PHYSICAL REVIEW B **88**, 045131 (2013)

Short-range entangled bosonic states with chiral edge modes and T duality of heterotic strings

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(Received 30 April 2013; published 26 July 2013)

We consider states of bosons in two dimensions that do not support anyons in the bulk, but nevertheless have stable chiral edge modes that are protected even without any symmetry. Such states must have edge modes with central charge $c = 8k$ for integer k. While there is a single such state with $c = 8$, there are, naively, two such states with $c = 16$, corresponding to the two distinct even unimodular lattices in 16 dimensions. However, we show that these two phases are the same in the bulk, which is a consequence of the uniqueness of signature $(8k + n, n)$ even unimodular lattices. The bulk phases are stably equivalent, in a sense that we make precise. However, there are two different phases of the edge corresponding to these two lattices, thereby realizing a novel form of the bulk-edge correspondence. Two distinct fully chiral edge phases are associated with the same bulk phase, which is consistent with the uniqueness of the bulk since the transition between them, which is generically first order, can occur purely at the edge. Our construction is closely related to T duality of toroidally compactified heterotic strings. We discuss generalizations of these results.

DOI: 10.1103/PhysRevB.88.045131 PACS number(s): 11.15.Yc, 21.60.Fw, 71.27.+a

I. INTRODUCTION

The last decade has seen enormous progress in the understanding of topological phases (see Ref. 1, and references therein) and of symmetry-protected topological (SPT) phases.[2–5] SPT phases are gapped phases of matter that do not have nontrivial excitations in the bulk, have vanishing topological entanglement entropy[6,7] or, equivalently, have short-ranged entanglement (SRE), but have gapless excitations at the edge in the presence of a symmetry. In the case of the most famous and best-understood example, "topological insulators" (see Refs. 8–17, and references therein), the symmetry is time reversal. Topological phases (without a modifier) are gapped phases of matter that are stable to arbitrary perturbations, support anyons in the bulk, and have nonzero topological entanglement entropy or, equivalently, have long-ranged entanglement (LRE). They may or may not (depending on the topological phase) have gapless edge excitations.[45]

However, there is a third possibility: phases of matter that do not support anyons but nevertheless have gapless excitations even in the *absence of any symmetry*. Thus they lie somewhere between topological phases and symmetry-protected topological phases but are neither. Integer quantum Hall states of fermions are a well-known example. Their gapless edge excitations[18,19] are stable to arbitrary weak perturbations even though they do not support anyons and only have SRE. Although the existence and stability of SRE integer quantum Hall (QH) states might seem to be a special feature of fermions, such states also exist in purely bosonic systems, albeit with some peculiar features. We emphasize that unlike in other proposals of bosonic QH states,[20] the edge modes of the states we discuss are stable without imposing any symmetry. Note that according to an alternate definition of SRE states—adiabatic continuability to a local product state with finite-depth local unitary transformations[2]—integer quantum Hall states of fermions and the bosonic states discussed in this paper would be classified as LRE states.

For any integer N, there is an integer quantum Hall state of fermions with SRE, electrical Hall conductance $\sigma_{xy} = N\frac{e^2}{h}$, and thermal Hall conductance $\kappa_{xy} = N\frac{\pi^2 k_B^2 T}{3h}$.[21] In fact, there is only one such state for each N: any two SRE states of fermions at the same filling fraction N can be transformed into each other without encountering a phase transition.[46] (This is true in the bulk; see Sec. VII B for the situation at the edge.) Therefore the state with N filled Landau levels of noninteracting fermions is representative of an entire universality class of SRE states. As a result of its N chiral Dirac fermion edge modes, this is a distinct universality class from ordinary band insulators. These edge modes, which have Virasoro central charge $c = N$ if all of the velocities are equal, are stable to *all* perturbations. If we do not require charge conservation symmetry, then some Hamiltonians in this universality class may not have $\sigma_{xy} = N\frac{e^2}{h}$, but they will all have $\kappa_{xy} = c\frac{\pi^2 k_B^2 T}{3h} = N\frac{\pi^2 k_B^2 T}{3h}$.

Turning now to bosons, there are SRE states of bosons with similarly stable chiral edge modes, but only for central charges $c = 8k$. As we discuss, they correspond to even, positive-definite, unimodular lattices. Moreover, while there is a unique such state with $c = 8$, there appear to be two with $c = 16$, twenty four with $c = 24$, and more than ten million with $c = 32$.[22] Thus we are faced with the possibility that there are many SRE bosonic states with the same thermal Hall conductance κ_{xy}, presumably distinguished by a more subtle invariant. In this paper, we show that this is not the case for $c = 16$. The two SRE bosonic states with $c = 16$ edge excitations are equivalent in the bulk: their partition functions on arbitrary closed manifolds are equal. However, there are two distinct chiral edge phases of this unique bulk state. They are connected by an edge reconstruction: a phase transition must be encountered at the edge in going from one state to the other, but this transition can occur solely at the edge and the gap need not close in the bulk. Although we focus on the $c = 16$ case, the logic of our analysis readily generalizes. Therefore we claim that there is essentially a unique bulk bosonic phase for each $c = 8k$ given by k

copies of the so-called E_8 state.[4,5] However, there are two distinct *fully chiral* edge phases with $c = 16$, twenty four with $c = 24$, more than ten million with $c = 32$, and even more for larger c.

One important subtlety arises in our analysis. The two $c = 16$ phases do not, initially, appear to be identical. However, when combined with a trivial insulating phase, the two bulk partition functions can be mapped directly into each other by a change of variables. This is a physical realization of the mathematical notion of *stable equivalence*. In general, an effective description of a phase of matter will neglect many gapped degrees of freedom (e.g., the electrons in inner shells). However, the sequence of gapped Hamiltonians that interpolates between two gapped Hamiltonians may involve mixing with these usually forgotten gapped degrees of freedom. Therefore it is natural, in considering a phase of matter, to allow an arbitrary enlargement of the Hilbert space by trivial gapped degrees of freedom (i.e., by SRE phases without gapless edge excitations). This is useful when, for instance, comparing a trivial insulating phase with p bands with another trivial insulating phase with $q > p$ bands. They can be adiabatically connected if we are allowed to append $q - p$ trivial insulating bands to the latter system. This notion is also natural when connecting different phases of gapless edge excitations. The edge of a gapped bulk state will generically have gapped excitations that we ordinarily ignore. However, they can become gapless—which is a form of edge reconstruction—and interact with the other gapless degrees of freedom, driving the edge into a different phase. However, this does not require any change in the bulk. As we will see, such a purely edge phase transition connects the two seemingly different chiral gapped edges with $c = 16$. By combining a $c = 16$ state with a trivial insulator, we are able to take advantage of the uniqueness of signature $(8k + n,n)$ even unimodular lattices,[23] from which it follows that the two phases are the same. This is closely related to the fact that T duality exchanges toroidal compactifications of the $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ versions of the heterotic string, as explained by Ginsparg.[24]

In the remainder of this paper, we describe the equivalence of the two candidate phases at $k = 2$ from two complementary perspectives. To set the stage, we begin in Sec. II with a short introduction to the K-matrix formalism that we use to describe the phases of matter studied in this paper. In Sec. III, we provide a bulk description of the equivalence of the two candidate phases at $k = 2$. We then turn to the edge, where we show that there are two distinct chiral phases of the edge. We first discuss the fermionic description of the edge modes in Sec. IV and then turn to the bosonic description in Sec. V. There is an (purely) edge transition between these two phases. We discuss the phase diagram of the edge, which is rather intricate, and its relation to the bulk. In Sec. VI, we summarize how the phase diagram can change when some of the degrees of freedom are electromagnetically charged so that a U(1) symmetry is preserved. We then conclude in Sec. VII and discuss possible generalizations of this picture.

In Appendix A, we collect basic definitions and explain the notation used throughout the text. In Appendix B, we provide some technical details for an argument used in the main text.

II. K-MATRIX FORMALISM

A. Chern-Simons theory

We will consider $2+1$-dimensional phases of matter governed by bulk effective field theories of the form

$$\mathcal{L} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}K_{IJ}a^I_\mu\partial_\nu a^J_\rho + j^\mu_I a^I_\mu, \tag{1}$$

where a^I_μ, for $I = 1,\ldots,N$ and $\mu = 0,1,2$. See Refs. 25 and 26 for a pedagogical introduction to such phases. K_{IJ} is a symmetric, nondegenerate $N \times N$ integer matrix. (Repeated indices should be summed over unless otherwise specified.) We normalize the gauge fields a^I_μ and sources j^μ_I so that fluxes that are multiples of 2π are unobservable by the Aharonov-Bohm effect. Consequently, if we take the sources to be given by prescribed nondynamical classical trajectories $x^\mu_m(\tau)$ that serve as sources of a^I_μ flux, they must take the form

$$j^\mu_I = \sum_m n^{(m)}_I \delta\left[x^\mu - x^\mu_m(\tau)\right]\partial_\tau x^\mu_m, \tag{2}$$

for integers $n^{(m)}_I$. The sum over m is a sum over the possible sources x_m.

Therefore, each excitation m of the system is associated with an integer vector $n^{(m)}_I$. These integer vectors can be associated with the points of a lattice as follows. Let λ_a for $a = 1,\ldots,N$ be the eigenvalues of $(K^{-1})^{IJ}$ with f^I_a the corresponding eigenvectors. We normalize the f^I_a so that $(K^{-1})^{IJ} = \eta^{ab} f^I_a f^J_b$, where $\eta^{ab} = \mathrm{sgn}(\lambda_a)\delta^{ab}$. Now suppose that we view the f^I_a as the components of a vector $\mathbf{f}^I \in \mathbb{R}^{N_+,N_-}$ [i.e., of $\mathbb{R}^N$ with a metric $\eta_{ab} = \mathrm{sgn}(\lambda_a)\,\delta_{ab}$ of signature (N_+,N_-)], where K^{-1} has N_+ positive eigenvalues and N_- negative ones. In other words, the unit vector $\hat{\mathbf{x}}_a = (0,\ldots,0,1,0,\ldots,0)^{\mathrm{tr}}$ with a 1 in the ath entry and zeros otherwise is an orthonormal basis of $\mathbb{R}^{N_+,N_-}$ so that $\hat{\mathbf{x}}_a \cdot \hat{\mathbf{x}}_b \equiv (\hat{\mathbf{x}}_a)^c \eta_{cd} (\hat{\mathbf{x}}_b)^d = \eta_{ab}$. Then we can define $\mathbf{f}^I \equiv f^I_a \hat{\mathbf{x}}_a$. Thus the eigenvectors $\mathbf{f}^I$ define a lattice Γ in $\mathbb{R}^{N_+,N_-}$ according to $\Gamma = \{m_I \mathbf{f}^I | m_I \in \mathbb{Z}\}$; this lattice determines the allowed excitations of the system.[27,28]

The lattice Γ enters directly into the computation of various physical observables. For example, consider two distinct excitations corresponding to the lattice vectors $\mathbf{u} = m_I \mathbf{f}^I$ and $\mathbf{v} = n_J \mathbf{f}^J$ in Γ. If one excitation is taken fully around the other, then the resulting wave function differs from its original value by the exponential of the Berry's phase $2\pi (K^{-1})^{IJ} m_I n_J = 2\pi \mathbf{u}\cdot\mathbf{v}$. When the excitations are identical, $\mathbf{u} = \mathbf{v}$, a half-braid is sufficient and a phase equal to $\pi \mathbf{u}\cdot\mathbf{u}$ is obtained.

Of course, any basis of the lattice Γ is equally good; there is nothing special about the basis $\mathbf{f}^I$. We can change to a different basis $\mathbf{f}^I = W^I{}_J \tilde{\mathbf{f}}^J$, where $W \in SL(N,\mathbb{Z})$. (W must have integer entries since it relates one set of lattice vectors to another. Its inverse must also be an integer matrix since either set must be able to serve as a basis. However, since $\det(W) = 1/\det(W^{-1})$, W and W^{-1} can both be integer matrices only if $\det(W) = \pm 1$.) This lattice change of basis can be interpreted as the field redefinitions, $\tilde{a}^I_\mu = W^I{}_J a^J_\mu$ and $\tilde{j}^\mu_I W^I{}_J = j^\mu_J$, in terms of which the Lagrangian (1) becomes

$$\mathcal{L} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}\tilde{K}_{IJ}\tilde{a}^I_\mu\partial_\nu \tilde{a}^J_\rho + \tilde{j}^\mu_I \tilde{a}^I_\mu, \tag{3}$$

where $K = W^T \tilde{K} W$. Therefore two theories are physically identical if their K matrices are related by such a similarity transformation.

We note that the low energy phases described here may be further subdivided according to their coupling to the electromagnetic field, which is determined by the N-component vector t_I:

$$\mathcal{L} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho} K_{IJ} a^I_\mu \partial_\nu a^J_\rho + j^\mu_I a^I_\mu - \frac{1}{2\pi}\epsilon^{\mu\nu\rho} t_I A_\mu \partial_\nu a^I_\rho. \quad (4)$$

It is possible for two theories with the same K matrix to correspond to different phases if they have different t_I vectors since they may have different Hall conductances $\sigma_{xy} = \frac{e^2}{h}(K^{-1})^{IJ} t_I t_J$. (It is also possible for discrete global symmetries, such as time reversal, to act differently on theories with the same K matrix in which case they can lead to different SPT phases if that symmetry is present.)

In this paper, we will be interested in states of matter in which all excitations have bosonic braiding properties, i.e., in which any exchange of identical particles or full braid of distinguishable particles leads to a phase that is a multiple of 2π. Hence, we are interested in lattices for which $\mathbf{f}^I \cdot \mathbf{f}^J$ is an integer for all I,J and is an even integer if $I = J$. Hence, K^{-1} is a symmetric integer matrix with even entries on the diagonal. By definition, K must also an integer matrix. Since both K and K^{-1} are integer matrices, their determinant must be ± 1. Because $\mathbf{f}^I \cdot \mathbf{f}^I \in 2\mathbb{Z}$ (no summation on I) and $\det(\mathbf{f}^I \cdot \mathbf{f}^J) = \pm 1$, the lattice Γ is said to be an even unimodular lattice.

It is convenient to introduce the (dual) vectors $e^a_I = K_{IJ}\eta^{ab} f^J_b$. If, as above, we view the e^a_I as the components of a vector $\mathbf{e}_I \in \mathbb{R}^{N_+,N_-}$ according to $\mathbf{e}_I \equiv e^a_I \hat{\mathbf{x}}_a$, then $K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$. Moreover, $\mathbf{e}_I$ is the basis of the dual lattice Γ^* defined by $\mathbf{f}^I \cdot \mathbf{e}_J = \delta^I{}_J$. Since the lattice Γ is unimodular, it is equal to Γ^*, up to an $SO(N_+,N_-)$ rotation, from which we see that K must be equivalent to K^{-1}, up to an $SL(N,\mathbb{Z})$ change of basis. (In fact, the required change of basis is provided by the defining relation $e^a_I = K_{IJ}\eta^{ab} f^J_b$.)

Now consider the Lagrangian (5) on the spatial torus. For convenience, we assume there are no sources so $\mathbf{j}^\mu = 0$. We can rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho}\mathbf{e}_I \cdot \mathbf{e}_J a^I_\mu \partial_\nu a^J_\rho + j^\mu_I \mathbf{f}^I \cdot \mathbf{e}_J a^J_\mu \quad (5)$$

$$= \frac{1}{4\pi}\epsilon^{\mu\nu\rho}\mathbf{a}_\mu \cdot \partial_\nu \mathbf{a}_\rho + \mathbf{j}^\mu \cdot \mathbf{a}_\mu, \quad (6)$$

where we have defined $\mathbf{a}_\mu \equiv \mathbf{e}_I a^I_\mu$ and $\mathbf{j}^\mu \equiv \mathbf{f}^I j^\mu_I$. Choosing the gauge $\mathbf{a}_0 = 0$, $\partial_i \mathbf{a}_i = 0$, the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2\pi}\mathbf{a}_1 \cdot \partial_t \mathbf{a}_2. \quad (7)$$

Therefore $\mathbf{a}_1$ and $\mathbf{a}_2$ are canonically conjugate. Although we have gauge-fixed the theory for small gauge transformations, under a large gauge transformation, $a^I_k \to a^I_k + n^I_{(k)}$, where $n^I_{(k)}$ are integers (so that physical observables such as the Wilson loop $e^{i\oint_{C_k} a^I_k}$ about the 1-cycle C_k remains invariant). Therefore we must identify $\mathbf{a}_j$ and $\mathbf{a}_j + n^I_{(k)}\mathbf{e}_I$ since they are related by a gauge transformation.

Suppose that we write a ground state wave function in the form $\Psi[\mathbf{a}_1]$. Then $\mathbf{a}_1$ will act by multiplication and its canonical conjugate $\mathbf{a}_2$ will act by differentiation. To display the full gauge invariance of the wave function, $\Psi[\mathbf{a}_1] = \Psi[\mathbf{a}_1 + n^I \mathbf{e}_I]$, it is instructive to expand it in the form

$$\Psi[\mathbf{a}_1] = \mathcal{N}\sum_{m_I}\Psi_{m_I} e^{2\pi i m_I \mathbf{f}^I \cdot \mathbf{a}_1}, \quad (8)$$

where $m_I \in \mathbb{Z}$. This is an expansion in eigenstates of $\mathbf{a}_2$, with the m_I term having the eigenvalue $2\pi i m_I \mathbf{f}^I$. However, by gauge invariance, $\mathbf{a}_1$ takes values in $\mathbb{R}^N/\Gamma^*$. Therefore we should restrict m_I such that $m_I \mathbf{f}^I$ lies inside the unit cell of Γ^*. In other words, the number of ground states on the torus is equal to the number of sites of Γ that lie inside the unit cell of Γ^*. This is simply the ratio of the volumes of the unit cells, $|\det(K)|^{1/2}/|\det(K)|^{-1/2} = |\det(K)|$. It may be shown that this result generalizes to a ground state degeneracy $|\det K|^g$ on a genus g surface.[29] Therefore the theories on which we focus in this paper have nondegenerate ground states on an arbitrary surface, which is another manifestation of the trivial braiding properties of its excitations.

One further manifestation of the trivial braiding properties of such a phase's excitations is the bipartite entanglement entropy of the ground state.[6,7] If a system with action (1) with $j^\mu_I = 0$ is divided into two subsystems A and B and the reduced density matrix ρ_A for subsystem A is formed by tracing out the degrees of freedom of subsystem B, then the von Neumann entropy $S_A = -\mathrm{tr}[\rho_A \ln(\rho_A)]$ takes the form

$$S_A = \alpha L - \ln\sqrt{|\det(K)|} + \cdots. \quad (9)$$

Here, α is a nonuniversal constant that vanishes for the action (1) but is nonzero if we include irrelevant subleading terms in the action (e.g., Maxwell terms for the gauge fields). L is the length of the boundary between regions A and B. The . . . denote terms with subleading L dependence. For the theories that we will consider in this paper, the second term, which is universal, vanishes. For this reason, such phases are called "short-range entangled."

The discussion around Eq. (8), though essentially correct as far as the ground state degeneracy is concerned, swept some subtleties under the rug. A more careful treatment[30] uses holomorphic coordinates $a = \mathbf{a}_1 + iK \cdot \mathbf{a}_2$, in terms of which the wave functions are ϑ functions. Moreover, the normalization $\mathcal{N}$ must account for the fact that the wave function Ψ is a function only on the space of $\mathbf{a}_i$ with vanishing field strength (which the $\mathbf{a}_0 = 0$ gauge constraint requires), not on arbitrary $\mathbf{a}_i$. Consequently, it depends on the modular parameter of the torus as $\mathcal{N} = (\eta(\tau))^{-N_+}(\eta(\overline{\tau}))^{-N_-}$ where $N_\pm$ are the number of positive and negative eigenvalues of K_{IJ}; the torus is defined by the parallelogram in the complex plane with corners at 0, 1, τ, $\tau + 1$ and opposite sides identified; and is $\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^\infty(1-q^n)$ is the Dedekind η function, where $q = e^{2\pi i\tau}$. Consequently, the ground-state wave function transforms nontrivially under the mapping class group of the torus (i.e., under diffeomorphisms of the torus that are disconnected from the identity, modulo those that can be deformed to the identity), which is equal to the modular group $SL(2,\mathbb{Z})$ generated by $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$. Under T, which cuts open the torus along its longitude, twists one end of the resulting cylinder by 2π, and then rejoins the two ends of the cylinder to reform the torus, thereby enacting $\tau \to \tau + 1$, the ground state transforms according to $\Psi \to e^{-2\pi i(N_+ - N_-)/24}\,\Psi$. Therefore, so long as $N_+ - N_- \not\equiv 0 \pmod{24}$, the bulk is not really trivial.

B. Edge excitations

The nontrivial nature of these states is reflected in more dramatic fashion on surfaces with a boundary, where there may be gapless edge excitations. For simplicity, consider the disk D with no sources in its interior.[31,32] The action (1) is invariant under gauge transformations $a^I_\mu \to a^I_\mu + -i(g^I)^{-1}\partial_\mu g^I$, where $g^I \in [U(1)]^N$, so long as $g^I = 1$ at the boundary ∂D. In order to fully specify the theory on a disk, we must fix the boundary conditions. Under a variation of the gauge fields δa^J_μ, the variation of the action $S = \int_{\mathbb{R}\times D} L$ (here, $\mathbb{R}$ is the time direction) is

$$\delta S = \frac{1}{2\pi}\int_{\mathbb{R}\times D} \delta a^I_\mu\, K_{IJ}\epsilon^{\mu\nu\rho}\partial_\nu a^J_\rho + \frac{1}{4\pi}\int_{\mathbb{R}\times\partial D} \epsilon^{\mu\nu r} K_{IJ} a^I_\mu \delta a^J_\nu. \quad (10)$$

Here, r is the radial coordinate on the disk. The action will be extremized by $K_{IJ}\epsilon^{\mu\nu\rho}\partial_\nu a^J_\rho = 0$ (i.e., there will not be extra boundary terms in the equations of motion) so long as we take boundary conditions such that $\epsilon^{\mu\nu r}K_{IJ}a^I_\mu\delta a^J_\nu = 0$. We can take boundary condition $K_{IJ}a^I_0 + V_{IJ}a^I_x = 0$, where x is the azimuthal coordinate. Here, V_{IJ} is a symmetric matrix that is determined by nonuniversal properties of the edge such as how sharp it is. The Lagrangian (1) is invariant under all transformations $a^J_\mu(x) \to a^J_\mu(x) - i(g^J)^{-1}(x)\partial_\mu g^J(x)$ that are consistent with this boundary condition. Only those with $g^J = 1$ at the boundary are gauge symmetries. The rest are ordinary symmetries of the theory. Therefore, although all bulk degrees of freedom on the disk are fixed by gauge invariance and the Chern-Simons constraint, there are local degrees of freedom at the boundary.

The Chern-Simons constraint $K_{IJ}\epsilon_{ij}\partial_i a^J_j = 0$ can be solved by taking $a^I_i = (U^I)^{-1}\partial_i U^I$ or, writing $U^I = e^{i\phi}$, $a^I_i = \partial_i\phi$, where $\phi \equiv \phi + 2\pi$. This gauge field is pure gauge everywhere in the interior of the disk (i.e., we can locally set it to zero in the interior with a gauge transformation), but it is nontrivial on the boundary because we can only make gauge transformations that are consistent with the boundary condition. Substituting this expression into the action (1), we see that the action is a total derivative, which can be integrated to give a purely boundary action:

$$S = \frac{1}{4\pi}\int dt\,dx\,(K_{IJ}\,\partial_t\phi^I\,\partial_x\phi^J - V_{IJ}\,\partial_x\phi^I\,\partial_x\phi^J). \quad (11)$$

The Hamiltonian associated with this action will be positive semidefinite if and only if V_{IJ} has nonnegative eigenvalues. If we define $\mathbf{X} \equiv \mathbf{e}_J\phi^J$ or, in components, $X^a \equiv e^a_J\phi^J$, then we can rewrite this in the form

$$S = \frac{1}{4\pi}\int dt\,dx\,(\eta_{ab}\partial_t X^a\,\partial_x X^b - v_{ab}\,\partial_x X^a\,\partial_x X^b), \quad (12)$$

where $v_{ab} \equiv V_{IJ}f^I_a f^J_b$. We see that the velocity matrix v_{ab} parameterizes density-density interactions between the edge modes. Note that the fields X^a satisfy the periodicity conditions $X^a \equiv X^a + 2\pi e^a_I n^I$ for $n^I \in \mathbb{Z}$.

This theory has N different dimension-1 fields $\partial_x\phi^I$. The theory also has "vertex operators," or exponentials of these fields that must be consistent with their periodicity conditions: $e^{im_I\phi^I}$ or, equivalently, $e^{im_I\mathbf{f}^I\cdot\mathbf{X}}$ or, simply, $e^{i\mathbf{u}\cdot\mathbf{X}} = e^{i\eta_{ab}u^a X^b}$ for $\mathbf{u} \in \Gamma$. They have correlation functions

$$\langle e^{i\mathbf{u}\cdot\mathbf{X}}e^{-i\mathbf{u}\cdot\mathbf{X}}\rangle = \prod_{b=1}^{N_+}\frac{1}{(x - v_b t)^{y_b}}\prod_{b=N_++1}^{N}\frac{1}{(x + v_b t)^{y_b}}. \quad (13)$$

In this equation, $y_b \equiv \sum_{a,c,d,e} u_a S_{ab}\eta_{bc}(S^T)_{cd}\eta_{de}u_e$, where S_{ab} is an SO(N) matrix that diagonalizes $\eta_{ab}v_{bc}$. Its first N_+ columns are the normalized eigenvectors corresponding to positive eigenvalues of $\eta_{ab}v_{bc}$ and the next N_- columns are the normalized eigenvectors corresponding to negative eigenvalues of $\eta_{ab}v_{bc}$. The velocities v_b are the absolute values of the eigenvalues of $\eta_{ab}v_{bc}$. Therefore this operator has scaling dimension

$$\Delta_{\mathbf{u}} = \frac{1}{2}\sum_{b=1}^{N} y_b. \quad (14)$$

The scaling dimensions of an operator in a nonchiral theory generally depend upon the velocity matrix v_{ab}. For a fully chiral edge, however, $\eta_{ab} = \delta_{ab}$, so $\Delta_{\mathbf{u}} = \frac{1}{2}|\mathbf{u}|^2$.

If the velocities all have the same absolute value, $|v_a| = v$ for all a, then the theory is a conformal field theory with right and left Virasoro central charges $c = N_+$ and $\bar{c} = N_-$. Consequently, we can separately rescale the right- and left-moving coordinates: $(x - vt) \to \lambda(x - vt)$ and $(x + vt) \to \lambda'(x + vt)$. The field $\partial_x X^a$ has right- and left-scaling dimension (1,0) for $a = 1,2,\ldots,N_+$ and dimension (0,1) for $a = N_+ + 1,\ldots,N$. Meanwhile, $e^{i\mathbf{u}\cdot\mathbf{X}}$ has scaling dimension

$$\left(\Delta^R_{\mathbf{u}},\Delta^L_{\mathbf{u}}\right) = \left(\frac{1}{2}\sum_{b=1}^{N_+} y_b, \frac{1}{2}\sum_{b=N_++1}^{N} y_b\right), \quad (15)$$

which simplifies for the case of a fully chiral edge to $(\Delta^R_{\mathbf{u}},\Delta^L_{\mathbf{u}}) = (\frac{1}{2}\mathbf{u}\cdot\mathbf{u},0)$.

In a slight abuse of terminology, we will call the state of matter described by Eq. (1) in the bulk and Eq. (11) on the edge a $c = N_+$, $\bar{c} = N_-$ bosonic SRE phase. In the case of fully chiral theories that have $\bar{c} = 0$, we will sometimes simply call them $c = N$ bosonic SRE phases. Strictly speaking, the gapless edge excitations are only described by a conformal field theory when the velocities are all equal. However, we will continue to use this terminology even when the velocities are not equal, and we will use it to refer to both the bulk and edge theories.

In the case of a $c > 0, \bar{c} = 0$ bosonic SRE phase, all possible perturbations of the edge effective field theory Eq. (11)—or, equivalently, Eq. (12)—are chiral. Since such perturbations cannot open a gap, completely chiral edges are stable. A nonchiral edge may have a vertex operator $e^{i\mathbf{u}\cdot\mathbf{X}}$ with equal right- and left-scaling dimensions. If its total scaling dimension is less than 2, it will be relevant and can open a gap at weak coupling. More generally, we expect that a bosonic SRE will have stable gapless edge excitations if $c - \bar{c} > 0$. Some of the degrees of freedom of the theory (11) will be gapped out, but some will remain gapless in the infrared (IR) limit and the remaining degrees of freedom will be fully chiral with $c_{IR} = c - \bar{c}$ and $\bar{c}_{IR} = 0$. Therefore, even if such a phase is not, initially, fully chiral, the degrees of freedom that remain stable to arbitrary perturbations is fully chiral. Therefore positive-definite even unimodular lattices correspond to $c > 0$, $\bar{c} = 0$ bosonic SRE phases with stable chiral edge excitations, in spite of the absence of anyons in the bulk.

C. The cases $c - \bar{c} = 0,8,16$

Positive-definite even unimodular lattices only exist in dimension $8k$ for integer k,[23] so bosonic SRE phases with stable chiral edge excitations must have $c = 8k$. There is a unique positive-definite even unimodular lattice in dimension 8, up to an overall rotation of the lattice. There are two positive-definite even unimodular lattices in dimension 16; there are 24 in dimension 24; there are more than 10^7 in dimension 32, and even more in higher dimensions. If we relax the condition of positive definiteness, then there are even unimodular lattices in all even dimensions; there is a unique one with signature $(8k + n,n)$ for $n \geqslant 1$.

In dimension 2, the unique even unimodular lattice in $\mathbb{R}^{1,1}$, which we will call U, has basis vectors $\mathbf{e}_1 = \frac{1}{r}(\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2)$, $\mathbf{e}_2 = \frac{r}{2}(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)$, and the corresponding K matrix is

$$K_U = \mathbf{e}_1 \cdot \mathbf{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

This matrix has signature (1,1). (Within this discussion, r is an arbitrary parameter. It will later develop a physical meaning and play an important role in the phase transition we describe.) The even unimodular lattice of signature (n,n) has a block diagonal K matrix with n copies of K_U along the diagonal:

$$K_{U\oplus U\oplus\cdots\oplus U} = \begin{pmatrix} K_U & 0 & 0 & \dots \\ 0 & K_U & 0 & \\ 0 & 0 & K_U & \\ \vdots & & & \ddots \end{pmatrix}. \tag{17}$$

The unique positive-definite even unimodular lattice in dimension 8 is the lattice generated by the roots of the Lie algebra of E_8. We call this lattice Γ_{E_8}. The basis vectors for Γ_{E_8} are given in Appendix A, and the corresponding K matrix takes the form

$$K_{E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{18}$$

The two positive-definite even unimodular lattices in dimension 16 are the lattices generated by the roots of $E_8 \times E_8$ and $\text{spin}(32)/\mathbb{Z}_2$. (The latter means that a basis for the lattice is given by the roots of SO(32), but with the root corresponding to the vector representation replaced by the weight of one of the spinor representations.) We will call these lattices $\Gamma_{E_8} \oplus \Gamma_{E_8}$ and $\Gamma_{\text{Spin}(32)/\mathbb{Z}_2}$. They are discussed further in Appendix A. The corresponding K matrices take the form

$$K_{E_8\times E_8} = \begin{pmatrix} K_{E_8} & 0 \\ 0 & K_{E_8} \end{pmatrix}, \tag{19}$$

[for later convenience, we permute the rows and columns of the second copy of E_8 in Eq. (A5) so that it looks superficially different from the first] and

$$K_{\text{spin}(32)/\mathbb{Z}_2} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4
\end{pmatrix}. \tag{20}$$

The even unimodular lattice with signature $(8+n,n)$ has K matrix

$$K_{E_8\oplus U\oplus\cdots\oplus U} = \begin{pmatrix} K_{E_8} & 0 & 0 & \dots \\ 0 & U & 0 & \\ 0 & 0 & U & \\ \vdots & & & \ddots \end{pmatrix}. \tag{21}$$

The even unimodular lattice with signature $(16+n,n)$ has K matrix

$$K_{E_8\times E_8\oplus U\oplus\cdots\oplus U} = \begin{pmatrix} K_{E_8} & 0 & 0 & \dots \\ 0 & K_{E_8} & 0 & \\ 0 & 0 & U & \\ \vdots & & & \ddots \end{pmatrix}. \tag{22}$$

These lattices are unique, so the matrix

$$K_{\text{spin}(32)/\mathbb{Z}_2\oplus U\oplus\cdots\oplus U} = \begin{pmatrix} K_{\text{spin}(32)/\mathbb{Z}_2} & 0 & \dots \\ 0 & U & \\ \vdots & & \ddots \end{pmatrix} \tag{23}$$

is equivalent to Eq. (22) under an $SL(16+2n,\mathbb{Z})$ basis change. This fact will play an important role in the sections that follow.

III. EQUIVALENCE OF THE TWO $c = 16$ BOSONIC SRE PHASES

In the previous section, we saw that two theories of the form (1) with different $N\times N$ K matrices are equivalent if the two K matrices are related by an $SL(N,\mathbb{Z})$ transformation or, equivalently, if they correspond to the same lattice. But if two K matrices are not related by an $SL(N,\mathbb{Z})$ transformation, is there a more general notion that may relate the theories? A more general notion might be expected if the difference in the number of positive and negative eigenvalues of the two K matrices coincide. Consider, for instance, the case of an $N_1\times N_1$ K matrix and an $N_2\times N_2$ K matrix with $N_1<N_2$. Could there be a relation between them, even though they clearly cannot be related be related by an $SL(N_1,\mathbb{Z})$ or $SL(N_2,\mathbb{Z})$ similarity transformation?

The answer is yes, for the following reason. Consider the theory associated with K_U, defined in Eq. (16). Its partition function is equal to 1 on an arbitrary three-manifold, M_3, as was shown in Ref. 33:

$$Z(M_3) \equiv \int \mathcal{D}a_I e^{i\int \frac{1}{4\pi}\epsilon^{\mu\nu\rho}(K_U)_{IJ}a^I_\mu\partial_\nu a^J_\rho} = 1. \tag{24}$$

One manifestation of the triviality of this theory in the bulk is that it transforms trivially under modular transformations, as we saw earlier. Furthermore, a state with this K matrix can be smoothly connected to a trivial insulator by local unitary transformations if no symmetries are maintained.[2] We shall not do so here, but it is important to note that, if we impose a symmetry on the theory, then we can guarantee the existence of gapless (nonchiral) excitations that live at the edge of the system.[2,5] (We emphasize that we focus, in this section, on the bulk and, in this paper, on properties that do not require symmetry.)

Therefore we can simply replace it with a theory with no degrees of freedom. We will denote such a theory by $K=\emptyset$ to emphasize that it is a 0×0 K matrix in a theory with 0 fields and *not* a theory with a 1×1 K matrix that vanishes. Similarly, the partition function for a theory with arbitrary K matrix K_A on any three-manifold M_3 is equal to the partition function of $K_{A\oplus U}$:

$$\begin{aligned} &\int \mathcal{D}a_I e^{\frac{i}{4\pi}\int \epsilon^{\mu\nu\rho}(K_A)_{IJ}a^I_\mu\partial_\nu a^J_\rho} \\ &\quad = \int \mathcal{D}a_I\,\mathcal{D}a'_I \left[e^{\frac{i}{4\pi}\int \epsilon^{\mu\nu\rho}(K_A)_{IJ}a^I_\mu\partial_\nu a^J_\rho} e^{\frac{i}{4\pi}\int \epsilon^{\mu\nu\rho}(K_U)_{IJ}a'^I_\mu\partial_\nu a'^J_\rho}\right] \\ &\quad = \int \mathcal{D}a_I e^{\frac{i}{4\pi}\int \epsilon^{\mu\nu\rho}(K_{A\oplus U})_{IJ}a^I_\mu\partial_\nu a^J_\rho}. \end{aligned} \tag{25}$$

Therefore all of the theories corresponding to even, unimodular lattices of signature (n,n) are, in fact, equivalent when there is no symmetry preserved. There is just a single completely trivial gapped phase. We may choose to describe it by a very large K matrix (which is seemingly perverse) but it is still the same phase. Moreover, any phase associated with a K matrix can equally well be described by a larger K matrix to which we have added copies of K_U along the block diagonal. This is an expression of the physical idea that no phase transition will be encountered in going from a given state to one in which additional trivial, gapped degrees of freedom have been added. Of course, in this particular case, we have added zero local degrees of freedom to the bulk and we have not enlarged the Hilbert space at all. So it is an even more innocuous operation. However, when we turn to the structure of edge excitations, there will be more heft to this idea.

At a more mathematical level, the equivalence of these theories is related to the notion of "stable equivalence," according to which two objects are the same if they become isomorphic after augmentation by a "trivial" object. In physics, stable equivalence has been used in the K-theoretic classification of (noninteracting) topological insulators.[34] In the present context, we will be comparing gapped phases and the trivial object that may be added to either phase is a topologically trivial band insulator. Heuristically, stable equivalence says that we may add some number of topologically trivial bands to our system in order to effectively enlarge the parameter space and, thereby, allow a continuous interpolation between two otherwise different states.

We now turn to the two $c=16$ bosonic SRE phases. Their bulk effective field theories are of the form of Eq. (1) with K matrices given by $K_{E_8\times E_8}$ and $K_{\text{spin}(32)/\mathbb{Z}_2}$. Their bulk properties are seemingly trivial, but not entirely so since, as we noted in Sec. II, they transform nontrivially under modular transformations.

These two nontrivial theories are, at first glance, distinct. They are associated with different lattices. For instance, $\Gamma_{E_8}\oplus\Gamma_{E_8}$ is the direct sum of two eight-dimensional lattices while $\Gamma_{\text{spin}(32)/\mathbb{Z}_2}$ is not. The two K matrices are not related by an $SL(16,\mathbb{Z})$ transformation.

Suppose, however, that we consider the K matrices $K_{E_8\times E_8}\oplus U$ and $K_{\text{spin}(32)/\mathbb{Z}_2}\oplus U$, which describe "enlarged" systems. (We use quotation marks because, although we now have theories with 18 rather than 16 gauge fields, the physical Hilbert space has not been enlarged.) These K matrices are, in fact, related by an $SL(18,\mathbb{Z})$ transformation:

$$W_G^T\, K_{\text{spin}(32)/\mathbb{Z}_2\oplus U}\, W_G = K_{E_8\times E_8\oplus U}, \tag{26}$$

where W_G is given by

$$W_G = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -4 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -5 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -6 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -7 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -8 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\ -9 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1\\ -10 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2\\ -11 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3\\ -12 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & -4\\ -13 & 0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 5 & -5\\ -14 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 & -6\\ -7 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & -3\\ -8 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & -4\\ -2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 2 \end{pmatrix}. \tag{27}$$

We will explain how W_G is derived in Sec. V. Here, we focus on its implication: these two theories are equivalent on an arbitrary closed manifold. There is a unique bulk $c = 16$ bosonic SRE phase of matter. However, there appear to be two possible distinct effective field theories for the edge of this unique bulk phase, namely the theories (11) with $K_{E_8\times E_8}$ and $K_{\text{spin}(32)/\mathbb{Z}_2}$. In the next section, we explain the relation between these edge theories.

IV. FERMIONIC REPRESENTATIONS OF THE TWO $c = 16$ SRE BOSONIC PHASES

In Sec. III, we saw that there is a unique bulk $c = 16$ bosonic SRE phase of matter. We now turn our attention to the two corresponding edge effective field theories, namely Eq. (11) with K_{IJ} given by either $K_{E_8\times E_8}$ or $K_{\text{spin}(32)/\mathbb{Z}_2}$. These two edge theories are distinct, although the difference is subtle. To understand this difference, it is useful to consider fermionic representations[35,36] of these edge theories.

Consider 32 free chiral Majorana fermions:

$$S = \int dx d\tau\, \psi_j\,(-\partial_\tau + v_a i\partial_x)\,\psi_j, \tag{28}$$

where $j = 1,\ldots,32$. If the velocities v_a are all the same, then this theory naively has SO(32) symmetry, up to a choice of boundary conditions. We could imagine such a $1+1$-dimensional theory as the edge of a 32-layer system of electrons, with each layer in a spin-polarized $p+ip$ superconducting state. We will assume that the order parameters in the different layers are coupled by interlayer Josephson tunneling so that the superconducting order parameters are locked together. Consequently, if a flux $hc/2e$ vortex passes through one of the layers, it must pass through all 32 layers. Then all 32 Majorana fermion edge modes have the same boundary conditions. When two vortices in a single-layer spin-polarized $p+ip$ superconducting state are exchanged, the resulting phase is $e^{-i\pi/8}$ or $e^{3i\pi/8}$, depending on the fusion channel of the vortices (i.e., the fermion parity of the combined state of their zero modes). Therefore a vortex passing through all 32 layers (which may be viewed as a composite of 32 vortices, one in each layer) is a boson. These bosons carry 32 zero modes, so there are actually 2^{16} states of such vortices—2^{15} if we require such a vortex to have even fermion parity. (Of course, the above construction only required 16 layers if our goal was to construct the minimal dimension SRE chiral phase of bosons.)[4]

Now suppose that such vortices condense. (Without loss of generality, we suppose that the vortices are in some particular internal state with even fermion parity.) Superconductivity is destroyed and the system enters an insulating phase. Although individual fermions are confined since they acquire a minus sign in going around a vortex, a pair of fermions, one in layer i and one in layer j, is an allowed excitation. The dimension-1 operators in the edge theory are of the form $i\psi_i\psi_j$ where $1 \leqslant i < j \leqslant 32$. There are $\frac{1}{2}\times 32\times 31 = 496$ such operators. We may choose $i\psi_{2a-1}\psi_{2a}$, with $a = 1,2,\ldots,16$ as a maximal commuting subset, i.e., as the Cartan subalgebra of SO(32). The remaining 480 correspond to the vectors of (length)$^2 = 2$ in the lattice Γ_{16}. To see this, it is useful to bosonize the theory (28). We define the Dirac fermions $\Psi_I \equiv \psi_{2a-1} + i\psi_{2a}$, with $a = 1,2,\ldots,16$ and represent them with bosons: $\Psi_I = e^{iX_a}$. Then the Cartan subalgebra consists of the 16 dimension-1 operators ∂X_a. The operators $e^{i\mathbf{v}\cdot\mathbf{X}}$ with $\mathbf{v} \in \Gamma_{\mathrm{SO}(32)} \subset \Gamma_{\mathrm{spin}(32)/\mathbb{Z}_2}$ and $|\mathbf{v}|^2 = 2$ correspond to the vectors of (length)$^2 = 2$ in the SO(32) root lattice: $\pm\hat{\mathbf{x}}_a \pm \hat{\mathbf{x}}_b$ with $1 \leqslant a < b \leqslant 16$. In the fermionic language, we see that the relevant perturbations of $i\psi_i\psi_k$ can be gauged away with a spatially dependent SO(32) rotation and, therefore, do not affect the basic physics of the state.

To complete the description of the spin(32)/$\mathbb{Z}_2$ theory, recall that a vortex in a single layer braids nontrivially with the composite vortex that condenses. Such single vortices are confined after condensation of the composite. Therefore it is impossible to change the boundary conditions of just one of the fermions ψ_i by inserting a single vortex into the bulk; all of the fermions must have the same boundary conditions. The fermion boundary conditions can be changed from anti-periodic to periodic by the operator $e^{i\mu_s\cdot\mathbf{X}} = \exp[i(X_1 + X_2 + \cdots + X_{16})/2]$, where μ_s is the weight of one of the spinor representations of SO(32). This is a dimension-2 operator.

Note that the group spin(32) is a double-cover of SO(32) that has spinor representations. By disallowing one of the spinor representations and the vector representation (i.e., the odd fermion parity sector), the theory is associated with spin(32)/$\mathbb{Z}_2$ but the $\mathbb{Z}_2$ that is moded out is not the $\mathbb{Z}_2$ that leads back to SO(32). Thus it is the inclusion of μ_s along with the vectors $\mathbf{v}$ of SO(32) mentioned above that is essential to the description of the fermionic representation of the spin(32)/$\mathbb{Z}_2$ theory. If we had chosen not to include μ_s, i.e., if we had not condensed the composite vortex, the resulting theory would have had topological order with a torus ground state degeneracy equal to four. [The SO(32) root lattice has unit cell volume equal to four while the unit cell volume of the spin(32)/$\mathbb{Z}_2$ lattice is unity.]

Now suppose that the first 16 layers are coupled by interlayer Josephson tunneling so that their order parameters are locked and the remaining 16 layers are coupled similarly, but the first 16 layers are not coupled to the remaining 16. Then there are independent vortices in the first 16 layers and in the remaining 16 layers. Suppose that both types of vortices condense. Each of these 16-vortex composites is a boson, and superconductivity is again destroyed. Individual fermions are again confined and, moreover, the fermion parity in each half of the system must be even. Therefore the allowed dimension-1 operators in the theory are $i\psi_i\psi_j$ with $1 \leqslant i < j \leqslant 16$ or $17 \leqslant i < j \leqslant 32$. There are $2 \times \frac{1}{2} \times 16 \times 15 = 240$ such dimension-1 operators. As above, 16 of them correspond to the Cartan subalgebra. The other 224 correspond to lattice vectors $e^{i\mathbf{v}\cdot\mathbf{X}}$ with $\mathbf{v} = \pm\hat{\mathbf{x}}_a \pm \hat{\mathbf{x}}_b$ and $1 \leqslant a < b \leqslant 8$ or $9 \leqslant a < b \leqslant 16$. Unlike in the case of spin(32)/$\mathbb{Z}_2$, the boundary-condition changing operators $\exp[i(\pm X_1 \pm X_2 \pm \cdots \pm X_8)/2]$ and $\exp[i(\pm X_9 \pm X_{10} \pm \cdots \pm X_{16})/2]$ are dimension-1 operators. There are $2 \times 2^7 = 256$ such operators with even fermion parity in each half of the system (i.e., an even number of $+$ signs in the exponential). The corresponding vectors $\mathbf{v} = (\pm\hat{\mathbf{x}}_1 \pm \hat{\mathbf{x}}_2 \pm \cdots \pm \hat{\mathbf{x}}_8)/2$ and $\mathbf{v} = (\pm\hat{\mathbf{x}}_9 \pm \hat{\mathbf{x}}_{10} \pm \cdots \pm \hat{\mathbf{x}}_{16})/2$ with an even number of $+$ signs together with $\mathbf{v} = \pm\hat{\mathbf{x}}_a \pm \hat{\mathbf{x}}_b$ are the 480 different (length)$^2 = 2$ vectors in the $E_8 \times E_8$ root lattice. Consequently, this is the fermionic representation of the $E_8 \times E_8$ theory.

It is unclear, from this fermionic description, how to adiabatically connect the two bulk theories. The most obvious route between them, starting from the $E_8 \times E_8$ theory, is to restore superconductivity, couple the order parameters of the two sets of 16 layers, and then condense 32-layer vortices to destroy superconductivity again. This route takes the system across three phase transitions while the analysis in the previous section showed that they are, in fact, the same phase and, therefore, it should be possible to go from one to the other without crossing any bulk phase boundaries.

As we saw above, there are 480 vectors $\mathbf{u}$ with $|\mathbf{u}|^2 = 2$ in both $\Gamma_{E_8\times E_8}$ and $\Gamma_{\mathrm{spin}(32)/\mathbb{Z}_2}$. In fact, a result of Milnor[37] (related to hearing the shape of a drum) states that the two lattices have the same number of vectors of *all lengths*: for every $\mathbf{u} \in \Gamma_{E_8\times E_8}$, there is a unique partner $\mathbf{v} \in \Gamma_{\mathrm{spin}(32)/\mathbb{Z}_2}$ such that $|\mathbf{v}|^2 = |\mathbf{u}|^2$. (See Ref. 36 for an elegant presentation of this fact following Ref. 23.) Therefore the $E_8 \times E_8$ and spin(32)/$\mathbb{Z}_2$ edge theories have identical spectra of operator scaling dimensions $\Delta_{\mathbf{u}} = \frac{1}{2}|\mathbf{u}|^2$. Thus it is impossible to distinguish these two edge theories by measuring the possible exponents associated with two-point functions. However, in the fermionic realization described above, consider one of the 496 dimension-1 operators, which we will call J_i, $i = 1,2,\ldots,496$. They are given by ∂X_a and $e^{i\mathbf{u}\cdot\mathbf{X}}$ with $|\mathbf{u}|^2 = 2$ for $\mathbf{u} \in \Gamma_{E_8\times E_8}$ or $\Gamma_{\mathrm{spin}(32)/\mathbb{Z}_2}$. In the limit that all of the velocities are equal, these are conserved currents corresponding to the 496 generators of either $E_8 \times E_8$ or spin(32)/$\mathbb{Z}_2$, but we will use the notation J_i even when the velocities are not equal. It is clear that, in the spin(32)/$\mathbb{Z}_2$ phase, there are J_is that involve both halves of the system, but not in the $E_8 \times E_8$ phase. In other words, in the spin(32)/$\mathbb{Z}_2$ phase, there are two-point functions involving both halves of the system that decay as $\langle J_i(x,0)J_i(0,0)\rangle \propto 1/x^2$. In the $E_8 \times E_8$ phase, such operators J_i only exist acting entirely within the top half or the bottom half of the system.

Moreover, the n-point functions for $n \geqslant 3$ of the two theories are different. Consider, for the sake of concreteness, the following 3-point function:

$$\langle J_a(x_1,t_1)J_b(x_2,t_2)J_c(x_3,t_3)\rangle, \tag{29}$$

where J_a, J_b, J_c are dimension-1 operators. In the spin(32)/$\mathbb{Z}_2$ phase, there are 4960 such nonvanishing 3-point functions (up to permutations of the labels a,b,c) since this correlation function will only be nonvanishing if $J_a = i\psi_j\psi_k$, $J_b = i\psi_k\psi_l$, $J_b = i\psi_l\psi_j$, with $1 \leqslant j < k < l \leqslant 32$. (Here, we are using the fermionic representation for simplicity, but the same conclusion can be reached using the bosonic representation.) In the $E_8 \times E_8$ phase, on the other hand, there are 8288 such nonvanishing 3-point functions. There are 1120 correlation functions (up to permutations of the labels a,b,c) with $J_a = i\psi_j\psi_k$, $J_b = i\psi_k\psi_l$, $J_b = i\psi_l\psi_j$, with $1 \leqslant j < k < l \leqslant 16$ or $17 \leqslant j < k < l \leqslant 32$. There are also 7168 3-point functions involving the twist fields, such as the 3-point function with $J_a = \exp[i(X_1 + X_2 + X_3 + \cdots + X_8)/2]$, $J_b = \exp[i(X_1 + X_2 - X_3 - \cdots - X_8)/2]$, and $J_c = \exp[i(X_1 + X_2)]$.

V. PHASE DIAGRAM OF THE $c - \bar{c} = 16$ EDGE

Since there is a unique bulk $c = 16$ bosonic SRE phase of matter, the two different edge theories corresponding to $K_{E_8\times E_8}$ or $K_{\text{spin}(32)/\mathbb{Z}_2}$ must be different edge phases that can occur at the boundary of the same bulk phase. For this scenario to hold, it must be the case that the transition between these two edge theories is purely an edge transition—or, in other words, an "edge reconstruction"—that can occur without affecting the bulk. Such a transition can occur as follows. The gapless modes in the effective theory (11) are the lowest energy excitations in the system. However, there will generically be gapped excitations at the edge of the system that we usually ignore. So long as they remain gapped, this is safe. However, these excitations could move downward in energy and begin to mix with the gapless excitations, eventually driving a phase transition. Such gapped excitations must be nonchiral and can only support bosonic excitations.

A perturbed nonchiral Luttinger liquid is the simplest example of such a gapped mode:

$$S_{\text{LL}} = \frac{1}{4\pi}\int dt\,dx\left[2\partial_t\varphi\,\partial_x\theta - \frac{v}{g}(\partial_x\theta)^2 - vg(\partial_x\varphi)^2 + u_1^{(m)}\cos(m\theta) + u_2^{(n)}\cos(n\varphi)\right], \tag{30}$$

with Luttinger parameter g and integers m,n. The φ and θ fields have period 2π. The first line is the action for a gapless Luttinger liquid. The second line contains perturbations that can open a gap in the Luttinger liquid spectrum. The couplings $u_1^{(m)}$ and $u_2^{(n)}$ have scaling dimensions $2 - \frac{m^2}{2}g$ and $2 - 2n^2g^{-1}$, respectively. Let us concentrate on the lowest harmonics which are the most relevant operators with couplings $u_1^{(1)} \equiv u_1$ and $u_2^{(1)} \equiv u_2$. The first operator is relevant if $g < 4$ and the second one is relevant if $g > 1$. At least one of these is always relevant. Given our parametrization of the Luttinger Lagrangian, a system of hard-core bosons on the lattice with no other interactions or in the continuum with infinite δ-function repulsion has $g = 1$ (see Ref. 38).

When considering one-dimensional bosonic systems, the above cosine perturbations can be forbidden by, respectively, particle-number conservation and translational invariance. Here, however, we do not assume that there is any symmetry present, so these terms are allowed. The Luttinger action can be rewritten in the same way as the edge theory (11):

$$S_{\text{LL}} = \frac{1}{4\pi}\int dt\,dx\,[(K_U)_{IJ}\partial_t\phi^I\,\partial_x\phi^J - V_{IJ}\partial_t\phi^I\,\partial_x\phi^J + u_1\cos(\phi_{17}) + u_2\cos(\phi_{18})], \tag{31}$$

where $I,J = 17,18$ in this equation and $\phi_{17} = \theta$ and $\phi_{18} = \varphi$. Therefore we see that the action for a perturbed Luttinger liquid is the edge theory associated with the trivial bulk theory with K matrix given by K_U that we discussed in Sec. III. It is gapped unless u_1 and u_2 are fine-tuned to zero or forbidden by a symmetry. However, augmenting our system with this trivial one does increase the number of degrees of freedom at the edge and expands the Hilbert space, unlike in the case of the bulk.

Hence, we consider the edge theory

$$S = \frac{1}{4\pi}\int dtdx\left[\left(K_{E_8\times E_8\oplus U}\right)_{IJ}\partial_t\phi^I\,\partial_x\phi^J - V_{IJ}\,\partial_x\phi^I\,\partial_x\phi^J + u_1\cos(\phi_{17}) + u_2\cos(\phi_{18}) + \cdots\right]. \tag{32}$$

We can integrate out the trivial gapped degrees of freedom ϕ^{17} or ϕ^{18}, leaving the gapless chiral edge theory associated with $K_{E_8\times E_8}$. The ... represents other nonchiral terms that could appear in the Lagrangian (i.e., cosines of linear combinations of the fields ϕ^I), they are all irrelevant for $V_{I,17} = V_{I,18} = 0$ for $I = 1,\ldots,16$, or more accurately, they are less relevant than u_1 or u_2 and so we ignore them to first approximation. However, if we vary the couplings V_{IJ}, then u_1, u_2 could both become irrelevant and some other term could become relevant, driving the edge into another phase.

To further analyze the possible transition, it is useful to rewrite the action in terms of the fields $\mathbf{X} = \mathbf{e}_J\phi^J$:

$$S = \frac{1}{4\pi}\int dt\,dx\left\{\eta_{ab}\partial_t X^a\,\partial_x X^b - v_{ab}\,\partial_x X^a\,\partial_x X^b + u_1\cos\left[\frac{r}{2}(X^{17} + X^{18})\right] + u_2\cos\left[\frac{1}{r}(X^{17} - X^{18})\right] + \cdots\right\}, \tag{33}$$

where $v_{ab} \equiv V_{IJ}f_a^I f_b^J$, $f_a^I e_J^a = \mathbf{f}^I\cdot\mathbf{e}_J = \delta^I{}_J$ and $\eta_{ab} = (1^{16},1,-1)$. Here, $\mathbf{e}_J$ for $J = 1,\ldots,16$ is a basis of $\Gamma_{E_8}\oplus\Gamma_{E_8}$ given explicitly in Appendix A and c^n refers to the n-component vector where each component equals c. We take $\mathbf{e}_{17} = (0^{16},\frac{1}{r},\frac{1}{r})$ and $\mathbf{e}_{18} = (0^{16},\frac{r}{2},-\frac{r}{2})$ so that $\mathbf{e}_{17}\cdot\mathbf{e}_{17} = \mathbf{e}_{18}\cdot\mathbf{e}_{18} = 0$ and $\mathbf{e}_{17}\cdot\mathbf{e}_{18} = 1$. When $v_{a,17} = v_{a,18} = 0$ for $a = 1,\ldots,16$ (or, equivalently, when $V_{I,17} = V_{I,18} = 0$ for $I = 1,\ldots,16$), the parameter r is related to the Luttinger parameter according to $g = r^2/2$ and u_1, u_2 have renormalization group (RG) equations:

$$\frac{du_1}{d\ell} = \left(2 - \frac{r^2}{4}\right)u_1, \quad \frac{du_2}{d\ell} = (2 - r^{-2})u_2. \tag{34}$$

Hence one of these two perturbations is always relevant when $v_{a,17} = v_{a,18} = 0$ for $a = 1,\ldots,16$ and, consequently, $X^{17,18}$ become gapped. The arguments of the cosine follow from the field redefinition $\phi^I = \mathbf{f}^I\cdot\mathbf{X} = (K^{-1})^{IJ}\mathbf{e}_J\cdot\mathbf{X}$. The field $\mathbf{X}$ satisfies the periodicity conditions $\mathbf{X} \equiv \mathbf{X} + 2\pi\mathbf{u}$ for $\mathbf{u}\in\Gamma_{E_8}\oplus\Gamma_{E_8}\oplus U$. Again, the ... refers to other possible perturbations, i.e., cosines of other linear combinations of the X^as.

In a nearly identical manner, we can construct a theory for spin$(32)/\mathbb{Z}_2\oplus U$ in which a nonchiral gapped mode is added to the spin$(32)/\mathbb{Z}_2$ edge theory and allowed to interact with it. The only difference is in the parametrization of the U lattice. We choose $\tilde{\mathbf{e}}_{17} = (0^{16},-r,r)$ and $\tilde{\mathbf{e}}^{18} = (0^{16},-\frac{1}{2r},-\frac{1}{2r})$. The action

$$S = \frac{1}{4\pi}\int dt\,dx\left\{\eta_{ab}\partial_t\tilde{X}^a\,\partial_x\tilde{X}^b - \tilde{v}_{ab}\,\partial_x\tilde{X}^a\,\partial_x\tilde{X}^b + \tilde{u}_1\cos\left[\frac{1}{2r}(\tilde{X}^{17} - \tilde{X}^{18})\right] + \tilde{u}_2\cos[r(\tilde{X}^{17} + \tilde{X}^{18})] + \cdots\right\}. \tag{35}$$

Again, the ... refers to cosines of other linear combinations of the $\tilde{X}^a$s. When $\tilde{v}_{17,18} = \tilde{v}_{a,17} = \tilde{v}_{a,18} = 0$ for $a = 1,\ldots,16$, the parameter r is related to the Luttinger parameter according to $g = r^{-2}/2$ and $\tilde{u}_1$, $\tilde{u}_2$ have RG equations:

$$\frac{d\tilde{u}_1}{d\ell} = \left(2 - \frac{1}{4r^2}\right)\tilde{u}_1, \quad \frac{d\tilde{u}_2}{d\ell} = (2 - r^2)\tilde{u}_2. \tag{36}$$

Hence one of these two perturbations is always most relevant when $\tilde{v}_{a,17} = \tilde{v}_{a,18} = 0$ for $a = 1,\ldots,16$ and, consequently, $X^{17,18}$ become gapped. The fields $\tilde{\mathbf{X}}$ satisfy the periodicity conditions $\tilde{\mathbf{X}} \equiv \tilde{\mathbf{X}} + 2\pi\mathbf{v}$ for $\mathbf{v} \in \Gamma_{\mathrm{spin}(32)/\mathbb{Z}_2} \oplus U$.

We now make use of the fact there is a unique signature (17,1) even unimodular lattice. It implies that there is an SO(17,1) rotation O_G that transforms $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ into $\Gamma_{\mathrm{spin}(32)/\mathbb{Z}_2} \oplus U$. Therefore the fields $O_G\mathbf{X}$ satisfy the periodicity condition $O_G\mathbf{X} \equiv O_G\mathbf{X} + 2\pi\mathbf{v}$ for $\mathbf{v} \in \Gamma_{\mathrm{spin}(32)/\mathbb{Z}_2} \oplus U$ or, in components, $(O_G)^a{}_b X^b \equiv (O_G)^a{}_b X^b + 2\pi n^I \tilde{e}^a_I$ for $n^I \in \mathbb{Z}$. Thus we identify $\tilde{X}^a = (O_G)^a{}_b X^b$. The explicit expression for O_G is provided in Appendix A. [As an aside, having identified X^a and $\tilde{X}^b$ through the SO(17,1) transformation O_G, we can now explain how the SL(18,$\mathbb{Z}$) transformation W_G is obtained. The desired transformation is read off from the relation,

$$\tilde{\phi}^J = \tilde{f}^J_a (O_G)^a{}_b e^b_I \phi^I =: (W_G)_{IJ}\phi^I, \tag{37}$$

which follows from equation relating the $\Gamma_{E_8} \oplus \Gamma_{E_8}$ and $\Gamma_{\mathrm{spin}(32)/\mathbb{Z}_2}$ bases,

$$(O_G)^a{}_b e^b_I = \sum_K m^K_I \tilde{e}^a_K, \tag{38}$$

where the m^K_I are a collection of integers. Multiplying both sides of Eq. (38) by $\tilde{f}^J_c$ allows us to read off the elements of W_G.]

Therefore by substituting $\tilde{X}^a = (O_G)^a{}_b X^b$, the action (35) could equally well be written in the form

$$S = \frac{1}{4\pi}\int dt\, dx \Big(\eta_{ab}\partial_t X^a \partial_x X^b - \tilde{v}_{ab}(O_G)^a{}_c (O_G)^b{}_d \times \partial_x X^c\, \partial_x X^d + \tilde{u}_1 \cos\left\{\frac{1}{2r}\left[(O_G)^{17}_a X^a - (O_G)^{18}_a X^a\right]\right\} + \tilde{u}_2 \cos\left\{r\left[(O_G)^{17}_a X^a + (O_G)^{18}_a X^a\right]\right\} + \cdots\Big), \tag{39}$$

where $\mathbf{X} \equiv \mathbf{X} + 2\pi\mathbf{u}$ for $\mathbf{u} \in \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$. [We have used the defining property, $(O_G)^a{}_b \eta_{ac} (O_G)^c{}_d = \eta_{bd}$, in rewriting the first term in the action (35).]

Having rewritten the augmented spin(32)/$\mathbb{Z}_2$ action Eq. (35) in terms of the $\Gamma_{E_8} \oplus \Gamma_{E_8}$ fields, let us add in two of the available mass perturbations u_1, u_2 written explicitly in Eq. (33):

$$\begin{aligned} S = \frac{1}{4\pi}\int dt\, dx \Big(&\eta_{ab}\partial_t X^a\, \partial_x X^b \\ &- \tilde{v}_{ab}(O_G)^a{}_c (O_G)^b{}_d \partial_x X^c\, \partial_x X^d \\ &+ \tilde{u}_1 \cos\left\{\frac{1}{2r}\left[(O_G)^{17}_a X^a - (O_G)^{18}_a X^a\right]\right\} \\ &+ \tilde{u}_2 \cos\left\{r\left[(O_G)^{17}_a X^a + (O_G)^{18}_a X^a\right]\right\} \\ &+ u_1 \cos\left[\frac{r}{2}(X^{17} + X^{18})\right] \\ &+ u_2 \cos\left[\frac{1}{r}(X^{17} - X^{18})\right] + \cdots\Big). \end{aligned} \tag{40}$$

So far we have only rewritten Eq. (35) and included additional mass perturbations implicitly denoted by "...". If $\tilde{v}_{17,18} = \tilde{v}_{a,17} = \tilde{v}_{a,18} = 0$ for $a = 1,\ldots,16$, then either $\tilde{u}_1$ or $\tilde{u}_2$ is the most relevant operator and the $\tilde{X}^{17}$ and $\tilde{X}^{18}$ fields are gapped out. The remaining gapless degrees of freedom are those of the spin(32)/$\mathbb{Z}_2$ edge theory. On the other hand, if $v_{cd} = \tilde{v}_{ab}(O_G)^a{}_c (O_G)^b{}_d$ with $v_{17,18} = v_{a,17} = v_{a,18} = 0$, either u_1 or u_2 is the most relevant operator. At low energies, X^{17} and X^{18} are gapped with the remaining degrees of freedom being those of the $E_8 \times E_8$ theory. We see that the transition between the chiral $E_8 \times E_8$ and spin(32)/$\mathbb{Z}_2$ is mediated by O_G given a starting velocity matrix—this is an interaction driven transition.

Given O_G, we can define a one-parameter family of SO(17,1) transformations as follows. As discussed in Appendix A, O_G can be written in the form $O_G = \eta W(A)\eta W(A')$, where $W(A), W(A')$ are SO(17,1) transformations labeled by the vectors A, A', which are defined in Appendix A as well and η is a reflection. We define $O_G(s) = \eta W(sA)\eta W(sA')$. This family of SO(17,1) transformations, parameterized by $s \in [0,1]$ interpolates between $O_G(0) = I$, the identity, and $O_G(1) = O_G$ or, in components, $(O_G(0))^a{}_b = \delta^a{}_b$, the identity, and $(O_G(1))^a{}_b = (O_G)^a{}_b$. This one-parameter family of transformations defines a one-parameter family of theories:

$$\begin{aligned} S_4(s) = \frac{1}{4\pi}\int dt\, dx \Big(&\eta_{ab}\partial_t X^a\, \partial_x X^b \\ &- v_{ab}[O_G(s)]^a{}_c [O_G(s)]^b{}_d \partial_x X^c \partial_x X^d \\ &+ \tilde{u}_1 \cos\left\{\frac{1}{2r}\left[(O_G)^{17}_a X^a - (O_G)^{18}_a X^a\right]\right\} \\ &+ \tilde{u}_2 \cos\left\{r\left[(O_G)^{17}_a X^a + (O_G)^{18}_a X^a\right]\right\} \\ &+ u_1 \cos\left[\frac{r}{2}(X^{17} + X^{18})\right] \\ &+ u_2 \cos\left[\frac{1}{r}(X^{17} - X^{18})\right] + \cdots\Big). \end{aligned} \tag{41}$$

These theories are parametrized by s, which determines a one-parameter family of velocity matrices $v_{ab}[O_G(s)]^a{}_c [O_G(s)]^b{}_d$ (this is the only place where s enters the action). We call this action $S_4(s)$ because there are four potentially mass-generating cosine perturbations. Note that the $\tilde{u}_{1,2}$ terms have $O_G = O_G(1)$ in the arguments of the cosines, not $O_G(s)$. As our starting point, we take $v_{17,18} = v_{a,17} = v_{a,18} = 0$ for $a = 1,\ldots,16$. (For instance, we can take diagonal v_{ab}.) Then, for $s = 0$, this theory is of the form of Eq. (33) with two extra mass perturbations parameterized by $\tilde{u}_1$ and $\tilde{u}_2$; however, either u_1 or u_2 is the most relevant, and the remaining gapless degrees of freedom are those of the chiral $E_8 \times E_8$ edge theory. For $s = 1$, this theory is of the form of Eq. (40), which we know is equivalent to Eq. (35) with two extra mass perturbations parameterized by u_1 and u_2; now, either $\tilde{u}_1$ or $\tilde{u}_2$ is the most relevant, and the remaining gapless degrees of freedom are those of the spin(32)/$\mathbb{Z}_2$ edge theory. For intermediate values

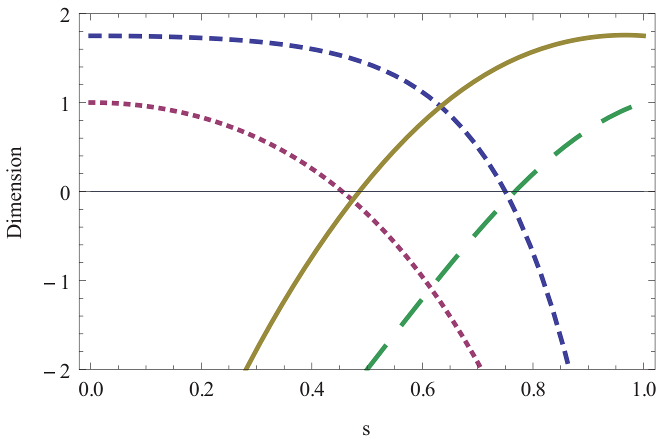

FIG. 1. (Color online) The scaling dimensions of $u_{1,2}$ (densely dashed and dotted) and $\tilde{u}_{1,2}$ (thick and dashed), plotted as a function of s at $r = 1$. The $E_8 \times E_8$ phase lives roughly within $0 \leqslant s < 0.625$ and the spin$(32)/\mathbb{Z}_2$ phase between $0.625 < s \leqslant 1$.

of s, the RG equations for u_1, u_2, $\tilde{u}_1$, $\tilde{u}_2$ are

$$\frac{du_1}{d\ell} = \left\{2 - \frac{[2s^2 + r^2(1 - s^2 + 4s^4)]^2}{4r^2}\right\}u_1,$$
$$\frac{du_2}{d\ell} = \left[2 - \frac{(1 + 2r^2s^2)^2}{r^2}\right]u_2,$$
$$\frac{d\tilde{u}_1}{d\ell} = \left\{2 - \frac{[4 - 7s + 4s^2 + 2r^2(s-1)^2(1 + s + 4s^2)]^2}{4r^2}\right\}\tilde{u}_1,$$
$$\frac{d\tilde{u}_2}{d\ell} = \left\{2 - \frac{[2(s-1)^2 + r^2(1 + s + 3s^2 - 8s^3 + 4s^4)]^2}{r^2}\right\}\tilde{u}_2. \quad (42)$$

The expressions in square brackets on the right-hand sides of these equations, which are equal to $\frac{1}{u_{1,2}}\frac{du_{1,2}}{d\ell}$ and $\frac{1}{\tilde{u}_{1,2}}\frac{d\tilde{u}_{1,2}}{d\ell}$, are the scaling dimensions of $u_{1,2}$ and $\tilde{u}_{1,2}$ near the $u_{1,2} = \tilde{u}_{1,2} = 0$ fixed line.

We plot the weak-coupling RG flows of these operators in Figs. 1–3 for three different choices of r. First, we notice that, depending upon r, either u_1 or u_2 is most relevant at $s = 0$. At $s = 1$, either $\tilde{u}_1$ or $\tilde{u}_2$ is most relevant. At intermediate values of s, there are several possibilities. Assuming that the most relevant operator determines the flow to low energy [which must have the same value $c - \overline{c} = 16$ as the action (41)], we conclude that when either of these two sets of operators is most relevant we expect a mass to be generated for, respectively, the $X^{17,18}$ or $\tilde{X}^{17,18}$ modes, thereby leaving behind either the $E_8 \times E_8$ or spin$(32)/\mathbb{Z}_2$ edge theories at low energies. If there are no relevant operators, then the edge is not fully chiral, it has $c = 17$, $\overline{c} = 1$.

Thus we see that the two different positive-definite even unimodular lattices in 16 dimensions correspond to two different fully chiral phases at the edge of the same bulk phase. In the model in Eq. (41), the transition between them can occur in two possible ways: either a direct transition (naively, first-order, as we argue below) or or via two Kosterlitz-Thouless-like phase transitions, with an intermediate $c = 17$, $\overline{c} = 1$ phase between the two fully chiral phases. The former possibility occurs (again, assuming that the most relevant operator determines the flow to low energy) when there is always at least one relevant operator. The system is in the minimum of the corresponding cosine, but when another operator becomes more relevant, the system jumps to this minimum as s is tuned through the crossing point. Precisely at the point where two operators are equally relevant (e.g., u_1 and $\tilde{u}_1$ at $r = 1, s \approx 0.6$ as shown in Fig. 1) the magnitudes of the two couplings become important. At a mean-field level, the system will be in the minimum determined by the larger coupling and there will be a first-order phase transition at the point at which these two couplings are even in magnitude.

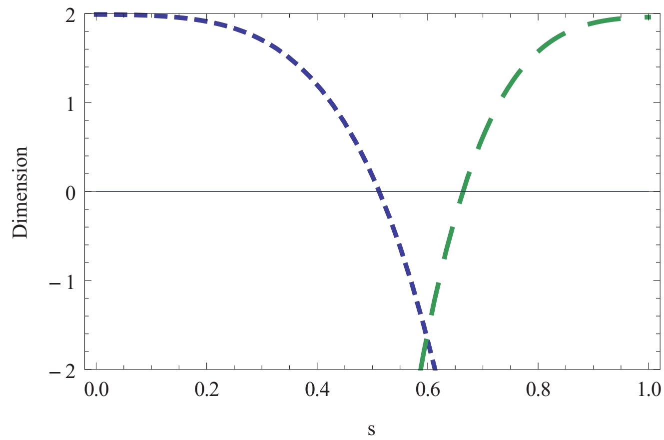

FIG. 2. (Color online) The scaling dimensions of u_1 (densely dashed) and $\tilde{u}_2$ (dashed), plotted as a function of s at at $r = 0.2$. The scaling dimensions of u_2 and $\tilde{u}_1$ lie outside the range of the plot and are not displayed. The system is not fully chiral phase between approximately $s = 0.5$ and 0.625.

If the most relevant operator is in the set $u_1, u_2, \tilde{u}_1, \tilde{u}_2$, then this means that the crossing point between the larger of $\frac{1}{u_{1,2}}\frac{du_{1,2}}{d\ell}$ and the larger of $\frac{1}{\tilde{u}_{1,2}}\frac{d\tilde{u}_{1,2}}{d\ell}$ occurs when both are positive so that the system goes directly from $E_8 \times E_8$ to spin$(32)/\mathbb{Z}_2$ theory. However, if there is a regime in which there are no relevant operators, then there will be a stable $c = 17$, $\overline{c} = 1$

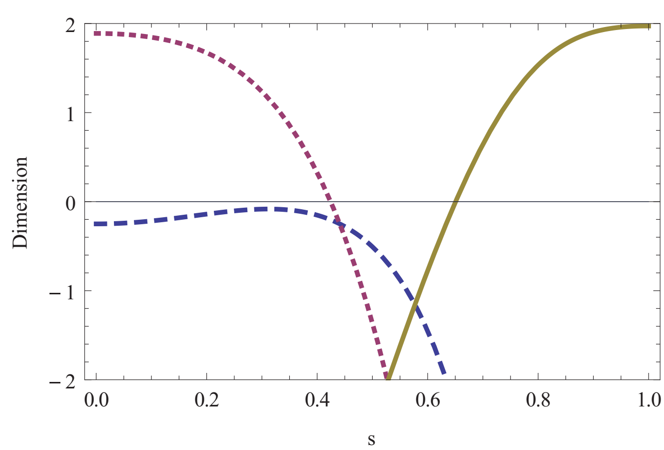

FIG. 3. (Color online) The scaling dimensions of $u_{1,2}$ (densely dashed and dotted) and $\tilde{u}_1$ (thick), plotted as a function of s at at $r = 3$. The scaling dimension of $\tilde{u}_2$ lies outside the range of the plot and is not displayed. The system is in the not fully chiral phase between approximately $s = 0.425$ and 0.625.

phase. [Note that we adhere to a slightly weaker definition of stability than used in the recent papers;[39,40] we say that an edge is unstable to gapping out some subset of its modes if a null vector[41] of the K matrix exists and that the associated operator is relevant in the RG sense. A null vector is simply an integer vector n_I satisfying $n_I(K^{-1})^{IJ}n_J = 0$ or, equivalently, a lattice vector k_a satisfying $k_a\eta^{ab}k_b = 0$.] If the crossing point between the larger of $\frac{1}{u_{1,2}}\frac{du_{1,2}}{d\ell}$ and the larger of $\frac{1}{\tilde{u}_{1,2}}\frac{d\tilde{u}_{1,2}}{d\ell}$ occurs when both are negative, then there may be a stable $c = 17$, $\bar{c} = 1$ phase.

However, the model of Eq. (41) is not the most general possible model; it is a particular slice of the parameter space in which the only perturbations of the quadratic theory are $u_{1,2}$ and $\tilde{u}_{1,2}$. A more general model will have many potentially mass-generating perturbations:

$$S_{\text{gen}}(s) = S_4(s) + \int dt\,dx \sum_{\mathbf{v}\in\Gamma_{E_8}\oplus\Gamma_{E_8}\oplus U} \delta_{|\mathbf{v}|^2,0}\, u_{\mathbf{v},s}\cos(\mathbf{v}\cdot\mathbf{X}), \tag{43}$$

where the sum is over vectors $\mathbf{v} \in \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ that have zero norm. This guarantees that these are spin-0 operators that are mass-generating if relevant. In Eq. (41), we have chosen 4 particular operators of this form and set the coefficients of the others to zero.[47] However, to determine if there is a stable nonchiral phase, it behooves us to consider a more general model in order to determine whether the nonchiral phase requires us to set more than one of the potentially mass-generating operators in Eq. (43) to zero by hand and so any such critical point is multicritical.

Of course, there are many possible $\mathbf{v} \in \Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ with $|\mathbf{v}|^2 = 0$. But most of them give rise to operators that are highly irrelevant over most of the range of the parameters r and s. However, there are two sets of operators that cannot be ignored. In one set, each operator is highly relevant in the vicinity of a particular value of s (which depends on the operator) in the $r \to 0$ limit and, in the other set, each operator is highly relevant in the vicinity of a particular value of s in the $r \to \infty$ limit. Consider the operators

$$\cos\left(\alpha \tilde{f}^{17}_a R^a{}_b X^b\right),\quad \cos\left(\beta \tilde{f}^{18}_a R^a{}_b X^b\right), \tag{44}$$

where R is an arbitrary SO(17,1) transformation. These operators have spin-0 since $\tilde{\mathbf{f}}^{17,18}$ have vanishing norm, which R preserves. Although they have spin-0 and can, therefore, generate a mass gap, there is no particular reason to think that either one is relevant. Moreover, it is not even likely that either one is an allowed operator. For an arbitrary SO(17,1) transformation, $\tilde{f}^{17}_a R^a{}_b$ will not lie in the $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ lattice spanned by the f^Is, so this operator will not be allowed. However, there is a special class of R for which these operators are allowed and are relevant in the vicinity of special points. Let us suppose that $R = O_G(p/q)$ and let us consider $\alpha = q^4$, $\beta = q^2$.[48] Consider the action

$$S_4\left(s = \frac{p}{q}\right) + u_{18,\frac{p}{q}} \int dt\,dx\, \cos\left\{q^2 \tilde{f}^{18}_a\,[O_G(p/q)]^a{}_b\, X^b\right\}. \tag{45}$$

This is a spin-0 perturbation. Moreover, it is an allowed operator for the following reason. We can write

$$q^2 \tilde{f}^{18}_a[(O_G(p/q)]^a{}_b = q^2[W(p/q)]_{18,J} f^J_a \tag{46}$$

where $[W(s)]_{IJ}$ is defined in analogy with W_G: $[W(s)]_{IJ} = \tilde{f}^J_a[O_G(s)]^a{}_b e^b_I$. The vector $q^2[W(p/q)]_{18,J}$ has integer entries, so $q^2\tilde{f}^{18}_a[(O_G(p/q)]^a{}_b$ is in the lattice $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$. At the point $s = p/q$, its scaling dimension is the same as the scaling dimension of $q^2\tilde{f}^{18}_a X^a$ at $s = 0$:

$$\frac{d}{d\ell} u_{18,\frac{p}{q}} = (2 - q^4 r^2) u_{18,\frac{p}{q}}. \tag{47}$$

Therefore, for $r < \sqrt{2}/q^2$, the coupling $u_{18,\frac{p}{q}}$ is a relevant mass-generating interaction at $s = p/q$ and, over some range of small r, it is relevant for s sufficiently near p/q. By a similar analysis, $u_{17,\frac{p}{q}}$ is a relevant mass-generating interaction at $s = p/q$ for $r > q^4/(2\sqrt{2})$ and, over some range of large r, it is relevant for s sufficiently near p/q. Therefore, when these couplings are nonzero, the nonchiral phase survives in a much smaller region of the phase diagram. (Making contact with our previous notation, we see that $u_{17,1} = \tilde{u}_1$ and $u_{18,1} = \tilde{u}_2$.)

When one of these interactions gaps out a pair of counter-propagating modes, we are left with a fully chiral $c = 16$ edge theory corresponding to either $E_8 \times E_8$ to spin(32)/$\mathbb{Z}_2$. To see which phase we get, consider, for the sake of concreteness, the coupling $u_{18,\frac{p}{q}}$. When it generates a gap, it locks the combination of fields $q^2\tilde{f}^{18}_a[O_G(p/q)]^a{}_b X^b = q^2[W(p/q)]_{18,J} f^J_a X^a$. In the low-energy limit, we may set this combination to zero. Only fields that commute with this combination remain gapless. (Moreover, since we have set this combination to zero, any fields that differ by a multiple of it are equal to each other at low energy.) Therefore the vertex operators that remain in the theory are of the form $\exp(n_I f^J_a X^a)$ where n_I satisfies $n_I(K^{-1})^{IJ}[W(p/q)]_{18,J} = 0$. We note that $[W(p/q)]_{18,J}$ is nonzero only for $J = 8,16,17,18$. Therefore $[W(p/q)]_{18,J} f^J_a$ is orthogonal to $\mathbf{e}_1,\ldots,\mathbf{e}_7$ and $\mathbf{e}_9,\ldots,\mathbf{e}_{15}$.

Much as in our discussion in Sec. IV of the difference between the $E_8 \times E_8$ and spin(32)/$\mathbb{Z}_2$ edge theories, we again make use of the basic observation that $E_8 \times E_8$ is a product while spin(32)/$\mathbb{Z}_2$ has a single component in order to identify the low energy theory. If the vectors $n_I f^I_a$ with $n_I(K^{-1})^{IJ}[W(p/q)]_{18,J} = 0$ (and two vectors differing by a multiple of $q^2[W(p/q)]_{18,J} f^J_a$ identified) form the spin(32)/$\mathbb{Z}_2$ lattice, then there must be a vector $\mathbf{c} = c_I\mathbf{f}^I$ in the lattice with $|\mathbf{c}|^2 = 2$ such that $\mathbf{c}\cdot\mathbf{e}_1 = -\mathbf{c}\cdot\mathbf{e}_7 = \mathbf{c}\cdot\mathbf{e}_9 = 1$ and $\mathbf{c}\cdot\mathbf{e}_2 = \mathbf{c}\cdot\mathbf{e}_3 = \cdots = \mathbf{c}\cdot\mathbf{e}_6 = 0$ and $\mathbf{c}\cdot e_{10} = \mathbf{c}\cdot e_{11} = \cdots = \mathbf{c}\cdot e_{15} = 0$. This is because there exists a set of Cartesian coordinates $\hat{\mathbf{y}}_a$ such that all the vectors in spin(32)/$\mathbb{Z}_2$ with $(\text{length})^2 = 2$ are of the form $\pm\hat{\mathbf{y}}_a \pm \hat{\mathbf{y}}_b$ with $a,b = 1,\ldots,16$, while for $E_8 \times E_8$, vectors of the form $\pm\hat{\mathbf{y}}_a \pm \hat{\mathbf{y}}_b$ must have $a,b = 1,\ldots,8$ or $a,b = 9,\ldots,16$. In $E_8 \times E_8$, vectors of $(\text{length})^2 = 2$ cannot "connect" the two halves of the system. If the equations $c_I(K^{-1})^{IJ}(W(p/q))_{18,J} = 0$ and $c_I(K^{-1})^{IJ}c_J = 2$ with $c_1 = -c_7 = c_9 = 1$ and $c_2 = c_3 = \cdots = c_6 = c_{10} = c_{11} = \cdots = c_{15} = 0$ have integer solutions, then the remaining gapless degrees of freedom are in the spin(32)/$\mathbb{Z}_2$ phase. Otherwise, they are in the $E_8 \times E_8$ phase. We could choose $\mathbf{e}_1$, $-\mathbf{e}_7$, and $\mathbf{e}_9$ as the vectors with unit product with $\mathbf{c}$ because such a $\mathbf{c}$ must exist in spin(32)/$\mathbb{Z}_2$.

(Note that we could have taken c_7 to be arbitrary, and we would have found that solutions to these equations must necessarily have $c_7 = -1$.) The phase is $E_8 \times E_8$ if and only if such a vector **c** is not in the lattice. Of course, it is essential that we can restrict our attention to the two possibilities, $E_8 \times E_8$ and spin$(32)/\mathbb{Z}_2$, since these are the only two unimodular self-dual lattices in dimension 16.

With the aid of MATHEMATICA, we have found that solutions to the above equations must be of the form $c_I = (1,0^5, -1,c_8,1,0^6,c_8-1,q/p(2c_8-1),-p/q(2c_8-1))$. Since c_I must be an integer vector, both p and q must be odd since $2c_8 - 1$ is odd. Here, as above, we have assumed that p and q are relatively prime. Further, we see that this solution requires $2c_8 = pqm + 1$ for odd m.

This means that the chiral spin$(32)/\mathbb{Z}_2$ theory is left behind at low energies when both p and q are odd and $u_{18,\frac{p}{q}}$ is the most relevant operator that generates a mass gap for two counter-propagating edge modes. When either p or q is even, the remaining gapless modes of the edge are in the $E_8 \times E_8$ phase. We find the identical behavior for the low energy theory when $u_{17,\frac{p}{q}}$ is the most relevant operator.

When these operators have nonzero coefficients in the Lagrangian, they eliminate a great deal of the nonchiral phase shown in the $u_{1,2}, \tilde{u}_{1,2}$-only phase diagram in Fig. 4. The effect is most noticeable as $r \to 0$ and $r \to \infty$ as shown in Fig. 5.

However, there still remain pockets of the nonchiral phase at intermediate values of r and s, where these operators are irrelevant. However, we find that these regions of nonchiral phase are not stable when we include a larger set of operators in the Lagrangian. Consistent with our expectations, it is possible to find a relevant operator in the region around any given point (r,s) in the phase diagram such that the low energy theory remaining after a pair of counter-propagating modes gaps out is $E_8 \times E_8$ or spin$(32)/\mathbb{Z}_2$.

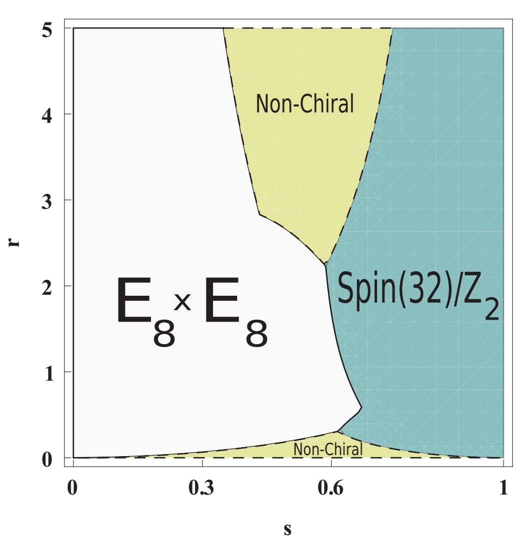

FIG. 4. (Color online) Phase diagram of our edge theory as a function of s and r for the theory $S_4(s)$ in which the only nonzero perturbations are $u_{1,2}$ and $\tilde{u}_{1,2}$. The light region is in the $E_8 \times E_8$ phase. The darkest region is in the spin$(32)/\mathbb{Z}_2$ phase. The system is not fully chiral in the intermediately shaded region. The dashed phase boundary line indicates a KT transition. The solid lines denote regions where there are two equally relevant couplings; the phase is determined by their ratio.

To see how this works, consider, for instance, the point $(r,s) = (3,3/5)$ that exists in the putative region of nonchiral phase according to Fig. 4. The couplings $u_{17,\frac{p}{q}}$, $u_{18,\frac{p}{q}}$ are all irrelevant there so the system remains nonchiral even when these couplings are turned on. However, we can find a relevant spin-0 operator at this point as follows. It must take the form $\cos(p_a X^a)$, with $p_a \in \Gamma_8 \oplus \Gamma_8 \oplus U$, where $\eta^{ab} p_a p_b = 0$ (this is the spin-0 condition). To compute its scaling dimension, we observe that it can be written in the form $\cos[q_a X^a(s)]$, where $X^a(s) \equiv [O_G(s)]^a_b X^b$ and $p_b = q_a [O_G(s)]^a_b$. In terms of this field, the quadratic part of the action is diagonal in the $X^a(s)$ fields, so their correlation functions (and, therefore, their scaling dimensions can be computed straightforwardly). Since the operator in question has spin-0, its total scaling dimension $\delta^{ab} q_a q_b$ is twice their left-moving dimension or, simply, $|q_{18}|^2$. Therefore such an operator is relevant if $|q_{18}|^2 < 2$.

$O_G^{-1}(s)$ is simply a boost along some particular direction in the 17-dimensional space combined with a spatial rotation. The eigenvalues of such a transformation are either complex numbers of modulus 1 (rotation) or contraction/dilation by $e^{\pm\alpha}$ (Lorentz boost). Consequently, even if $\delta^{ab} p_a p_b$ is large—which means that $\cos(p_a X^a)$ is highly irrelevant at $s = 0$–$\delta^{ab} q_a q_b$ can be smaller by as much as $e^{-2\alpha}$, thereby making $\cos(p_a X^a)$ a relevant operator at this value of s (and of r). The maximum possible contraction, $e^{-\alpha}$, occurs when p_a is antiparallel to the boost. (The maximum dilation, $e^{-\alpha}$, occurs when p_a is parallel to the boost, and there is no change in the scaling dimension when p_a is perpendicular to the boost.) For a given r,s, we can choose a lattice vector p_a that is arbitrarily close to the direction of the boost, but at the cost of making $\delta^{ab} p_a p_b$ very large. Then $\delta^{ab} q_a q_b \approx e^{-2\alpha} \delta^{ab} p_a p_b$ may not be sufficiently small to be relevant. (The $\approx$ will be an $=$ sign if p_a is precisely parallel to the direction of the boost, however, we are not guaranteed to be able to find an element of the lattice that is precisely parallel.) Alternatively, we can choose a smaller $\delta^{ab} p_a p_b$, but the angle between p_a and the boost may not larger. As explained through an example in Appendix B, we can balance these two competing imperatives and find a p_a so that neither $\delta^{ab} p_a p_b$ nor the angle between p_a and the boost is too large. Then $\frac{1}{2}\delta^{ab} q_a q_b \approx \frac{1}{2} e^{-2\alpha} \delta^{ab} p_a p_b < 2$, so that the corresponding operator is relevant.

The following simple ansatz leads to a relevant operator

$$p_a = n f^7_a + (m - 2n) f^8_a + m f^{16}_a + n_{17} f^{17}_a + n_{18} f^{18}_a \quad (48)$$

at all candidate nonchiral points in the (r,s) phase diagram that we have checked. We do not have a proof that there is not some region in parameter space where a nonchiral phase is stable, but we have explicitly excluded nearly all of it, as may be seen from the phase diagram in Fig. 6 where we have included a selection of the possible operators described here that become relevant at the set of points $(r,s) = (6, p/q)$ for $q = 5$, and we anticipate that this ansatz will enable us to do so for any other point not already excluded. Thus we expect the nonchiral phase to be entirely removed by this collection of operators combined with those discussed earlier.

Therefore the phase diagram has a quite rich and intricate structure. From our experience with the above operators, our general expectation is that in the neighborhood of any

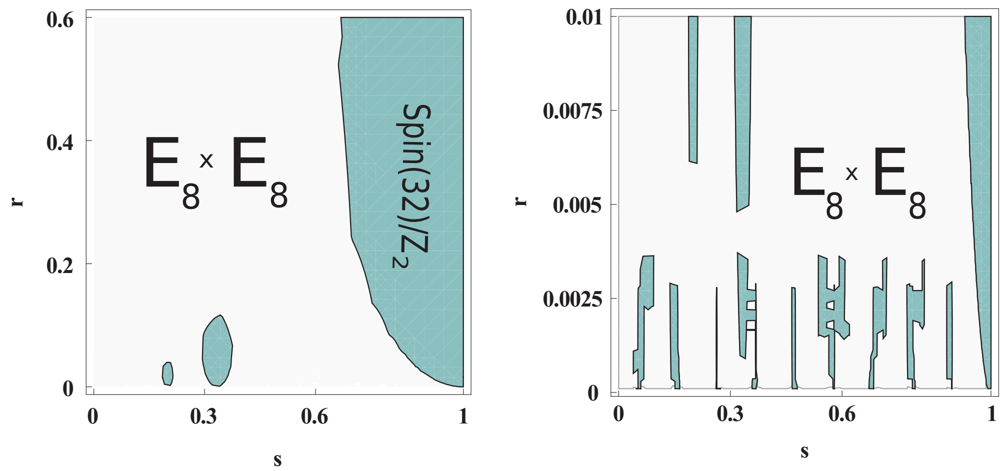

FIG. 5. (Color online) The small-r region of the phase diagram of our edge theory as a function of s and r for the theory with nonzero $u_{1,2}$, $\tilde{u}_{1,2}$; $u_{17,\frac{p}{q}}$, $u_{17,\frac{p}{q}}$ for all $p,q \leqslant 57$; and several $\cos(p_a X^a)$ operators with p_a nearly aligned with the direction of the boost $O_G(s)$, as described in the text. The light region is in the $E_8 \times E_8$ phase. The darker region is in the spin(32)/$\mathbb{Z}_2$ phase. All phase boundary lines denote regions where there are two equally relevant couplings; the phase is determined by the ratio of these couplings. The left panel shows the $r < 0.6$ region of the phase diagram, where we see that regions of the two phases are interspersed with each other along the s axis. In the right panel, we zoom in on the $r < 0.01$ region of the phase diagram and see an even richer intermingling of these two phases as we sweep over s.

point $(r, p/q)$, there exists a relevant operator that gaps out a pair of modes leading to the fully chiral $E_8 \times E_8$ theory if p or q is even, while spin(32)/$\mathbb{Z}_2$ remains if p and q are odd.

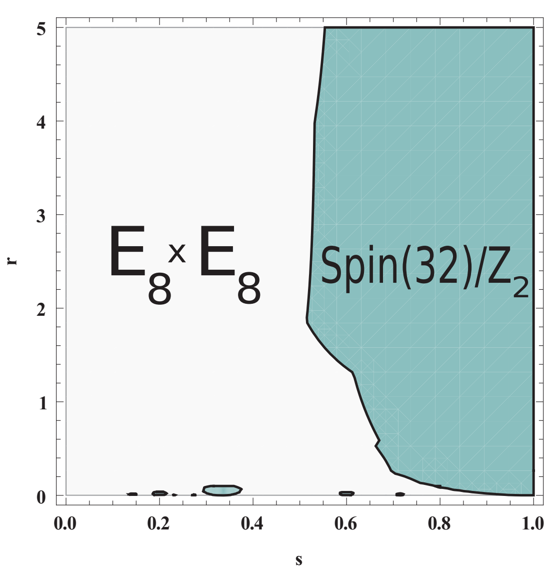

FIG. 6. (Color online) Phase diagram of our edge theory as a function of s and r for the theory $S_4(s)$ in which the only nonzero perturbations are $u_{1,2}$ and $\tilde{u}_{1,2}$; $u_{17,\frac{p}{q}}$, $u_{17,\frac{p}{q}}$ for all $p,q \leqslant 57$; and several $\cos(p_a X^a)$ operators with p_a nearly aligned with the direction of the boost $O_G(s)$, as described in the text. The latter operators were specifically chosen to remove the remaining points of nonchiral phase at $r = 6$, $s = p/q$ for $q = 5$. This set of operators was sufficient to remove all the nonchiral phase displayed previously in Fig. 4. The light region is in the $E_8 \times E_8$ phase. The darker region is in the spin(32)/$\mathbb{Z}_2$ phase. Solid phase boundary lines denote regions where there are two equally-relevant couplings; the phase is determined by the ratio of these couplings.

VI. CHARGED SYSTEMS

We return to our $c - \bar{c} = 16$ theories and consider the case in which some of the degrees of freedom are charged as a result of coupling to an external electromagnetic field as in Eq. (4). Now, there are many phases for a given K, distinguished by different t. They may, as a consequence, have different Hall conductances $\sigma_{xy} = \frac{e^2}{h} t_I (K^{-1})^{IJ} t_J$, which must be even integer multiples of $\frac{e^2}{h}$ since K^{-1} is an integer matrix with even entries on the diagonal.

Let us focus on the minimal possible nonzero Hall conductance, $\sigma_{xy} = 2\frac{e^2}{h}$. We will not attempt to systematically catalog all of these states here, but will examine a few examples with $c = 16$ that are enlightening. By inspection, we see that we have three distinct $\sigma_{xy} = 2\frac{e^2}{h}$ states with K matrix $K = K_{E_8 \times E_8}$: (1) $t_I = \delta_{I6}$, (2) δ_{I9}, and (3) $-2\delta_{I1} + \delta_{I2}$. These states have stable edge modes even if the $U(1)$ symmetry of charge conservation is violated (e.g., by coupling the system to a superconductor), in contrast to the $\sigma_{xy} = 2\frac{e^2}{h}$ bosonic quantum Hall states discussed in Ref. 20.

As before, we adjoin a trivial system to our system so that the K matrices are $K = K_{E_8 \times E_8 \oplus U}$. Under the similarity transformation W_G, these states are equivalent to the states with $K_{\mathrm{spin}(32)/\mathbb{Z}_2 \oplus U}$ and, respectively, $t = (0,0,0,0,1,-2,0,0,0,0,0,0,0,0,0,4,-2,2)$, $t = 0,0,0,0,0,0,0,0,-2,1,0,0,0,0,0,0,4,-2,2$, and $t_I = \delta_{I1}$. Consider the first of these, $K_{\mathrm{spin}(32)/\mathbb{Z}_2 \oplus U}$, $t = (0,0,0,0,1,-2,0,0,0,0,0,0,0,0,0,4,-2,2)$. It is *not* equal to $K_{\mathrm{spin}(32)/\mathbb{Z}_2}$ with an additional trivial system adjoined to it because $\tilde{\phi}_{17}$ and $\tilde{\phi}_{18}$ are both charged. In other words, there is a right-moving neutral edge mode $\tilde{\phi}_{17} + \tilde{\phi}_{18}$ and a left-moving charged edge mode $\tilde{\phi}_{17} - \tilde{\phi}_{18}$. This is nontrivial, and there is no charge-conserving perturbation, which will give a gap to these modes. The same is true of the second state. In the case of the third state, both ϕ_{17}, ϕ_{18} and $\tilde{\phi}_{17}$, $\tilde{\phi}_{18}$ are neutral. Therefore there are perturbations that could gap out either of them.

In addition, operators defined by lattice vectors of the form, Eq. (48), are neutral. Consequently, we conclude that $K = K_{E_8\times E_8}$, $t_I = -2\delta_{I1} + \delta_{I2}$ and $K_{\text{spin}(32)/\mathbb{Z}_2}$, $t_I = \delta_{I1}$ are stably equivalent bulk states with an edge theory phase diagram similar to that in Fig. 6. We remark that neutrality of both $\exp(i\tilde{\phi}_{17})$ and $\exp(i\tilde{\phi}_{18})$ implies that operators defined by lattice vectors of the form, Eq. (48), are neutral as well.

VII. DISCUSSION

A. Summary

Bosonic SRE states with chiral edge modes are bosonic analogues of fermionic integer quantum Hall states: they do not support anyons in the bulk, but they have completely stable chiral edge modes. Together, they populate an "intermediate" class of phases that are completely stable and do not require symmetry protection, however, they lack nontrivial bulk excitations. Unlike in the fermionic case, such states can only occur when the number of edge modes is a multiple of 8. As we have seen in this paper, the scary possibility that the number of edge modes does not uniquely determine such a state is not realized, at least for the first case in which it can happen, namely, when there are 16 edge modes. The two phases that are naively different are, in fact, the same phase. This is consistent with the result that all 3-manifold invariants associated with the two phases are the same,[30] and we have gone further and shown that it is possible to go directly from one state to the other without crossing a phase boundary in the bulk. However, there are actually two distinct sets of edge excitations corresponding to these adiabatically connected bulk states. We have shown that the phase transition between them can occur purely at the edge, without closing the bulk gap. However, both edge phases are fully chiral, unlike the "T-unstable" states considered in Refs. 41 and 42. There is no sense in which one of these two phases is inherently more stable in a topological sense than the other; it is simply that, for some values of the couplings, one or the other is more stable.

Our construction is motivated by the observation that there is a unique even, unimodular lattices with signature $(8k+n,n)$. Consequently, enlarging the Hilbert spaces of seemingly different phases associated with distinct even, unimodular lattices with signature $(8k,0)$ by adding trivial insulating degrees of freedom associated with even, unimodular lattices with signature (n,n) leads to the same bulk phase. Since the edge is characterized by additional data, the corresponding edge theories are distinct but are separated by a phase transition that can occur purely on the edge without closing the bulk gap. The details of our construction draw on a similar one by Ginsparg[24] who showed explicitly how to interpolate between toroidal compactifications of $E_8 \times E_8$ and spin$(32)/\mathbb{Z}_2$ heterotic string theories.

B. Future directions

Let us describe a few possible directions for future study. (1) We have considered one possible interpolation between the $E_8 \times E_8$ and spin$(32)/\mathbb{Z}_2$ theories and, therefore, have only considered a small region of possible parameter space determined by r and s. It would be interesting to carve out in more detail the full 153-dimensional phase space.

(2) The last phase diagram displayed in Fig. 6 includes only a subset of the possible operators that may be added to the edge theory. The operators that have been added are sufficient to lift the nonchiral phase that is naively present and displayed in Fig. 4 when only four operators are included. It is possible that consideration of all allowed operators could result in an even more complex phase diagram with a rich topography of interspersed $E_8 \times E_8$ and spin$(32)/\mathbb{Z}_2$ phases.

(3) The uniqueness of even, unimodular lattices with signature $(8k+n,n)$ implies that a similar route can be taken to adiabatically connect states associated to different positive-definite even unimodular lattices of dimension $8k = 24,32,\ldots$. However, in these cases, it is possible for states corresponding to different lattices to have different spectra of operator scaling dimensions at the edge, unlike in the $c = 16$ case, so the situation may be more subtle. The 24-dimensional case may be particularly interesting as the ground state transforms trivially (as reviewed at the end of Sec. II A) under modular transformation of the torus.

(4) It is possible to have an edge in which the interaction varies along the edge so that u_1 is the only relevant operator for $x < 0$ and $\tilde{u}_1$ is the only relevant operator for $x > 0$. The edge will then be in the $E_8 \times E_8$ phase to the left of the origin and the spin$(32)/\mathbb{Z}_2$ phase to the right of the origin. It would be interesting to study the defect that will be located at the origin.

(5) Unimodular lattices occur in the study of four-manifold topology as the intersection form of $H^2(M,\mathbb{Z})$, where M is a four-manifold and $H^2(M,\mathbb{Z})$ is the second cohomology group over the integers. (We assume that M is closed.) In the circumstances when de Rham cohomology can be defined, we can think of the intersection form as follows. Consider all pairs of 2-forms, ω_I,ω_J and construct the matrix, $K_{IJ} = \int_M \omega_I \wedge \omega_J \in \mathbb{Z}$. Even when de Rham cohomology does not make sense, the above matrix can be defined. K_{IJ} is unimodular and symmetric. Interestingly, the cases for which K_{IJ} is even (and, therefore, provide intersection forms of the type studied in this paper) correspond to nonsmooth four-manifolds. The first instance is the so-called E_8 manifold whose intersection form is the E_8 Cartan matrix. Likewise, there exist two distinct four-manifolds, $E_8 \times E_8$ and the Chern manifold, with $E_8 \times E_8$ and spin$(32)/\mathbb{Z}_2$ intersection form, respectively.[43] While these two four-manifolds are not equivalent or homeomorphic, they are cobordic: there exists a five-manifold whose two boundary components correspond to these two four-manifolds. The cobordism can be understood as taking the direct sum of each four-manifold with $S^2 \times S^2$, which has intersection matrix equal to U. A series of surgeries then relates these two connected augmented four-manifolds. In other words, our paper has been a physical implementation of the above cobordism. Is there a deeper connection between four-manifold topology and integer quantum Hall states? We might go further and imagine that any such relation could be generalized to fractional and, possibly, non-Abelian states. Further, the introduction of symmetry-protected topological phases in $2+1d$ could inform the study of four-manifolds, i.e., the stabilizing symmetry of any phase could further refine the possible invariants characterizing any manifold.

(6) We have concentrated on bosonic systems in this paper, but very similar considerations apply to fermionic SRE systems with chiral edge modes, which correspond to

positive-definite odd unimodular lattices. The conventional integer quantum Hall states correspond to the hypercubic lattices $\mathbb{Z}^N$. However, there is a second positive-definite odd unimodular lattice in dimensions greater than 8, namely, $K_{E_8} \oplus I_{N-8}$. In dimensions greater than 11, there is also a third one, and there are still more in higher dimensions. However, there is a unique unimodular lattice with indefinite signature. Therefore, by a very similar construction to the one that we have used here, these different lattices correspond to different edge phases of the $\nu \geqslant 9$ integer quantum Hall states.

(7) Finally, stable equivalence is not restricted to topologically ordered states in $2+1d$; it would be interesting to see explicitly how it manifests itself in the study of topological phases in other dimensions.

ACKNOWLEDGMENTS

We would like to thank Parsa Bonderson, Matthew Fisher, Michael Freedman, Tarun Grover, Max Metlitski, Ashvin Vishwanath, and Jon Yard for discussions. C.N. has been partially supported by the DARPA QuEST program and AFOSR under Grant FA9550-10-1- 0524.

APPENDIX A: LATTICES AND MATRICES

In this Appendix, we collect formulas for the various lattice vectors and matrices we use throughout the main text. To fix some notation, consider the standard basis for $\mathbf{R}^N$,

$$\hat{x}_I = (0\cdots 010\cdots 0)^t, \tag{A1}$$

where the 1 appears in the Ith row for $I = 1,\ldots,N$. The root lattice Γ_G of any rank N Lie group G is defined in terms of linear combinations of the $\hat{x}_I$. Given a basis $\mathbf{e}_I$ for the lattice, we may construct the Cartan matrix or K matrix, $(K_G)_{IJ} = e_I^a \eta_{ab} e_J^b$ where η is the diagonal matrix diag$(\mathbf{1}^M,-\mathbf{1}^{N-M})$ and $\mathbf{1}^P$ is the P-component vector with every entry equal to unity. The Cartan matrix summarizes the minimal data needed to specify a Lie group. Geometrically, a diagonal entry $(K_G)_{II}$ is equal to the length-squared of the root I and an off-diagonal entry $(K_G)_{IJ}$ gives the dot product between roots I and J and so can be interpreted as being proportional to the cosine of the angle (in $\mathbf{R}^N$) between the two roots. Given the inverse $(K_G^{-1})^{IJ}$, we may define dual lattice vectors $f_a^I = (K_G^{-1})^{IJ}\eta_{ab}e_J^b$ that satisfy $f_a^I e_J^a = \delta_J^I$.

1. Γ_{E_8}

A basis for the root lattice Γ_{E_8} of the rank 8 group E_8 is given by

$$\begin{aligned} &\mathbf{e}_I = \hat{\mathbf{x}}_I - \hat{\mathbf{x}}_{I+1}, \quad \text{for} \quad I = 1,\ldots 6, \\ &\mathbf{e}_7 = -\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2, \quad \mathbf{e}_8 = \tfrac{1}{2}(\hat{\mathbf{x}}_1 + \cdots + \hat{\mathbf{x}}_8). \end{aligned} \tag{A2}$$

The associated K matrix takes the form

$$K_{E8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{A3}$$

The inner product is Euclidean so $\eta_{ab} = \delta_{ab}$.

2. $\Gamma_{E_8} \oplus \Gamma_{E_8}$

The rank 16 Lie group $E_8 \times E_8$ is equal to two copies of E_8. We take as our lattice basis for $\Gamma_{E_8} \oplus \Gamma_{E_8}$,

$$\begin{aligned} &\mathbf{e}_I = \hat{\mathbf{x}}_I - \hat{\mathbf{x}}_{I+1}, \quad \text{for} \quad I = 1,\ldots 6, \\ &\mathbf{e}_7 = -\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2, \quad \mathbf{e}_8 = \tfrac{1}{2}(\hat{\mathbf{x}}_1 + \cdots + \hat{\mathbf{x}}_8), \\ &\mathbf{e}_{8+I} = \hat{\mathbf{x}}_{9+I} - \hat{\mathbf{x}}_{10+I}, \quad \text{for} \quad I = 1,\ldots,6, \\ &\mathbf{e}_{15} = \hat{\mathbf{x}}_{15} + \hat{\mathbf{x}}_{16}, \quad \mathbf{e}_{16} = -\tfrac{1}{2}(\hat{\mathbf{x}}_9 + \cdots + \hat{\mathbf{x}}_{16}). \end{aligned} \tag{A4}$$

The associated K matrix takes the form

$$K_{E_8\oplus E_8} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The inner product is again taken to be $\eta_{ab} = \delta_{ab}$.

3. $\Gamma_{\text{spin}(32)/Z_2}$

A basis for the root lattice $\Gamma_{\text{spin}(32)/\mathbb{Z}_2}$ of the rank 16 Lie group $\text{spin}(32)/\mathbb{Z}_2$ is given by

$$\tilde{e}_I = \hat{\mathbf{x}}_{I+1} - \hat{\mathbf{x}}_{I+2}, \quad \text{for} \quad I = 1, \ldots, 14, \qquad \tilde{e}_{15} = \hat{\mathbf{x}}_{15} + \hat{\mathbf{x}}_{16}, \quad \tilde{e}_{16} = -\tfrac{1}{2}(\hat{\mathbf{x}}_1 + \cdots + \hat{\mathbf{x}}_{16}). \tag{A5}$$

The associated K matrix is

$$K_{\text{spin}(32)/\mathbb{Z}_2} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4
\end{pmatrix}.$$

The inner product is given by $\eta_{ab} = \delta_{ab}$.

4. $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$

To write a basis for the $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ lattice, we must enlarge the dimension of our previous $\Gamma_{E_8} \oplus \Gamma_{E_8}$ lattice by two. Thus we take as our lattice basis the following:

$$\mathbf{e}_I = \hat{\mathbf{x}}_I - \hat{\mathbf{x}}_{I+1}, \;\; \text{for} \;\; I = 1, \ldots 6, \quad \mathbf{e}_7 = -\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2, \qquad \mathbf{e}_8 = \frac{1}{2}(\hat{\mathbf{x}}_1 + \cdots + \hat{\mathbf{x}}_8), \qquad \mathbf{e}_{8+I} = \hat{\mathbf{x}}_{9+I} - \hat{\mathbf{x}}_{10+I}, \;\; \text{for} \;\; I = 1, \ldots, 6,$$
$$\mathbf{e}_{15} = \hat{\mathbf{x}}_{15} + \hat{\mathbf{x}}_{16}, \quad \mathbf{e}_{16} = -\frac{1}{2}(\hat{\mathbf{x}}_9 + \ldots + \hat{\mathbf{x}}_{16}), \quad \mathbf{e}_{17} = \frac{1}{r}\hat{\mathbf{x}}_{17} + \frac{1}{r}\hat{\mathbf{x}}_{18}, \quad \mathbf{e}_{18} = \frac{r}{2}\hat{\mathbf{x}}_{17} - \frac{r}{2}\hat{\mathbf{x}}_{18}. \tag{A6}$$

The associated K matrix takes the form

$$K_{E_8 \oplus E_8 \oplus U} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$

The inner product is taken with respect to $\eta_{ab} = (\mathbf{1}^{17}, -1)$.

5. $\Gamma_{\text{spin}(32)/\mathbb{Z}_2} \oplus U$

We must again enlarge the dimension of $\Gamma_{\text{spin}(32)/\mathbb{Z}_2}$ by two in order to write a basis for $\Gamma_{\text{spin}(32)/\mathbb{Z}_2} \oplus U$,

$$\begin{aligned} &\tilde{e}_I = \hat{\mathbf{x}}_{I+1} - \hat{\mathbf{x}}_{I+2}, \quad \text{for} \quad I = 1,\ldots,14, \qquad \tilde{e}_{15} = \hat{\mathbf{x}}_{15} + \hat{\mathbf{x}}_{16}, \quad \tilde{e}_{16} = -\frac{1}{2}(\hat{\mathbf{x}}_1 + \ldots + \hat{\mathbf{x}}_{16}), \\ &\tilde{e}_{17} = -r\hat{\mathbf{x}}_{17} + r\hat{\mathbf{x}}_{18}, \quad \tilde{e}_{18} = -\frac{1}{2r}\hat{\mathbf{x}}_{17} - \frac{1}{2r}\hat{\mathbf{x}}_{18}. \end{aligned} \tag{A7}$$

The associated K matrix is

$$K_{\text{spin}(32)/\mathbb{Z}_2 \oplus U} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The inner product is taken with respect to $\eta_{ab} = (\mathbf{1}^{17}, -1)$.

6. SO(17,1) and SL(18,$\mathbb{Z}$) transformations

There exist two distinct even, self-dual 16-dimensional lattices, $\Gamma_{E_8} \oplus \Gamma_{E_8}$ and $\Gamma_{\text{spin}(32)/\mathbb{Z}_2}$, that cannot be rotated into each other via an SO(16) transformation.[23] However, if we augment each lattice by U, we obtain a Lorentzian lattice of signature (17,1), i.e., the augmented lattice has the inner product $\eta_{ab} = \text{diag}(\mathbf{1}^{17}, -1)$. Such lattices are unique up to an SO(17,1) rotation. Following,[24] the SO(17,1) transformation relating the $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ and $\Gamma_{\text{spin}(32)/\mathbb{Z}_2} \oplus U$ lattices is given by

$$O_G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2r} - \frac{-1+r^2}{2r} & -\frac{1}{2r} - \frac{1+r^2}{2r} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2r} - \frac{-1+r^2}{2r} & \frac{1}{2r} - \frac{1+r^2}{2r} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{r} & -\frac{1}{r} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{r}{2} & \frac{r}{2} & \frac{r}{2} & \frac{r}{2} & \frac{r}{2} & \frac{r}{2} & \frac{r}{2} & -\frac{r}{2} + \frac{1-r^2}{r} & r - \frac{1-r^2}{r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} + \frac{(1-r^2)(-1+r^2)}{r^2} & -\frac{1}{2} - r^2 + \frac{1-r^2}{r^2} \\ -\frac{r}{2} & -\frac{r}{2} & -\frac{r}{2} & -\frac{r}{2} & -\frac{r}{2} & -\frac{r}{2} & -\frac{r}{2} & \frac{r}{2} + \frac{1+r^2}{r} & -r - \frac{1+r^2}{r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} + r^2 - \frac{1+r^2}{r^2} & \frac{1}{2} + \frac{(1+r^2)^2}{r^2} \end{pmatrix}.$$

O_G acts on basis vectors as

$$O_G{}^a{}_b e_I^b = \sum_J m_I^J \tilde{e}_J^a, \tag{A8}$$

where m_I^J are a collection of integers.

Because O_G lies in the component of SO(17,1) connected to the identity transformation, we may build O_G from a series of infinitesimal transformations beginning at **1**. First, we rewrite

$$O_G = \eta W(A) \eta W(A'), \tag{A9}$$

where

$$W(A) = \exp\left[\frac{1}{2}\begin{pmatrix} 0 & A & -A \\ -A^t & 0 & 0 \\ -A^t & 0 & 0 \end{pmatrix}\right], \quad \text{with} \tag{A10}$$

$$A = \frac{2}{r}(0^7, -1, 1, 0^7), \tag{A11}$$

$$A' = -2r\left(\left(\frac{1}{2}\right)^8, 0^8\right). \tag{A12}$$

We then introduce the (infinitesimal) parameter s by rescaling $A, A' \to sA, sA'$ and defining

$$O_G(s) = \eta W(sA) \eta W(sA'). \tag{A13}$$

(While the resulting matrix does not fit between the margins of this page, the expression is not beautiful.)

Substituting the transformation Eq. (A8) into the periodicity condition, $X^a \equiv X^a + 2\pi n^I e_I^a$, for the $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ lattice, we find

$$(O_G)^a{}_b X^b \equiv (O_G)^a{}_b X^b + 2\pi \tilde{n}^J \tilde{e}_J^a, \tag{A14}$$

where we have defined the integer vector $\tilde{n}^J = \sum_I n^I m_I^J$. However, Eq. (A14) is simply the periodicity obeyed by $\tilde{X}^a$. Therefore we identify $\tilde{X}^a = (O_G)^a{}_b X^b$. Having identified X^a and $\tilde{X}^b$ through the SO(17,1) transformation O_G, we can obtain the SL(18,$\mathbb{Z}$) transformation W_G that relates $K_{\mathrm{spin}(32)/\mathbb{Z}_2 \oplus U}$ and $K_{E_8 \oplus E_8 \oplus U}$ by conjugation. The desired transformation is read off from the relation

$$\tilde{\phi}^J = \tilde{f}_a^J (O_G)^a{}_b e_I^b \phi^I =: (W_G)_{IJ} \phi^I, \tag{A15}$$

which follows immediately from Eq. (A8). We find

$$W_G = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-4 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-7 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-9 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
-10 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \\
-11 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 \\
-12 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & -4 \\
-13 & 0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5 & -5 \\
-14 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 6 & -6 \\
-7 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & -3 \\
-8 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & -4 \\
-2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & 2
\end{pmatrix}.$$

This matrix satisfies $W_G^T K_{\mathrm{spin}(32)/\mathbb{Z}_2 \oplus U} W = K_{E_8 \oplus E_8 \oplus U}$.

APPENDIX B: "DIMENSION CONTRACTION" AND RELEVANT MASS-GENERATING OPERATORS AT INTERMEDIATE r,s

We consider spin-0 operators that take the form $\cos(p_a X^a)$, with $p_a \in \Gamma_8 \oplus \Gamma_8 \oplus U$ and $\eta^{ab} p_a p_b = 0$. Even if $\frac{1}{2}\delta^{ab} p_a p_b > 2$, which means that $\cos(p_a X^a)$ is irrelevant at $s = 0$, this operator may become relevant at an intermediate value of s. At general s, the scaling dimension of the operator is $\frac{1}{2}\delta^{ab} q_a q_b = |q_{18}|^2$, where $q_b = p_a[O_G^{-1}(s)]^a{}_b$. In writing the scaling dimension in terms of q_{18} only, we have used the fact that q_b is a null vector in $\mathbb{R}^{17,1}$ ($\eta^{ab} q_a q_b = q_1^2 + \cdots + q_{17}^2 - q_{18}^2 = 0$). Thus $\cos(p_a X^a)$ will become relevant at s if $p_a(O_G^{-1}(s))^a_{18}$ is sufficiently Lorentz contracted so that $q_{18}^2 < 2$.

If the direction of the boost $O_G^{-1}(s)$ happened to be along the 1-direction, then we know that the only components of p_a affected by the boost are the 1st and 18th component; they are contracted/dilated according to

$$\begin{pmatrix} p_1 \\ p_{18} \end{pmatrix} \mapsto \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} p_1 \\ p_{18} \end{pmatrix}. \tag{B1}$$

Therefore multiples of the eigenvectors $(1,\pm 1)^T$ with eigenvalues $\exp(\mp\alpha)$ have components that are maximally contracted/dilated. If the boost took the above simple form, it would be simple to choose a vector p_a whose 18th component after the boost was maximally contracted. This vector would determine the most relevant operator at a given point in the (r,s) phase diagram.

Unfortunately, $O_G^{-1}(s)$ is defined in terms of a rather complicated combination of rotations and boosts, and so it is not *a priori* obvious which spatial direction to choose in order to maximize the possible contraction, i.e., it is difficult to know the direction $\vec{v}$ of the boost. However, we know that we can view the $O_G^{-1}(s)$ transformation as: $O_G^{-1}(s) = M^T \Lambda M$, where M is a rotation that aligns $\vec{v}$ along the 1-direction and Λ is a boost along the 1-direction. (Both of these transformations, of course, depend upon the initially chosen r and s.) To find null vectors whose components maximally contract, we need only consider the eigenvector of $O_G^{-1}(s)$ given by $M^{\rm tr}(1,0^{16},1)^{\rm tr}$ with eigenvalue $\exp(-\alpha)$, for some constant α depending upon r and s. For $(r,s) = (3,3/5)$, we find that this maximally contracting eigenvector takes the simple (approximate) form:

$$p_a = 0.3 f_a^7 + (0.1 - 0.6) f_a^8 + 0.1 f_a^{16} + f_a^{17} - 0.9 f_a^{18}. \quad \text{(B2)}$$

While the components of this vector are maximally contracted under $O_G^{-1}(s)$ in the sense discussed above, it is certainly not an element of $\Gamma_{E_8} \oplus \Gamma_{E_8} \oplus U$ since the coefficients are not integral. We can find a vector with very large components that is nearly parallel to this vector, but it will be irrelevant because $O_G^{-1}(s)$ cannot contract it by enough at $(r,s) = (3,3/5)$.

However, we can find a shorter lattice vector that is sufficiently aligned with the maximally contracting vector, but of lower starting dimension so that we obtain a relevant operator at the point of interest. Indeed, if we take the ansatz

$$p_a = n f_a^7 + (m - 2n) f_a^8 + m f_a^{16} + n_{17} f_a^{17} + n_{18} f_a^{18}, \quad \text{(B3)}$$

it is straightforward to find n, m, n_{17} and n_{18} determining a relevant spin-0 operator at (r,s). At $(r,s) = (3,3/5)$, we may take $n = 1, m = 2, n_{17} = 2$, and $n_{18} = -3$. We lack a proof that this ansatz is sufficient to exclude all possible nonchiral points in the (r,s) phase diagram. However, we have yet to find a point (r,s) for which this ansatz is unsuccessful. Thus we expect the nonchiral phase to be entirely removed by this collection of operators combined with those discussed earlier. [Note, we expect the resulting chiral phase for this operator to be spin$(32)/\mathbb{Z}_2$.]

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[45] We note that SPT phases can all be adiabatically connected to a trivial ground state if we do not require that the associated symmetry be preserved. Topological phases cannot be. However, if we restrict to Hamiltonians that respect a symmetry then, just as the trivial phase splits into many SPT phases, a nontrivial topological phase could split into multiple phases that could be distinguished,

for instance, by their edge excitations. For a discussion of such "symmetry-enhanced topological phases," see Ref. 44.

[46]Of course, it may be possible to take a route from one to the other that does cross a phase transition but such a transition can always be avoided. For instance, if we restrict to S_z-conserving Hamiltonians, then a phase transition must be encountered in going from a spin-singlet $N = 2$ state to a spin-polarized one. If we do not make this restriction, however, then this phase transition can be avoided and the two states can be adiabatically connected.

[47]To lowest order in $u_{1,2}$ and $\tilde{u}_{1,2}$, this is consistent, but at higher order, these four operators will generate some others, and we must consider a more general theory. However, it does not appear that these operators generate any spin-0 operators other than multiples of themselves, which are less relevant than they are.

[48]This choice of α and β is a sufficient one for generic $s = p/q$; however, certain q accommodate smaller α and β so that the resulting operators are well defined. For example, when q is even, we may take $\alpha = q^2/2$ and $\beta = q^4/4$.