

Thermally induced spin polarization of a two-dimensional electron gas

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Spin polarization of a two-dimensional electron gas with Rashba spin-orbit interaction, induced by a thermocurrent, is considered theoretically. It is shown that a temperature gradient gives rise to an in-plane spin polarization of the electron gas, which is normal to the temperature gradient. The low-temperature spin polarization changes sign when the Fermi level crosses the bottom edge of the upper electronic subband. We also compare the results with spin polarization induced by an external electric field (current).

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I. INTRODUCTION

Spin-orbit interaction is responsible for mixing of orbital and spin degrees of freedom. This mixing, in turn, gives rise to a variety of interesting phenomena which are observable experimentally.¹⁻⁵ For instance, it is well known that insulating materials having sufficiently low symmetry can reveal linear magneto-electric phenomena,⁶ which follow from direct coupling of magnetic and electric degrees of freedom. Such materials can be magnetized electrically, and the induced magnetic moment is linear in the electric field. In turn, the magnetic field leads then to a linear electric polarization.

The situation is more complex in conducting materials, where the electric field is inevitably associated with a charge current. It was predicted a long time ago that the electric current in a system with spin-orbit (SO) interaction can induce a spin polarization of conduction electrons,⁷ with the polarization vector perpendicular to the direction of the current and electric field. This phenomenon was studied later in various systems exhibiting SO interaction.⁸⁻¹³ Edelstein⁸ has considered the spin polarization induced by the electric field (current) in a two-dimensional (2D) electron gas with Rashba spin-orbit interaction, and found that the spin polarization is in the plane of the system and is normal to the electric field, as shown schematically in Fig. 1 (left). The current-induced spin polarization in a 2D electron gas for a general case of SO interaction including both Rashba and Dresselhaus terms has been studied theoretically in a recent paper.¹² Moreover, the current-induced spin polarization has also been proved experimentally.^{1-3,14-17}

On the other hand, it is well known that electric current can also be driven by a temperature gradient, like, for instance, in the Seebeck effect. Thus, combining the above two phenomena, one can expect that spin polarization of a system can be induced by the gradient of temperature (by the thermo-current) as well, as shown schematically in Fig. 1 (right). Physical origin of spin polarization, however, is now different, as the electric field and the temperature gradient affect the carrier distribution in quite different ways. As the applied electric field shifts the Fermi surface of electrons in the momentum space, the temperature gradient makes different “smearing” of the Fermi surface at different temperatures. Spin polarization of a two-dimensional electron gas, induced

by a temperature gradient, was considered theoretically by Wang *et al.*¹⁸ in terms of the Boltzmann kinetic equation, who showed that the induced spin polarization is in the layer plane and normal to the temperature gradient. It is worth noting, that spin-related thermoelectric phenomena, like, for instance, spin Seebeck and spin Nernst effects, are currently of great interest.¹⁹⁻²²

In this paper the spin polarization induced by a temperature gradient is considered within the linear response theory and the temperature (Matsubara) Green function formalism. We analyze in detail variation of the induced spin polarization with position of the Fermi level. Moreover, we analyze the temperature dependence of the effect in the temperature regime where the spin relaxation time can be considered as constant (independent of temperature). As in Ref. 18, we show that the spin polarization is in the plane of the 2D electron gas. However, we found some new features of the spin polarization induced by temperature gradient. More specifically, we show that the spin polarization in the low-temperature regime changes sign when the Fermi level crosses the bottom edge of the upper subband. Furthermore, as the spin polarization for the Fermi level above this edge is linear in the chemical potential and spin-orbit parameter, this linearity does not hold when the Fermi level is below the edge. For completeness, we also apply the same method to spin polarization induced by the electric field (charge current).

The paper is organized as follows. In Sec. II we present some general considerations on the system and method used to calculate spin polarization induced by temperature gradient. Description of the two-dimensional electron gas with Rashba interaction is presented in Sec. III, where also some analytical formulas for the spin polarization are derived. Numerical results on the spin polarization induced by a temperature gradient are presented and discussed in Sec. IV. In turn, the spin polarization induced by electric field is considered in Sec. V, whereas summary and final conclusions are in Sec. VI.

II. GENERAL CONSIDERATIONS

To describe energy transport, a fictitious “gravitational” potential has been introduced a long time ago by Luttinger.²³ Gradient of this potential is a driving force for the energy

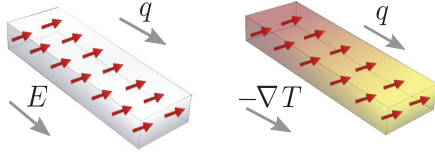


FIG. 1. (Color online) Schematic illustration of the electric-field-induced (left side) and thermally induced (right side) spin polarization.

current— analogously as the gradient of electrostatic potential is a driving force for charge current. It was shown that the transport coefficients corresponding to the “gravitational” potential coincide with those appropriate for the temperature gradient.²³ The concept of gravitational potential was explored in a couple of papers.^{24,25} In this paper we use another approach, similar to that developed in Refs. 26 and 27. Instead of the “gravitational” potential, we introduce here an auxiliary (“gravitational”) vector field, which may be considered as an analog of the vector potential in the theory of electromagnetism.

Thus, we consider a system that can be described by the Hamiltonian $H = H_0 + V$, where H_0 includes kinetic and spin-orbit interaction terms, while V is a perturbation induced by the “gravitational” field, which we write in the form,

$$V = -\hat{\mathbf{j}}_Q \cdot \mathbf{A}. \quad (1)$$

Here, \mathbf{A} is a vector field, which can also be understood as an auxiliary field, response to which gives the heat current, $\hat{\mathbf{j}}_Q = -\partial H / \partial \mathbf{A}$. In turn, the heat current operator $\hat{\mathbf{j}}_Q$ in Eq. (1) is defined as

$$\hat{\mathbf{j}}_Q = \frac{1}{2} [(H_0 - \mu), \mathbf{v}]_+, \quad (2)$$

where $[A, B]_+ = AB + BA$ and \mathbf{v} is the velocity operator.

Applying the formalism to the well-known results on the Seebeck effect, one may identify $A_i(\omega)$ as $A_i(\omega) = \frac{i \nabla_i T}{\omega}$. Thus, applying the imaginary-time (Matsubara) Green function formalism, one can write the thermally induced spin polarization of the system, linear in $\nabla T / T$, in the form,

$$S_j(\omega_m) = -\frac{1}{2} \frac{\nabla_j T}{T} \frac{iT}{\omega_m} \text{Tr} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \sum_n \sigma_j \times G_{\mathbf{k}}^0(\varepsilon_n + \omega_m) [(H_0 - \mu), v_j]_+ G_{\mathbf{k}}^0(\varepsilon_n), \quad (3)$$

where σ_j are the Pauli matrices, $G_{\mathbf{k}}^0(\varepsilon_n)$ is the temperature Green’s function corresponding to the Hamiltonian H_0 , the sum goes over imaginary discrete Matsubara energies $\varepsilon_n = (2n + 1)\pi T$ while $\omega_m = 2m\pi T$ for $(m, n \in \mathbb{Z})$. Upon calculating the sum over Matsubara energies in Eq. (3) and making analytical continuation to the whole complex plane, one takes the limit $\omega \rightarrow 0$, which allows one to find the static spin polarization.²⁸ Below we apply the above-described approach to the thermally induced spin polarization of a 2D electron gas with Rashba SO interaction.

III. SYSTEM AND SOLUTION

In the following we will use the units with $\hbar = 1$. Hamiltonian H_0 of the 2D electron gas with Rashba interaction in the

basis of plane waves can be written in the form,

$$H_0 = \varepsilon_k + \alpha(k_y \sigma_x - k_x \sigma_y), \quad (4)$$

where α is the Rashba SO coupling parameter, and $\varepsilon_k = k^2 / 2m$. We assume that the temperature gradient is along the axis y , and calculate the spin polarization along the axis x (the other components vanish). Let us consider first the case of a finite temperature, when both subbands of the electron states described by the Hamiltonian (4) are populated for arbitrary chemical potential μ . The spin polarization can be calculated from Eq. (3), with the Green function $G_{\mathbf{k}}^0$ given by

$$G_{\mathbf{k}}^0(\varepsilon_n) = \frac{\varepsilon_n - \varepsilon_k + \mu + \alpha(k_y \sigma_x - k_x \sigma_y)}{(\varepsilon_n - E_{1k} + \mu)(\varepsilon_n - E_{2k} + \mu)}. \quad (5)$$

Here $E_{1,2k} = \varepsilon_k \pm \alpha k$ are the dispersion relations of the two (upper and lower) electron subbands corresponding to the Hamiltonian H_0 .

Upon inserting Eq. (5) into Eq. (3), we calculate the trace and then calculate the sum over Matsubara energies ε_n by integrating over an appropriate contour in the complex plane. Then we make analytical continuation to the complex plane, and take the limit $\omega \rightarrow 0$. As a result, one finds the dominant contribution to the induced spin density S_x in the form,

$$S_x = \frac{\nabla T}{T} \int \frac{dk}{2\pi} \tau_k \varepsilon_k (\varepsilon_k - \mu) [f'(E_{1k}) - f'(E_{2k})] + \frac{\alpha \nabla T}{2T} \int \frac{dk}{2\pi} \tau_k k (2\varepsilon_k - \mu) [f'(E_{1k}) + f'(E_{2k})], \quad (6)$$

where $f'(\varepsilon)$ is the first derivative of the distribution function. In order to take into account relaxation processes, we have also replaced $\delta \rightarrow 1/2\tau_k$, with τ_k being the relevant relaxation time (k dependence means effectively the energy dependence). Here, δ is an infinitesimally small number which emerges from integration over the contour in the complex plane.²⁸ Generally, τ_k also depends on temperature T , which is not indicated explicitly.

Since α is rather small, the derivative of the distribution functions can be expanded for $\alpha \ll T$, which leads to the following formula for spin density,

$$S_x = 2\alpha \frac{\nabla T}{T} \int \frac{dk}{2\pi} \tau_k k \varepsilon_k (\varepsilon_k - \mu) f''(\varepsilon_k) + \alpha \frac{\nabla T}{T} \int \frac{dk}{2\pi} \tau_k k (2\varepsilon_k - \mu) f'(\varepsilon_k). \quad (7)$$

This formula clearly shows that the leading term in spin polarization for $\alpha \ll T$ is linear in α , and that the spin polarization vanishes for $\alpha = 0$.

The above expansion, however, is not valid in the limit of low temperatures, where the derivatives of the distribution functions are sharp. We derive now some approximate formula for low- T spin polarization by replacing the derivatives in Eq. (6) by the corresponding delta-Dirac functions. We will distinguish the cases of positive and negative chemical potentials, $\mu > 0$ and $\mu < 0$. In the former case both electron subbands are then occupied with electrons, while in the later case only the lower subband, E_{2k} , is populated. Let us consider first the case of $\mu > 0$. From Eq. (6) one finds the dominant

contribution (linear in α) in the form,

$$S_x = \alpha \mu \frac{\nabla T}{T} \frac{m}{2\pi} \tau_{k_F}. \quad (8)$$

Here, τ_{k_F} is the relaxation time at the Fermi wave vector k_F corresponding to zero spin-orbit coupling, $k_F = \sqrt{2m\mu}$. In turn, when $\mu < 0$, only the lower band, E_{2k} , is populated, but there are two Fermi wavevectors, $k_{F2}^{\pm} = \alpha m \pm \sqrt{\alpha^2 m^2 + 2m\mu}$. From Eq. (6) one finds then the dominant contribution in the form,

$$S_x = \alpha \mu \frac{\nabla T}{T} \frac{m}{2\pi} \frac{n^*}{n} \tau_0, \quad (9)$$

where $\tau_0 = \tau_{k_F=0}$ and $\mu \geq -\alpha^2 m/2$. The latter condition follows from the position of the lower band edge. Apart from this, n is the electron concentration corresponding to the Fermi level μ , while n^* is the corresponding electron density when $\mu = 0$. For instance, assuming parameters typical of GaAs-based quantum wells, i.e., $\alpha = 2 \times 10^{-9}$ eV cm, $k_F = 10^7$ cm $^{-1}$, $\tau = 10^{-11}$ s, $T = 6.5$ K, and $\Delta T = 5$ K at the sample of length 0.1 cm, one can estimate the spin polarization S_x to be of an order of $S_x \simeq 2 \times 10^9$ cm $^{-2}$.

IV. NUMERICAL RESULTS

In the low-temperature regime we have found above some approximate analytical solutions. For higher temperatures, however, one needs to find the integrals in Eq. (6) or Eq. (7). To do this, we need to know the explicit form of the relaxation time τ_k . For simplicity, the numerical results will be presented for a specific case, where τ_k is independent of k , $\tau_k = \tau$, at the energy scale of the order of kT around the Fermi level. Since the maximum temperature and chemical potential are comparable ($kT \sim |\mu|$), we assume that τ is constant in the considered range of T and μ . Such a situation takes place, for instance, in the case when the dominant contribution to relaxation time comes from scattering on nonionized impurities,²⁹ where one finds $1/\tau = 4\pi^2 e^4 N_i m / \epsilon_0^2 \kappa_0^2$. Here, N_i is the impurity concentration, ϵ_0 is the dielectric constant, and κ_0 is the Thomas-Fermi momentum. For $kT \sim |\mu| < 10$ meV, and for sufficiently large impurity concentration, this approximation is reasonable.

When the relaxation time is constant, $\tau_k = \tau$, the low-temperature approximate solutions are then given by Eqs. (8) and (9) with $\tau_{k_F} = \tau$ and $\tau_0 = \tau$, respectively. In turn, in the finite temperature regime the integrals in Eq. (6) or (7) can be easily calculated by changing the integration variable from k to ϵ_k . The dependence on temperature is now more complex as additionally the distribution functions contribute to this dependence. Numerical results on the spin polarization induced by temperature gradient are shown in Fig. 2. For convenience, we normalized there the spin polarization to $\tau(\nabla T/T)$ and the normalized spin polarization is presented as a function of the chemical potential μ for different temperatures T . Consider first the upper part [part (a)] of this figure, which describes the range of positive μ , $\mu > 0$, i.e., the range which is most relevant experimentally. According to Eq. (8), the spin polarization for low temperatures grows linearly with increasing μ , i.e., with increasing electron concentration (at constant $\nabla T/T$). This linear behavior does

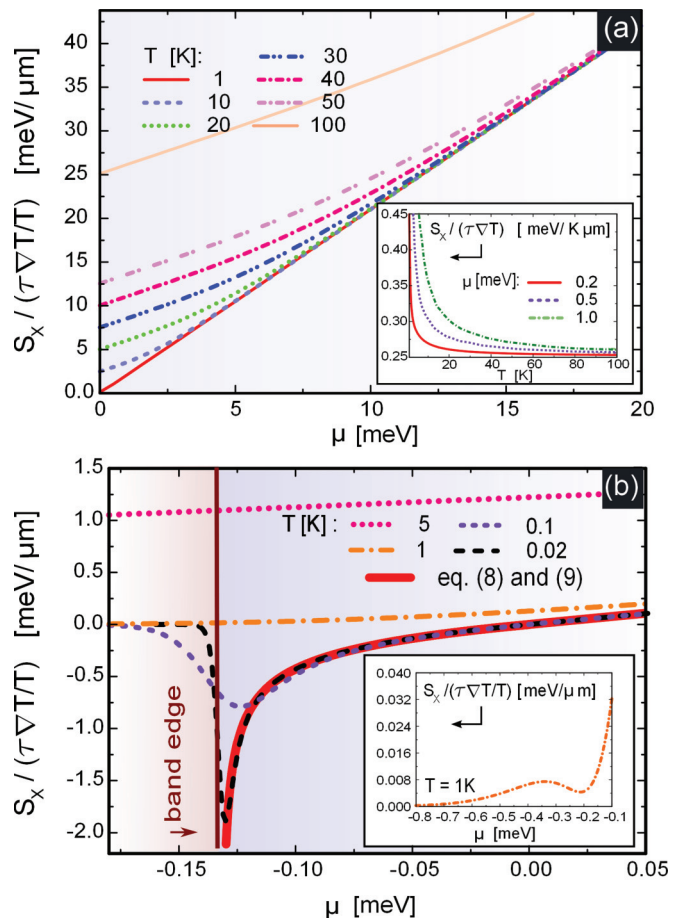


FIG. 2. (Color online) Spin polarization induced by temperature gradient, normalized to $\tau(\nabla T/T)$ and shown as a function of chemical potential μ for indicated values of temperature. Here, (a) presents the situation for $\mu > 0$, while (b) shows spin polarization for low chemical potentials, down to values below the band edge. The inset in (a) shows the temperature dependence of the spin polarization, while the inset in (b) corresponds to $T = 1$ K. The other parameters are $\alpha = 2 \times 10^{-11}$ eV m, $m = 0.05m_0$, where m_0 is free electron mass.

not hold at higher temperatures [see Fig. 2(a)]. Apart from this, keeping constant temperature gradient one can conclude that the spin polarization decreases with increasing temperature. This is clearly shown in the inset in Fig. 2(a), where to emphasize the temperature dependence, S_x is normalized to $\tau \nabla T$ instead of $\tau(\nabla T/T)$.

Figure 2(b), in turn, shows the normalized spin polarization in the range of low chemical potentials—down to values below the bottom of the lower subband (band edge). We remind the readers that the band edge of the upper subband E_{2k} is shifted by spin-orbit interaction to a negative energy, indicated in Fig. 2(b) as the band edge. The approximate solution for the low-temperature spin polarization, given by Eqs. (8) and (9), is also shown in Fig. 2(b). As one can easily note, the formulas (8) and (9) describe very well the low-temperature spin polarization, except for chemical potentials very close to the band edge. Apart from this, spin polarization at low temperatures changes sign at $\mu = 0$, and becomes negative for negative μ . When μ is below the band edge, both subbands are empty at $T = 0$ (electrons are localized at the donor states), but

for a nonzero T some electrons are excited to the 2D subbands and a tail in the spin polarization appears for μ below the band edge [see Fig. 2(b)].

The above numerical results clearly show that spin polarization can be induced by a temperature gradient. As mentioned already in the introductory part, the spin polarization can also be induced by a charge current (electric field), as shown already by Edelstein.⁸ Below we calculate the current-induced spin polarization assuming the same conditions as above. This will allow us to compare the results on spin polarization induced by a temperature gradient with those obtained with an electric field.

V. SPIN POLARIZATION INDUCED BY ELECTRIC FIELD

Now we derive some formulas for spin polarization due to electric current flowing through the system. Instead of a temperature gradient, however, there is now an electric field $\mathbf{E} = (0, E_y, 0)$. Accordingly, Hamiltonian (1) is now replaced with $V = -e\mathbf{v} \cdot \mathbf{A}$, where \mathbf{A} is the vector potential for the electromagnetic field. Following the same methodology as above, the x component of spin polarization induced by the electric field can be written as

$$S_x(\omega_m) = -eE_y \frac{iT}{\omega_m} \text{Tr} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \sum_n \sigma_x G_{\mathbf{k}}^0(\varepsilon_n + \omega_m) v_y G_{\mathbf{k}}^0(\varepsilon_n). \quad (10)$$

For finite temperatures and constant relaxation time τ , the above formula leads to the following expression for static S_x :

$$S_x = eE_y \tau \int \frac{dk}{2\pi} \varepsilon_k [f'(E_{1k}) - f'(E_{2k})] + \alpha eE_y \int \frac{dk}{2\pi} k [f'(E_{1k}) + f'(E_{2k})], \quad (11)$$

which can be used for numerical calculations. In turn, the spin polarization at $T = 0$, can be obtained in a similar way as the analytical formulas (8) and (9) in the case of a temperature gradient. For positive chemical potentials one arrives then at the following formula:

$$S_x = \alpha eE_y \frac{m}{2\pi} \tau, \quad (12)$$

which coincides with the known expression for spin polarization induced by electric current.^{8,11,13} In turn, when $\mu < 0$ the corresponding formula reads

$$S_x = \alpha eE_y \frac{m}{2\pi} \frac{n}{n^*} \tau. \quad (13)$$

Numerical results for current-induced spin polarization are shown in Fig. 3, where (a) corresponds to positive μ , while (b) presents spin polarization for low chemical potentials, down to values below the band edge, when the subbands are populated for nonzero temperatures. As before, the spin polarization increases with increasing μ . There is, however, no sign change of the spin polarization, contrary to the case of thermally induced spin polarization. Furthermore, the spin polarization decreases with increasing temperature, except for the chemical potentials in the vicinity of the band edge.

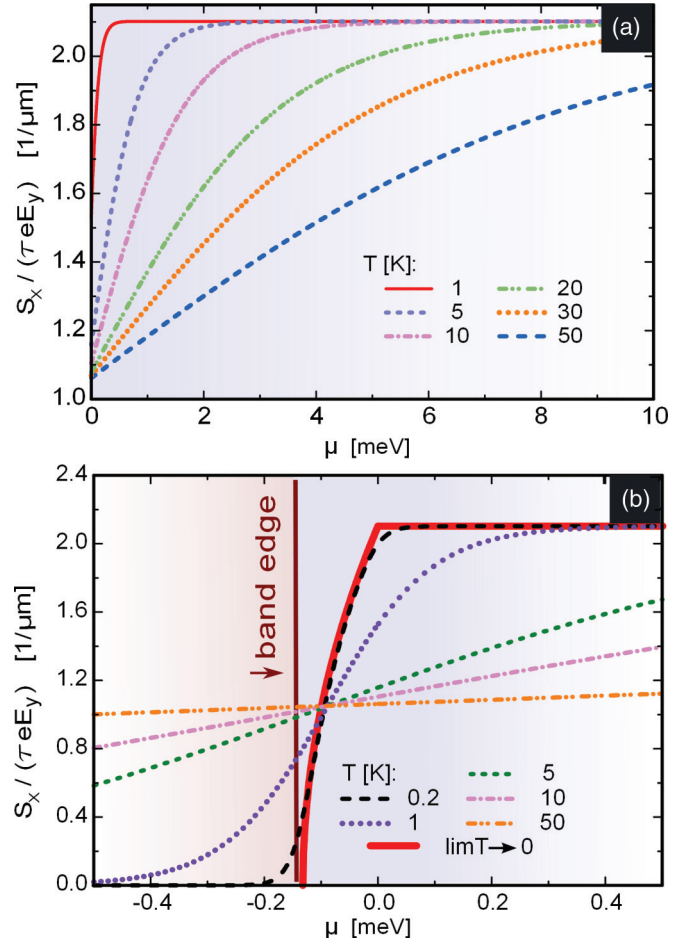


FIG. 3. (Color online) Spin polarization induced by electric field, normalized to $\tau e E_y$ and shown as a function of chemical potential for indicated temperatures. Here, (a) corresponds to positive chemical potentials $\mu > 0$, while (b) shows spin polarization for low chemical potentials, down to values below the band edge. The other parameters are as in Fig. 2.

VI. SUMMARY

We have calculated spin polarization of a 2D electron gas with Rashba spin-orbit coupling, induced by a temperature gradient as well as by electric field. We have shown that the thermo-current can effectively induce spin polarization in the plane of the electron gas and normal to the direction of the temperature gradient. The method we applied is based on the concept of “gravitational” potential, but we used the Green function formalism requiring rather an auxiliary vector potential. We have shown that the thermally induced spin polarization changes sign when the Fermi level crosses the band edge of the upper subband. The results also show that the thermally induced spin polarization decreases with increasing temperature. Additionally, we have calculated the electric-field-induced spin polarization as a function of chemical potential and temperature. Contrary to the case of thermally induced spin polarization, there is now no change of the spin polarization sign. We note that the analytical results are valid for arbitrary dependence of the relaxation time on energy and temperature. Numerical results, however, have been presented for a constant relaxation time, which allows to emphasize those

features of the spin polarization, which are independent of the specific energy and temperature dependence of the relaxation time.

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- ¹Y. K. Kato, R. C. Myers, A. C. Gossard, and D. D. Awschalom, *Phys. Rev. Lett.* **93**, 176601 (2004).
- ²A. Yu. Silov, P. A. Blaynov, J. H. Wolter, R. Hey, K. H. Ploog, and N. S. Averkiev, *Appl. Phys. Lett.* **85**, 5929 (2004).
- ³V. Sih, R. C. Myers, Y. K. Kato, W. H. Lau, A. C. Gossard, and D. D. Awschalom, *Nat. Phys.* **1**, 31 (2005).
- ⁴A. Chernyshov, M. Overby, X. Liu, J. K. Furdyna, Y. Lyanda-Geller, and L. P. Rokhinson, *Nat. Phys.* **5**, 656 (2009).
- ⁵T. Jungwirth, J. Wunderlich, and K. Olejník, *Nat. Mater.* **11**, 382 (2012).
- ⁶M. Fiebig, *J. Phys. D: Appl. Phys.* **38**, R123 (2005).
- ⁷M. I. Dyakonov and V. I. Perel, *Phys. Lett. A* **35**, 459 (1971).
- ⁸V. M. Edelstein, *Sol. State Commun.* **73**, 233 (1990).
- ⁹A. G. Aronov and Y. B. Lynda-Geller, *JETP Lett* **50**, 431 (1989).
- ¹⁰M.-H. Liu, S.-H. Chen, and C.-R. Chang, *Phys. Rev. B* **78**, 165316 (2008).
- ¹¹C. Gorini, P. Schwab, M. Dzierzawa, and R. Raimondi, *Phys. Rev. B* **78**, 125327 (2008).
- ¹²C. M. Wang, H. T. Cui, and Q. Lin, *Phys. Status Solidi B* **246**, 2301 (2009).
- ¹³P. Schwab, R. Raimondi, and C. Gorini, *Europhysics Lett.* **90**, 67004 (2010).
- ¹⁴C. L. Yang, H. T. He, L. Ding, L. J. Cui, Y. P. Zeng, J. N. Wang, and W. K. Ge, *Phys. Rev. Lett.* **96**, 186605 (2006).
- ¹⁵N. P. Stern, S. Ghosh, G. Xiang, M. Zhu, N. Samarth, and D. D. Awschalom, *Phys. Rev. Lett.* **97**, 126603 (2006).
- ¹⁶W. F. Koehl, M. H. Wong, C. Poblenc, B. Swenson, U. K. Mishra, J. S. Speck, and D. D. Awschalom, *Appl. Phys. Lett.* **95**, 072110 (2009).
- ¹⁷S. Kuhlen, K. Schmalbuch, M. Hagedorn, P. Schlamme, M. Patt, M. Lepsa, G. Güntherodt, and B. Beschoten, *Phys. Rev. Lett.* **109**, 146603 (2012).
- ¹⁸C. M. Wang and M. Q. Pang, *Solid State Communications* **150**, 1509 (2010).
- ¹⁹K. Uchida, S. Takahashi, K. Harii, J. Ieda, W. Koshibae, K. Ando, S. Maekawa, and E. Saitoh, *Nature (London)* **455**, 778 (2008).
- ²⁰T. S. Nunner and F. von Oppen, *Phys. Rev. B* **84**, 0250405(R) (2011).
- ²¹J. Borge, C. Gorini, and R. Raimondi, *Phys. Rev. B* **87**, 085309 (2013).
- ²²K. Tauber, M. Gradhand, D. V. Fedorov, and I. Mertig, *Phys. Rev. Lett.* **109**, 026601 (2012).
- ²³J. M. Luttinger, *Phys. Rev.* **135**, A1505 (1964).
- ²⁴L. Smrcka and P. Streda, *J. Phys. C: Solid State Phys.* **10**, 2153 (1977).
- ²⁵N. R. Cooper, B. I. Halperin, and I. M. Ruzin, *Phys. Rev. B* **55**, 2344 (1997).
- ²⁶G. Strinati and C. Castellani, *Phys. Rev. B* **36**, 2270 (1987).
- ²⁷D. R. Niven and R. A. Smith, *Phys. Rev. B* **71**, 035106 (2005).
- ²⁸A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Dover, New York, 1963).
- ²⁹T. Ando, A. B. Fowler, and F. Stern, *Rev. Mod. Phys.* **54**, 437 (1982).