Interaction effect on topological classification of superconductors in two dimensions

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We introduce a class of superconductors (SCs) in two spatial dimensions with time-reversal symmetry and reflection (i.e., mirror) symmetry. In the absence of interactions, topological classes of these SCs are distinguished by an integer-valued ($\mathbb{Z}$) topological invariant. When interactions are included, we show that the topological classification is modified to $\mathbb{Z}_8$. This clearly demonstrates that interactions can have a qualitative effect on topological classifications of gapped states of matter in more than one dimension.

I. INTRODUCTION

Topological phases are fully quantum mechanical states of matter which are not characterized by classical symmetry breaking.[1] While gapped in the bulk, quite often, they are accompanied by *gapless* excitations at their boundary, signaling a highly entangled nature of their ground state. Since the discovery of the integer quantum Hall effect (IQHE),[2] the list of topological phases in nature has been expanded, in particular, by the recent discovery of topological insulators in two and three dimensions (2D and 3D) in systems with strong spin-orbit coupling,[3–12] and the identification of ^{3}He B as a topological SC (superfluid).[13] Unlike the IQHE, the topological character of these topological insulators and SCs (i.e., the stable gapless edge or surface modes) is protected by time-reversal symmetry (TRS). The presence or absence of a topological distinction among gapped phases for a given set of symmetries and for given spatial dimensions can be studied systematically, and is summarized in the "Periodic Table" of topological insulators and SCs for noninteracting[14] fermions.[15–17]

Since interactions are ubiquitous in real materials, a natural question is whether and how interactions could modify topological classifications obtained for noninteracting systems.[18] In other words, are there cases where interactions qualitatively affect topological classifications? Even though this question has not been fully or thoroughly answered, exciting progress has been made in Ref. 19, in which Fidkowski and Kitaev explicitly showed that the putative $\mathbb{Z}$ classification of one-dimensional (1D) "noninteracting" SCs with unusual time-reversal symmetry ("BDI class") is modified to $\mathbb{Z}_8$ when interactions are included. This $\mathbb{Z}_8$ classification of the 1D BDI SCs with interactions is further illustrated from the study of the entanglement spectrum.[20,21] In more than one dimension, stability of topological classifications in a number of classes of noninteracting topological insulators and SCs is argued from the perspective of low-energy topological response theory.[12,22,23] For bosonic (or spin) systems, a systematic approach of constructing symmetry protected topological phases in general spatial dimensions was recently proposed.[24] However, how interactions in fermionic systems in 2D and 3D can dramatically modify their topological classification obtained from noninteracting fermions remains largely unexplored.[25]

In this paper, we partly fill this gap by considering 2D SCs with TRS ($\mathcal{T}^2 = -1$) and reflection symmetry (RS) ($\mathcal{R}^2 = -1$). Note that $\mathcal{T}$ is antiunitary, while $\mathcal{R}$ is unitary. We call SCs with these symmetry properties "DIII + R" SCs. First, we show that the topological classes of these DIII + R SCs are classified by an integer-valued ($\mathbb{Z}$) topological invariant at the quadratic level. Such topologically nontrivial SCs are characterized by helical Majorana modes on the system's edge. The number of helical Majorana edge states is protected by the symmetries at the noninteracting level.

We then ask whether those helical gapless Majorana edge modes are stable against interactions while preserving those relevant symmetries in question. As we show later in detail, the helical Majorana edge states in the case of a $\mathbb{Z} = 8$ topological invariant are unstable against (even weak) interactions—the gapless helical Majorana edge states become gapped while no symmetry is broken in the bulk or the edges of the system. It is worth stressing that all the relevant symmetries are fully preserved while the gap opens in the edge. In other words, the edge of the putative $\mathbb{Z} = 8$ system is qualitatively the same as $\mathbb{Z} = 0$ systems when symmetry-preserving interactions are included. Because the system's edge we consider respects the same set of symmetries as its bulk, robust edge states could fully encode topological properties of the bulk due to the bulk-edge correspondence. Consequently, the putative $\mathbb{Z}$ topological classification of noninteracting DIII + R superconductors in 2D is reduced $\mathbb{Z}_8$ when interactions are considered. A similar model, but with a different set of symmetries, was studied independently in Refs. 26 and 27.

II. DIII + R SCS

Two-dimensional SCs with TRS $\mathcal{T}^2 = -1$ (symmetry class DIII) have a $\mathbb{Z}_2$ topological invariant as shown in the Periodic Table. To see this explicitly, it is helpful to study their edge theory. For $\mathbb{Z}_2$ nontrivial DIII SCs in 2D, its generic edge theory is the 1D helical Majorana fermions:

$$H_{\text{free}} = \int dx [\psi_\uparrow i\, \partial_x \psi_\uparrow - \psi_\downarrow i\, \partial_x \psi_\downarrow], \tag{1}$$

where ψ_σ are left/right-moving edge Majorana fermion operators and $\sigma = \uparrow, \downarrow$ are spin index with the following properties under time-reversal transformation: $\mathcal{T}^{-1}\psi_\uparrow \mathcal{T} = \psi_\downarrow$ and $\mathcal{T}^{-1}\psi_\downarrow \mathcal{T} = -\psi_\uparrow$, which satisfies $\mathcal{T}^2 = -1$. It is clear that

the helical edge states above respect the TRS. Moreover, this TRS protects this putative gapless helical edge state since there is only one mass term $im\psi_\uparrow\psi_\downarrow$ which breaks the TRS in the edge. However, when there are two such helical gapless modes labeled by $a = 1, 2$, the putative gapless modes can be fully gapped by time-reversal invariant terms $im(\psi_\uparrow^1\psi_\downarrow^2 + \psi_\downarrow^1\psi_\uparrow^2)$. Consequently, the topological classification is $\mathbb{Z}_2$.

To possibly have a $\mathbb{Z}$ topological classification of DIII SCs in 2D, further discrete symmetry is needed. Since the helical edge states preserve not only TRS but also the RS $\mathcal{R}$ defined as $\mathcal{R}^{-1}\psi_\uparrow(x)\mathcal{R} = \psi_\downarrow(-x)$ and $\mathcal{R}^{-1}\psi_\downarrow(x)\mathcal{R} = -\psi_\uparrow(-x)$,[28] we consider 2D DIII SCs with RS, dubbed as the DIII + R class. Note that $\mathcal{R}^2 = -1$ as required for spin half-integer fermions. Now, we show that the DIII + R SCs in 2D are distinguished by an integer-valued ($\mathbb{Z}$) topological invariant. To see this, we consider N copies (or flavors) of the helical gapless Majorana edge states given by $\sum_{a=1}^N \int dx \, v_a[i\psi_\uparrow^a\partial_x\psi_\uparrow^a - i\psi_\downarrow^a\partial_x\psi_\downarrow^a]$, where $a = 1, \ldots, N$ are the "flavor" indices (which is preserved by both the time-reversal or reflection transformations) and v_a are the Fermi velocities. We then write down the most general mass term

$$i\psi_\uparrow^a M_{ab}\psi_\downarrow^b, \quad M_{ab}^* = M_{ab}, \tag{2}$$

and check if it can preserve both TRS and RS. (Note that terms proportional to $i\psi_\sigma^a\psi_\sigma^b$ for $\sigma = \uparrow$ or $\downarrow$ are irrelevant and only renormalize the Fermi velocities of edge fermions.) Time-reversal symmetry requires $M_{ab} = M_{ba}$, while RS requires $M_{ab} = -M_{ba}$. Consequently, $M = 0$ identically. In other words, it is impossible to gap the N pairs of helical gapless Majorana edge states by considering the noninteracting fermion-bilinear terms for arbitrary N. This class of SCs is then characterized by an integer-valued ($\mathbb{Z}$) topological invariant (see below).

III. MICROSCOPIC MODELS

Armed with insights from the edge theories, we now introduce an explicit mean-field SC Hamiltonian which respects both TRS and RS and which has nontrivial gapless helical Majorana edge states:

$$
\begin{aligned}
H_{\text{latt}} = &\sum_{\langle ij\rangle\sigma}[-tc_{i\sigma}^\dagger c_{j\sigma} + \text{H.c.}] - \mu\sum_{i\sigma}c_{i\sigma}^\dagger c_{i\sigma} \\
&+ \sum_i[\Delta(c_{i\uparrow}^\dagger c_{i+\hat{x}\uparrow}^\dagger + c_{i\downarrow}^\dagger c_{i+\hat{x}\downarrow}^\dagger) + \text{H.c.}] \\
&+ \sum_i[i\Delta(c_{i\uparrow}^\dagger c_{i+\hat{y}\uparrow}^\dagger - c_{i\downarrow}^\dagger c_{i+\hat{y}\downarrow}^\dagger) + \text{H.c.}], \tag{3}
\end{aligned}
$$

where $c_{i\sigma}^\dagger$ are the fermion creation operators on site i and $\sigma = \uparrow, \downarrow$ are spin indices. It is clear that the model above is invariant under RS $\mathcal{R}$: $c_{x,y,\sigma} \rightarrow (i\sigma^x)_{\sigma\sigma'}c_{-x,y,\sigma'}$ and TRS $\mathcal{T}$: $c_{i\sigma} \rightarrow (i\sigma^y)_{\sigma\sigma'}c_{i\sigma'}$.[28] This model describes a SC with spin up $p_x + ip_y$ pairing and spin down $-p_x + ip_y$ pairing, and can describe a thin film of ^{3}He-B. On a cylinder with edges parallel to the x direction, we (numerically) obtain the helical Majorana edge states, for $t = \Delta = 1$ and for $-4 < \mu < 0$ or $0 < \mu < +4$, say. These helical edge states are effectively described by the Hamiltonian (1). The N-flavor generalization of this model is straightforward.

IV. BULK TOPOLOGICAL INVARIANT

There is a bulk topological invariant which guarantees, when noninteracting and when there is a translation symmetry, the stability of the edge modes of DIII + R topological SCs for arbitrary N. The construction is similar in spirit to the mirror Chern number in 3D topological insulators protected by RS.[29] With the periodic boundary condition, the quadratic bulk Hamiltonian can be Fourier transformed as

$$
\begin{aligned}
H &= \sum_{0\leqslant k_x\leqslant\pi}\sum_{k_y}\Psi_{k_x}^\dagger(k_y)\mathcal{H}_{k_x}(k_y)\Psi_{k_x}(k_y), \\
&\Psi_{k_x}^\dagger(k_y) := (c_{\uparrow,k}^\dagger, c_{\downarrow,k}^\dagger, c_{\uparrow,-k}, c_{\downarrow,-k}).
\end{aligned}
\tag{4}
$$

We then note at the reflection symmetric points $k_x = \tilde{k}_x$ ($=0$ and π), the Bloch Hamiltonian $\mathcal{H}_{\tilde{k}_x}(k_y)$ commutes with $\mathcal{R}$, namely $[\mathcal{H}_{\tilde{k}_x}(k_y), J^x] = 0$, where $J^x = \text{diag}(i\sigma^x, -i\sigma^x)$. In other words, at $\tilde{k}_x$, the quadratic Hamiltonian conserves S^x. Combined with TRS, the quadratic Hamiltonian at $k_x = \tilde{k}_x$ can be written as

$$H(\tilde{k}_x) = \sum_{k_y}(c_{+,k}^\dagger, c_{-,-k})\bar{\mathcal{H}}_{\tilde{k}_x}(k_y)\begin{pmatrix} c_{+,k} \\ c_{-,-k}^\dagger \end{pmatrix}, \tag{5}$$

where the subscripts $\pm$ are eigenvalues of σ^x. Now, the Hamiltonian $\bar{\mathcal{H}}_{\tilde{k}_x}(k_y)$ above is in the AIII class with chiral symmetry.[15]

Following Ref. 15, gapped 1D quadratic Hamiltonians in symmetry class AIII are distinguished by an integer topological invariant, the winding number ν. Thus, at each reflection symmetric momentum $k_x = \tilde{k}_x$ ($\tilde{k}_x = 0$ and π), we can introduce an integer topological invariant ("the reflection winding number"), $\nu(\tilde{k}_x)$, the winding number of $\mathcal{H}_{\tilde{k}_x}(k_y)$. We thus have two integral topological invariants, $\tilde{\nu}(k_x = 0, \pi)$. The nonzero value of the invariant, $\tilde{\nu}(\tilde{k}_x) \neq 0$, guarantees the presence of $|\tilde{\nu}(k_x)|$ pairs of zero-energy Majorana states at $\tilde{k}_x$, when an edge is introduced along x direction. In particular, when the invariant is nonzero $\tilde{\nu}(\tilde{k}_x) \neq 0$ at one of the reflection symmetric momenta ($\tilde{k}_x = 0$, say) and it is zero at the other ($\tilde{k}_x = \pi$), this means there must be $|\tilde{\nu}(k_x)|$ branches of nonchiral edge modes. In general, the difference of the reflection winding number

$$\tilde{\nu}(k_x = 0) - \tilde{\nu}(k_x = \pi) \tag{6}$$

tells us the number of nonchiral edge modes.

V. INTERACTION EFFECT

At the noninteracting level (and without disorder), the DIII + R SCs have $\mathbb{Z}$ topological classification as shown from both bulk and edge theories. Now, we consider the effect of symmetry-preserving interactions in the mean-field BdG Hamiltonian and check if the putative topological classification of $\mathbb{Z}$ is modified or not. Since gapless helical edge states are the hallmark of those topologically nontrivial SCs, we believe that it would be sufficient to check if the gapless helical edge states are stable against interactions while requiring there is no symmetry breaking induced by interactions.[30]

For the case of topological invariant $N = 8$ [more generally $N \equiv 0 \pmod 8$], we try to identify certain interactions that can destabilize the gapless edge states while preserving the

symmetries of the bulk and the edge. For simplicity, we consider the helical edge modes of free fermions with the same Fermi velocities $v_a = v$,

$$H_{\text{free}} = v \int dx \sum_{a=1}^{8} [\psi_\uparrow^a i \partial_x \psi_\uparrow^a - \psi_\downarrow^a i \partial_x \psi_\downarrow^a], \qquad (7)$$

which is invariant under the global SO(8) rotations among the left-moving or right-moving Majorana fermions. Now, we consider interactions allowed by TRS and RS. One naturally starts with the following SO(8) symmetric interactions: $H_{\text{GN}} = H_{\text{free}} - g \int dx (\sum_{a=1}^{8} i \psi_\uparrow^a \psi_\downarrow^a)^2$, where g is the coupling constant. This theory is then the SO(8) Gross-Neveu (GN) model in $(1+1)$D, which is exactly solvable.[31] Especially, the interaction is marginally relevant for $g > 0$: for arbitrary small interaction strength g the ground state is gapped by spontaneously breaking the time-reversal (or $\mathbb{Z}_2$ chiral) symmetry with the order parameter $\langle i \psi_\uparrow^a \psi_\downarrow^a \rangle \sim e^{-\pi/(vg)}$. There are twofold degenerate ground states at the edge. When $g < 0$, the interaction is marginally irrelevant and the ground state remains gapless. In other words, the SO(8) symmetric GN interactions cannot result in a unique gapped ground state in the edge. We need to look for some other channel of interactions to fulfill this.

We follow the construction introduced in Ref. 19. The noninteracting edge is described by the conformal field theory of eight free Majorana fermions, which is equivalent to the SO(8)$_1$ Wess-Zumino-Witten model. The eight fermion operators ψ_σ^a ($\sigma = \uparrow$ or $\downarrow$) form the vector representation of SO(8). Moreover, the 16-dimensional spinor representation of SO(8) is reducible to two eight-dimensional irreducible ones formed by spinor operators η_σ^a and χ_σ^a, $a = 1, \ldots, 8$. The explicit forms of η_σ^a and χ_σ^a are given by

$$\exp\left[\frac{i}{2} \left(\pm\phi_\sigma^1 \pm \phi_\sigma^2 \pm \phi_\sigma^3 \pm \phi_\sigma^4 \right) \right], \qquad (8)$$

where ϕ_σ^a are boson fields obtained from bosonizing the system, $\psi_\sigma^{2a-1} + i\psi_\sigma^{2a} = e^{\pm i\phi_\sigma^a}$. The number of minus sign in the exponent of Eq. (8) is even for η^a but odd for χ^a. Accidently, ψ, η, and χ all form eight-dimensional representations of SO(8); they can actually be transformed into one another by the so-called triality symmetry of SO(8). It turns out that the spinor fields are useful to construct the interactions we desire.

To fully gap the edge states without spontaneously breaking any symmetry, we consider the following interaction:

$$H_{\text{int}} = -\int dx \left[A \left(\sum_{a=1}^{7} i\eta_\uparrow^a \eta_\downarrow^a \right)^2 \right.$$
$$\left. + B \left(\sum_{a=1}^{7} i\eta_\uparrow^a \eta_\downarrow^a \right) (i\eta_\uparrow^8 \eta_\downarrow^8) \right], \qquad (9)$$

which is SO(7) invariant and leaves η^8 fixed. This SO(7)-symmetric interaction is also local in terms of the original fermions ψ_σ^a. Indeed, a finite gap opens in the edge states while preserving the symmetries in question, as explicitly shown in Ref. 19 when $B < 0$ and $2A > B$. To understand this, let us first look at the limit $A \gg |B|$ which is the SO(7)

GN model plus free η_σ^8 fermions (when $B \to 0$). The chiral symmetry is broken by the SO(7) GN interactions with the order parameter $M = \langle i \sum_{a=1}^{7} i\eta_\uparrow^a \eta_\downarrow^a \rangle \neq 0$, which generates a mass term $i B M \eta_\uparrow^8 \eta_\downarrow^8$ for η^8 fermions. Now, the η^8 fermions can be mapped to the transverse field Ising model with field strength $(h - h_c) \propto \pm BM$, where h_c is the critical field strength in the transverse field Ising model which is equal to the Ising interaction strength. For $h > h_c$, its ground state is paramagnetic without any symmetry breaking; for $h < h_c$, the system spontaneously breaks the Ising symmetry resulting in twofold degenerate ground states. In other words, $B > 0$ and $B < 0$ lie in two different phases. Since $B = 2A > 0$ is the SO(8) GN model which spontaneously breaks the chiral symmetry having twofold degenerate ground states, it is then clear that $B < 0$ phase has a unique gapped ground state without breaking any symmetry.

It is worthwhile to understand more heuristically why $N = 8$ is special. The edge theory with $N = 8$ is qualitatively equivalent to the two-leg ladder electron model at half-filling. Since there are two electrons per unit cell for the two-leg ladder at half-filling, having a fully gapped ground state without breaking any symmetry is possible and expected.[32] To illustrate this, let us consider lattice Majorana fermions on 1D chains described by the Hamiltonian,[19] $H = uH_1 + wH_2$, where H_1 is quadratic in lattice real fermion operators: $H_1 = -(i/2) \sum_{a=1}^{8} \sum_j s_a \lambda_j^a \lambda_{j+1}^a$, where $\{\lambda_i^a, \lambda_j^b\} = 2\delta_{ij}^{ab}$ and $s_a = \{1, 1, -1, -1, 1, 1, -1, -1\}$. It is clear that H_1 is invariant under either TRS or RS, when we assume

$$\mathcal{R} : \lambda_j^a \to (-1)^a \lambda_{-j}^{a+2}, \quad \lambda_j^{a+2} \to -(-1)^a \lambda_{-j}^a, \qquad (10)$$

$$\mathcal{T} : \lambda_j^a \to \lambda_j^{a+2}, \quad \lambda_j^{a+2} \to -\lambda_j^a, \quad a = 1,2,5,6. \qquad (11)$$

H_2 is an interaction, and given by $H_2 = \sum_j W(\lambda_j^a)$, where $W(\lambda_j^a)$ is a four fermion interaction composed of eight Majorana fermions:

$$\begin{aligned} W(\lambda^a) &= +\lambda^1\lambda^2\lambda^3\lambda^4 + \lambda^1\lambda^2\lambda^5\lambda^6 + \lambda^1\lambda^2\lambda^7\lambda^8 \\ &+ \lambda^3\lambda^4\lambda^5\lambda^6 + \lambda^3\lambda^4\lambda^7\lambda^8 + \lambda^5\lambda^6\lambda^7\lambda^8 \\ &- \lambda^2\lambda^3\lambda^6\lambda^7 - \lambda^1\lambda^4\lambda^5\lambda^8 + \lambda^1\lambda^3\lambda^5\lambda^7 + \lambda^2\lambda^4\lambda^6\lambda^8 \\ &- \lambda^2\lambda^3\lambda^5\lambda^8 - \lambda^1\lambda^4\lambda^6\lambda^7 - \lambda^1\lambda^3\lambda^6\lambda^8 - \lambda^2\lambda^4\lambda^5\lambda^7. \end{aligned} \qquad (12)$$

In the absence of the interaction term $w = 0$, the lattice model is gapless, whose continuum limit is given by the Hamiltonian (7). When we switch on $w \neq 0$, the edge theory is gapped with unique ground state without breaking symmetries. This can be understood in the following way: the interaction W can be written in terms of complex fermions,

$$\begin{aligned} (\lambda^1 + i\lambda^2)/2 &= c_{1\uparrow}, \quad (\lambda^3 - i\lambda^4)/2 = c_{1\downarrow}, \\ (-\lambda^5 + i\lambda^6)/2 &= c_{2\uparrow}, \quad (\lambda^7 + i\lambda^8)/2 = c_{2\downarrow}, \end{aligned} \qquad (13)$$

as follows:

$$W(\lambda_j^a) = 16 \mathbf{S}_{j,1} \cdot \mathbf{S}_{j,2} + 2(n_{j,1} - 1)^2 + 2(n_{j,2} - 1)^2 - 2, \qquad (14)$$

where $\mathbf{S}_{1,2}$ and $n_{1,2}$ are the spin and the fermion number operator for $c_{1,2s}$ ($s = \uparrow/\downarrow$). With this interaction, the charge

degrees of freedom will be frozen by Mott physics. The exchange interaction $S_1 \cdot S_2$ (the "rung-exchange" interaction) realizes the rung-single phase which has a unique ground state without breaking any symmetry.

VI. DISCUSSION

We have discussed interaction effects on topological SCs in 2D protected by TRS and RS. This is a nontrivial example in 2D where topological classification of ground states is dramatically altered by the interaction effect. In a separate paper, we plan on a more systematic study on topological insulators and SCs protected by RS in addition to other possible discrete symmetries.[33,34]

We close with discussion on disorder effects. In the tenfold classification of topological insulators and SCs, it has been proved useful to consider the boundary (edge, surface, etc.) Anderson localization problem: for a topological bulk, one should find a boundary mode which is completely immune to disorder. In turn, once one finds such "Anderson delocalization" at the boundary, it means there is a topologically nontrivial bulk. Not only this bulk-boundary correspondence can be used to find and classify bulk topological phases *in the absence of disorder*, it immediately tells us such topological phases are stable against disorder. For topological phases protected by a set of spatial symmetries, stability against disorder is, in general, not trivial, since spatial inhomogeneity does not respect the spatial symmetries. One can still consider, however, situations where the spatial symmetries are preserved *on average*. The effects of disorder in N-channel quantum wires in symmetry class DIII, which are from our point of view the edge theory of the DIII + R topological SC with the topological integer ($=N$), have been studied.[35] It is known that there is an even-odd effect in N: the mean conductance decreases algebraically as $L^{-1/2}$ with the length of the wire L for odd N, whereas it decays exponentially with L for even N. Correspondingly, for symmetry class DIII with N odd the density of states shows the Dyson singularity. This implies that disorder simply reduces the $\mathbb{Z}$ topological classification of DIII + R SCs to the $\mathbb{Z}_2$ classification, which is the same as the topological classification of 2D DIII SCs in the Periodic Table. In a separate paper[33] we will report the other cases where the $\mathbb{Z}$ topological classification is reduced to the $\mathbb{Z}_2$ classification, which is not related to the existing topological class in the Periodic Table.

Note added: Recently, some other papers, a couple of works that deal with similar topic have appeared.[36–38]

ACKNOWLEDGMENTS

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