

Quantum-critical pairing in electron-doped cuprates

Yuxuan Wang and Andrey Chubukov

Department of Physics, University of Wisconsin, Madison, Wisconsin 53706, USA

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We revisit the problem of spin-mediated superconducting pairing at the antiferromagnetic quantum-critical point with the ordering momentum $(\pi, \pi) = 2\mathbf{k}_F$. The problem has been previously considered by one of the authors [P. Krotkov and A. V. Chubukov, *Phys. Rev. Lett.* **96**, 107002 (2006); *Phys. Rev. B* **74**, 014509 (2006)]. However, it was later pointed out [D. Bergeron *et al.*, *Phys. Rev. B* **86**, 155123 (2012)] that the analysis neglected umklapp processes for the spin polarization operator. We incorporate umklapp terms and reevaluate the normal state self-energy and the critical temperature of the pairing instability. We show that the self-energy has a Fermi-liquid form and obtain the renormalization of the quasiparticle residue Z , the Fermi velocity, and the curvature of the Fermi surface. We argue that the pairing is a BCS-type problem, but go one step beyond the BCS theory and calculate the critical temperature T_c with the prefactor. We apply the results to electron-doped cuprates near optimal doping and obtain $T_c \geq 10$ K, which matches the experimental results quite well.

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I. INTRODUCTION

Superconductivity near a quantum-critical point (QCP) in a metal is one of the most studied and debated topics in the physics of strongly correlated electron systems.¹⁻⁴ When a metal is brought, by doping, external field, or pressure, to a near vicinity of a phase transition into a state with either spin or charge density wave order, corresponding collective bosonic excitations become soft and the fermion-fermion interaction, mediated by collective bosons, gets enhanced. Quite often, this interaction turns out to be attractive in one or the other pairing channel, and gives rise to enhanced superconducting T_c near a QCP. Examples of the pairing mediated by soft bosonic fluctuations include the pairing in double-layer composite fermion metals,⁵ pairing due to singular momentum-dependent interaction,⁶ color superconductivity of quarks mediated by gluon exchange,⁷ pairing by singular gauge fluctuations,^{8,9} pairing by near-critical ferromagnetic and antiferromagnetic spin fluctuations,¹⁰⁻¹⁷ and phonon-mediated pairing at vanishing Debye frequency.¹⁸

The pairing at a QCP in dimensions $D \leq 3$ is generally a nontrivial problem because near a QCP fermions do not display a conventional Fermi-liquid (FL) behavior down to the smallest frequencies, at least in some (hot) regions on the Fermi surface (FS). To obtain the pairing instability in this situation one has to go beyond the leading logarithmic approximation as the summation of the leading logarithms does not lead to an instability,^{12,19} except for a special case of a color superconductivity.⁷

Pairing near the $\mathbf{Q} = (\pi, \pi)$ antiferromagnetic QCP attracted most of the attention in recent years, particularly in $D = 2$, because of its relation to d -wave superconductivity in the cuprates.^{3,13-15,20-24} The FS in hole-doped cuprates around optimal doping is an open electron FS (closed hole FS) which contains four pairs of hot spots (points for which \mathbf{k}_F and $\mathbf{k}_F + \mathbf{Q}$ are both on the FS). The hot spots are located near $(0, \pi)$ and other symmetry-related points. At each hot spot, fermionic self-energy at a QCP has a non-FL form $\Sigma(\omega) \propto \omega^a$ with $a \approx 1/2$, down to the lowest frequencies.^{3,14,26,27} The pairing of these hot fermions, the relative role of quasiparticles with non-FL and FL forms of the self-energy, and the interplay

between the pairing and the bond-order instabilities near a QCP are intriguing phenomena which are not fully understood yet, but for which a strong progress has been made theoretically in the last few years.^{14,15,19}

The present work is devoted to the analysis of superconductivity at the antiferromagnetic QCP, but in the special case when \mathbf{Q} coincides with $2\mathbf{k}_F$ along the diagonals in the Brillouin zone (see Fig. 2).⁸ In this situation, there are only two pairs of hot spots, located along the two diagonals of the Brillouin zone, and for each pair Fermi velocities at \mathbf{k}_F and $\mathbf{k}_F + \mathbf{Q}$ are strictly antiparallel. This case is applicable to electron-doped cuprates $\text{Nd}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_7$ and $\text{Pr}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_7$.²⁹ The FS of these materials near the onset of spin density wave (SDW) order has four pairs of hot spots located close enough to zone diagonals. This makes our model a good starting point. As T_c only increases as hot spots move apart (see below), our result places the lower boundary on T_c in these materials.

The phase diagram of electron-doped cuprates is somewhat better understood than that of hole-doped materials in the sense that the pseudogap physics of these materials is most likely due to magnetic precursors.²⁸⁻³³ This in turn implies that magnetic fluctuations are strong and well may be relevant to superconductivity. An earlier random phase approximation (RPA) study has found³⁴ that the electron doping, at which magnetic order emerges, is close to the one at which hot spots merge along zone diagonals. At larger electron doping, there are no hot spots and no magnetism. At smaller dopings, magnetic order emerges and, simultaneously, each pair of diagonal hot spots splits into two pairs which move away from diagonals in two different directions.

The quantum-critical pairing near a $2\mathbf{k}_F$ QCP has been earlier considered by one of us in Refs. 35 and 36. It was found that there is a sizable attraction in the $d_{x^2-y^2}$ channel, despite that at $2\mathbf{k}_F$ QCP the strongest interaction connects the points where $d_{x^2-y^2}$ gap vanishes. This result stands. However, the value of T_c has to be reconsidered because the analysis in Refs. 35 and 36 used for the normal state self-energy at the hot spots the non-FL form $\Sigma(\omega) = \omega^{3/4}$, which was obtained in Ref. 35 and in earlier work⁸ by neglecting umklapp processes. Recent study³⁷ has found that umklapp processes

severely reduce the self-energy, such that quasiparticles remain coherent even at a QCP. The imaginary part of the self-energy at the QCP scales as $\omega^{3/2} \ln \omega$, i.e., the Fermi liquid is noncanonical in the notations of Ref. 38, but, still, $\text{Im}\Sigma \ll \omega$ at small frequency, and the Fermi-liquid criteria of long-lived quasiparticles near the FS remains valid. Whether the $\omega^{3/2}$ self-energy is responsible for the observed $T^{1.6}$ behavior of resistivity in overdoped $\text{La}_{2-x}\text{Ce}_x\text{CuO}_4$ (Ref. 39) remains to be seen.

In this paper, we reconsider the problem of the pairing at the $2k_F$ QCP using the correct form of the self-energy. We show that the pairing is a FL phenomenon, i.e., it is fully determined by the coherent component of the quasiparticle Green's function and depends on the quasiparticle Z factor and the effective mass. Moreover, when spin-fermion coupling is small compared to the Fermi energy, the pairing can be analyzed within the weak coupling, BCS-type analysis. Our goal is to go one step beyond the BCS theory and compute T_c exactly, with the numerical prefactor.

The calculation we present here is similar in spirit to the one done in early days of BCS superconductivity for a model of fermions with a constant attractive interaction,⁴⁰ but is more involved, as in our case the pairing interaction contains the propagator of low-energy collective bosons which strongly depends on the transferred momentum and the transferred frequency. We show that these two dependencies make calculations somewhat tricky, but still doable.

We consider the low-energy model of fermions located near Brillouin zone diagonals and assume that fermions interact by exchanging near-critical soft collective fluctuations in the spin channel (the spin-fermion model). The model contains two parameters: the overall energy scale \bar{g} , which is the effective fermion-fermion interaction mediated by spin fluctuations, and the dimensionless coupling λ , which determines mass renormalization and the renormalization of the quasiparticle Z factor.^{3,14} In our case, the coupling λ is one-third power of the ratio of \bar{g} and the effective Fermi energy of quasiparticles near Brillouin zone diagonal:

$$\lambda = \left(\frac{\bar{g}}{4\pi E_F} \right)^{1/3}. \quad (1)$$

We approximate the fermionic dispersion near Brillouin zone diagonals by $\epsilon_k = v_F(k_x + \kappa k_y^2/2)$, where k_x and k_y are the directions along and transverse to zone diagonals, measured relative to a hot spot, v_F is the Fermi velocity along zone diagonal, and κ is the curvature of the FS. In terms of these parameters $E_F = v_F/(2\kappa)$.

In the normal state we found that the quasiparticle residue Z , the Fermi velocity v_F , and the curvature κ differ from free-fermion values by corrections of order λ :

$$Z = 1 - 0.7\lambda, \quad v_F^* = v_F(1 + 0.05\lambda), \quad \kappa^* = \kappa(1 - 1.45\lambda). \quad (2)$$

Observe that the renormalizations of Z and v_F^* do not satisfy $Zv_F/v_F^* = 1$, which holds in Eliashberg-type theories (ET's) of electron-phonon¹⁸ and electron-electron^{3,12} interactions. A similar result was obtained in the calculation of self-energy in the AFM state.²⁵

The reason for the discrepancy with ET's is quite fundamental: in ET's the self-energy $\Sigma(\mathbf{k}, \omega_m) \approx i\lambda\omega_m$ depends only on frequency and comes from intermediate fermions with energies comparable to ω_m , other corrections are relatively small in Eliashberg parameter $[(\omega_D/E_F)^{1/2}$ for electron-phonon interaction]. In our case, Eliashberg parameter is of order one, and there are two contributions to $\Sigma(\mathbf{k}, \omega_m)$ of comparable strength. One comes from intermediate fermions with energies comparable to ω_m and scales as ω_m , i.e., depends only on frequency. The other comes from intermediate fermions with energies much larger than ω_m , and scales as $i\omega_m + v_F(k - k_F)$. Because the two contributions are of the same order, $\partial\Sigma/\partial(i\omega_m)$ and $(1/v_F)\partial\Sigma/\partial k$ are comparable. As a result, the renormalizations of Z and of Fermi velocity are comparable in strength (both are of order λ), but the prefactors are different.

We used normal state results as an input for the pairing problem and computed superconducting T_c at the QCP. We found

$$T_c = 0.0013 \frac{\bar{g}}{\lambda} e^{-\frac{C}{\lambda}}, \quad C = 0.6874. \quad (3)$$

The exponential dependence on λ and the overall factor \bar{g}/λ can be obtained within logarithmical BCS-type treatment. However, to obtain the overall numerical factor in (3), we had to go beyond the logarithmical approximation and use the fact that the boson-mediated interaction is dynamical and decays at large frequencies. This part of our analysis is similar to the calculation of T_c in ETs.^{12,18} However, as we said, the similarity is only partial because the renormalized Green's function in our case is different from the one in ET. In another distinction from ET, in our case we need to include into consideration nonladder diagrams which account for Kohn-Luttinger-type renormalization of the pairing interaction.⁴¹ These diagrams do not affect the exponent but contribute $O(1)$ to the numerical prefactor in T_c . In ET theories, the contribution from Kohn-Luttinger renormalizations is small in the Eliashberg parameter.

We plot T_c/\bar{g} as a function of λ in Fig. 1. We see that T_c increases with λ at small λ , passes through a broad maximum at $\lambda \sim 0.5$, and slowly decreases at larger λ . The actual value of λ can be extracted from the data. The energy \bar{g} can be deduced from optical measurements in the magnetically ordered Mott-Heisenberg state at half-filling⁴² where \bar{g} coincides with the optical gap. The data yield $\bar{g} \sim 1.6\text{--}1.7$ eV, which is essentially the same as in hole-doped cuprates (this scale

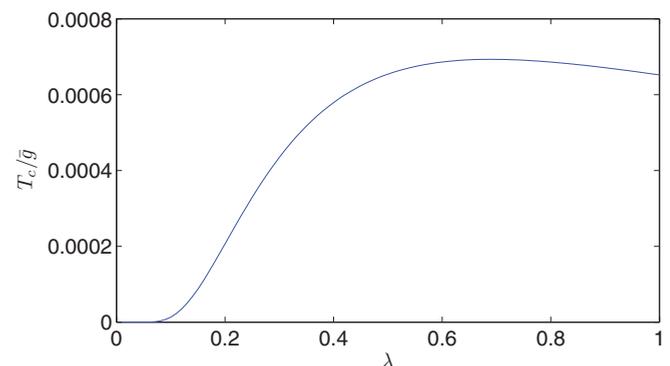


FIG. 1. (Color online) The critical temperature T_c/\bar{g} as a function of the coupling λ .

is close to the charge-transfer energy U in the effective Hubbard model). The Fermi velocity v_F and the curvature κ can be extracted from the ARPES measurements of optimally doped $\text{Nd}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_7$.⁴³ The fit yields $v_F = 0.87$ eV, $\kappa = 0.31$, and the effective Fermi energy $E_F = v_F/(2\kappa) \sim 1.4$ eV. Substituting into (2) we obtain $\lambda \sim 0.46$. For such λ , $Z \approx 0.7$, i.e., weak coupling approximation is reasonably well justified. Using $\lambda = 0.46$ and $\bar{g} = 1.7$ eV, we obtained $T_c = 0.0006\bar{g} \sim 12$ K. This value is actually the lower theoretical boundary on T_c by two reasons. One is theoretical: we found that T_c gets enhanced if we keep the fermionic bandwidth (the upper limit of the low-energy theory) finite. The second is practical: in optimally doped $\text{Nd}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_4$ and $\text{Pr}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_4$ hot spots are located close to but still at some distance from zone diagonals.²⁹ When they are far apart, T_c is much higher^{19,45} and it is natural to expect that T_c gets higher when hot spots split.

Another feature of real materials, which may affect T_c a bit, is the observation³³ that SDW antiferromagnetism and superconductivity in $\text{Nd}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_{4+\delta}$ may be actually separated by weak first-order transition rather than coexist,⁴⁴ in which case the magnetic correlation length remains large but finite at optimal doping. Still, the value which we found theoretically is in quite reasonable agreement with the experimental $T_c \sim 10$ – 25 K in $\text{Nd}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_4$ and in $\text{Pr}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_4$ (Refs. 46 and 47). Also, our $\lambda = 0.46$ is close to the position of the broad maximum of $T_c(\lambda)$ in Fig. 1, hence T_c only slightly increases or decreases if the actual λ differs somewhat from our estimate.

The paper is organized as follows. In the next section we introduce the spin-fermion model as the minimal model to describe interacting fermions near the $2k_F$ spin-density-wave instability. In Sec. III, we present one-loop normal state calculations, which include umklapp scattering. We obtain the spin polarization operator, which accounts for the dynamics of collective excitations, and use it to obtain fermionic self-energy to first order in λ . In Sec. IV, we solve for T_c in one-loop (ladder) approximation. We show that T_c is exponentially small in λ and find the prefactor to one-loop order. In Sec. V, we analyze two-loop corrections to T_c and show that they change the numerical prefactor for T_c by a finite amount. We argue that higher-loop corrections are irrelevant as they only change T_c by a factor $1 + O(\lambda)$. Combining one-loop and two-loop results, we obtain the full expression for T_c at weak coupling [Eq. (63)]. We also discuss in Sec. V the effect on T_c from lowering the upper energy cutoff Λ of the theory. When Λ is of order bandwidth, W , T_c is essentially independent on Λ as long as $T_c \ll W$. However, if, by some reasons, Λ is smaller than W , T_c gets larger, and its increase becomes substantial when $\Lambda < \bar{g}^{1/3}W^{2/3}$. We compare our results with the experiments in Sec. VI and summarize our results in Sec. VII.

II. MODEL

We consider fermions on a square lattice at a density larger than one electron per site (electron doping) and use tight-binding form of electron dispersion with hopping to first and second neighbors. We choose doping at which free-fermion FS touches the magnetic Brillouin zone along the diagonals, as shown in Fig. 2. We assume, following earlier works,³⁴ that at around this doping the system develops a spin-density-wave

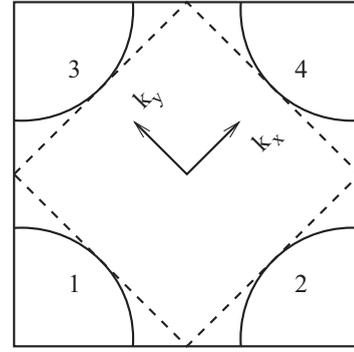


FIG. 2. Fermi surface at a particular electron doping when the Fermi surface touches the magnetic Brillouin zone along the zone diagonals. The four diagonal Fermi surface points (hot spots) are labeled by numbers. We assume that this doping is close to the one at which antiferromagnetic instability emerges at momentum $\mathbf{Q} = (\pi, \pi)$.

(SDW) order with momentum $\mathbf{Q} = (\pi, \pi)$. For convenience, throughout the paper we will use a rotated reference frame with (k_x, k_y) shown in Fig. 2. In this frame, the SDW ordering momentum is $(0, \sqrt{2}\pi)$ or $(\sqrt{2}\pi, 0)$.

We analyze the physics near the $\mathbf{Q} = 2\mathbf{k}_F$ antiferromagnetic QCP within the semiphenomenological spin-fermion model.^{3,8,12,14,15,35,37} The model assumes that antiferromagnetic correlations develop already at high energies, comparable to bandwidth, and mediate interactions between low-energy fermions. In the context of superconductivity, spin-mediated interaction then plays the same role of a pairing glue as phonons do in conventional superconductors. The static part of the spin-fluctuation propagator comes from high energies and should be treated as an input for low-energy theory. However, the dynamical part of the propagator should be self-consistently obtained within the model as it comes entirely from low-energy fermions.

The Lagrangian of the model is given by^{3,14,15}

$$S = - \int_k^\Lambda G_0^{-1}(k) \psi_{k,\alpha}^\dagger \psi_{k,\alpha} + \frac{1}{2} \int_q^\Lambda \chi_0^{-1}(q) \mathbf{S}_q \cdot \mathbf{S}_{-q} + g \int_{k,q}^\Lambda \psi_{k+q,\alpha}^\dagger \sigma_{\alpha\beta} \psi_{k,\beta} \cdot \mathbf{S}_{-q}, \quad (4)$$

where \int_k^Λ stands for the integral over d dimensional \mathbf{k} (up to some upper cutoff Λ) and the sum runs over fermionic and bosonic Matsubara frequencies. The $G_0(k) = G_0(\mathbf{k}, \omega_m) = 1/(i\omega_m - \epsilon_{\mathbf{k}})$ is the bare fermion propagator and $\chi_0(\mathbf{q} + \mathbf{Q}) = \chi_0/(\mathbf{q}^2 + \xi^{-2})$ is the static propagator of collective bosons, in which ξ^{-1} measures the distance to the QCP, and \mathbf{q} is measured with respect to \mathbf{Q} . At the QCP, $\xi^{-1} = 0$. The fermion-boson coupling g and χ_0 appear in theory only in combination $\bar{g} = g^2\chi_0$, which has the dimension of energy.

The interaction between fermions and collective spin bosons gives rise to fermionic self-energy $\Sigma(\mathbf{k}, \omega_m)$, which modifies fermionic propagator to $G^{-1}(\mathbf{k}, \omega_m) = G_0^{-1}(\mathbf{k}, \omega_m) + \Sigma(\mathbf{k}, \omega_m)$, and bosonic self-energy which modifies bosonic propagator to $\chi^{-1}(\mathbf{q} + \mathbf{Q}, \Omega_m) = (\mathbf{q}^2 + \xi^{-2} + \Pi(\mathbf{q} + \mathbf{Q}, \Omega_m))/\chi_0$. We focus on the hot regions on the FS, which are most relevant to the pairing when \bar{g} is smaller than the cutoff Λ . In our case, there are four hot regions

around Brillouin zone diagonals, which we labeled 1 to 4 in Fig. 2. The physics in one hot region is determined by the interaction with another hot region, separated by \mathbf{Q} . This creates two “pairs”: (1, 4) and (2, 3). However, the pairs can not be fully separated because \mathbf{Q} and $-\mathbf{Q}$ differ by inverse lattice vector, hence umklapp processes are allowed.³⁷ As a result, the interaction between fermions in regions 1 and 4 is mediated by $\chi(\mathbf{q} + \mathbf{Q}, \Omega_m)$ whose polarization operator $\Pi(\mathbf{q} + \mathbf{Q}, \Omega_m)$ has contribution from fermions in the same regions 1 and 4, but also from fermions in the regions 2 and 3.

We define fermion momenta relative to their corresponding hot spots. The dispersion of a fermion is linear in transverse momentum (the one along the Brillouin zone diagonal) and quadratic in the momentum transverse to the diagonal. Specifically, the dispersion relation in region 1 is

$$\epsilon_k = -v_F \left(k_x + \kappa \frac{k_y^2}{2} \right), \quad (5)$$

where κ is the curvature of the FS.

III. NORMAL STATE ANALYSIS

The spin polarization operator $\Pi(\mathbf{q} + \mathbf{Q}, \Omega_m)$ is the dynamical part of the particle-hole bubble with external momentum near \mathbf{Q} . The dressed dynamical spin fluctuations give rise to fermionic self-energy $\Sigma(\mathbf{k}, \omega_m)$, which in turn affects the form of $\Pi(\mathbf{q} + \mathbf{Q}, \Omega_m)$. This mutual dependence generally implies that $\Sigma(\mathbf{k}, \omega_m)$ and $\Pi(\mathbf{q} + \mathbf{Q}, \Omega_m)$ form a set of two coupled equations.

When hot spots are far from zone diagonals and Fermi velocities at \mathbf{k}_F and $\mathbf{k}_F + \mathbf{Q}$ are not antiparallel (like in hole-doped cuprates), the two equations decouple because the bosonic polarization operator has the Landau damping form $\Pi(\mathbf{q} + \mathbf{Q}, \Omega_m) = \gamma |\Omega_m|$, and the prefactor γ does not depend on fermionic self-energy as long as the latter predominantly depends on frequency. Evaluating fermionic Σ with the Landau overdamped $\chi(\mathbf{q} + \mathbf{Q}, \Omega_m)$ one can in turn verify^{3,14} that Σ predominantly depends on frequency near a QCP, i.e., equations for $\Sigma(\mathbf{k}, \omega_m) \approx \Sigma(\omega_m)$ and $\Pi(\mathbf{q} + \mathbf{Q}, \Omega_m)$ do indeed decouple. This decoupling allows one to compute the Landau damping using free-fermion propagator, even when $\Sigma(\omega_m)$ is not small, and use the dynamical $\chi(\mathbf{q} + \mathbf{Q}, \Omega)$ with Landau damping term in the calculations of the fermionic self-energy.⁵¹ In our case, the dynamical part of χ is not Landau damping because Fermi velocities at \mathbf{k}_F and $\mathbf{k}_F + \mathbf{Q}$ are strictly antiparallel, and $\Pi(\mathbf{q} + \mathbf{Q}, \Omega_m)$ does depend on fermionic self-energy.³⁶ This generally requires full self-consistent analysis of the coupled set of nonlinear equations for $\Sigma(\mathbf{k}, \omega_m)$ and $\Pi(\mathbf{q} + \mathbf{Q}, \Omega_m)$. Fortunately, in our case the system preserves a FL behavior even at the QCP, and for $\bar{g} < W$, which we assume to hold, calculations can be done perturbatively rather than self-consistently.

Below we will obtain lowest-order (one-loop) expressions for the polarization bubble and fermionic self-energy and use them to compute T_c in the ladder approximation. We then discuss how these expressions are affected by two-loop diagrams.

A. One-loop bosonic and fermionic self-energies

The one-loop Feynman diagram for the polarization operator is shown in Fig. 3. The polarization operator contains

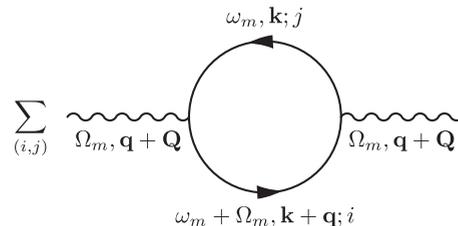


FIG. 3. The one-loop polarization bubble. Labels i and j denote fermions at different hot spots. The full expression is the sum of direct and umklapp processes (1,4), (4,1), (2,3), (3,2).

contributions from *direct* and *umklapp* scattering between hot spots separated by either \mathbf{Q} or $-\mathbf{Q}$ [the combinations of (i, j) of internal fermions can be (1,4), (4,1), (2,3), and (3,2)]. For external momenta near $\mathbf{Q} = (\pi, \pi)$, the processes (1,4) and (4,1) are direct and (2,3) and (3,2) are umklapp. Only direct processes have been considered in Refs. 8 and 35, but, as was pointed out in Ref. 37, all four processes should be included into Π . The authors of Ref. 37 obtained

$$\begin{aligned} \Pi(\mathbf{q} + \mathbf{Q}, \Omega) &= \Pi^{(0)}(q_x, q_y, \Omega) + \Pi^{(0)}(-q_x, q_y, \Omega) \\ &+ \Pi^{(0)}(q_y, q_x, \Omega) + \Pi^{(0)}(-q_y, q_x, \Omega), \end{aligned} \quad (6)$$

where^{8,35}

$$\Pi^{(0)}(q_x, q_y, \Omega) = \frac{\bar{g}}{\pi \sqrt{2} v_F^3 \kappa} \sqrt{\sqrt{\Omega^2 + E_q^2} + E_q}, \quad (7)$$

and

$$E_q = v_F \left(q_x - \kappa \frac{q_y^2}{4} \right). \quad (8)$$

The full dynamical spin susceptibility is

$$\chi(\mathbf{q} + \mathbf{Q}, \Omega) = \frac{\chi_0}{\mathbf{q}^2 + \Pi(\mathbf{q} + \mathbf{Q}, \Omega)}. \quad (9)$$

We now use $\chi(\mathbf{q}, \Omega)$ from (9) and calculate one-loop self-energy of an electron. We will see that relevant \mathbf{q}^2 and $\Pi(\mathbf{q} + \mathbf{Q}, \Omega)$ are of the same order, i.e., the spin polarization operator evaluated at one-loop order is not a small perturbation of the bare static $\chi_0(\mathbf{q} + \mathbf{Q})$. Higher-loop terms in $\Pi(\mathbf{q} + \mathbf{Q}, \Omega)$ are, however, small in λ and can be treated perturbatively. The one loop self-energy is presented diagrammatically in Fig. 4.

Specifically, close to the hot spot, labeled by 1, the analytical expression is

$$\begin{aligned} \Sigma(\mathbf{k}, \omega_m) &= -\frac{3\bar{g}}{8\pi^3} \int \frac{d\mathbf{q} d\Omega_m}{i(\omega_m + \Omega_m) - \epsilon_{\mathbf{k}+\mathbf{q}+\mathbf{Q}}} \frac{1}{\mathbf{q}^2 + \Pi(\mathbf{q} + \mathbf{Q}, \Omega_m)}, \end{aligned} \quad (10)$$

where, as before, the momenta \mathbf{k} and $\mathbf{k} + \mathbf{q}$ are taken close to hot spots 1 at $(-k_F, 0)$, and 4 at $(k_F, 0)$, and the dispersion

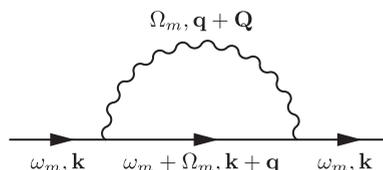


FIG. 4. One-loop electron self-energy.

$\epsilon_{\mathbf{k}+\mathbf{Q}}$ is

$$\epsilon_{\mathbf{k}+\mathbf{Q}} = -v_F \left(-k_x + \kappa \frac{k_y^2}{2} \right) = \epsilon_{-\mathbf{k}}. \quad (11)$$

The coefficient 3 comes from summation over the x , y , and z components of the spin susceptibility.

For definiteness, we consider the self-energy right at the QCP, when $\xi^{-1} = 0$. We assume and then verify that FL behavior is preserved at the QCP, i.e., at small ω_m , $\Sigma(0, \omega_m) = i\lambda\omega_m$ and $\Sigma(k_x, 0) \propto v_F k_x$, $\Sigma(k_y, 0) \propto k_y^2$. We will need both frequency- and momentum-dependent components of the self-energy for the calculation of T_c .

To compute Σ , it is convenient to subtract from $\Sigma(\mathbf{k}, \omega_m)$ its expression at $\mathbf{k} = 0$ and $\omega_m = 0$ as the latter vanishes by symmetry for our approximate form of the dispersion $\epsilon_{\mathbf{k}}$. This subtraction can actually be done even if one does not approximate $\epsilon_{\mathbf{k}}$, as, in the most general case, $\Sigma(0, 0)$ accounts only for the renormalization of the chemical potential by fermions with energies above Λ , which we have to neglect anyway to avoid double counting as such renormalization is already included into $\epsilon_{\mathbf{k}}$. After the subtraction, the self-energy becomes

$$\begin{aligned} \Sigma(\mathbf{k}, \omega_m) &= \frac{3\bar{g}}{8\pi^3} \int d\mathbf{q} d\Omega_m \frac{1}{\mathbf{q}^2 + \Pi(\mathbf{q} + \mathbf{Q}, \Omega_m)} \\ &\times \frac{i\omega_m - [\epsilon_{-(q+k)} - \epsilon_{-q}]}{(i\Omega_m - \epsilon_{-q})[i(\omega_m + \Omega_m) - \epsilon_{-(q+k)}]}. \end{aligned} \quad (12)$$

It is tempting to set ω_m and k to zero in the denominator of (12) but this would lead to an incorrect result as the integrand in (12) contains two close poles, and the contribution from the region between the poles is generally of order one.³ To obtain the correct self-energy one has to explicitly integrate over internal frequency and momenta without setting ω_m and k to zero. The order of integration does not matter because the integral is convergent in the ultraviolet limit. We choose to integrate over q_x first as this will allow us to make a comparison with hole-doped cuprates.

The integral in (12) above can be simplified if we introduce the dimensionless small parameter

$$\lambda \equiv \left(\frac{\bar{g}\kappa}{2\pi v_F} \right)^{1/3} = \left(\frac{\bar{g}}{4\pi E_F} \right)^{1/3} \quad (13)$$

and rescale

$$\begin{aligned} \Sigma &\rightarrow \frac{\bar{g}\tilde{\Sigma}}{\pi\lambda}, \quad q \rightarrow \frac{\bar{g}\tilde{q}}{v_F\pi\lambda}, \quad k \rightarrow \frac{\bar{g}\tilde{k}}{v_F\pi\lambda}, \\ \Omega_m &\rightarrow \frac{\bar{g}\tilde{\Omega}_m}{\pi\lambda}, \quad \omega_m \rightarrow \frac{\bar{g}\tilde{\omega}_m}{\pi\lambda}. \end{aligned} \quad (14)$$

$$\frac{1}{\mathbf{q}^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(\tilde{\mathbf{q}}, \tilde{\Omega}_m)} = \frac{1}{\tilde{q}_y^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(\tilde{q}_x=0, \tilde{q}_y, \tilde{\Omega}_m)} + \left[\frac{1}{\mathbf{q}^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(\tilde{\mathbf{q}}, \tilde{\Omega}_m)} - \frac{1}{\tilde{q}_y^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(q_x=0, q_y, \tilde{\Omega}_m)} \right]. \quad (18)$$

The pole contribution $\tilde{\Sigma}_1$ comes from the first term in the right-hand side of (18), the branch cut contribution $\tilde{\Sigma}_2$ comes from the second term.

The expression for $\tilde{\Sigma}_1$ is

$$\Sigma_1(\mathbf{k}, \omega_m) = \frac{3\lambda}{8\pi^2} \int d\tilde{q}_y \int d\tilde{\Omega}_m \int d\tilde{q}_x \frac{i\tilde{\omega}_m - \tilde{k}_x + \lambda^2(\tilde{k}_y^2 + 2\tilde{k}_y\tilde{q}_y)}{(i\tilde{\Omega}_m - \tilde{q}_x) [i(\tilde{\Omega}_m + \tilde{\omega}_m) - \tilde{q}_x - \tilde{k}_x + \lambda^2(\tilde{k}_y^2 + 2\tilde{k}_y\tilde{q}_y)]} \frac{1}{\tilde{q}_y^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(\tilde{q}_x=0, \tilde{q}_y, \tilde{\Omega}_m)}. \quad (19)$$

In the new variables,

$$\epsilon_{q,k} = \frac{\bar{g}}{\pi\lambda} \tilde{\epsilon}_{\tilde{q},\tilde{k}}, \quad \mathbf{q}^2 = \left(\frac{\bar{g}}{\pi v_F \lambda} \right)^2 \tilde{\mathbf{q}}^2,$$

$$\begin{aligned} \Pi(\mathbf{q} + \mathbf{Q}, \Omega_m) &= \left(\frac{\bar{g}}{\pi v_F \lambda} \right)^2 [\tilde{\Pi}^{(0)}(\tilde{q}_x, \tilde{q}_y, \tilde{\Omega}_m) \\ &+ \tilde{\Pi}^{(0)}(-\tilde{q}_x, \tilde{q}_y, \tilde{\Omega}_m) + \tilde{\Pi}^{(0)}(\tilde{q}_y, \tilde{q}_x, \tilde{\Omega}_m) \\ &+ \tilde{\Pi}^{(0)}(-\tilde{q}_y, \tilde{q}_x, \tilde{\Omega}_m)], \end{aligned} \quad (15)$$

where

$$\begin{aligned} \tilde{\epsilon}_{\tilde{q}} &= -\tilde{q}_x - \lambda^2 \tilde{q}_y^2, \\ \tilde{\Pi}^{(0)}(\tilde{q}_x, \tilde{q}_y, \tilde{\Omega}_m) &= \frac{1}{2} \sqrt{\tilde{\Omega}_m^2 + \tilde{E}_{\tilde{q}}^2} + \tilde{E}_{\tilde{q}}, \\ \tilde{E}_{\tilde{q}} &= \tilde{q}_x - \frac{\lambda^2 \tilde{q}_y^2}{2}. \end{aligned} \quad (16)$$

Substituting these expressions into (12) we find that the rescaled self-energy is the function of a single parameter λ :

$$\begin{aligned} \tilde{\Sigma}(\tilde{\mathbf{k}}, \tilde{\omega}_m) &= \frac{3\lambda}{8\pi^2} \int \frac{d\tilde{q}_x d\tilde{q}_y d\tilde{\Omega}_m}{\tilde{\mathbf{q}}^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(\tilde{\mathbf{q}}, \tilde{\Omega}_m)} \\ &\times \frac{i\tilde{\omega}_m - \tilde{k}_x + \lambda^2(\tilde{k}_y^2 + 2\tilde{k}_y\tilde{q}_y)}{(i\tilde{\Omega}_m - \tilde{q}_x) [i(\tilde{\Omega}_m + \tilde{\omega}_m) - \tilde{q}_x - \tilde{k}_x + \lambda^2(\tilde{k}_y^2 + 2\tilde{k}_y\tilde{q}_y)]}, \end{aligned} \quad (17)$$

where $\tilde{\Pi}_a^{(0)}(\tilde{\mathbf{q}}, \tilde{\Omega}_m) = \frac{1}{2} \sqrt{(\tilde{\Omega}_m^2 + \tilde{q}_x^2)^{1/2} + a\tilde{q}_x} + \frac{1}{2} \sqrt{(\tilde{\Omega}_m^2 + \tilde{q}_y^2)^{1/2} + a\tilde{q}_y}$. In (17) we shifted \tilde{q}_x by $\lambda^2 \tilde{q}_y^2$ and dropped all irrelevant λ^2 terms. We, however, keep \tilde{k}_y^2 term as it accounts for the renormalization of the FS curvature.

The integrand in (17), viewed as a function of \tilde{q}_x , contains two closely located poles coming from fermionic Green's functions, and the poles and branch cuts coming from spin susceptibility. The two contributions can be separated as the first one comes from small \tilde{q}_x of order $\tilde{\omega}_m$ (and $\tilde{\Omega}_m \sim \tilde{\omega}_m$), while the one from poles and branch cuts in $\chi(\mathbf{q}, \Omega_m)$ comes from \tilde{q}_x of order one. We label the first contribution as $\tilde{\Sigma}_1$ and the second as $\tilde{\Sigma}_2$.

To separate the two contributions it is convenient to divide the magnetic susceptibility into two parts as

The evaluation of $\tilde{\Sigma}_1$ is straightforward: the integral over $d\tilde{q}_x$ comes from the region where the two poles are in different half-planes of complex \tilde{q}_x . This happens when $\tilde{\Omega}_m$ is sandwiched between $-\tilde{\omega}_m$ and 0. Then both \tilde{q}_x and $\tilde{\Omega}_m$ are small, and one can safely set $\tilde{\Omega}_m = 0$ in the polarization operator. The remaining integration is straightforward, and restoring to original variables we obtain

$$\Sigma_1 = i c_1 \lambda \omega_m, \quad (20)$$

where

$$c_1 = \frac{3}{2\pi} \int_0^\infty d\tilde{q}_y \frac{1}{\tilde{q}_y^2 + \sqrt{\tilde{q}_y}/2} = \frac{2^{4/3}}{3^{1/2}} = 1.45. \quad (21)$$

We call this part a “*nonperturbative*” contribution because it comes from internal $\tilde{\Omega}_m$ and \tilde{q}_x comparable to external $\tilde{\omega}_m$, i.e., one can not obtain this term by expanding in $\tilde{\omega}_m$.

Observe that the nonperturbative contribution only depends on $\tilde{\omega}_m$, but not on \mathbf{k} . If this would be the full self-energy, then the effective mass m^* and quasiparticle residue Z would be simply related as $Zm^*/m = 1$ as in ET.

We see from (17) that the nonperturbative contribution to the self-energy remains finite at the QCP and, moreover, is small as long as λ is small. The nondivergence of $d\Sigma_1/d\omega_m$ at the QCP is the consequence of including umklapp processes into Π , along with direct processes. Out of four terms in $\Pi_a^{(0)}$, the ones with $a\tilde{q}_x$ under the square root are direct processes and the ones with $a\tilde{q}_y$ are umklapp processes. When $\tilde{q}_x = \tilde{\Omega}_m = 0$, the direct component of $\tilde{\Pi}_a^{(0)}$ vanishes, and if we would keep only this term, we would obtain that the integral over q_y in (21) diverges as $\int d\tilde{q}_y/\tilde{q}_y^2$ and has to be cut by external ω_m . In this situation, the nonperturbative self-energy would scale as $\omega_m^{3/4}$ (Refs. 8 and 35). Umklapp processes add another contribution to $\tilde{\Pi}_a^{(0)}$, which behaves as $\sqrt{2|\tilde{q}_y|}$, and the presence of such term makes the integral in (21) infrared convergent.³⁷

The second contribution to self-energy, $\tilde{\Sigma}_2$, comes from poles and branch cuts in the bosonic propagator. We dub the contribution as “*perturbative*” because typical internal \tilde{q}_x and $\tilde{\Omega}_m$ for this term are much larger than external $\tilde{\omega}_m$ and $\tilde{\mathbf{k}}$, hence one can safely expand in external momentum and frequency. We have

$$\begin{aligned} \tilde{\Sigma}_2(\tilde{\mathbf{k}}, \tilde{\omega}_m) &= \frac{3\lambda}{8\pi^2} \int d\tilde{q}_y \int d\tilde{\Omega}_m \int d\tilde{q}_x \frac{i\tilde{\omega}_m - \tilde{k}_x + \lambda^2(\tilde{k}_y^2 + 2\tilde{k}_y\tilde{q}_y)}{(i\tilde{\Omega}_m - \tilde{q}_x) [i(\tilde{\Omega}_m + \tilde{\omega}_m) - \tilde{q}_x - \tilde{k}_x + \lambda^2(\tilde{k}_y^2 + 2\tilde{k}_y\tilde{q}_y)]} \\ &\times \left[\frac{1}{\tilde{\mathbf{q}}^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(\tilde{\mathbf{q}}, \tilde{\Omega}_m)} - \frac{1}{\tilde{q}_y^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(q_x = 0, q_y, \tilde{\Omega}_m)} \right]. \end{aligned} \quad (22)$$

Expanding in $\tilde{\omega}_m$, \tilde{k}_x , and \tilde{k}_y we obtain

$$\begin{aligned} \tilde{\Sigma}_2(\tilde{\mathbf{k}}, \tilde{\omega}_m) &= \frac{3\lambda}{8\pi^2} \int d\tilde{q}_y \int d\tilde{\Omega}_m \int d\tilde{q}_x \left[\frac{i\tilde{\omega}_m - \tilde{k}_x + \lambda^2\tilde{k}_y^2}{(i\tilde{\Omega}_m + \tilde{q}_x)^2} - \frac{4\lambda^4\tilde{q}_y^2\tilde{k}_y^2}{(i\tilde{\Omega}_m + \tilde{q}_x)^3} \right] \\ &\times \left[\frac{1}{\tilde{\mathbf{q}}^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(\tilde{\mathbf{q}}, \tilde{\Omega}_m)} - \frac{1}{\tilde{q}_y^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(q_x = 0, q_y, \tilde{\Omega}_m)} \right]. \end{aligned} \quad (23)$$

One can easily make sure that the three-dimensional (3D) integral converges in the infrared and ultraviolet limits, hence all three integrals can be taken from minus to plus infinity. The term with $1/(i\tilde{\Omega}_m + \tilde{q}_x)^3$ vanishes after integration over \tilde{q}_x and $\tilde{\Omega}_m$ because it is odd in these variables, but the term with $1/(i\tilde{\Omega}_m + \tilde{q}_x)^2$ yields a finite contribution. Restoring to original variables, we obtain

$$\Sigma_2 = c_2 \lambda \left[i\omega_m - v_F \left(k_x - \kappa \frac{k_y^2}{2} \right) \right], \quad (24)$$

where

$$c_2 = \frac{3}{8\pi^2} \int d\tilde{q}_y d\tilde{\Omega}_m \int \frac{d\tilde{q}_x}{(i\tilde{\Omega}_m + \tilde{q}_x)^2} \left[\frac{1}{\tilde{\mathbf{q}}^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(\tilde{\mathbf{q}}, \tilde{\Omega}_m)} - \frac{1}{\tilde{q}_y^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(\tilde{q}_y, \tilde{\Omega}_m)} \right]. \quad (25)$$

The numerical evaluation of the integral yields

$$c_2 = -0.75. \quad (26)$$

Combining Σ_1 and Σ_2 , we finally obtain that in hot region 1,

$$\Sigma(\mathbf{k}, \omega_m) = 0.7\lambda i\omega_m + 0.75\lambda v_F \left(k_x - \kappa \frac{k_y^2}{2} \right). \quad (27)$$

Obviously, $\partial\Sigma/\partial\omega$, $(1/v_F)\partial\Sigma/\partial k_x$, and $(1/v_F)\partial\Sigma/\partial k_y^2$ are of the same order, and all three components of the self-energy have to be kept.

Substituting $\Sigma(\mathbf{k}, \omega_m)$ into the Green's function $G^{-1}(\mathbf{k}, \omega_m) = i\omega_m - \epsilon_{\mathbf{k}} + \Sigma(k_x, \omega_m)$ we find that at hot region 1,

$$G(\mathbf{k}, \omega_m) = \frac{Z}{i\omega_m + v_F^* \left(k_x + \kappa^* \frac{k_y^2}{2} \right)}, \quad (28)$$

where, to first order in λ ,

$$\begin{aligned} Z &= 1 - 0.7\lambda, \quad v_F^* = v_F \frac{m}{m^*} = 1 + 0.05\lambda, \\ \kappa^* &= \kappa(1 - 1.45\lambda). \end{aligned} \quad (29)$$

We see that the dominant effect of the self-energy is the renormalization of the quasiparticle residue Z and the renormalization of the curvature κ . The renormalization of the Fermi velocity is much smaller.

The imaginary part of the self-energy has been calculated in Ref. 37. At low frequencies it scales as $\omega^{3/2} \ln \omega$. The frequency dependence is stronger than in a ‘‘conventional’’ FL, but still, $\text{Im}\Sigma(k, \omega) \ll \omega/Z$ at small enough frequencies, hence quasiparticles near the FS remain well defined.

IV. PAIRING PROBLEM

The straightforward way to analyze whether a fermionic system becomes superconducting below some T_c is to introduce an infinitesimally small pairing vertex $\Phi_{\alpha\beta}^{(0)}(k)\psi_\alpha(k)\psi_\beta(-k)$, where k stands for a three-component vector (\mathbf{k}, ω_m) , renormalize it by the pairing interaction, and verify whether the pairing susceptibility diverges at some T . The divergence of susceptibility at some $T = T_c$ implies that, below this temperature, the system is unstable against a spontaneous generation of a nonzero $\Phi_{\alpha\beta}(k)$, even if we set $\Phi_{\alpha\beta}^{(0)}(k) = 0$. For spin-singlet superconductivity, the spin dependence of the pairing vertex is $\Phi_{\alpha\beta}(k) = i\sigma_{\alpha\beta}^y \Phi(k)$.

To obtain T_c with logarithmic accuracy at weak coupling (small λ), one can restrict with only ladder diagrams for $\Phi(k)$. Each additional ladder insertion contains $a\lambda \ln \Lambda/T$, where $a = O(1)$. Ladder series are geometrical, and summing them up one obtains $T_c \sim \Lambda e^{-a/\lambda}$. More efforts are required, however, to get the prefactor. Which diagrams have to be included depends on what theory is applicable. In Eliashberg-type theories, all nonladder diagrams have additional smallness (in ω_D/E_F for electron-phonon interaction) and can be neglected. In this situation, one still can restrict with ladder diagrams, but has to solve for the full dynamical $\Phi(k)$ beyond logarithmic accuracy, and also include fermionic self-energy to order λ .

As an example, consider momentary Eliashberg theory for the pairing by a single Einstein phonon. The attractive electron-phonon interaction depends on transferred frequency Ω as $\lambda/[1 + (\Omega/\omega_D)^2]$. At small λ , the normal state self-energy is $\Sigma = i\lambda\omega_m$, and the frequency dependence of the pairing vertex can be approximated as $\Phi(k) = \Phi(\mathbf{k}, \omega) = \Phi_0/[1 + (\omega/\omega_D)^2]$ (see Appendix A). Summing up ladder series, one then obtains,^{48,50} up to corrections $O(\lambda)$,

$$T_c = 1.13e^{-1/2}\omega_D e^{-\frac{1+\lambda}{\lambda}} = 0.25\omega_D e^{-\frac{1}{\lambda}}. \quad (30)$$

This result, rather than frequently cited BCS expression $T_c = 1.13\omega_D e^{-\frac{1}{\lambda}}$, is the correct T_c for weak electron-phonon interaction.

In our case, the Eliashberg parameter is of order one, and we have to include on equal footing (a) ladder diagrams, which have to be taken beyond logarithmic accuracy by including the frequency dependence of the interaction, (b) the renormalization of quasiparticle Z , v_F , and κ , (c) vertex correction to the spin polarization bubble $\Pi(\mathbf{q} + \mathbf{Q}, \Omega_m)$, and (d) Kohn-

Luttinger-type exchange renormalization of the pairing interaction, which in our case includes vertex corrections to spin-mediated pairing interaction and exchange diagram with two crossed spin-fluctuation propagators. To order $O(\lambda)$, which we will need to get the prefactor in T_c , these four contributions add up and can be evaluated independently. On the other hand we do not need to subtract from the gap equation the contribution with $k_F = 0$, as it was done in Ref. 40, and add the subtracted part to the renormalization of the coupling \bar{g} into the scattering amplitude. Such contributions come from energies above the upper cutoff of our low-energy theory Λ and are already incorporated into the spin-fermion coupling \bar{g} , which, by construction, incorporates all renormalizations from fermions with energies larger than Λ . In momentum space, the scale Λ roughly corresponds to $|k - k_F| \sim k_F$, but can be smaller.

A. Ladder diagrams

We begin with ladder diagrams. We consider spin-singlet pairing between fermions with k and $-k$, located in opposite hot regions along the same diagonal (see Fig. 5). Because the pairing vertex is a spin singlet, $\Phi_{\alpha\beta}(k) = i\sigma_{\alpha\beta}^y \Phi(k)$. We denote by $\Phi_0(k)$ and $\Phi_Q(k)$ the pairing vertices with momenta near \mathbf{k} and $\mathbf{k} + \mathbf{Q}$, respectively, and treat k as a small deviation from the corresponding hot spot. Each ladder diagram renormalizes the pairing vertex by $\int_q G(q)G(-q)\chi(k - q + Q)$, where $\sum_q = T \sum_{\omega_m} \int d^2q/(2\pi)^2$ and $Q = (\mathbf{Q}, 0)$ in 3D notations. The ladder diagrams are readily summed up and at $T = T_c$ give rise to the integral equation for $\Phi(k)$ in the form

$$\Phi_0(k) = -3g^2 \sum_q \Phi_Q(q)G(q)G(-q)\chi(k - q + Q), \quad (31)$$

where χ is given by Eq. (9). The overall factor 3 comes from the convolutions of Pauli matrices at each vertex $\sigma_{\alpha'\beta'}^y \sigma_{\alpha\alpha'}^i \sigma_{\beta\beta'}^i = -3\sigma_{\alpha\beta}^y$, and the overall minus sign reflects the repulsive nature of the interaction. Superconducting instability is then possible only when $\Phi_0(k) = -\Phi_Q(k)$. The spin-singlet nature of pairing requires $\Phi(k)$ to be an even function of the actual 2D momentum, which in our notations implies that $\Phi_0(\mathbf{k}, \omega_m) = \Phi_Q(-\mathbf{k}, \omega_m)$. Combined with $\Phi_0(\mathbf{k}, \omega_m) = -\Phi_Q(\mathbf{k}, \omega_m)$, this requires $\Phi_0(-\mathbf{k}, \omega_m) = -\Phi_0(\mathbf{k}, \omega_m)$ and the same for Φ_Q . Obviously then, the pairing amplitude passes through zero along the diagonals, i.e., the pairing symmetry is d wave. For simplicity, below we replace $\Phi_0(k)$ by $\Phi(k)$ and $\Phi_Q(k)$ by $-\Phi(k)$ and treat $\Phi(k)$ as an odd function of momentum (which, we remind, is counted from the corresponding hot spot along the diagonal). With this substitution, there will be no minus sign in the right-hand side of the gap equation.

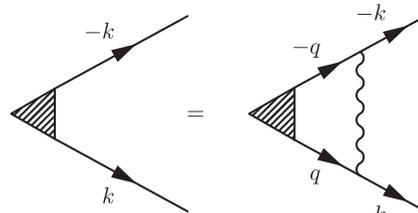


FIG. 5. Ladder diagrams for the pairing vertex. The wavy line denotes the interaction mediated by spin fluctuations.

To simplify the calculation, we introduce the same set of rescaled dimensionless momenta and frequencies as before [Eq. (14)], and also rescale the temperature:

$$q \rightarrow \frac{\bar{g}\tilde{q}}{v_F\pi\lambda}, \quad \omega_m \rightarrow \frac{\bar{g}\tilde{\omega}_m}{\pi\lambda}, \quad \tilde{T} = \frac{\pi T\lambda}{\bar{g}}. \quad (32)$$

In the new variables we have, instead of (31),

$$\begin{aligned} & \Phi(\tilde{\mathbf{k}}, \tilde{\omega}_m) \\ &= \frac{3\lambda}{4\pi} \tilde{T} \sum_{\tilde{\omega}'_m} \int \frac{d\tilde{\mathbf{q}}}{\tilde{\omega}'_m{}^2 + \tilde{q}_x^2} \chi^{(0)}(\tilde{\mathbf{k}} - \tilde{\mathbf{q}}, \tilde{\omega}_m - \tilde{\omega}'_m) \Phi(\tilde{\mathbf{q}}, \tilde{\omega}'_m), \end{aligned} \quad (33)$$

where

$$\begin{aligned} & \chi^{(0)}(\tilde{\mathbf{k}} - \tilde{\mathbf{q}}, \tilde{\omega}_m - \tilde{\omega}'_m) \\ &= \frac{1}{(\tilde{\mathbf{k}} - \tilde{\mathbf{q}})^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(\tilde{\mathbf{k}} - \tilde{\mathbf{q}}, \tilde{\omega}_m - \tilde{\omega}'_m)}. \end{aligned} \quad (34)$$

As we did before, we have shifted \tilde{q}_x by $\lambda^2 \tilde{q}_y^2$ in (33) and dropped λ^2 terms in the spin susceptibility $\chi^{(0)}$. To simplify the presentation, below we will drop the tilde from the intermediate momentum and frequencies.

We focus our attention on the pairing between fermions in regions 1 and 4. The corresponding pairing vertex is an odd function of k_y (the momentum component along the FS at a hot spot). For small k_y , we approximate the pairing vertex by $\Phi(\mathbf{k}, \omega'_m) = k_y \Phi(k_x, \omega'_m)$. The only other place in (33) where the dependence on k_y is present is the spin susceptibility. However, it depends only on the relative momentum transfer $\mathbf{k} - \mathbf{q}$ and is an even function of the latter. In this situation, the integration over q_y gives the result proportional to k_y , consistent with our approximation that $\Phi(\mathbf{k}, \omega_m) = k_y \Phi(k_x, \omega_m)$. In explicit form, we have

$$\begin{aligned} & \Phi(k_x, \omega_m) \\ &= \frac{3\lambda}{4\pi} \tilde{T} \sum_{\omega'_m} \int \frac{dq_x \chi^{(0)}(k_x - q_x, \omega_m - \omega'_m)}{\omega_m^2 + q_x^2} \Phi(q_x, \omega'_m), \end{aligned} \quad (35)$$

where

$$\begin{aligned} & \chi^{(0)}(k_x - q_x, \omega_m - \omega'_m) \\ &= \int \frac{dk_y}{k_y^2 + (k_x - q_x)^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(k_x - q_x, k_y, \omega_m - \omega'_m)}. \end{aligned} \quad (36)$$

The function $\chi^{(0)}(k_x - q_x, \omega_m - \omega'_m)$ plays the role of the pairing kernel. Like for electron-phonon systems, it tends to a finite value when $k_x - q_x$ and $\omega_m - \omega'_m$ vanish:

$$\chi^{(0)} \equiv \int_{-\infty}^{\infty} \frac{dk_y}{k_y^2 + \sqrt{|k_y|}/2} = \frac{2^{10/3}\pi}{3^{3/2}} = 6.09. \quad (37)$$

To obtain the exponential term in T_c , we can just pull this constant out of $\tilde{T} \sum_{\omega'_m} \int dq_x$ and evaluate the rest to logarithmical accuracy. We obtain

$$T_c \propto e^{-\frac{4\pi}{3\lambda\chi^{(0)}}} = e^{-\frac{0.6874}{\lambda}}. \quad (38)$$

To find the contribution from the ladder diagram to the prefactor, we use the same strategy as for the electron-phonon case (see Appendix A) and write

$$\chi^{(0)}(k_x - q_x, \omega_m - \omega'_m) = \chi^{(0)}(k_x, \omega_m) + \delta\chi^{(0)}(k_x, q_x; \omega_m, \omega'_m), \quad (39)$$

where $\delta\chi^{(0)}(k_x, 0; \omega_m, 0) = 0$. Substituting into Eq. (35), we obtain

$$\begin{aligned} \Phi(k_x, \omega_m) &= \frac{3\lambda}{4\pi} \tilde{T} \sum_{\omega'_m} \int dq_x \frac{1}{\omega_m^2 + q_x^2} [\chi^{(0)}(k_x, \omega_m) \\ &+ \delta\chi^{(0)}(k_x, q_x; \omega_m, \omega'_m)] \Phi(\omega'_m, q_x). \end{aligned} \quad (40)$$

Of the two terms in the last line, the first one contains $\lambda \ln \tilde{T} \sim O(1)$, while the second one [with $\delta\chi^{(0)}(k_x, q_x; \omega_m, \omega'_m)$] is convergent in the infrared limit and is of order of λ . The structure on the right-hand side is reproduced on the left-hand side if we take the pairing vertex in the form

$$\Phi(k_x, \omega_m) = \Phi^0[\chi^{(0)}(k_x, \omega_m) + \lambda\delta\Phi(k_x, \omega_m)]. \quad (41)$$

Substituting this form back into Eq. (35) for Φ , we obtain

$$\begin{aligned} & \chi^{(0)}(k_x, \omega_m) + \lambda\delta\Phi(k_x, \omega_m) \\ &= \frac{3\lambda}{4\pi} \tilde{T} \sum_{\omega'_m} \int dq_x \frac{\chi^{(0)}(k_x, \omega_m) + \delta\chi^{(0)}(k_x, q_x; \omega_m, \omega'_m)}{q_x^2 + \omega_m^2} [\chi^{(0)}(q_x, \omega'_m) + \lambda\delta\Phi(q_x, \omega'_m)] \\ &= \frac{3\lambda^2}{4\pi} \tilde{T} \sum_{\omega'_m} \int dq_x \frac{\chi^{(0)}(k_x, \omega_m)}{q_x^2 + \omega_m^2} \delta\Phi(q_x, \omega'_m) + \frac{3\lambda}{4\pi} \tilde{T} \sum_{\omega'_m} \int dq_x \frac{\chi^{(0)}(q_x, \omega'_m) \chi^{(0)}(k_x - q_x, \omega_m - \omega'_m)}{q_x^2 + \omega_m^2} + O(\lambda^2). \end{aligned} \quad (42)$$

To order λ we then have

$$\delta\Phi(k_x, \omega_m) = \chi^{(0)}(k_x, \omega_m) [P + R(k_x, \omega_m)], \quad (43)$$

where

$$P = \frac{3\lambda}{4\pi} \tilde{T} \sum_{\omega'_m} \int dq_x \frac{\delta\Phi(q_x, \omega'_m)}{q_x^2 + \omega_m^2}, \quad (44)$$

$$R(k_x, \omega_m) = \frac{3}{4\pi} \tilde{T} \sum_{\omega'_m} \int dq_x \frac{\chi^{(0)}(q_x, \omega'_m) \chi^{(0)}(k_x - q_x, \omega_m - \omega'_m)}{\chi^{(0)}(k_x, \omega_m) (q_x^2 + \omega_m^2)} - \frac{1}{\lambda}. \quad (45)$$

Substituting Eq. (43) into the right-hand side of Eq. (44) we obtain

$$\begin{aligned} & \left[1 - \frac{3\lambda}{4\pi} \tilde{T} \sum_{\omega'_m} \int dq_x \frac{\chi^{(0)}(q_x, \omega'_m)}{q_x^2 + \omega'_m{}^2} \right] P \\ &= \frac{3\lambda}{4\pi} \tilde{T} \sum_{\omega'_m} \int dq_x \frac{\chi^{(0)}(q_x, \omega'_m)}{q_x^2 + \omega'_m{}^2} R(q_x, \omega'_m). \end{aligned} \quad (46)$$

We now use the fact that $P = O(1)$, while the expression in the brackets of the left-hand side is of order $O(\lambda)$. The left-hand side of (46) is then of $O(\lambda)$. The right-hand side is $R(0,0)[1 + O(\lambda)]$. Obviously then $R(0,0) = O(\lambda)$, i.e.,

$$\frac{3\lambda\chi^{(0)}}{4\pi} \tilde{T} \sum_{\omega'_m} \int dq_x \frac{[\chi^{(0)}(q_x, \omega'_m)/\chi^{(0)}]^2}{(q_x^2 + \omega'_m{}^2)} = 1 + O(\lambda^2). \quad (47)$$

Evaluating the integral over q_x and the sum over Matsubara frequencies, we obtain

$$\frac{4\pi}{3\chi^{(0)}} = \lambda \ln \frac{0.00874}{\tilde{T}_c} + O(\lambda^2) \quad (48)$$

or, in original notations, $T_c = T_{c1}$ from ladder diagrams is

$$T_{c1} = 0.00249 \frac{\bar{g}}{\lambda} e^{-\frac{0.6874}{\lambda}}. \quad (49)$$

B. Effect of fermionic self-energy

The self-energy corrections to ladder diagrams can be easily incorporated because in the FL regime their only role is to renormalize the quasiparticle Z , the Fermi velocity v_F , and the FS curvature κ . All three renormalizations can be absorbed into the renormalization of λ :

$$\lambda = \left(\frac{\bar{g}\kappa}{2\pi v_F} \right)^{1/3} \rightarrow \left(\frac{\bar{g}Z^2\kappa^*}{2\pi v_F^*} \right)^{1/3} = \lambda(1 - 0.95\lambda). \quad (50)$$

Substituting this renormalization into Eq. (49) we obtain

$$T_{c2} = 0.00130 \frac{\bar{g}}{\lambda} e^{-\frac{0.6874}{\lambda}}. \quad (51)$$

If Eliashberg theory was applicable to our problem, this would be the full result for T_c . However, as we already discussed, in our case the Eliashberg parameter is of order one, and other renormalizations also play a role. Specifically, there are two extra contributions: from vertex corrections to the polarization operator and from Kohn-Luttinger renormalization of the irreducible pairing interaction. The two contributions add up and we consider them separately.

C. Correction to T_c due to the renormalization of the polarization operator

The exponential factor in the expressions for T_{c1} and T_{c2} is proportional to the integral

$$\begin{aligned} \chi^{(0)} &= \int dq_y \chi(q_x = 0, q_y, \omega_m = 0) \\ &= \int dq_y \frac{1}{q_y^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(0, q_y, 0)}. \end{aligned} \quad (52)$$

In evaluating this integral, we used the free-fermion form of the polarization operator, in which case $\sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(0, q_y, 0) =$

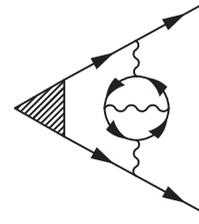


FIG. 6. Ladder diagrams with the effective interaction which includes vertex correction to the polarization bubble.

$\sqrt{|q_y|/2}$ and $\chi^{(0)} = 6.09$. However, to get the prefactor in T_c , we need to know the exponential factor with accuracy $O(\lambda)$. Self-energy contributions to $\tilde{\Pi}_a^{(0)}(0, q_y, 0)$ are incorporated into the renormalization of λ in (50) and are accounted for in Eq. (51). However, the vertex correction to $\tilde{\Pi}_a^{(0)}(0, q_y, 0)$ also contributes the term of order λ , and this term has to be included.

The effective pairing interaction with vertex correction to the polarization bubble included is shown in Fig. 6. For a generic \mathbf{q} and Ω_m , the computation of the vertex renormalization is rather messy.³⁶ For our purposes, however, we will only need the vertex correction for a static polarization bubble at $\mathbf{q} = (0, q_y)$, and only for the *umklapp* process (i.e., in our case, the contribution to Π from virtual fermions in hot regions 3 and 2). We present the computation of the vertex correction to Π in Appendix B and here state the result: this renormalization changes $\chi^{(0)}$ by an $O(\lambda)$ term:

$$\chi^{(0)} \rightarrow \chi^{(0)}(1 - 0.042\lambda). \quad (53)$$

Including this renormalization into the expression for T_c [Eq. (49)], we find

$$T_{c3} = 0.00126 \frac{\bar{g}}{\lambda} e^{-\frac{0.6874}{\lambda}}. \quad (54)$$

D. Kohn-Luttinger-type corrections to effective interaction

Finally, we consider the effect on T_c from Kohn-Luttinger-type second-order corrections to the effective pairing interaction. We show the corresponding diagrams in Fig. 7. Compared to the original Kohn-Luttinger work,⁴¹ we have dropped one diagram since in our case it is already included into $\chi^{(0)}(k)$.

Similar to what we did before with the dynamical part of the interaction in ladder series, we analyze the equation for the pairing vertex using the full χ which we split into the original and the Kohn-Luttinger terms. We have

$$\Phi(\mathbf{k}, \omega_m) = \frac{3\lambda}{4\pi} \tilde{T} \sum_{\omega'_m} \int \frac{d\mathbf{q}}{\omega_m^2 + q_x^2} \chi(\mathbf{k}, \omega_m; \mathbf{q}, \omega'_m) \Phi(\mathbf{q}, \omega'_m), \quad (55)$$

where

$$\begin{aligned} \chi(\mathbf{k}, \omega_m; \mathbf{q}, \omega'_m) &= \chi^{(0)}(k_x - q_x, k_y - q_y, \omega_m - \omega'_m) \\ &+ \lambda \chi^{(1)}(k_x, k_y, \omega_m; q_x, q_y, \omega'_m); \end{aligned} \quad (56)$$

$\chi^{(0)}(k - q)$, defined in Eq. (34), accounts for the first graph in Fig. 7, and $\lambda \chi^{(1)}(k, q)$ accounts for the other three terms.

The first term in $\chi(k, q)$ depends on the momentum transfer $k_y - q_y$. For this reason we could shift, in the right-hand side of (50), the integration over q_y by the integral over $q_y - k_y$ and obtain the pairing vertex as a linear function of k_y . This

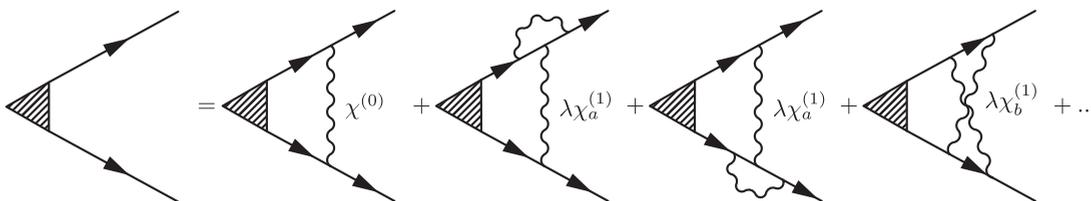


FIG. 7. Kohn-Luttinger diagrams for the irreducible pairing interaction. The diagram with internal particle-hole bubble is already included into $\chi^{(0)}(k)$ and has to be dropped to avoid double counting. Each of the remaining three Kohn-Luttinger diagrams gives contribution of order λ .

simple scaling with k_y does not extend to Kohn-Luttinger terms because $\chi^{(1)}(k, q)$ depends separately on k and on q . Accordingly, we write

$$\Phi(k_x, k_y, \omega_m) = \Phi^0[k_y \chi^{(0)}(k_x, \omega_m) + \lambda \delta \Phi(k_x, k_y, \omega_m)]. \quad (57)$$

Because we are interested in $O(\lambda)$ terms, we can safely neglect the difference between continuous and discrete Matsubara frequencies and treat ω_m as continuous variable.

Plugging (57) back to the pairing equation and formally setting $k_x = \omega_m = 0$, we obtain

$$\begin{aligned} k_y \chi^{(0)} + \lambda \delta \Phi(0, k_y, 0) &= \frac{3\lambda}{4\pi} \tilde{T} \sum \int \frac{dq_x}{(\omega'_m)^2 + q_x^2} k_y (\chi^{(0)}(q_x, \omega'_m))^2 \\ &+ \frac{3\lambda^2}{4\pi} \tilde{T} \sum \int \frac{dq_x}{(\omega'_m)^2 + q_x^2} \int dq_y \chi^{(1)}(0, k_y, 0; q_x, q_y, \omega'_m) q_y \chi^{(0)}(q_x, \omega'_m) \\ &+ \frac{3\lambda^2}{4\pi} \tilde{T} \sum \int \frac{dq_x}{(\omega'_m)^2 + q_x^2} \int dq_y \chi^{(0)}(q_x, k_y - q_y, \omega'_m) \delta \Phi(q_x, q_y, \omega'_m). \end{aligned} \quad (58)$$

Evaluating the integrals, we obtain

$$\begin{aligned} k_y \chi^{(0)} + \lambda \delta \Phi(0, k_y, 0) &= k_y \frac{3\lambda}{4\pi} \chi^{(0)2} \ln \frac{0.00874}{\tilde{T}} + \frac{3\lambda^2}{4\pi} \chi^{(0)} \ln \frac{a}{\tilde{T}} \int dq_y \chi^{(1)}(0, k_y, 0; 0, q_y, 0) q_y \\ &+ \frac{3\lambda^2}{4\pi} \ln \frac{b}{\tilde{T}} \int dq_y \chi^{(0)}(0, k_y - q_y, 0) \delta \Phi(0, q_y, 0) + O(\lambda^2), \end{aligned} \quad (59)$$

where a and b are constants of order one.

Simplifying the notations and rearranging, we re-express (59) as

$$\frac{4\pi}{3\lambda \chi^{(0)}} = \ln \frac{0.00874}{\tilde{T}} + \frac{A(k_y) + C(k_y)}{k_y (\chi^{(0)})^2} + O(\lambda^2), \quad (60)$$

where we defined

$$A(k_y) = \int dq_y \left[\frac{\chi^{(0)}(k_y - q_y)}{\chi^{(0)}} \delta \Phi(0, q_y, 0) \right] - \delta \Phi(0, k_y, 0), \quad (61)$$

$$C(k_y) = \int dq_y q_y \chi^{(1)}(0, k_y, 0; 0, q_y, 0),$$

and $\chi^{(0)}(k_y - q_y) = 1/[(k_y - q_y)^2 + \sqrt{|k_y - q_y|/2}]$. One can easily make sure that $A(0) = C(0) = 0$, hence $[A(k_y) + \xi^{(1)}(k_y)]/k_y$ is not singular when k_y vanishes. However, Eq. (59) sets a more stringent requirement: $A(k_y) + C(k_y)$ must be equal to $B k_y (\chi^{(0)})^2$, where B is a constant. Then $T_c = T_{c3} e^B$, where T_{c3} is given by Eq. (54).

The $A(k_y)$ in Eq. (61) contains $\delta \Phi(0, k_y, 0)$ and the integral over q_y of $\delta \Phi(0, q_y, 0)$, weighted with a kernel. The condition

$$A(k_y) + C(k_y) = B k_y (\chi^{(0)})^2 \quad (62)$$

then sets the integral equation on $\delta \Phi(0, k_y, 0)$. We solve this equation in Appendix C and obtain B in the form $B = C(-\Lambda)/(\chi^{(0)})^2$, where Λ is the dimensionless momentum cutoff along the FS. The value of B depends on the interplay between λ and $1/\Lambda$.

For a generic FS, the momentum cutoff Λ is of order k_F , in which case $\lambda \Lambda \sim (W/\bar{g})^{1/3} \gg 1$, where W is the fermionic bandwidth. In this situation, the pairing interaction dies off at momenta k_y , which are parametrically smaller than the cutoff, and $C(-\Lambda)$ is small in $1/(\lambda \Lambda)$. Then B is also small in $1/(\lambda \Lambda)$, hence $e^B \approx 1$, i.e., the correction to T_c from Kohn-Luttinger diagrams can be neglected. In this situation, the fully renormalized T_c coincides with T_{c3} and is given by

$$T_c = 0.00126 \frac{\bar{g}}{\lambda} e^{-\frac{0.6874}{\lambda}}. \quad (63)$$

Equation (63) is the main result of this paper.

If, by some reasons, Λ is numerically much smaller than k_F , such that $\lambda \Lambda$ is actually a small number, the renormalization of T_c due to Kohn-Luttinger diagrams becomes relevant. We present the calculation of B for this case in Appendix C. We find that, at small $\lambda \Lambda$, B is logarithmically singular: $B = (10/3) \ln 1/(\lambda \Lambda) + \dots$ where the ellipsis stand for terms $O(1)$. As a result, T_c is enhanced by the factor $(1/\lambda \Lambda)^{10/3}$

compared to Eq. (63). The outcome is that Eq. (63) provides the lower boundary for T_c : the actual T_c gets enhanced by Kohn-Luttinger contributions. How strong the enhancement is depends on the actual band structure, which sets the value of Λ .

E. Comparison with the experiments on electron-doped cuprates

We now compare our theoretical T_c [Eq. (63)] with the data for near-optimally doped $\text{Nd}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_4$ and $\text{Pr}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_4$, in which doping creates extra electrons. The parameters of the quasiparticle dispersion, v_F and κ , can be extracted from the ARPES measurements on $\text{Nd}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_4$.⁴³ We found $v_F = 0.87$ eV and $\kappa = 0.31$ (in units where the lattice constant $a = 1$). Similar parameters have been obtained in Ref. 24. The only other input parameter for the theory is the strength of spin-fermion coupling \bar{g} . For hole-doped cuprates, the fits to ARPES and NMR data in the normal state yielded $\bar{g} \leq 2$ eV (Ref. 3). This \bar{g} is consistent with the value of the charge-transfer gap in the effective Hubbard model in the Mott-Heisenberg regime at half-filling⁵² as calculations in the ordered state of the spin-fermion model⁵³ place the gap to be exactly \bar{g} ; quantum corrections cancel out. This consistency is not an anticipated result as \bar{g} extracted from the optics is the coupling at high energies, comparable to E_F , while the one used in the comparison with ARPES and NMR is the coupling at low energies (below our Λ), where, strictly speaking, the spin-fermion model is only valid. The (rough) agreement between the two likely implies that renormalizations between E_F and Λ do reduce \bar{g} , but only by a small fraction. For electron-doped cuprates, the detailed fits of $\Sigma(k, \omega)$ in the spin-fermion model to the self-energy, extracted from ARPES data, have not been done yet, but optical measurements⁴² show that the charge-transfer gap is about 1.7 eV. Assuming that the situation in electron-doped cuprates is the same as in hole-doped cuprates, i.e., that spin-fermion couplings, extracted from ARPES and optics, are not far each other, we just take \bar{g} to be equal to this 1.7 eV.

Using the numbers for \bar{g} , v_F , and κ , we obtain $\lambda \sim 0.46$, which implies that weak coupling analysis should be applicable. Substituting $\lambda = 0.46$ and $\bar{g} \approx 20\,000$ K, we find $T_c \sim 0.0006\bar{g} \sim 12$ K. This is reasonably close to the experimental $T_c = 20\text{--}24$ K in optimally doped $\text{Nd}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_4$,^{46,47} particularly given that our theoretical T_c [Eq. (63)] is the lower boundary for the actual T_c because (i) as we found above, T_c goes up once we include corrections due to a finite upper cutoff of the theory, and (ii) in real situation, hot spots at optimal doping are still located at some distance from each other, in which case the value of T_c should move a bit towards the one when hot spots are well separated, and the latter is much higher: when hot spots are near $(0, \pi)$ and symmetry-related points $T_c \sim 0.04\bar{g}$.^{19,45}

The transition temperature in the similar range of 10–20 K has been found in FLEX calculations,²⁴ and the agreement between our and FLEX results in an encouraging sign. The authors of Refs. 24 and 54 considered the model with a static interaction V , extracted V by fitting the value of the magnetization in the antiferromagnetically ordered state, and used BCS formula for T_c . Amazingly, their T_c is quite

similar to the one we obtained. The two-particle self-consistent approach, applied to the Hubbard model with nearest-neighbor hopping only and values of the interaction U typical of electron-doped systems, yields a much higher optimal $T_c \sim 200$ K (Refs. 4 and 24). However, given the sensitivity of T_c to the specifics of the FS,⁵⁵ the value of T_c in this approach has to be reanalyzed using the model for the hopping consistent with the measured FS.

V. SUMMARY

In this work, we revisited the issue of normal state renormalizations and superconducting T_c in electron-doped cuprates near optimal doping. We used the spin-fermion model to model electronic interactions and assumed that the doping at which magnetic order with $\mathbf{Q} = (\pi, \pi)$ sets in is close to the one at which the Fermi surface touches the magnetic Brillouin zone boundary along the zone diagonals (this case is often labeled as $\mathbf{Q} = 2\mathbf{k}_F$).

Quantum-critical fluctuations and the pairing instability in the $\mathbf{Q} = 2\mathbf{k}_F$ case have been studied before.^{8,35} However, recent work³⁷ has shown that earlier analysis did not include umklapp processes and, as a result, severely overestimated the strength of quantum-critical fluctuations. Once umklapp processes are properly accounted for, the real part of the self-energy at a QCP scales as ω and the imaginary part behaves as $\omega^{3/2} \ln \omega$, i.e., fermionic coherence is preserved at the lowest energies.

The goal of this work was to revisit the calculation of T_c . We found that the argument³⁵ that the $d_{x^2-y^2}$ superconductivity survives when hot spots merge along Brillouin zone diagonals holds. However, the value of T_c has to be reconsidered. The calculation of T_c requires one to know the renormalization of the quasiparticle propagator in the normal state, and in the first part of the paper we computed the real part of the fermionic self-energy (which was not considered in Ref. 37). We found that Eliashberg approximation is not valid because the Eliashberg parameter is of order one, and the only way to proceed with calculations is to perform a direct perturbative loop expansion. We found that $\text{Re } \Sigma(k, \omega)$ is a regular function of momentum and frequency, and the renormalizations of the quasiparticle residue Z , Fermi velocity v_F , and the FS curvature κ hold on powers of the single dimensionless parameter λ . We treated λ as a small parameter and obtained Z , v_F , and κ to order $O(\lambda)$.

We then used normal state results as an input and computed superconducting T_c by solving the $(2+1)$ -dimensional gap equation in momentum and frequency. To logarithmical accuracy, the solution of the linearized gap equation is similar to that in BCS theory, and $T_c \propto e^{-a/\lambda}$, where in our case $a = 0.6874$. We, however, computed T_c with the prefactor, which required us to go one step beyond BCS approximation and include the frequency dependence of the interaction, the renormalizations of Z , v_F , and κ , vertex corrections to particle-hole polarization bubble, and Kohn-Luttinger (nonladder) corrections to the irreducible pairing interaction. Using the parameters extracted from the data on optimally electron-doped cuprates, we found $T_c \geq 10$ K, which is in a reasonably good agreement with the experimental values. The agreement is particularly striking because our result is the lower boundary of the actual T_c .

One issue brought about by our work in comparison with earlier works on the Hubbard model^{4,24,28,55} is the origin of the difference between hole- and electron-doped cuprates. The reasoning displayed in Refs. 4, 28, 24, 55, and 56 is that the interaction U is somewhat smaller in electron-doped cuprates than in hole-doped cuprates such that in electron-doped materials correlations are relevant, but Mott physics does not develop. This is certainly a valid point as, e.g., the magnetic T_N is smaller in half-filled $\text{Pr}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_4$, and $\text{Pr}_{2-x}\text{Ce}_x\text{Cu}_3\text{O}_4$ than in undoped La and Y based materials). Our results, however, point on a complementary reason for the difference between near-optimally hole and electron-doped cuprates. Namely, even if interaction (our \bar{g}) is the same, there is still a substantial difference between the magnitude of fermionic self-energy and of superconducting T_c due to the difference in the geometry of the electronic FS.

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APPENDIX A: T_c AT WEAK COUPLING IN A PHONON SUPERCONDUCTOR

A portion of our calculation of T_c is similar to the calculation of T_c in the weak-coupling limit of the Eliashberg theory for a phonon superconductor for the case when a phonon propagator can be approximated by a single Einstein mode:

$$\chi_{ph}(\Omega_m) = \frac{\chi_0}{\Omega_m^2 + \omega_D^2}. \quad (\text{A1})$$

In Eliashberg theory, T_c is the temperature at which the linearized equation for the pairing vertex $\Phi(\Omega_m)$ has a nonzero solution. The equation for $\Phi(\Omega_m)$ is well known^{18,48–50} and in rescaled variables $\bar{T} = T/\omega_D$, $\bar{\omega}_m = \omega_m/\omega_D = \pi\bar{T}(2m+1)$ reads as

$$\Phi(\bar{\Omega}_m) = \pi\bar{T}\lambda^* \sum_m \frac{\Phi(\bar{\omega}_m)}{|\bar{\omega}_m|} \chi(\bar{\omega}_m, \bar{\Omega}_m), \quad (\text{A2})$$

where $\chi(\bar{\omega}_m, \bar{\Omega}_m) \equiv \frac{1}{1+(\bar{\omega}_m-\bar{\Omega}_m)^2}$ and $\lambda^* = \lambda/(1+\lambda)$ [λ is dimensionless effective electron-phonon coupling and the factor $(1+\lambda)$ comes from mass renormalization]. The formula for T_c in Eliashberg theory at weak coupling has been discussed several times in the past.^{48–50} However, until now, there is some confusion about the interplay between the weak coupling limit of the Eliashberg theory and the BCS theory.¹⁸ Within BCS theory (extended to include $1+\lambda$ mass renormalization), the pairing vertex is approximated by a constant and the dependence on the external $\bar{\Omega}_m$ in the bosonic propagator is neglected. The equation for T_c then reduces to

$$1 = \lambda^* \sum_m \frac{1}{|2m+1|} \frac{1}{1 + \pi^2 \bar{T}^2 (2m+1)^2}. \quad (\text{A3})$$

The sum in the right-hand side converges at the largest m and is expressed in terms of digamma functions. At small \bar{T} it

reduces to $\ln 1.13/\bar{T}$. From (A4) we then obtain, in original notations,

$$T_c^{\text{BCS}} = 1.13\omega_D e^{-\frac{1+\lambda}{\lambda}}. \quad (\text{A4})$$

The point made in Refs. 48–50 is that this expression is *not* the correct T_c in the small λ limit of the Eliashberg theory. The correct formula, obtained first in Ref. 48 (see also Refs. 49 and 50), is

$$T_c = 1.13e^{-1/2}\omega_D e^{-\frac{1+\lambda}{\lambda}} = 0.69\omega_D e^{-\frac{1+\lambda}{\lambda}}. \quad (\text{A5})$$

The reason for the discrepancy between Eqs. (A4) and (A5) is that in Eliashberg theory the numerical prefactor in T_c comes from fermions with energies of order ω_D [$\bar{\omega}_m = O(1)$], and for such fermions the dependence of the pairing vertex $\Phi(\bar{\omega}_m)$ on $\bar{\omega}_m$ can not be neglected.

The computational procedure presented in Ref. 48 and in subsequent work⁵⁰ uses the iteration method and is somewhat involved. Below we present an alternative computation procedure to obtain Eq. (A5). We use the same procedure in the calculations of T_c for our case of electron-doped cuprates. We reexpress $\chi(\bar{\omega}_m, \bar{\Omega}_m) \equiv \frac{1}{1+(\bar{\omega}_m-\bar{\Omega}_m)^2}$ in Eq. (A2) as

$$\begin{aligned} \chi(\bar{\omega}_m, \bar{\Omega}_m) &\equiv \chi(\bar{\Omega}_m) + \delta\chi(\bar{\omega}_m, \bar{\Omega}_m), \\ \chi(\bar{\Omega}_m) &= \frac{1}{1 + \bar{\Omega}_m^2}, \\ \delta\chi(\bar{\omega}_m, \bar{\Omega}_m) &= \frac{\bar{\omega}_m}{1 + \bar{\Omega}_m^2} \frac{2\bar{\Omega}_m - \bar{\omega}_m}{1 + (\bar{\omega}_m - \bar{\Omega}_m)^2}. \end{aligned} \quad (\text{A6})$$

Plugging this expression back to Eq. (A2), we find that the first term in Eq. (A6) gives $\lambda^* \ln \bar{T} \sim 1$, while the second term is free of logarithm and is of order λ^* . We then search for the solution of Eq. (A2) in the form

$$\Phi(\bar{\Omega}_m) = \Phi_0[\chi(\bar{\Omega}_m) + \lambda^* \delta\Phi(\bar{\Omega}_m)] + O(\lambda^{*2}), \quad (\text{A7})$$

where $\delta\Phi(\bar{\Omega}_m)$ is assumed to be independent of λ^* up to corrections of order λ^* . Substituting Eq. (A7) into Eq. (A2) and neglecting nonlogarithmical terms of order $(\lambda^*)^2$, we obtain

$$\delta\Phi(\bar{\Omega}_m) = \chi(\bar{\Omega}_m)[P + R(\bar{\Omega}_m)], \quad (\text{A8})$$

where

$$P = \lambda^* \pi T_c \sum_m \frac{\delta\Phi(\bar{\omega}_m)}{|\bar{\omega}_m|}, \quad (\text{A9})$$

$$R(\bar{\Omega}_m) = \frac{\pi T_c}{\chi(\bar{\Omega}_m)} \sum_m \frac{1}{|\bar{\omega}_m|} \chi(\bar{\omega}_m) \chi(\bar{\omega}_m, \bar{\Omega}_m) - \frac{1}{\lambda^*}. \quad (\text{A10})$$

Observe that Eq. (A8) is not an integral equation on $\delta\Phi(\bar{\Omega}_m)$ because the integral term P does not depend on $\bar{\Omega}_m$.

Constructing P in the right-hand side of Eq. (A8), we obtain

$$\left[1 - \lambda^* \pi \bar{T}_c \sum_m \frac{\chi(\bar{\omega}_m)}{|\bar{\omega}_m|} \right] P = \lambda^* \pi \bar{T}_c \sum_m \frac{\chi(\bar{\omega}_m)}{|\bar{\omega}_m|} R(\bar{\omega}_m). \quad (\text{A11})$$

Because $1 - \lambda^* \pi \bar{T}_c \sum_m \frac{1}{|\bar{\omega}_m|} \chi(\bar{\omega}_m) = O(\lambda^*)$ and P is at most of order $O(1)$, the left-hand side of Eq. (A11) is $O(\lambda^*)$. On the other hand, the right-hand side of Eq. (A11) becomes

$$\lambda^* \pi \bar{T}_c \sum_m \frac{1}{|\bar{\omega}_m|} \chi(\bar{\omega}_m) R(\bar{\omega}_m) = R(0)[1 + O(\lambda^*)]. \quad (\text{A12})$$

Matching both sides, we find that

$$R(0) = O(\lambda^*), \quad (\text{A13})$$

hence

$$\lambda^* \pi T_c \sum \frac{1}{|\bar{\omega}_m|} \frac{1}{(1 + \bar{\omega}_m^2)^2} = 1 + O(\lambda^{*2}). \quad (\text{A14})$$

Evaluating the sum (it is expressed in terms of digamma functions) and taking the limit $\bar{T} \ll 1$, we reproduce Eq. (A5).

Another way to obtain this result is to formally set $\bar{\Omega}_m = 0$ in Eq. (A8), which gives $\delta\Phi(0) = P + R(0)$. At the same time, from Eq. (A9) we have $P = \delta\Phi(0) + O(\lambda^*)$. Matching the two expressions, we reproduce $R(0) = O(\lambda^*)$.

APPENDIX B: EVALUATION OF THE CONTRIBUTION TO T_c FROM VERTEX CORRECTIONS TO THE POLARIZATION BUBBLE

In this appendix we present the evaluation of the contribution to T_c from vertex correction to the polarization

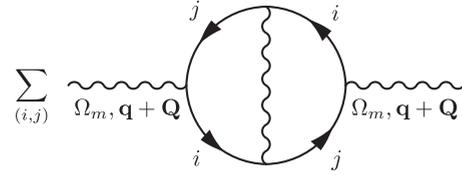


FIG. 8. Vertex correction to polarization bubble. Labels i and j denote fermions at different hot spots. The full expression is the sum of direct and umklapp processes (1,4), (4,1), (2,3), (3,2).

bubble $\Pi^{(1)}(\mathbf{q} + \mathbf{Q}, \Omega_m)$. The diagram for $\Pi^{(1)}(\mathbf{q} + \mathbf{Q}, \Omega_m)$ is shown in Fig. 8. As we discussed, vertex correction to the polarization bubble contributes to T_c by renormalizing $\chi^{(0)}$. For our purposes, one can easily make sure that we only need to include the renormalization of the spin susceptibility and at transferred momentum $\mathbf{q} = (0, q_y)$. For momentum along the y axis, the contribution from umklapp process dominates because in direct process q_y dependence is suppressed by λ . We denote the contribution from umklapp process at $(q_x = 0, q_y, \Omega_m = 0)$ as $\Pi_{um}^{(1)}(0, q_y, 0)$. We have

$$\begin{aligned} \Pi_{um}^{(1)}(0, q_y, 0) = & -2\bar{g}^2 \int \frac{d\mathbf{k}_1 d\omega_{m1}}{(2\pi)^3} \int \frac{d\mathbf{k}_2 d\omega_{m2}}{(2\pi)^3} \frac{1}{(\mathbf{k}_1 - \mathbf{k}_2)^2 + \Pi(\mathbf{k}_1 - \mathbf{k}_2, \omega_{m1} - \omega_{m2})} \\ & \times \frac{1}{i\omega_{m1} - \bar{\epsilon}_{k_1}} \frac{1}{i\omega_{m1} - \bar{\epsilon}_{-k_1+q}} \frac{1}{i\omega_{m2} - \bar{\epsilon}_{k_2}} \frac{1}{i\omega_{m2} - \bar{\epsilon}_{-k_2+q}} + (\mathbf{q} \rightarrow -\mathbf{q}), \end{aligned} \quad (\text{B1})$$

where

$$\bar{\epsilon}_k = v_F \left(k_y - \kappa \frac{k_x^2}{2} \right), \quad (\text{B2})$$

where the coefficient (-2) comes from Pauli matrix algebra. Using a set of rescaling variables similar to the one introduced in Eq. (14) in the main text, we obtain

$$\Pi_{um}^{(1)}(0, q_y, 0) = \lambda \left(\frac{\bar{g}}{\pi v_F \lambda} \right)^2 \tilde{\Pi}_{um}^{(1)}(0, \tilde{q}_y, 0), \quad (\text{B3})$$

where

$$\begin{aligned} \tilde{\Pi}_{um}^{(1)}(0, \tilde{q}_y, 0) = & -4\pi^2 \int \frac{du dx dy}{(2\pi)^3} \int \frac{dU dX dY}{(2\pi)^3} \frac{1}{X^2 + Y^2 + \sum_a \tilde{\Pi}_a^{(0)}(X, Y, U)} \\ & \times \frac{1}{-i(u - \frac{U}{2}) + (y - \frac{Y}{2} + \frac{\tilde{q}_y}{2}) - x^2} \frac{1}{i(u - \frac{U}{2}) + (y - \frac{Y}{2} - \frac{\tilde{q}_y}{2}) + x^2} \\ & \times \frac{1}{i(u + \frac{U}{2}) + (y + \frac{Y}{2} + \frac{\tilde{q}_y}{2}) + x^2} \frac{1}{-i(u + \frac{U}{2}) + (y + \frac{Y}{2} - \frac{\tilde{q}_y}{2}) - x^2}. \end{aligned} \quad (\text{B4})$$

Equation (B4) is a 6D integral, for which numerical schemes designed to evaluate multidimensional integrals, such as Monte Carlo, do not yield satisfactory results. Fortunately, the integration over u , x , and y , which are rescaled frequency and two momentum components in the fermionic loop, can be done analytically. The remaining integration over U, X, Y , can be done numerically to a good precision.

To perform the integration, we first write

$$\tilde{\Pi}_{um}^{(1)}(0, \tilde{q}_y, 0) = -4\pi^2 I(\tilde{q}_y), \quad (\text{B5})$$

where

$$\begin{aligned}
I(\tilde{q}_y) &= \int \frac{dU dX dY}{(2\pi)^3} \int \frac{du dx dy}{(2\pi)^3} \frac{1}{X^2 + Y^2 + \sum_a \tilde{\Gamma}_a^{(0)}(X, Y, U)} \\
&\times \frac{1}{\underbrace{-i(u - \frac{U}{2}) + (y - \frac{Y}{2} + \frac{\tilde{q}_y}{2}) - x^2}_{(1)}} \frac{1}{\underbrace{i(u - \frac{U}{2}) + (y - \frac{Y}{2} - \frac{\tilde{q}_y}{2}) + x^2}_{(2)}} \\
&\times \frac{1}{\underbrace{i(u + \frac{U}{2}) + (y + \frac{Y}{2} + \frac{\tilde{q}_y}{2}) + x^2}_{(3)}} \frac{1}{\underbrace{-i(u + \frac{U}{2}) + (y + \frac{Y}{2} - \frac{\tilde{q}_y}{2}) - x^2}_{(4)}}. \tag{B6}
\end{aligned}$$

We use the residue theorem for the integration over dy . There are four poles, from terms labeled (1), (2), (3), and (4). We split I into two parts, $I(\tilde{q}_y) = I_1(\tilde{q}_y) + I_2(\tilde{q}_y)$. I_1 comes from the range where the poles in (1) and (4) are in the same half-plane, while I_2 comes from the range where the poles in (2) and (4) are in the same half-plane.

For I_1 we obtain

$$\begin{aligned}
I_1(\tilde{q}_y) &= 2\pi i \int \frac{dU dX dY}{(2\pi)^3} \int_{|u| > \frac{|U|}{2}} du \operatorname{sgn}(u) \int \frac{dx}{(2\pi)^3} \frac{1}{X^2 + Y^2 + \sum_a \tilde{\Gamma}_a^{(0)}(X, Y, U)} \\
&\times \frac{1}{2x^2 + Y + 2iu} \frac{1}{2x^2 - \tilde{q}_y + 2i(u - \frac{U}{2})} \frac{1}{-iU + Y - \tilde{q}_y} + (\tilde{q}_y \rightarrow -\tilde{q}_y). \tag{B7}
\end{aligned}$$

Rearranging the second line and combining the integrals over positive and negative u , we obtain

$$\begin{aligned}
I_1(\tilde{q}_y) &= \int \frac{dU dX dY}{16\pi^5} \frac{1}{X^2 + Y^2 + \sum_a \tilde{\Gamma}_a^{(0)}(X, Y, U)} \int_{\frac{|U|}{2}}^{\infty} du \int_{-\infty}^{\infty} dx \\
&\times \operatorname{Im} \left\{ \left[\frac{1}{2x^2 + Y + 2iu} - \frac{1}{2x^2 - \tilde{q}_y + 2i(u - \frac{U}{2})} \right] \frac{1}{Y^2 - (\tilde{q}_y + iU)^2} \right\} + (\tilde{q}_y \rightarrow -\tilde{q}_y). \tag{B8}
\end{aligned}$$

We next perform the integration over x and over u using the same steps as we did in the calculation of the one-loop polarization bubble in the main text. Carrying out the integrations, we obtain

$$\begin{aligned}
I_1(\tilde{q}_y) &= - \int \frac{dU dX dY}{16\pi^4 \sqrt{2}} \frac{1}{X^2 + Y^2 + \sum_a \tilde{\Gamma}_a^{(0)}(X, Y, U)} \\
&\times \operatorname{Re} \left[(\sqrt{Y + i|U|} - \sqrt{-\tilde{q}_y - iU + i|U|}) \frac{1}{Y^2 - (\tilde{q}_y + iU)^2} \right] + (\tilde{q}_y \rightarrow -\tilde{q}_y). \tag{B9}
\end{aligned}$$

Folding the integration over U to positive U , we rewrite (B9) as

$$\begin{aligned}
I_1(\tilde{q}_y) &= \int_0^{\infty} dU \int \frac{dX dY}{16\pi^4 \sqrt{2}} \frac{1}{X^2 + Y^2 + \sum_a \tilde{\Gamma}_a^{(0)}(X, Y, U)} \\
&\times \operatorname{Re} \left[(-2\sqrt{Y + iU} + \sqrt{-\tilde{q}_y} + \sqrt{\tilde{q}_y + 2iU}) \frac{1}{Y^2 - (\tilde{q}_y + iU)^2} \right] + (\tilde{q}_y \rightarrow -\tilde{q}_y). \tag{B10}
\end{aligned}$$

Similarly, for I_2 we obtain

$$\begin{aligned}
I_2(\tilde{q}_y) &= 2\pi i \int_{-\infty}^{\infty} dU \operatorname{sgn}(U) \int \frac{dX dY}{(2\pi)^3} \int_{-\frac{|U|}{2}}^{\frac{|U|}{2}} du \int \frac{dx}{(2\pi)^3} \frac{1}{X^2 + Y^2 + \sum_a \tilde{\Gamma}_a^{(0)}(X, Y, U)} \\
&\times \frac{1}{2x^2 - Y + 2iu} \frac{1}{2x^2 - \tilde{q}_y + 2i(u - \frac{U}{2})} \frac{1}{iU + Y + \tilde{q}_y} + (\tilde{q}_y \rightarrow -\tilde{q}_y). \tag{B11}
\end{aligned}$$

Rearranging the second line, folding the integration over U to positive U , and integrating over x and u we obtain

$$\begin{aligned}
I_2(\tilde{q}_y) &= \int_0^{\infty} dU \int \frac{dX dY}{16\pi^4 \sqrt{2}} \frac{1}{X^2 + Y^2 + \sum_a \tilde{\Gamma}_a^{(0)}(X, Y, U)} \\
&\times \operatorname{Re} \left[(\sqrt{Y + iU} - \sqrt{Y - iU} - \sqrt{-\tilde{q}_y} + \sqrt{-\tilde{q}_y - 2iU}) \frac{1}{Y^2 - (\tilde{q}_y + iU)^2} \right] + (\tilde{q}_y \rightarrow -\tilde{q}_y). \tag{B12}
\end{aligned}$$

Combining I_1 and I_2 and substituting into (B5), we obtain

$$\begin{aligned} \tilde{\Pi}_{um}^{(1)}(0, \tilde{q}_y, 0) = & - \int_0^\infty dU \int \frac{dXdY}{4\pi^2\sqrt{2}} \frac{1}{X^2 + Y^2 + \sum_a \tilde{\Pi}_a^{(0)}(X, Y, U)} \\ & \times \text{Re} \left[(-\sqrt{Y-iU} - \sqrt{Y+iU} + \sqrt{-\tilde{q}_y-2iU} + \sqrt{\tilde{q}_y+2iU}) \frac{1}{Y^2 - (\tilde{q}_y + iU)^2} \right] + (\tilde{q}_y \rightarrow -\tilde{q}_y). \end{aligned} \quad (\text{B13})$$

The integration over U, X, Y has been done numerically. Just as we did in the calculation of the one-loop polarization operator, we subtract from $\tilde{\Pi}_{um}^{(1)}(0, \tilde{q}_y, 0)$ its value at zero momentum $\tilde{\Pi}_{um}^{(1)}(0, 0, 0)$ to make the integral infrared convergent. The term we subtract only shifts the position of the QCP and is not of interest to us. After the subtraction, the integration in (B13) can be extended to an infinite range.

We plot $\tilde{\Pi}_{um}^{(1)}(0, \tilde{q}_y, 0) - \tilde{\Pi}_{um}^{(1)}(0, 0, 0)$ in Fig. 9. We see that numerically $\tilde{\Pi}_{um}^{(1)}(0, \tilde{q}_y, 0) - \tilde{\Pi}_{um}^{(1)}(0, 0, 0)$ is small even when $\tilde{q}_y = 1$.

The $O(\lambda)$ correction to $\chi^{(0)}$, which we need for the calculation of the right prefactor for T_c , is related to $\tilde{\Pi}_{um}^{(1)}(0, \tilde{q}_y, 0) - \tilde{\Pi}_{um}^{(1)}(0, 0, 0)$ as

$$\chi^{(0)} \rightarrow \chi^{(0)} - \lambda \int_{-\infty}^{\infty} dz \frac{\tilde{\Pi}_{um}^{(1)}(0, z, 0) - \tilde{\Pi}_{um}^{(1)}(0, 0, 0)}{(z^2 + \sqrt{|z|/2})^2}. \quad (\text{B14})$$

Using the numerical results for $\tilde{\Pi}_{um}^{(1)}(0, \tilde{q}_y, 0) - \tilde{\Pi}_{um}^{(1)}(0, 0, 0)$, we found that the renormalization of $\chi^{(0)}$ by the vertex correction in the polarization operator is

$$\chi^{(0)} \rightarrow \chi^{(0)}(1 - 0.042\lambda). \quad (\text{B15})$$

We cited this result in Eq. (53) in the main text.

APPENDIX C: CONTRIBUTION TO T_c FROM KOHN-LUTTINGER DIAGRAMS AT FINITE MOMENTUM CUTOFF

We found in the main text that Kohn-Luttinger diagrams renormalize the transition temperature T_c to

$$T_c = T_{c3} \exp B, \quad (\text{C1})$$

where T_{c3} , given by Eq. (54), is the transition temperature without Kohn-Luttinger contributions, and B is the prefactor

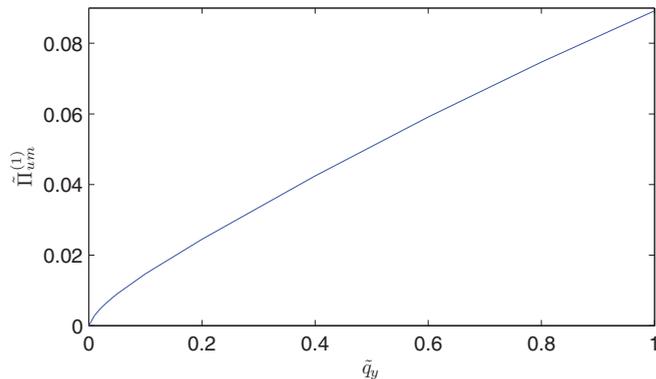


FIG. 9. (Color online) The dependence of $\tilde{\Pi}_{um}^{(1)}(0, \tilde{q}_y, 0) - \tilde{\Pi}_{um}^{(1)}(0, 0, 0)$ on \tilde{q}_y .

for the k_y term in the integral equation

$$A(k_y) + C(k_y) = Bk_y(\chi^{(0)})^2, \quad (\text{C2})$$

where A and C are given by

$$\begin{aligned} A(k_y) = & \int_{-\Lambda}^{\Lambda} dq_y \left[\frac{\chi^{(0)}(q_y)}{\chi^{(0)}} \delta\Phi(k_y + q_y) \right] - \delta\Phi(k_y), \\ C(k_y) = & \int_{-\Lambda}^{\Lambda} dq_y q_y \chi^{(1)}(k_y, q_y), \end{aligned} \quad (\text{C3})$$

and we remind that $\chi^{(0)}(q_y) = 1/(q_y^2 + \sqrt{|q_y|/2})$, $\chi^{(0)}$ is the number (=6.09), and $\chi^{(1)}(k_y, q_y)$ accounts for the contributions from Kohn-Luttinger diagrams. We also used shorthand notations $\delta\Phi(k_y) \equiv \delta\Phi(k_x = 0, k_y, \omega_m = 0)$ and $\chi^{(1)}(k_y, q_y) \equiv \chi^{(1)}(k_x = 0, k_y, \omega_m = 0; q_x = 0, q_y, \omega'_m = 0)$ and we explicitly restricted the momentum integration to $|q_y| < \Lambda$, where Λ is the upper momentum cutoff for the low-energy theory.

To obtain B , we take the first and second derivatives of Eq. (C2):

$$A'(k_y) + C'(k_y) = B(\chi^{(0)})^2, \quad (\text{C4})$$

$$A''(k_y) + C''(k_y) = 0. \quad (\text{C5})$$

Integrating Eq. (C5) from its lower limit $-\Lambda$ to k_y , we find

$$A'(k_y) + C'(k_y) = A'(-\Lambda) + C'(-\Lambda). \quad (\text{C6})$$

Hence

$$B(\chi^{(0)})^2 = A'(-\Lambda) + C'(-\Lambda). \quad (\text{C7})$$

Now, let us explicitly write

$$A'(-\Lambda) = \int_{-\Lambda}^{\Lambda} dq_y \frac{\chi^{(0)}(q_y)}{\chi^{(0)}} [\delta\Phi'(q_y - \Lambda) - \delta\Phi'(-\Lambda)]. \quad (\text{C8})$$

Typical values of q_y are set by $\chi^{(0)}(q_y)$ and are order $O(1)$. Because $\Lambda \gg 1$ by construction, $\delta\Phi'(q_y - \Lambda) - \delta\Phi'(-\Lambda) = O(1/\Lambda)$, i.e., $A'(-\Lambda)$ is small and can be neglected. Then

$$B(\chi^{(0)})^2 = C'(-\Lambda) = \int_{-\Lambda}^{\Lambda} dq_y q_y \left. \frac{\partial \chi^{(1)}(k_y, q_y)}{\partial k_y} \right|_{k_y = -\Lambda}. \quad (\text{C9})$$

We now need the explicit expression for $\chi^{(1)}(k_y, q_y)$. Evaluating explicitly the three Kohn-Luttinger contributions in Fig. 7, we obtain, for zero external frequency and x component

of momenta $\chi^{(1)}(k_y, q_y) = 2\chi_a^{(1)}(k_y, q_y) + \chi_b^{(1)}(k_y, q_y)$, where

$$\chi_a^{(1)}(k_y, q_y; \lambda) = -\frac{1}{8\pi^2} \int \frac{du dx dy}{iu - x - \lambda^2(k_y - y)^2} \frac{1}{iu + x - \lambda^2(q_y - y)^2} \times \frac{1}{x^2 + y^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(x, y, u)} \frac{1}{(k_y - q_y)^2 + \sqrt{|k_y - q_y|}/2} \quad (\text{C10})$$

and

$$\chi_b^{(1)}(k_y, q_y; \lambda) = -\frac{3}{8\pi^2} \int \frac{du dx dy}{iu - x - \lambda^2(k_y + y)^2} \frac{1}{iu + x - \lambda^2(q_y - y)^2} \times \frac{1}{x^2 + y^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(x, y, u)} \frac{1}{x^2 + (y + k_y - q_y)^2 + \sum_{a=\pm 1} \tilde{\Pi}_a^{(0)}(x, y + k_y - q_y, u)}. \quad (\text{C11})$$

The difference in prefactors comes from different structures in Pauli matrices convolution and is trivial to verify. In these two integrals the λ^2 terms in the denominator, together with external momenta k_y and q_y , serve as an infrared cutoff.

One can easily make sure that the magnitude of $C'(-\Lambda)$ depends on the value $\lambda^2 \Lambda^2$. One can show that, if the momentum cutoff along the FS is of order k_F , $\lambda^2 \Lambda^2 \sim (W/\bar{g})^{2/3} \gg 1$, where W is the fermionic bandwidth. In this case we find that $B(\chi^{(0)})^2 = C'(-\Lambda) \ll 1$. Then $B \ll 1$, and the renormalization due to Kohn-Luttinger-type diagrams are small.

If, however, the momentum cutoff Λ is much smaller such that $\lambda \Lambda$ is actually a small number, the integral for $C'(-\Lambda)$ in (C9) contains $\ln \frac{1}{\lambda \Lambda}$. Collecting the contributions from (C10) and (C11), we obtain at $\lambda \Lambda \ll 1$

$$C'(-\Lambda) = \frac{10}{3} (\chi^{(0)})^2 \ln \frac{1}{\lambda \Lambda} + O(1). \quad (\text{C12})$$

Using Eqs. (C9) and (C1), we find that T_c in this case will be enhanced by a factor of $\exp B \sim (\lambda \Lambda)^{-10/3}$.

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