsilon \Psi, \tag{2}$$

where $\hat{H}_0 = \hat{\tau}_3[(\hat{\mathbf{p}} - \hat{\tau}_3\mathbf{A})^2 - p_F^2]/2m$, $\Psi = (U,V)$, U and V are the particle- and holelike parts of the fermionic quasiparticle wave function, $\hat{\mathbf{p}} = -i\nabla$, $\hat{\tau}_i$ are the Pauli matrices in a particle-hole space, the gap operator is $\hat{\Delta} = \{\Delta(\hat{\mathbf{r}}), e^{i\chi\theta_p}\}$, where $\chi = \pm 1$ is chirality for $p_x \pm ip_y$ wave and $\chi = 0$ for s wave, $\hat{\mathbf{r}}$ is a coordinate operator, $\Delta(\mathbf{r})$ describes the spatial dependence of the gap function, and $\{A,B\} = (AB + BA)/2$ is an anticommutator which provides the gauge invariance of $\hat{\Delta}$. The phase of the order parameter depends on the direction of the electron momentum in xy plane: $\mathbf{p} = p(\cos\theta_p, \sin\theta_p)$. The magnetic field is directed along the z axis $\mathbf{B} = B\mathbf{z}$ and for extreme type-II superconductors we can consider the magnetic field to be homogeneous on the spatial scale of intervortex distance and take the gauge $\mathbf{A} = [\mathbf{B} \times \mathbf{r}]/2$.

Further calculations require expansion of the wave functions $\Psi(\mathbf{r})$ by the basis of localized fermionic states of isolated vortices. It can be implemented using the quasiclassical approximation and the convenient formalism[14,15] of the so-called s-θ_p representation. It allows one to express the quasiparticle wave function in momentum representation in the following form:

$$\Psi(\mathbf{p}) = \frac{1}{p_F}\int_{-\infty}^{+\infty} ds\, e^{-i(|\mathbf{p}|-p_F)s/\hbar}\psi(s,\theta_p)\,. \tag{3}$$

The inner product can be expressed through the envelope functions

$$\langle\Psi_1|\Psi_2\rangle = \frac{\pi}{p_F}\int_{-\infty}^{\infty} ds \int_0^{2\pi} d\theta_p \psi_1^+\psi_2(s,\theta_p). \tag{4}$$

The BdG equation for $\psi(s,\theta_p)$ reads $\hat{H}\psi = E\psi$, where

$$\hat{H} = -iV_F\hat{\tau}_3\frac{\partial}{\partial s} + \begin{pmatrix} 0 & \hat{\Delta} \\ \hat{\Delta}^+ & 0 \end{pmatrix}. \tag{5}$$

$V_F = p_F/m$ is Fermi velocity. Here s is a coordinate along the quasiclassical trajectory and θ_p is the angle which determines its direction. The Hamiltonian (5) takes account of noncommutability of angular momentum z axis projection operator $\hat{\mu} = -i\partial/\partial\theta_p$ and θ_p. Therefore, it involves the angular momentum quantization. Hence the spatial coordinate in the gap operator in Eq. (5) is quantum variable in s-θ_p representation $\hat{\mathbf{r}} = s\mathbf{p}_F/p_F + \{[\mathbf{p}_F \times \mathbf{z}], \hat{\mu}\}/p_F^2$.

Let us consider now an isolated vortex positioned at $\mathbf{r} = 0$ and described by the gap function $\Delta(\mathbf{r}) = \Delta_v(r)e^{i\theta}$. The gap function operator in s-θ_p representation is given by

$$\hat{\Delta} = \Delta_v(s)e^{i(1+\chi)\theta_p}\left[\frac{s}{|s|} - \frac{1}{|s|p_F}\left(\frac{\partial}{\partial\theta_p} - \frac{\chi+1}{2i}\right)\right]. \tag{6}$$

Then Eq. (5) has a standard solution corresponding to the low energy CdGM levels,

$$\psi(s,\theta_p) = C(\theta_p)\psi_v(s,\theta_p), \tag{7}$$

$$\psi_v(s,\theta_p) = e^{i\hat{\tau}_3(1+\chi)\theta_p/2}\begin{pmatrix} 1 \\ -i \end{pmatrix}\frac{e^{-K(s)}}{\sqrt{\Lambda}}, \tag{8}$$

where $K(s) = V_F^{-1}|\int_0^s \Delta_v(t)dt|$ and Λ is a normalizing factor so that $\langle\Psi|\Psi\rangle = 1$. Such wave function ansatz implies periodic boundary conditions $C(\theta_p + 2\pi) = C(\theta_p)$ for $p_x + ip_y$ and $C(\theta_p + 2\pi) = -C(\theta_p)$ for s wave symmetry. Substituting it to the BdG Hamiltonian (5) we obtain that $C(\theta_p) = \exp[i(n + \gamma)\theta_p]$, which gives the CdGM spectrum (1) with the interlevel spacing,

$$\omega = \Lambda^{-1}\int_{-\infty}^{\infty} ds\, e^{-2K(s)}\Delta_v(s)/|s| \simeq \Delta_0/(p_F\xi), \tag{9}$$

where Δ_0 is the gap value far from the vortex core and $\xi = V_F/\Delta_0$ is the superconducting coherence length.

The generalization of the above procedure for vortex clusters with finite probability of intervortex tunneling of quasiparticles was obtained in Refs. 14 and 15. In particular for the generic problem of two vortices placed at the distance d the low energy fermionic spectrum has the form

$$\varepsilon = \omega[\pm\arccos(\sqrt{1-e^{-\alpha}}\sin\beta)/\pi + n + 1/2 + \gamma], \tag{10}$$

where $\gamma = 1/2$ for s wave and $\gamma = 0$ for $p_x + ip_y$ wave superconductivity, $\alpha = 2\pi e^{-2d/\xi}p_F\xi^2/d$, and $\beta = p_Fd + \alpha\ln(p_F\xi^2/d) + \arg[\Gamma(1 - i\alpha)] + \pi/4$, where $\Gamma(x)$ is the Γ function. The spectrum (10) contains two series of levels with the interlevel distance ω. Equation (10) demonstrates that the perturbation of energy levels due to the intervortex quasiparticle tunneling in general is not described by the plain tight binding theory used, e.g., in Ref. 16. Indeed, the shift of energy levels with respect to the isolated vortex spectrum becomes larger than the energy level spacing ω when the intervortex distance is smaller than the critical one $d < d_c$, where[14,15]

$$d_c \simeq (\xi/2)\ln(p_F\xi). \tag{11}$$

Note that $d_c \gg \xi$, since $p_F\xi \gg 1$. For the typical parameter $p_F\xi = 100$ corresponds to magnetic fields being larger than that of the order $0.1H_{c2}$.

In this paper we consider the next benchmark in the theory by solving the problem of electronic states in the periodic Abrikosov lattice. Here we address the case of fully gapped systems. The lattice spectrum in gapless d-wave superconductors was considered before[22] (see also review[23]).

III. ELECTRONIC BLOCH WAVES IN ABRIKOSOV LATTICE

The periodicity of vortex lattice is determined

$$\Delta(\mathbf{r} + \mathbf{d}) = e^{i[\mathbf{B}\times\mathbf{d}]\mathbf{r} + i\varphi_{\mathbf{d}}}\Delta(\mathbf{r}), \tag{12}$$

$$\mathbf{A}(\mathbf{r} + \mathbf{d}) = \mathbf{A}(\mathbf{r}) + [\mathbf{B} \times \mathbf{d}]/2, \tag{13}$$

where $\mathbf{d} = n_a\mathbf{a} + n_b\mathbf{b}$ is the translation of the vortex lattice, n_a, n_b are integer numbers, and $\varphi_{\mathbf{d}}$ is an arbitrary constant phase shift. By choosing the Wigner-Seitz elementary cell of vortex lattice and placing the origin $\mathbf{r} = 0$ at the vortex center in this cell we immediately obtain that $\varphi_{\mathbf{a}} = \varphi_{\mathbf{b}} = \pi$.

The translational properties of $\Delta(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ make Eq. (2) commute with the magnetic translation operator

$$T^h_{\mathbf{d}} = \hat{\tau}_3 e^{i\hat{\tau}_3[\mathbf{B}\times\mathbf{d}]\mathbf{r}/2}T_{\mathbf{d}}, \tag{14}$$

so that $T^h_{\mathbf{d}}\hat{H}T^{h+}_{\mathbf{d}} = \hat{H}$ where $T_{\mathbf{d}}$ is the usual translation by the lattice vector $\mathbf{d}$. Consequently, the solutions of the BdG Eq. (2) can be classified according to the eigenstates of the magnetic translation operator. An important point is that the magnetic

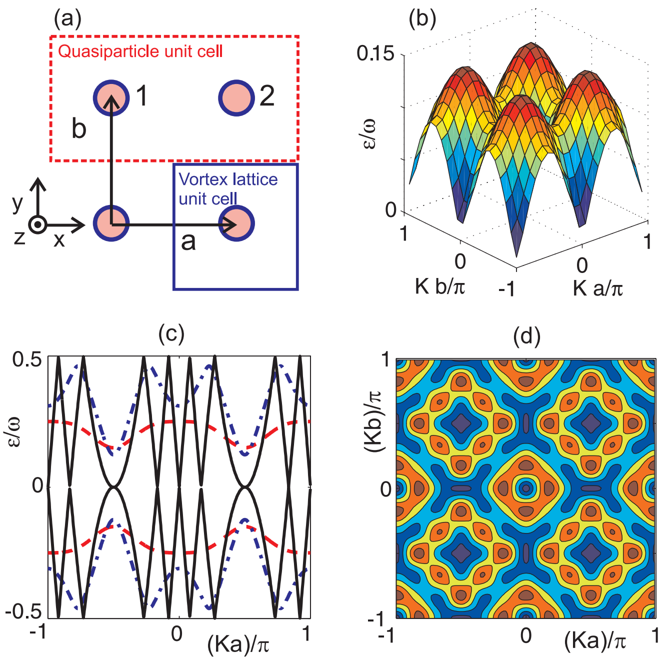

FIG. 1. (Color online) (a) Vortex lattice unit cell and corresponding quasiparticle unit cell containing the nodes marked by 1 and 2 in square vortex lattice. The vortex positions are marked with red filled circles which form the 2D Brave lattice with the basis $(\mathbf{a},\mathbf{b})$. (b) The 3D plot of magnetic Bloch band $\varepsilon_{0+}(\mathbf{K})$ given by (29) $\chi = \pi/4$ and $I_{\mathbf{a}} = I_{\mathbf{a}} = 0.14$. (c) The Bloch bands $\varepsilon_{0\pm}(K_x, K_y = 0)$ for $I_{\mathbf{a}} = I_{\mathbf{b}} = 1.4, 2.8$, and 8.9 shown by red dashed, blue dash-dotted, and black solid lines correspondingly. (d) The contour plot of $\varepsilon_{0+}(\mathbf{K})$ for $\chi = \pi/4$ and $I_{\mathbf{a}} = I_{\mathbf{b}} = 4.3$. The plots (b), (c), and (d) correspond for the square vortex lattice.

flux through the vortex lattice unit cell is one-half of the flux quantum $\mathbf{z}[\mathbf{a}\times\mathbf{b}]B = \pi$ so that the magnetic translations by lattice vectors anticommute $T^h_{\mathbf{a}} T^h_{\mathbf{b}} = -T^h_{\mathbf{b}} T^h_{\mathbf{a}}$. Therefore, we should introduce the unit cell for the quasiparticle functions consisting of two vortex lattice unit cells, for example shifted by the vector $\mathbf{a}$. For the case of square vortex lattice, this choice is illustrated in Fig. 1(a). Then the magnetic translation subgroup is formed by vectors $\mathbf{d_m} = 2n_a\mathbf{a} + n_b\mathbf{b}$ and the solution of Eq. (2) in general has the form

$$\Psi_{\mathbf{K}} = \sum_{\mathbf{d_m}} e^{i\mathbf{K}\mathbf{d_m}} T^h_{\mathbf{d_m}}[\Psi_1(\mathbf{r}) + e^{i\mathbf{K}\mathbf{a}}\Psi_2(\mathbf{r})]. \tag{15}$$

The functions $\Psi_{1,2}(\mathbf{r})$ are localized in the centers of vortices forming the unit lattice cell for the quasiparticles (see Fig. 1).

The form of the node functions $\Psi_{1,2}(\mathbf{r})$ in Eq. (15) is determined by the states localized in isolated vortex. We consider the vortex lattice cite 1 centered at the origin $\mathbf{r} = 0$ and define the spatial dependence of gap function inside the unit cell as $\Delta(\mathbf{r}) = \Delta_v(r)e^{i\theta}$, where $\Delta_v(r=0) = 0$. Then eigenfunctions of the Hamiltonian (5) centered at lattice sites 1 and 2 have the form

$$\psi_1(s,\theta_p) = C_1(\theta_p)\psi_v(s,\theta_p), \tag{16}$$

$$\psi_2(s,\theta_p) = C_2(\theta_p)T^h_{\mathbf{a}}\psi_v(s,\theta_p), \tag{17}$$

where the function $\psi_v = \psi_v(s,\theta_p)$ is given by Eq. (8).

The wave function (15) is determined by quasimomentum $\mathbf{K}$ in the Brillouin zone of quasiparticles. The magnetic translation operator (14) in the s-θ_p representation has the form $T^h_{\mathbf{d}} = \hat{\tau}_3 e^{-i\varphi_h(\mathbf{d})\hat{\tau}_3/2} T_{\mathbf{d}}$, where $\varphi_h(\mathbf{d}) = Bd(s + \mathbf{nd})\sin(\theta_p - \theta_d)$, $\mathbf{n} = (\cos\theta_p, \sin\theta_p)$ and the angle θ_d defines the direction of $\mathbf{d} = d(\cos\theta_d, \sin\theta_d)$, and $T_{\mathbf{d}} = \exp[-i\mathbf{d}\mathbf{p_F}(1 - ip_F^{-1}\partial_s)]$ is the translation operator.[14]

We substitute the ansatz (15) to the BdG Eq. (2) and calculate the inner product with $\Psi_{1,2}(\mathbf{r})$ taking into account only overlap with neighbor sites to obtain the system of tight binding equations,

$$\langle\Psi_j|\hat{H}|\Psi_j\rangle = \omega\int_0^{2\pi} C_j^*\hat{\mu}C_j d\theta_p, \tag{18}$$

$$\langle\Psi_j|\hat{H}|T^h_{\mathbf{b}}\Psi_j\rangle = i(-1)^{j+1}J_{\mathbf{b}}\int_0^{2\pi} e^{-i\mathbf{b}\mathbf{p_F}}|C_j|^2 d\theta_p, \tag{19}$$

$$\langle\Psi_1|\hat{H}|\Psi_2\rangle = iJ_{\mathbf{a}}\int_0^{2\pi} e^{-i\mathbf{a}\mathbf{p_F}} C_1^* C_2 d\theta_p, \tag{20}$$

$$\langle\Psi_1|\hat{H}|T^h_{-2\mathbf{a}}\Psi_2\rangle = iJ_{\mathbf{a}}\int_0^{2\pi} e^{i\mathbf{a}\mathbf{p_F}} C_1^* C_2 d\theta_p, \tag{21}$$

with $j = 1,2$. To obtain these expressions we have used Eq. (4) for the inner product. The sign in Eq. (19) is determined by the magnetic flux through the unit cell. Here for simplicity we take into account the overlap with four neighboring vortices. The cases of more neighbors can be considered analogously. The main contribution to the inner products (19), (20), and (21) comes from the stationary points of the phases $(\mathbf{a}\mathbf{p_F})$ and $(\mathbf{b}\mathbf{p_F})$, that is $\theta_p^* = \theta_{a,b} + \pi n$. The stationary points θ_p^* correspond to the trajectories passing through both of the neighbor vortex cores which means that we can calculate the overlap factors as follows:

$$J_{\mathbf{d}} = \frac{\pi}{p_F}\int_{-\infty}^{\infty}[\Delta(s) - \tilde{\Delta}_v(s_1)]\tilde{\psi}_v^+(s)\hat{\tau}_2\tilde{\psi}_v(s_1)ds, \tag{22}$$

where $\mathbf{d} = \mathbf{a},\mathbf{b}$ and $s_1 = s - \mathbf{nd}$, $\tilde{\Delta}_v(s) = \Delta_v(s)\mathrm{sgn}(s)$, and $\tilde{\psi}_v(s) = \psi_v(s,\theta_p = 0)$. Then with good accuracy Eq. (22) yields an estimation $|J_{\mathbf{d}}| \approx \Delta_0\exp(-d/\xi)$.

With the help of the inner products (18) and (19) and taking into account that $J_{\mathbf{d}} = -J_{-\mathbf{d}}$, we obtain the equations

$$\begin{aligned}(\varepsilon - \omega\hat{\mu})C_1 &= F_{\mathbf{b}}(\theta_p)C_1 + F_{\mathbf{a}}(\theta_p)C_2,\\ (\varepsilon - \omega\hat{\mu})C_2 &= -F_{\mathbf{b}}(\theta_p)C_2 + F_{\mathbf{a}}(\theta_p)C_1,\end{aligned} \tag{23}$$

where $F_{\mathbf{d}}(\theta_p) = J_{\mathbf{d}}\sin[\mathbf{d}(\mathbf{p_F} + \mathbf{K})]$ and $\mathbf{d} = \mathbf{a},\mathbf{b}$. The system (23) should be solved together with periodic boundary conditions $C_{1,2}(\theta_p) = C_{1,2}(\theta_p + 2\pi)$ for $p_x + ip_y$ wave and $C_{1,2}(\theta_p) = -C_{1,2}(\theta_p + 2\pi)$ for s wave. Note that in Eqs. (23) we can take into account overlapping with next-to-neighbor vortices which will introduce the corrections of the relative order $e^{-d/\xi}$ to the coefficients $F_{\mathbf{a},\mathbf{b}}$. Here we neglect such corrections.

The spectrum of Eq. (23) $\varepsilon = \varepsilon(\mathbf{K})$ is defined in the magnetic Brillouin zone which can be chosen as $-\pi < \mathbf{K}\mathbf{b} \leqslant \pi$ and $-\pi/2 < \mathbf{K}\mathbf{a} \leqslant \pi/2$. Besides it has an additional symmetry $\varepsilon(\mathbf{K}) = \varepsilon(\mathbf{K} + \mathbf{b_r}/2)$, where $\mathbf{b_r} = 2\pi[\mathbf{z}\times\mathbf{a}]/(\mathbf{z}\cdot[\mathbf{a}\times\mathbf{b}])$ is the reciprocal lattice vector so that $(\mathbf{b_r}\mathbf{b}) = 2\pi$. This symmetry is the same as for the usual tight binding 2D electron model with half quantum flux through the unit cell.

To solve Eq. (23) we use the approximate method employed earlier for the system of two vortices.[14] That is besides the vicinity of the angles $\theta_p^* = \theta_{a,b} + \pi n$ the solution with good accuracy is $C_{1,2} \sim e^{i\varepsilon\theta_p/\omega}$. In the δ vicinity of the angles θ_p^* the system (23) can be diagonalzed to obtain the matching conditions $\mathbf{C}(\theta_p^* + \delta) = \hat{M}\mathbf{C}(\theta_p^* - \delta)$ for the vector $\mathbf{C} = (C_1, C_2)^T$. The matching matrices are

$$\hat{M}(\theta_b) = \exp(-i\hat{\tau}_3\chi_{b+}), \tag{24}$$

$$\hat{M}(\theta_a) = \cos\chi_{a+} - i\hat{\tau}_1 \sin\chi_{a+}, \tag{25}$$

$$\hat{M}(\theta_b + \pi) = \exp(-i\hat{\tau}_3\chi_{b-}), \tag{26}$$

$$\hat{M}(\theta_a + \pi) = \cos\chi_{a-} - i\hat{\tau}_1 \sin\chi_{a-}, \tag{27}$$

where $\chi_{d\pm} = I_{\mathbf{d}} \sin(\mathbf{Kd} \pm \vartheta_d)$ where $\vartheta_d = p_F d - \pi/4$ and $I_{\mathbf{d}} = (J_{\mathbf{d}}/\omega)\sqrt{\pi/p_F d}$.

The eigenvalues of energy ε are determined by periodic boundary conditions

$$\hat{M}(\theta_b)\hat{M}(\theta_a)\hat{M}(\theta_b + \pi)\hat{M}(\theta_a + \pi)e^{2\pi i\varepsilon/\omega} = \pm 1, \tag{28}$$

where the upper (lower) signs are for the $p_x + ip_y$ (s) wave symmetry. Solving Eq. (28) we obtain the Bloch waves $\varepsilon = \varepsilon_n(\mathbf{K})$ in a periodic Abrikosov flux lattice

$$\varepsilon_{n\pm}(\mathbf{K}) = \omega\left(\pm\frac{\arccos X}{2\pi} + n + \gamma\right), \tag{29}$$

where $\gamma = 1/2$ for s wave and $\gamma = 0$ for $p_x + ip_y$ wave, $X = \cos\chi_{a-}\cos\chi_{a+}\cos(\chi_{b+} + \chi_{b-}) - \sin\chi_{a-}\sin\chi_{a+}\cos(\chi_{b+} - \chi_{b-})$, and n is integer. The width of Bloch bands (29) is determined by the overall amplitude $\max(\omega, \Delta_0 e^{-d/\xi}/\sqrt{p_F d})$ and rapid oscillations with the period p_F^{-1} by the intervortex distances a,b, which were found before in finite vortex clusters[14,15] and vortices in mesoscopic superconductors.[24] The phase of oscillations is determined by the average magnetic field, e.g., $a = b = \sqrt{\Phi_0/B}$ for the square lattice, where Φ_0 is magnetic flux quantum for Cooper pairs.

The plot of the Bloch band $\varepsilon_{0+}(\mathbf{K})$ is shown in Fig. 1(b) for the parameters $\vartheta_a = \vartheta_b = \pi/4$ and $I_{\mathbf{a}} = I_{\mathbf{b}} = 0.14$. One can see that this band contains a small energy gap. As will be discussed below, in this case the Majorana zero energy state exists at the center of Brillouin zone $\mathbf{K} = 0$ only for the specific case $\vartheta_a = \pi l$ or $\vartheta_b = \pi l$ where l is integer. Indeed for $I_{\mathbf{d}} \ll 1$ Eq. (29) can be simplified. Taking into account the quadratic terms of the order $I_{\mathbf{d}}^2$ we obtain the gapless spectrum identical to the one level lattice model[19] with the hopping amplitude determined by

$$t_d = \omega I_{\mathbf{d}} \cos\vartheta_d \approx \frac{\Delta_0 e^{-d/\xi}}{\sqrt{p_F d/\pi}} \sin(p_F d + \pi/4), \tag{30}$$

where $\mathbf{d} = \mathbf{a}, \mathbf{b}$. The same hopping amplitude determines the spectrum modification in finite vortex clusters with large vortex separation.[14] Note that the phase of oscillations in (30) is different from that used in subsequent works.[16] The terms of the order $I_{\mathbf{d}}^4$ open the gap in the spectrum (29) which is beyond the accuracy of one level approximation and appears due to the mixing with higher levels. Note that overlap with next-to-neighbor vortices introduces correction smaller by the factor $(p_F\xi)^{-1/2}$ than the interaction with higher levels.

Decreasing the intervortex distance one obtains that for higher values of the overlap factors $I_{\mathbf{d}}$ the structure of Bloch bands becomes more complicated with rapid oscillations as function of quasimomentum with the characteristic period of the order $(I_{\mathbf{d}} d)^{-1}$. This evolution is shown in Fig. 1(c) with red dashed, blue dash-dotted, and black solid lines for the overlap factors $I_{\mathbf{a}} = I_{\mathbf{b}} = 1.4$, 2.8, and 8.9 correspondingly. Such complicated structure of Bloch band $\varepsilon_{0+}(\mathbf{K})$ for $\vartheta_d = \pi/4$ and $I_{\mathbf{a}} = I_{\mathbf{b}} = 4.3$ is shown in the contour plot Fig. 1(d).

Let us consider the condition for the zero energy state existence in $p_x + ip_y$ wave superconductor. Clearly even despite the splitting of Majorana state on two vortices they can appear in vortex lattice at some points of the Brillouin zone. From Eq. (29) we get condition $X = 1$ which gives two sets of zero-energy points,

$$\chi_{b(a)-} = \pi m, \quad \chi_{b(a)+} = \pi n, \quad \chi_{a(b)+} + \chi_{a(b)-} = \pi k, \tag{31}$$

$$\chi_{b(a)-} = \pi m + \pi/2, \quad \chi_{b(a)+} = \pi n + \pi/2, \quad \chi_{a(b)+} - \chi_{a(b)-} = \pi k, \tag{32}$$

where m,n,k are integer numbers so that $m + n + k$ is even for Eq. (31) and odd for Eq. (32). One can see that at large intervortex distance the zero energy points are given by Eqs. (31) with $m = n = k = 0$. Thus zero energy state appears only at the center of Brillouin zone $\mathbf{K} = 0$ [and equivalent points; see Fig. 1(b)] if $\vartheta_a = \pi l$ or $\vartheta_b = \pi l$, where l is integer. Note that the corresponding wave function can be chosen so that $U = V^*$. Thus the zero energy state at $\mathbf{K} = 0$ is a Majorana one. For smaller intervortex distances $d < d_c$, where the critical distance d_c is given by Eq. (11), the overlap is larger $I_{\mathbf{d}} > \pi/2$. In this case Eqs. (31) and (32) yield additional zero energy points. Due to the lattice symmetry there is always an even number of zero energy states at $\mathbf{K} \neq 0$. Thus only that at the zone center in $p_x + ip_y$ superconductor is a Majorana one. On the contrary, at $\mathbf{K} \neq 0$ zero energy states exist both for $p_x + ip_y$ and s wave symmetry. In general, zero energy states appear only for the discrete set of intervortex distances determined by the values of phase ϑ_d when Eqs. (31) and (32) have solutions. For small intervortex distances when $I_d \gg 1$ the characteristic period of zero energy states formation is $d \sim (I_{\mathbf{d}} p_F)^{-1}$. Note that it is smaller than the Fermi wavelength and therefore the finite density of zero energy states can appear due to the fluctuations of vortex positions.

IV. DISCUSSION

Finally let us consider the possible experimental test of the suggested gapless spectrum of Majorana fermions (29). In addition to the variety of experiments proposed[25] the electronic thermal conductivity κ measurements have been proven as an effective tool to study the quasiparticle spectrum in the vortex phase.[15,26–28] The electronic states in magnetic Bloch bands (29) can carry the energy current in the direction $\perp \mathbf{B}$ due to the hopping of quasiparticles between neighboring vortices.

At large intervortex distances $d > d_c$ in $p_x + ip_y$ superconductor vortex lattice can contain single zero energy mode at $\mathbf{K} = 0$ which contributes to the thermal conductance in the limit $T \to 0$. In s wave superconductors there are no zero modes at $d > d_c$. Here the critical distance d_c is given by Eq. (11). At smaller distances $d < d_c$ many zero modes appear [see Fig. 1(c)]. Hence one should expect the threshold behavior of $\kappa(B)$ in the increasing magnetic field in the limit $T \to 0$. That is κ should be very small (or even zero) at $d > d_c$ and at $d < d_c$ it can be estimated by the textbook expression $\kappa_\perp \sim T V_g^2 \tau \nu$, where $\mathbf{V_g} = \partial\varepsilon/\partial\mathbf{K}$ is the group velocity of Bloch waves (29), τ is transport time, and $\nu = \omega^{-1} d^{-2}$ is the density of the vortex lattice. The typical value of group velocity determined by the intervortex hopping is $V_g \sim V_F e^{-d/\xi}\sqrt{d/p_F\xi^2}$ so that, assuming $d = \beta\xi\sqrt{H_{c2}/B}$ where $\beta \sim 1$, we obtain

$$\kappa_\perp/\kappa_N = \frac{\sqrt{B/H_{c2}}}{\beta p_F \xi} e^{-2\beta\sqrt{H_{c2}/B}}. \tag{33}$$

Besides that $\kappa_\perp$ also contains an oscillating part due to rapid oscillations of energy levels (29) with the period p_F^{-1} by the intervortex distance. However, these oscillations should be mostly canceled out due to the fluctuations of vortex positions. The obtained value of κ is valid in the superclean limit $\omega\tau > 1$ for fully gapped superconductors including s and $p_x + ip_y$ wave symmetries. The estimation (33) contains the small prefactor $(p_F\xi)^{-1}$ which can explain the experimentally observed small values of $\kappa_\perp$ at $B \ll H_{c2}$.[26] Qualitatively this small prefactor is determined by almost exact Andreev backscattering which suppresses the single particle transport.[29] Interestingly, the thermal conductivity $\kappa_\parallel$ in the direction $\parallel \mathbf{B}$ was observed in Ref. 27 to be described by Eq. (33) similar to $\kappa_\perp$. This behavior can be explained by the theory[15] applied to the spectrum (29) since the number of conducting modes along the vortex line is determined by the tunneling factors $I_{\mathbf{d}}$.

The band spectrum (29) rapid oscillations with characteristic length scale of Fermi wavelength p_F^{-1} were found before in finite vortex clusters[14,15] and vortices in mesoscopic superconductors.[24] They appear due to the effective interference of electronic waves with opposite wave vectors $\mathbf{p_F}$ and $-\mathbf{p_F}$. It is natural to expect that fluctuations of vortex positions will destroy this interference and the rapid oscillations will be smoothed out. However, this will not destroy the electronic Bloch bands. Indeed provided the vortex displacements are much smaller than coherence length ξ the periodic solution (10) with coefficients determined by Eqs. (23) is still applicable in the vicinity of the angles $\theta_p = \theta_{a,b} + \pi n$. Near these angles the quasiclassical trajectory of electronic wave is parallel to the main direction of the vortex lattice and passes through the periodic chain of vortices for in quasiclassical approximation the small vortex fluctuations can be neglected. Effectively, one can obtain the result for thermal conductivity in this case by averaging the group velocity V_g over the phases $\vartheta_{a,b}$ in the spectrum Eq. (10). This procedure removes the rapid oscillations of thermal conductivity and only the smooth exponential behavior remains (33).

V. CONCLUSION

To conclude, we developed band theory of electronic structure in Abrikosov vortex lattices. The results can be applied to lattices of arbitrary geometry. Electronic Bloch bands are formed due to the overlap of neighboring vortex core states. These bands can intersect the Fermi level forming zero energy states. We found that in $p_x + ip_y$ wave superconductor with sparse Abrikosov lattice the Majorana state appears at the center of Brillouin zone $\mathbf{K} = 0$ for the discrete set of intervortex distances. At smaller distances additional zero modes exist at $\mathbf{K} \neq 0$ both in s and $p_x + ip_y$ wave symmetries. We have shown that the obtained electronic Bloch bands can transmit the energy through the vortex lattice and therefore provide a natural mechanism for thermal conductivity in clean type-II superconductors at small magnetic fields. We argue that the presence of electronic zero modes can be tested by measuring electronic thermal conductivity.

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[1] G. E. Volovik, *The Universe in a Helium Droplet* (Oxford University Press, Oxford, 2009).
[2] G. E. Volovik, Zh. Eksp. Teor. Fiz. **94**, 123 (1988) [Sov. Phys. JETP **67**, 1804 (1988)]; G. E. Volovik and V. M. Yakovenko, J. Phys.: Condens. Matter **1**, 5263 (1989).
[3] A. P. Mackenzie and Y. Maeno, Rev. Mod. Phys. **75**, 657 (2003).
[4] M. A. Silaev and G. E. Volovik, Phys. Rev. B **86**, 214511 (2012).
[5] Y. Tsutsumi, M. Ichioka, and K. Machida, Phys. Rev. B **83**, 094510 (2011).
[6] G. Volovik, JETP Lett. **70**, 609 (1999).
[7] R. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. **59**, 1776 (1987); W. Pan, J.-S. Xia, V. Shvarts, D. E. Adams, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, *ibid.* **83**, 3530 (1999).
[8] L. Fu and C. L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
[9] J. Chaloupka, G. Jackeli, and G. Khaliullin, Phys. Rev. Lett. **105**, 027204 (2010); H. C. Jiang, Z. C. Gu, X. L. Qi, and S. Trebst, Phys. Rev. B **83**, 245104 (2011).
[10] A. Stern, Nature (London) **464**, 187 (2010).
[11] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
[12] C. Caroli, P. G. de Gennes, and J. Matricon, Phys. Lett. **9**, 307 (1964).
[13] C. Nayak, A. Simon, A. Stern, A. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
[14] A. S. Mel'nikov and M. A. Silaev, JETP Lett. **83**, 578 (2006).
[15] A. S. Mel'nikov, D. A. Ryzhov, and M. A. Silaev, Phys. Rev. B **78**, 064513 (2008).
[16] M. Cheng, R. M. Lutchyn, V. Galitski, and S. Das Sarma, Phys. Rev. Lett. **103**, 107001 (2009).

[17] A. Kitaev, Ann. Phys. (N.Y.) **321**, 2 (2006).
[18] C. R. Laumann, A. W. W. Ludwig, D. A. Huse, and S. Trebst, Phys. Rev. B **85**, 161301 (2012).
[19] E. Grosfeld and A. Stern, Phys. Rev. B **73**, 201303 (2006).
[20] Y. E. Kraus and A. Stern, New J. Phys. **13**, 105006 (2011).
[21] V. Lahtinen, A. W. W. Ludwig, J. K. Pachos, and S. Trebst, Phys. Rev. B **86**, 075115 (2012).
[22] A. S. Mel'nikov, J. Phys.: Condens. Matter **11**, 4219 (1999); JETP Lett. **71**, 327 (2000); Physica B **284**, 781 (2000); M. Franz and Z. Tesanovic, Phys. Rev. Lett. **84**, 554 (2000).
[23] A. S. Mel'nikov, in *Vortices in Unconventional Superconductors and Superfluids*, edited by R. P. Huebner, N. Schopohl, and G. E. Volovik, Springer Series in Solid-State Sciences Vol. 132 (Springer, New York, 2002), pp. 207–224.
[24] N. B. Kopnin, A. S. Mel'nikov, V. I. Pozdnyakova, D. A. Ryzhov, I. A. Shereshevskii, and V. M. Vinokur, Phys. Rev. Lett. **95**, 197002 (2005).
[25] Y. E. Kraus, A. Auerbach, H. A. Fertig, and S. H. Simon, Phys. Rev. Lett. **101**, 267002 (2008); N. R. Cooper and A. Stern, *ibid.* **102**, 176807 (2009); A. R. Akhmerov, J. Nilsson, and C. W. J. Beenakker, *ibid.* **102**, 216404 (2009); L. Fu and C. L. Kane, *ibid.* **102**, 216403 (2009).
[26] J. Lowell and J. B. Sousa, J. Low Temp. Phys. **3**, 65 (1970).
[27] P. H. Kes, J. P. M. van der Veeken, and D. de Kierk, J. Low Temp. Phys. **18**, 355 (1975).
[28] N. B. Kopnin, A. S. Mel'nikov, V. I. Pozdnyakova, D. A. Ryzhov, I. A. Shereshevskii, and V. M. Vinokur, Phys. Rev. B **75**, 024514 (2007).
[29] N. B. Kopnin, A. S. Mel'nikov, and V. M. Vinokur, Phys. Rev. B **68**, 054528 (2003).