

Aharonov-Bohm conductance of a disordered single-channel quantum ring

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We study the effect of weak disorder on tunneling conductance of a single-channel quantum ring threaded by magnetic flux. We assume that the temperature is higher than the level spacing in the ring and smaller than the Fermi energy. In the absence of disorder, the conductance shows sharp dips (antiresonances) as a function of magnetic flux. We discuss different types of disorder and find that the short-range disorder broadens antiresonances, while the long-range one leads to the appearance of additional resonant dips. We demonstrate that the resonant dips have essentially non-Lorentzian shape. The results are generalized to account for the spin-orbit interaction, which leads to splitting of the disorder-broadened resonant dips, and consequently, to coexisting of two types of oscillations (both having the form of sharp dips): Aharonov-Bohm oscillations with magnetic flux and Aharonov-Casher oscillations with the strength of the spin-orbit coupling. We also discuss the effect of the Zeeman coupling.

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I. INTRODUCTION

The Aharonov-Bohm (AB) effect¹ is one of the beautiful manifestations of the wave nature of electrons. The key physical issue—the sensitivity of the phase of an electronic wave function to a magnetic flux—enables the design of quantum AB interferometers^{2–20} that can be tuned by an external magnetic field. Such interferometers occupy a worthy place in the quantum interferometry based on low-dimensional electronic nanosystems. A *single-channel ballistic* ring tunnel-coupled to the leads and threaded by the magnetic flux is the simplest realization of the AB interferometer (see Fig. 1). The interference of clockwise and counterclockwise electron trajectories manifests itself in the oscillations of the ring conductance $G(\phi)$ with the period 1 (here ϕ is the magnetic flux measured in the units of the flux quantum hc/e).^{1,2}

At low temperature T and weak tunneling coupling, AB conductance exhibits narrow resonant peaks both in clean and disordered single-channel rings²¹ (see also Refs. 22–27 for discussion of disordered case). The peak arises each time when one of the field-dependent energy levels in the ring crosses the Fermi energy E_F . Hence the positions of the AB resonances depend on E_F .²¹ (AB resonances are also affected by the Coulomb blockade^{28,29}). Based on this physical picture, one could expect the suppression of the resonance structure at $T \gg \Delta$, where Δ is the level spacing in the ring. Remarkably, this naive expectation is incorrect and the interference effects are not entirely suppressed by the thermal averaging. Specifically, for $T \gg \Delta$, the conductance of the noninteracting ring with weak tunnel coupling to the contacts exhibits sharp narrow dips (antiresonances) at $\phi = 1/2 + n$, where n is an arbitrary integer number (see Fig. 2).^{30,31} It was also shown that the electron-electron interaction leads to the emergence of a fine structure of the antiresonances: each antiresonance splits into a series of narrow dips, which correspond to blocking of the tunneling current by the persistent one³¹ (in contrast to the Coulomb blockade, this effect is robust to increasing of temperature).

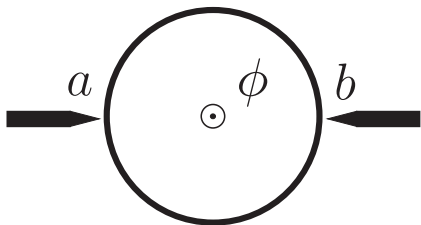
Additional physics comes into play in the presence of the spin-orbit (SO) interaction. In particular, the rotation of the electron spin in the built-in SO magnetic field results in a spin

phase shift between clockwise and counterclockwise waves. This phase is additional with respect to the AB phase and exists even at zero external magnetic field ($\phi = 0$) so that the zero-field conductance exhibits the Aharonov-Casher (AC) effect:^{32,33} periodic oscillations with the strength of the SO coupling. The AC oscillations were the focus of intensive theoretical^{34–50} research and their signatures were observed experimentally.^{51,52} Recently, we demonstrated that these oscillations are also not suppressed by thermal averaging,⁵³ in a full analogy with the AB ones. Specifically, at $T \gg \Delta$, SO interaction splits AB antiresonances into pairs of symmetrical (with respect to $\phi = 1/2 + n$) antiresonances. We also showed that the Zeeman interaction leads to the appearance of two additional negative peaks on each period.⁵³

What, to the best of our knowledge, has not been discussed in the literature is the effect of *disorder* on the tunneling conductance through a *single-channel* ring at relatively high temperatures, $T \gg \Delta$.^{54,55} The aim of the current study is to fill this gap.

In this paper, we study the tunneling transport of noninteracting electrons through a disordered single-channel quantum ring of length L threaded by a magnetic flux ϕ . We assume that T is much smaller than the Fermi energy E_F but large compared to $\Delta = 2\pi\hbar v_F/L$ (throughout the paper, we linearize electron spectrum near E_F , thus neglecting small variation of the electron velocity within the temperature band). The tunneling coupling characterized by tunneling probability γ is assumed to be weak, $\gamma \ll 1$, which implies that the ring is almost closed.

We discuss different types of disorder and find that the short-range disorder broadens antiresonances at $\phi = n + 1/2$ while the long-range one leads to the appearance of additional antiresonances at $\phi = n$. We also find that the resonant dips have essentially non-Lorentzian shape. The results are generalized to account for the spin-orbit interaction, which leads to splitting of the disorder-broadened resonant dips, and consequently to coexisting of two types of oscillations (both having the form of sharp periodic dips): Aharonov-Bohm oscillations with magnetic flux and Aharonov-Casher oscillations with the strength of the spin-orbit coupling. Additional

FIG. 1. The ring threaded by magnetic flux ϕ .

disorder-broadened resonant dips arise in the presence of the Zeeman coupling.

II. CLEAN RING

We start with a discussion of the high-temperature conductance of the clean ring following Refs. 31 and 53. This section aims to introduce basic notions and clarify our approach to the problem. Later, this approach will be generalized to describe the effect of disorder.

The conductance is given by the Landauer formula:

$$G(\phi) = \frac{e^2}{\pi\hbar} T(\phi), \quad (1)$$

where

$$T(\phi) = \langle T(\phi, E) \rangle_E = - \int T(\phi, E) \frac{\partial f}{\partial E} dE \quad (2)$$

is the thermal average of the transmission coefficient $T(\phi, E)$ and $f(E)$ is the Fermi-Dirac function (here we take into account double spin degeneracy).

We consider symmetrical setup (see Fig. 1) with identical point contacts described by the scattering matrix,

$$S = \begin{bmatrix} t_r & t_{\text{out}} & t_{\text{out}} \\ t & t_b & t_{\text{in}} \\ t & t_{\text{in}} & t_b \end{bmatrix}, \quad (3)$$

whose elements²¹

$$\begin{aligned} t_{\text{in}} &= \frac{1}{1+\gamma}, & t_b &= -\frac{\gamma}{1+\gamma}, \\ t = t_{\text{out}} &= \frac{\sqrt{2\gamma}}{1+\gamma}, & t_r &= -\frac{1-\gamma}{1+\gamma} \end{aligned} \quad (4)$$

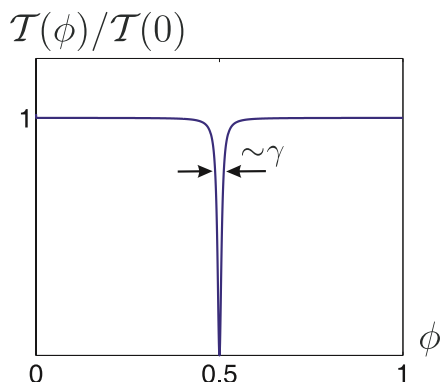


FIG. 2. (Color online) Antiresonance in high-temperature transmission coefficient in the absence of disorder.

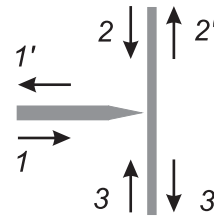


FIG. 3. Scattering on contacts: the amplitude t corresponds to processes $1 \rightarrow 2'$ and $1 \rightarrow 3'$, t_{out} : to $2 \rightarrow 1'$ and $3 \rightarrow 1'$, t_r : to $1 \rightarrow 1'$, t_{in} : to $2 \rightarrow 3'$ and $3 \rightarrow 2'$, and t_b : to $2 \rightarrow 2'$ and $3 \rightarrow 3'$.

represent amplitudes of scattering from three incoming channels (1,2,3) to three outgoing ones (1',2',3') (see Fig. 3). Here, γ is a real parameter characterizing the strength of the tunneling coupling to the contact: weak coupling corresponds to $\gamma \ll 1$, while an open contact is described by $\gamma \sim 1$.

The transmission amplitude can be calculated by summation of the amplitudes of all the trajectories connecting contact a and contact b , including the trajectories with backscatterings by contacts (the processes $2 \rightarrow 2'$ and $3 \rightarrow 3'$ on Fig. 3). Let us denote by n the number of times the electron passes the contact b , without exiting the ring ($n = 0, 1, \dots$).

The trajectories with a given n consist of the odd number $2n + 1$ of semicircles and thus have the same length $L_n = L(n + 1/2)$. The sum of the amplitudes of such trajectories can be written as $\beta_n \exp(ikL_n)$, where $k = \sqrt{2mE}/\hbar$ is the electron wave number. Hence the transmission amplitude is written as

$$t(\phi, E) = \sum_{n=0}^{\infty} \beta_n \exp(ikL_n). \quad (5)$$

Next, we separate contributions of trajectories ending with lower and upper semicircles thus writing $\beta_n = \beta_n^+ + \beta_n^-$.

Introducing vector β_n with two components, β_n^+ and β_n^- , one may easily derive the following recurrence relations:

$$\beta_{n+1} = \hat{A} \beta_n, \quad (6)$$

where the matrix \hat{A} is given by

$$\hat{A} = \begin{bmatrix} t_{\text{in}}^2 e^{-2\pi i \phi} + t_b^2 & t_b t_{\text{in}} (e^{-2\pi i \phi} + 1) \\ t_b t_{\text{in}} (e^{2\pi i \phi} + 1) & t_{\text{in}}^2 e^{2\pi i \phi} + t_b^2 \end{bmatrix} \quad (7)$$

$$= \frac{1}{(1+\gamma)^2} \begin{bmatrix} e^{-2\pi i \phi} + \gamma^2 & -\gamma(e^{-2\pi i \phi} + 1) \\ -\gamma(e^{2\pi i \phi} + 1) & e^{2\pi i \phi} + \gamma^2 \end{bmatrix}. \quad (8)$$

The element A_{ij} [multiplied by $\exp(ikL)$] is the sum of the amplitudes of the trajectories starting at the contact b and making a single return to the same contact (indices $i = \pm$ and $j = \pm$ specify, respectively, the final and initial directions of motion).

The components of the vector β_0 ,

$$\beta_0^+ = t t_{\text{out}} e^{-i\pi\phi}, \quad \beta_0^- = t t_{\text{out}} e^{i\pi\phi}, \quad (9)$$

yield contributions of shortest counterclockwise and clockwise trajectories, respectively.

Using Eq. (5), we express the transmission coefficient in terms of β_n :

$$T(\phi, E) = |t(\phi, E)|^2 = \sum_{n,m=0}^{\infty} \beta_n \beta_m^* e^{ik(L_n - L_m)}. \quad (10)$$

The terms with $n \neq m$ in Eq. (10) vanish after thermal averaging in the discussed case $T \gg \Delta$, so that the expression for the averaged transmission coefficient becomes

$$\mathcal{T}(\phi) = \sum_{n=0}^{\infty} |\beta_n|^2 = \sum_{n=0}^{\infty} |(\mathbf{e}, \hat{A}^n \boldsymbol{\beta}_0)|^2, \quad (11)$$

where vector \mathbf{e} has components (1, 1). The calculation of the sum entering Eq. (11) is quite cumbersome but straightforward (see Appendix A). Using Eq. (A5), we obtain³¹

$$\mathcal{T}(\phi) = \frac{2\gamma \cos^2 \pi \phi}{\gamma^2 + \cos^2 \pi \phi}. \quad (12)$$

The dependence $\mathcal{T}(\phi)$ is shown in Fig. 2. The physical explanation of the dip (antiresonance) at $\phi = 1/2$ is quite simple (here and below we consider the interval $0 < \phi < 1$). Let us demonstrate that at $\phi = 1/2$ the contribution of any trajectory is exactly canceled by the contribution of the trajectory mirrored with respect to the line connecting a and b . Indeed, the sum of the amplitudes of these two trajectories is proportional to $e^{ikL_n} [e^{i(2|m|+1)\pi\phi} + e^{-i(2|m|+1)\pi\phi}]$, where m is a difference between the number of clockwise and counterclockwise revolutions, $|m| \leq n$. At $\phi = 1/2$, this sum turns to zero for any k . Thus the antiresonance is due to the destructive interference of mirrored paths.

For weak tunneling coupling, $\gamma \ll 1$, the antiresonance is well approximated by the Lorentz-shape dip:

$$\mathcal{T}(\phi) \approx 2\gamma \frac{\pi^2 \delta\phi^2}{\gamma^2 + \pi^2 \delta\phi^2}, \quad (13)$$

where $\delta\phi = \phi - 1/2$.

It is worth noting that in the vicinity of antiresonance one can neglect the backscattering on the contacts. Indeed, at $|\phi - 1/2| \sim \gamma$, the off-diagonal elements of \hat{A} are proportional to $t_b \sim \gamma$ multiplied by small factors $1 + \exp(\pm i2\pi\phi) \sim \gamma$, which implies that backscattering is effectively suppressed by a factor γ .

Physically, the effective suppression of backscattering is explained by destructive interference of two processes. In the first process, an electron is reflected by a contact (say, contact b) and returns to this contact after one revolution around the ring.

The amplitude of such a process is $t_b t_{in} \exp(\pm i2\pi\phi)$ (the sign is prescribed by direction of the propagation) where the amplitude t_{in} appeared because the contact a was passed without reflection. In the second process, the electron passes the contact b without reflection and then is reflected by contact a and returns to b . The corresponding amplitude is given by $t_{in} t_b$. Evidently, for $\phi = 1/2$, the amplitudes of these processes exactly cancel each other.

It is worth noting that backscattering is important in vicinity of integer values of flux.³¹ In particular, by putting $t_b = 0$ in Eq. (7), we come to an incorrect conclusion that there are resonant peaks at $\phi = n$ in evident contradiction with Eq. (12).

The approach discussed above allows one to find the transmission coefficient for arbitrary γ and ϕ . However, it is technically cumbersome and lacks physical transparency. Below, we derive the main result of this section, Eq. (13), by using an alternative method. This method is valid only in the vicinity of $\phi = 1/2$, where backscattering by contacts can be

neglected. However, it has a number of advantages compared to the first one: it is more illustrative physically, and much more easily generalized to account for disorder.

The key idea is that for $\gamma \ll 1$ and $\delta\phi \ll 1$ the tunneling amplitude through the ring may be presented as a sum of the transition amplitudes through intermediate states corresponding to quasistationary levels of an almost closed ring. The appropriate analytical expression is derived in Appendix B and reads

$$\mathcal{T} \approx \hbar^2 v_F^2 t_{out}^2 \langle |G_{E+i\Gamma/2}(0, L/2)|^2 \rangle_E, \quad (14)$$

where

$$\begin{aligned} G_{E+i\Gamma/2}(0, L/2) &= \frac{1}{\hbar v_F} \sum_l C_l(E) \\ &= \sum_l \frac{\psi_l^*(0) \psi_l(L/2)}{E - \epsilon_l + i\Gamma/2}, \end{aligned} \quad (15)$$

$G_E(0, L/2)$ is the Green function of the closed ring, describing the transition from the contact a to contact b , E is the energy of the tunneling electron, and ϵ_l are the electron energies in the closed ring, corresponding to wave functions $\psi_l(x)$. The quantities

$$C_l(E) = \hbar v_F \frac{\psi_l^*(0) \psi_l(L/2)}{E - \epsilon_l + i\Gamma/2} \quad (16)$$

are the amplitudes of transition through the corresponding quasistationary states (see Fig. 4).

Here, Γ is the tunneling rate given by

$$\Gamma = \frac{2\Delta\gamma}{\pi}. \quad (17)$$

In the closed clean ring there are two types of the electron states, corresponding to counterclockwise and clockwise propagation, labeled below by indices $l = (n, +)$ and $l = (n, -)$,

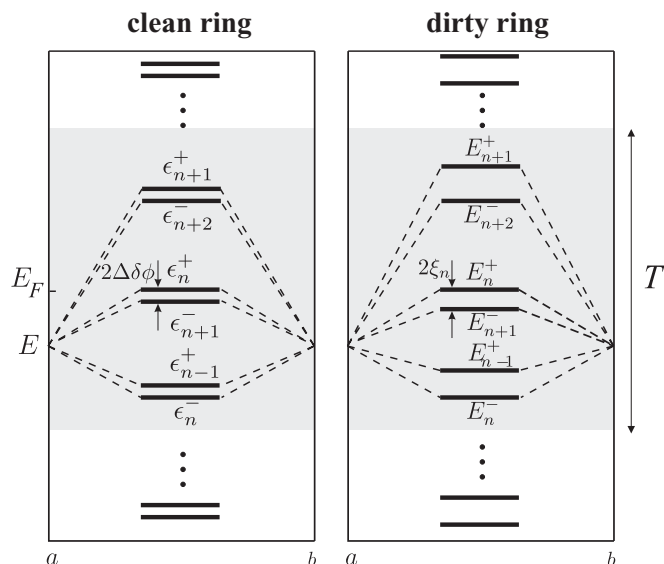


FIG. 4. Tunneling of an electron through pairs of close levels in the ring. For a clean ring, the distance between levels in all pairs is the same and is given by $2\Delta\delta\phi$. For a ring with disorder, this distance increases due to the repulsion of the levels in the disorder potential and becomes n dependent: $E_n^+ - E_{n+1}^- = 2\xi_n = 2\sqrt{\Delta^2\delta\phi^2 + |V_n|^2}$.

respectively. Wave functions and energies of these states read

$$\psi_n^\pm(x) = \frac{e^{\pm i2\pi nx/L}}{\sqrt{L}}, \quad \epsilon_n^\pm(\phi) = \Delta(n \pm \phi) + c, \quad (18)$$

where the constant $c = E_F - \Delta n_F$ arises in course of linearization of the spectrum near the Fermi energy [here n_F obeys the following equation: $E_F = 2\hbar^2\pi^2 n_F^2 / mL^2$]. As seen, for $\phi = 1/2$, each level is double degenerate, $\epsilon_n^+(1/2) = \epsilon_{n+1}^-(1/2)$. Finite $\delta\phi$ lifts the degeneracy of these levels:

$$\epsilon_n^+ = E_n + \Delta\delta\phi, \quad \epsilon_{n+1}^- = E_n - \Delta\delta\phi, \quad (19)$$

where $E_n = \Delta(n + 1/2) + c$.

Let us now demonstrate that Eqs. (14) and (15) yield $\mathcal{T}(1/2) = 0$. Indeed, from Eqs. (16) and (18), we easily find

$$C_n^+ = -C_{n+1}^-, \quad \text{for } \phi = 1/2, \quad (20)$$

so that $G_E(0, L/2)|_{\phi=1/2} \equiv 0$ for any E and the transmission coefficient turns to zero even before energy averaging. The minus sign in the right-hand side (r.h.s.) of Eq. (20) appeared due to the property

$$\psi_n^+(L/2) = -\psi_{n+1}^-(L/2). \quad (21)$$

This is the property that leads to destructive interference and formation of the dip in the tunneling conductance. Physically, this is an alternative way to describe the compensation of mirrored paths discussed above.

Next we derive Eq. (13). First, we rewrite Eq. (14) as follows:

$$\mathcal{T} = -t^2 t_{\text{out}}^2 \int \sum_{l,l'} C_l^*(E) C_{l'}(E) \frac{\partial f}{\partial E} dE. \quad (22)$$

The double sum in this equation contains ‘‘classical’’ terms proportional to $|C_l|^2$, as well as the interference ones, $C_l^* C_{l'}$ (with $l \neq l'$). The main contribution to the integral in Eq. (22) comes from the vicinities of poles (with the size on the order of Γ) of the amplitudes $C_l^*(E)$ and $C_{l'}(E)$. It is easy to see from Eqs. (16) and (22) that interference terms are comparable with classical ones only if $|\epsilon_l - \epsilon_{l'}| \lesssim \Gamma$. For $\delta\phi \ll 1$, energies ϵ_n^+ and ϵ_{n+1}^- are close to each other, $\epsilon_n^+ - \epsilon_{n+1}^- \approx 2\Delta\delta\phi$ (see Fig. 4), and differ from the energies of other levels by a much larger distance (Δ or larger). The interference contributions to Eq. (22) containing products of the amplitudes from different pairs can be neglected compared to the ‘‘classical’’ terms and to the interference terms, containing products of the amplitudes from the same pair. Therefore we can rewrite Eq. (22) as a sum over pairs of close levels:

$$\mathcal{T} \approx -t^2 t_{\text{out}}^2 \sum_n \int |C_n^+ + C_{n+1}^-|^2 \frac{\partial f}{\partial E} dE. \quad (23)$$

Hence the paths over which electron passes through the ring can be split into pairs of interfering paths, corresponding to quasidegenerate intermediate states ψ_n^+ and ψ_{n+1}^- (see Fig. 4).

Now we demonstrate that contributions of different pairs in Eq. (23) in fact differ by the thermal factor only, so that the problem can be reduced to the analysis of the transition through a single pair. First, we notice that the energy dependence of the thermal factor $\partial f / \partial E$ is smooth and in the n th term of the sum one may replace $\partial f / \partial E$ with $(\partial f / \partial E)|_{E=E_n}$. Next, we change the integration variable in this

term: $E \rightarrow \epsilon = E - E_n$. Then, dependence on n remains only in the factors $\psi_n^{\pm*}(0)\psi_n^\pm(L/2) = (-1)^n/L$. This dependence disappears after calculation of modulus squared in Eq. (23).

Finally, we calculate the sum over n , $\sum_n (-\partial f / \partial E)|_{E=E_n} \approx 1/\Delta$ (here, we use inequality $T \gg \Delta$), and arrive to the following equation:

$$\mathcal{T} = \frac{t^2 t_{\text{out}}^2 \Delta}{4\pi^2} \int_{-\infty}^{\infty} d\epsilon \left| \frac{1}{\epsilon - \epsilon^+ + i\Gamma/2} - \frac{1}{\epsilon - \epsilon^- + i\Gamma/2} \right|^2, \quad (24)$$

which expresses \mathcal{T} in terms of transition through a single pair of close levels with energies $\epsilon^\pm = \pm\Delta\delta\phi$. The minus sign in front of the second fraction in Eq. (24) appeared due to the property (21). One may separate in Eq. (24) a ‘‘classical’’ contribution, $\mathcal{T}_{\text{cl}}(\phi)$ (integral from the sum of the squared amplitudes), from the interference one, $\mathcal{T}_{\text{int}}(\phi)$. Performing integration, we find $\mathcal{T}_{\text{cl}}(\phi) = 2\gamma$ and $\mathcal{T}_{\text{int}}(\phi) = -2\gamma^3/(\gamma^2 + \pi^2\delta\phi^2)$. Summing these terms, we restore Eq. (13). We notice that, as expected, the interference term gives a significant contribution only in the region of small $\delta\phi$: $|\delta\phi| \lesssim \gamma$.

III. THE RING WITH IMPURITIES

In the above calculations, we considered the case of the clean ring. Now we discuss the effect of disorder on the high-temperature conductance of the ring.

A. Long-range disorder

One of the realizations of the disorder is a weak smooth random potential with the correlation length much exceeding the electron Fermi wavelength. In this case, backscattering by disorder is exponentially suppressed, so that the potential only leads to the additional phase shift between the right- and left-moving electron waves propagating from contact a to contact b along the upper and lower shoulder of an interferometer, respectively (with zero winding number). We denote the disorder-induced phase difference between these two waves as $\Psi(E)$. Such an interferometer is evidently equivalent to the clean one having two arms with lengths $(L - a)/2$ and $(L + a)/2$, where $a \approx \Psi(E_F)/k_F$. The conductance of the latter interferometer was calculated in Ref. 31. From Eq. (A3) of Ref. 31, we find

$$\mathcal{T}(\phi) = F[\sin(\pi\phi), \sin(\Psi/2)] + F[\cos(\pi\phi), \cos(\Psi/2)],$$

where

$$F(x, y) = 2\gamma \frac{x^2 y^2}{x^2 + \gamma^2 y^2}. \quad (25)$$

This equation is valid provided that $T(d\Psi/dE)_{E=E_F} \ll 1$. As seen, for $\Psi \neq 0$, there are two dips in the conductance (at $\phi = 1/2$ and at $\phi = 0$), the widths and the depths of the dips being oscillating functions of $\Psi = \Psi(E_F)$ [in particular, $\mathcal{T}(0) = 2\gamma \cos^2(\Psi/2)$, $\mathcal{T}(1/2) = 2\gamma \sin^2(\Psi/2)$]. Hence long-range disorder leads to appearance of the additional antiresonance in the conductance at $\phi = 0$ and modifies the antiresonance near $\phi = 1/2$.

We notice that averaging of the transmission coefficient over the phase Ψ (ensemble averaging) yields conductance

oscillations with the period $1/2$ in accordance with previous studies.^{22,23} In this case, the conductance is given by Eqs. (A5) and (A7) of Ref. 31. The oscillations with the period $1/2$ can be also observed in a single ring provided that $T(d\Psi/dE)_{E=E_F} \gg 1$.

B. Short-range disorder

1. Calculation of the transmission coefficient

Another realization of disorder is the potential created by weak short-range impurities, randomly distributed along the ring with the concentration n_i . Let us characterize the strength of disorder by the scattering rate in the infinite wire calculated by the golden rule. For short-range potential, transport and quantum scattering rates coincide and are given by $1/\tau = 2|r|^2 v_F n_i$, where r is the reflection amplitude for a single impurity ($|r| \ll 1$). Substituting in this equation $n_i = N/L$ (here N is the number of impurities in the ring), we get

$$\frac{1}{\tau} = \frac{N|r|^2 \Delta}{\pi \hbar}. \quad (26)$$

We restrict ourselves to the discussion of the ballistic case, $v_F \tau \gg L$, or, equivalently, $N|r|^2 \ll 1$.

In this case, backscattering events occur after many rotations around the ring, so that we are in the ballistic regime (as opposed to the regime of the strong localization, which develops for $v_F \tau \ll L$).

We will see that the main effect of the short-range potential is the broadening of the antiresonances. One could expect that scattering by disorder leads to an essential increase of the resonance width, when τ becomes shorter than the lifetime of the electron in the ring, \hbar/Γ , which implies $N|r|^2 \gg \gamma$. Another expectation is that in the regime $N|r|^2 \gg \gamma$, when an electron experiences many scatterings during its lifetime and therefore acquires a random phase, the interference is suppressed, and, consequently, the depth of the dip essentially decreases. However, we will show that the scattering on the impurities comes into play at a much smaller disorder strength when $N|r|^2 \sim \gamma^2$, so that for $\gamma^2 \ll N|r|^2 \ll 1$ the dip is essentially broadened. Also, in contrast to the naive expectation, its depth remains on the order of γ .

Now, we generalize the method introduced in the first part of the previous section. To do this, we should modify the matrix \hat{A} , taking into account the scattering on the impurities. This matrix becomes complicated, since it includes the amplitudes of all the trajectories with scatterings on both contacts and impurities, after which an electron returns to contact b . However, in the case $\delta\phi \ll 1$, $\gamma \ll 1$, and $N|r|^2 \ll 1$ the matrix \hat{A} can be simplified.

As a first step, we expand the matrix \hat{A} in a Taylor series up to the first order with respect to γ , denoting $\hat{A}_0 = \hat{A}|_{\gamma=0}$. For the clean ring ($r = 0$), we have

$$\hat{A} \approx (1 - 2\gamma)\hat{A}_0 \approx (1 - 2\gamma) \begin{bmatrix} e^{-2\pi i\phi} & 0 \\ 0 & e^{2\pi i\phi} \end{bmatrix}. \quad (27)$$

We neglected off-diagonal elements of \hat{A} , since, as we explained above, backscattering on the contacts is effectively suppressed at $\delta\phi \ll 1$. Now we write the expansion for a dirty ring in a way that reproduces Eq. (27) for $r = 0$:

$$\hat{A} = \hat{A}_0 - 2\gamma(\hat{A}_0 + \delta\hat{A}) + \dots \quad (28)$$

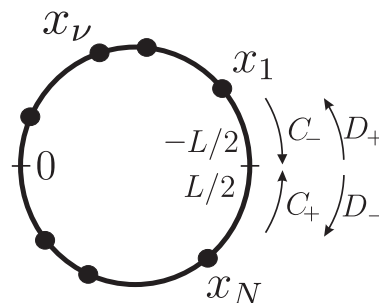


FIG. 5. A ring with impurities.

Here, both \hat{A}_0 and $\delta\hat{A}$ depend on r . The matrix $\delta\hat{A}$ should vanish at $r = 0$, so that $\delta\hat{A} \propto r$ at small r . Calculations using Eq. (A5) show that one can neglect the term $\gamma\delta\hat{A}$ in Eq. (28) (as well as terms on the order of γ^2 and higher). Physically, this implies neglecting of the processes involving both scattering by impurities and forward scattering by contacts during one revolution around the ring.

Let us now discuss the properties of the matrix \hat{A}_0 . According to the definition of the matrix \hat{A} (see previous section) the matrix $e^{ikL}\hat{A}_0$ relates the amplitudes C_{\pm} and D_{\pm} of the incoming and outgoing waves, respectively, at the point b (see Fig. 5):

$$\begin{bmatrix} C_+ \\ C_- \end{bmatrix} = e^{ikL}\hat{A}_0 \begin{bmatrix} D_+ \\ D_- \end{bmatrix}. \quad (29)$$

Hence, the matrix $e^{ikL}\hat{A}_0$ is the S matrix describing a complex scatterer consisting of N impurities, located at points x_v between $x = -L/2$ and $x = L/2$. Having in mind that this matrix should be unitary and taking into account the time-reversal symmetry, we write this matrix in the most general form:

$$\hat{A}_0 = e^{i\alpha} \begin{bmatrix} \sqrt{1 - |R|^2} e^{-2i\pi\phi} & R \\ -R^* & \sqrt{1 - |R|^2} e^{2i\pi\phi} \end{bmatrix}. \quad (30)$$

Here, α is the small forward scattering phase for a complex scatterer consisting of N impurities. This phase is added to the geometrical phase kL and, therefore, drops out after thermal averaging. The off-diagonal element R is, up to a phase factor, the reflection amplitude from a complex of N impurities. One can expand R with respect to r . In the lowest order in r , we obtain

$$R \approx r \sum_{v=1}^N e^{-2ikx_v}. \quad (31)$$

This expression takes into account only one backscattering on impurities during a revolution around the ring, and is valid in the case $N|r|^2 \ll 1$.

As we see, the matrix \hat{A} is now dependent on k , so that Eq. (11) does not generally follow from Eq. (10). However, if we assume that impurities are randomly distributed along the ring (some special nonrandom distributions will be discussed at the end of Sec. III B) the terms with $n \neq m$ in Eq. (10) do not survive the averaging over k . Therefore we can use Eq. (11) while performing averaging over k :

$$\mathcal{T} = \left\langle \sum_{n=0}^{\infty} |(\mathbf{e}, \hat{A}^n \boldsymbol{\beta}_0)|^2 \right\rangle_k. \quad (32)$$

Since the probability to scatter on an impurity before the first visit of the point b is small, we can neglect the scattering terms in the vector β_0 entering Eq. (32) and use Eq. (9). The sum in Eq. (32) can be calculated with the use of Eq. (A5), which simplifies after expansion of the numerator and denominator with respect to γ , r , and $\delta\phi$.

Calculations yield the following expression for the transmission coefficient:

$$\mathcal{T} \approx 2\gamma \left\langle \frac{\pi^2 \delta\phi^2 + |r|^2 (\sum_v \sin 2kx_v)^2 / 4}{\pi^2 \delta\phi^2 + \gamma^2 + |r|^2 |\sum_v e^{2ikx_v}|^2 / 4} \right\rangle_k. \quad (33)$$

This equation is valid provided that $\gamma \ll 1, N|r|^2 \ll 1$ and $\delta\phi \ll 1$. The relation between $\sqrt{N}|r|$ and γ can be arbitrary.

The approach, discussed above, can be also used to calculate the transmission coefficient for arbitrary ϕ . In particular, one can show that at $\phi = 0$, there appears a small dip with the amplitude on the order of $N|r|^2\gamma \ll \gamma$ (to obtain this result one should take into account backscattering by contacts). In the following discussion, we focus only on the antiresonance at $\phi = 1/2$.

Let us first consider a ring with a single impurity. As seen from Eq. (33), the transmission coefficient is given by the Lorentz-shape antiresonance:

$$\mathcal{T} \approx 2\gamma \frac{\pi^2 \delta\phi^2 + |r|^2 / 8}{\pi^2 \delta\phi^2 + \gamma^2 + |r|^2 / 4}. \quad (34)$$

We see that the transmission coefficient at $\phi = 1/2$ is no longer equal to zero and the antiresonance broadens so that its width becomes $\sqrt{\gamma^2 + |r|^2/4}$. We also find that the depth of the dip changes from 2γ to γ with increasing $|r|$. In other words, in contrast to the antiresonance width, its depth remains the same order of magnitude.

In order to perform the averaging over k in the case of many impurities, we notice, that for random impurity distribution the averaging over k is equivalent to averaging over x_v : $\langle \dots \rangle_k = \langle \dots \rangle_{x_1 \dots x_N}$. For two impurities, the average is easily calculated. The result reads

$$\mathcal{T} \approx \gamma \left[\frac{\pi^2 \delta\phi^2 - \gamma^2}{\sqrt{(\pi^2 \delta\phi^2 + \gamma^2)(\pi^2 \delta\phi^2 + \gamma^2 + |r|^2)}} + 1 \right]. \quad (35)$$

For the case $N > 2$, we rewrite Eq. (33) using the identities $x^{-1} \equiv \int_0^\infty \exp(-tx) dt$, $\exp(-x^2) \equiv \int_{-\infty}^\infty \exp(-y^2 + 2ixy) dy / \sqrt{\pi}$, and get the following expression:

$$\begin{aligned} \mathcal{T} \approx 2\gamma \int_0^\infty \frac{dt}{4\pi t} e^{-4(\pi^2 \delta\phi^2 + \gamma^2)t/|r|^2} \\ \times \int_{-\infty}^\infty d\xi d\eta e^{-\frac{\xi^2 + \eta^2}{4t}} \left(\frac{4\pi^2 \delta\phi^2}{|r|^2} - \frac{\partial^2}{\partial \xi^2} \right) J_0^N(\sqrt{\eta^2 + \xi^2}), \end{aligned} \quad (36)$$

or

$$\begin{aligned} \mathcal{T} \approx 2\gamma \int_0^\infty d\rho K_0 \left(\frac{2\rho \sqrt{\pi^2 \delta\phi^2 + \gamma^2}}{|r|} \right) \\ \times \left(\frac{4\pi^2 \delta\phi^2 \rho}{|r|^2} - \frac{\partial}{2\partial \rho} \rho \frac{\partial}{\partial \rho} \right) J_0^N(\rho), \end{aligned} \quad (37)$$

where K_0 is the modified Bessel function of the second kind.

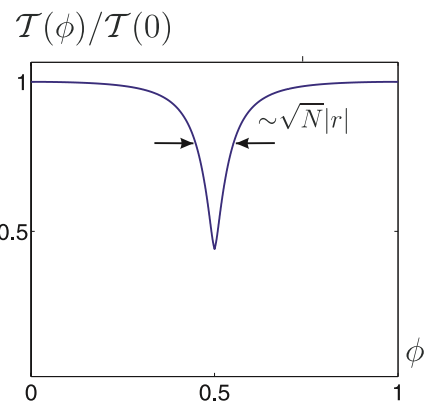


FIG. 6. (Color online) Antiresonance in high-temperature transmission coefficient in the presence of N randomly distributed impurities for $\sqrt{N}|r| \gg \gamma$.

Assuming now that the number of impurities is large, $N \gg 1$, we get $J_0^N(x) \approx \exp(-Nx^2/4)$ and after simple calculation obtain

$$\begin{aligned} \mathcal{T} \approx \frac{2\gamma}{s^2} \int_0^\infty dx \frac{\pi^2 \delta\phi^2 (1+x) + s^2/2}{(1+x)^2} \\ \times \exp\left(-x \frac{\pi^2 \delta\phi^2 + \gamma^2}{s^2}\right), \end{aligned} \quad (38)$$

where $s^2 = N|r|^2/4$. This dependence is plotted in Fig. 6. We see that similar to the case of a single impurity, the transmission coefficient at $\phi = 1/2$ is no longer equal to zero and the antiresonance broadens. It is also notable that for $N \geq 2$ the dip has a non-Lorentzian shape.

Let us discuss two limiting cases. For $\sqrt{N}|r| \ll \gamma$, the minimal value of conductance is given by $T|_{\delta\phi=0} \approx N|r|^2/4\gamma$, and the width of the antiresonances increases from γ to $\gamma' = \gamma + \delta$, where

$$\delta \sim \frac{N|r|^2}{\gamma}. \quad (39)$$

The relative contribution of the disorder to the resonance width, $\delta/\gamma \sim N|r|^2/\gamma^2 = \hbar/\gamma\Gamma\tau$, is enhanced by a factor $\gamma^{-1} \gg 1$ in comparison with naive expectation $\hbar/\Gamma\tau$. In the opposite limiting case, $\sqrt{N}|r| \gg \gamma$, we get $T|_{\delta\phi=0} \approx \gamma$, so that the depth of the antiresonance is two times smaller compared with the case of clean ring, while the width is given by

$$\gamma' \sim \sqrt{N}|r|. \quad (40)$$

2. Two-level approximation

Next, we discuss the obtained results in terms of transition amplitudes through intermediate quasistationary states of an almost closed ring. We recall that in the vicinity of the flux $\phi = 1/2$ the energy levels of the clean ring can be split into pairs of close levels corresponding to clockwise and counterclockwise propagations of the electron inside the ring. Energy distance between levels in the pairs equals to $2\Delta\delta\phi$ (see Fig. 4, left panel). For the case when impurity potential $V(x) = \sum_{v=1}^N U(x - x_v)$ [here, $U(x - x_v)$ is the potential of the v th impurity] is sufficiently weak (see corresponding

criterion below), it can be simply accounted for by perturbation theory for two close levels.

Calculation yields for energies and wave functions of potential-disturbed states:

$$E_n^+ = E_n + \xi_n, \quad E_{n+1}^- = E_n - \xi_n, \quad (41)$$

$$\xi_n = \sqrt{\Delta^2 \delta\phi^2 + |V_n|^2}, \quad (42)$$

$$\Psi_n^+ = \frac{\psi_n^+ + \psi_{n+1}^- V_n / W_n}{\sqrt{1 + |V_n|^2 / W_n^2}}, \quad (43)$$

$$\Psi_{n+1}^- = \frac{\psi_{n+1}^- - \psi_n^+ V_n^* / W_n}{\sqrt{1 + |V_n|^2 / W_n^2}}, \quad (44)$$

where $W_n = \Delta\delta\phi + \xi_n$, and

$$\begin{aligned} V_n &= \int dx \psi_{n+1}^{-*}(x) V(x) \psi_n^+(x) \\ &= \frac{i r \Delta}{2\pi} \sum_v e^{2\pi i(2n+1)x_v/L} \end{aligned} \quad (45)$$

is the matrix element of the impurity potential expressed in terms of reflection amplitude r . For $\delta\phi \ll 1$, the inequality that ensures validity of the two-level perturbation theory is given by $|V_n| \ll \Delta$. For randomly distributed impurities, the amplitude of the potential is estimated as $|V_n| \sim |r| \Delta \sqrt{N}$. Hence the perturbation theory applies for $|r| \sqrt{N} \ll 1$.

Let us first discuss the dependence of the conductance on $\delta\phi$ qualitatively. First we note that due to the repulsion between levels in the impurity potential, the minimal distance between levels in any pair is given by $|V_n|$ (this distance corresponds to $\delta\phi = 0$). For $|V_n| \ll \Gamma$ similar to the case of a clean ring, there exist a dip in the transmission coefficient, which arises due to the interference between transition amplitudes through pairs of close levels (see Fig. 4), while the ‘‘classical’’ term is featureless at $\phi = 1/2$.

In contrast, in the opposite limiting case, $|V_n| \gg \Gamma$, the energy distance between levels in any pair becomes much larger than Γ , and, consequently, the contribution of the interference terms to the transmission coefficient is small compared to the ‘‘classical’’ ones. This, however, does not lead to the disappearance of the dip in the transmission coefficient. It turns out that in the dirty ring, the ‘‘classical’’ terms acquire a sharp dependence on ϕ and decrease by a factor 2 within a narrow region $\delta\phi \sim \sqrt{N}|r|$.

Indeed, as seen from Eqs. (43) and (44), for $|\Delta\delta\phi| \gg |V_n|$, the wave functions in the n th pair are simply given by clockwise- and counterclockwise-moving waves ψ_n^+ and ψ_{n+1}^- . The ‘‘classical’’ contribution to the transmission coefficient from each of these levels, say level $(n,+)$, is proportional to $|\psi_n^+(0)|^2 |\psi_n^+(L/2)|^2 = 1/L^2$. In the opposite case, $|\Delta\delta\phi| \ll |V_n|$, disorder potential strongly mixes clockwise- and counterclockwise-propagating waves. Consider, for simplicity, the case $\delta\phi = 0$. From Eqs. (43) and (44), we see that $\Psi_n^+ = (\psi_n^+ + e^{i\varphi_n} \psi_{n+1}^-) / \sqrt{2}$ and $\Psi_{n+1}^- = (\psi_n^+ - e^{-i\varphi_n} \psi_{n+1}^-) / \sqrt{2}$, where $\exp(i\varphi_n) = V_n / |V_n|$. Averaging $|\Psi_n^+(0)|^2 |\Psi_n^+(L/2)|^2$ over random phase φ_n and taking into account Eq. (21), we obtain a twice smaller value, $1/2L^2$. This implies the existence of a dip of a width $\delta\phi \sim |V_n|/\Delta \sim$

$\sqrt{N}|r|$ in the transmission coefficient: \mathcal{T} decreases by a factor 2 within this width.

The transition from the interference to the ‘‘classical’’ picture of formation of the dip can be illustrated by an example of the ring with a single impurity. The conductance of such a ring is given by Eq. (34). The ‘‘classical’’ and interference contributions to this expression read

$$\mathcal{T}_{\text{cl}}(\phi) = \frac{2\gamma(\pi^2\delta\phi^2 + |r|^2/8)}{\pi^2\delta\phi^2 + |r|^2/4}, \quad (46)$$

$$\mathcal{T}_{\text{int}}(\phi) = -\frac{2\gamma^3(\pi^2\delta\phi^2 + |r|^2/8)}{(\pi^2\delta\phi^2 + |r|^2/4)(\pi^2\delta\phi^2 + \gamma^2 + |r|^2/4)}. \quad (47)$$

As seen, the interference contribution leads to formation of the dip at $\gamma \gg |r|$ and can be neglected for $\gamma \ll |r|$.

The rigorous calculations of the conductance can be performed in a way analogous to the case of the clean ring. The transmission coefficient is expressed via a Green function of the closed ring and can be approximately presented as a sum over pairs of interfering paths through intermediate states Ψ_n^+ and Ψ_{n+1}^- . Equations (16) and (23) still hold with the following replacement: $\psi_n^\pm \rightarrow \Psi_n^\pm$ and $\epsilon_n^\pm \rightarrow E_n^\pm$. In contrast to the case of the clean ring, the summands in Eq. (23) are now different not only due to the thermal factor $(\partial f / \partial E)|_{E=E_n}$ but also because of the strong dependence of V_n on n . However, for random distribution of impurities, the summation over n within the temperature window is equivalent to averaging over the impurity positions. After such averaging, the dependence on n remains only in the thermal factor and the problem is again reduced to the case of transition through a single pair of close levels. The transmission coefficient may be written as a sum of classical and interference terms:

$$\mathcal{T}(\phi) = \frac{t_{\text{out}}^2 t_{\text{in}}^2}{\Delta} \int_{-\infty}^{\infty} d\epsilon (\rho_{\text{cl}} + \rho_{\text{int}}). \quad (48)$$

Here, $\rho_{\text{cl}} = \langle |C_n^+|^2 + |C_{n+1}^-|^2 \rangle_{x_v}$, $\rho_{\text{int}} = 2\text{Re} \langle C_n^{+*} C_{n+1}^- \rangle_{x_v}$, where C_n^+, C_{n+1}^- are now expressed via energy ϵ in the following way: $C_n^+ = \hbar v_F \Psi_n^{+*}(0) \Psi_n^+(L/2) / (\epsilon - \xi_n + i\Gamma/2)$ and $C_{n+1}^- = \hbar v_F \Psi_{n+1}^{-*}(0) \Psi_{n+1}^-(L/2) / (\epsilon + \xi_n + i\Gamma/2)$. Using Eqs. (41)–(48), after some algebra, we arrive to Eq. (33).

3. Special impurity distributions

Above, we assumed that impurities are randomly distributed along the ring. If this is not the case, the results might be quite different. The reason is that there exist some impurity distributions for which the summation over n within the temperature band is not equivalent to the averaging over the impurity positions, so that the calculation should be performed more carefully. One may check that the method discussed in Sec. III B 2 reproduces Eq. (33) with the replacement $k \rightarrow k_n = 2\pi(n + 1/2)/L$ and $\langle \dots \rangle_k \rightarrow -\Delta \sum_n (\partial f / \partial E)_{k=k_n} \dots$.

Let us now give some examples of special impurity distributions for which Eqs. (35)–(38) are invalid. If the impurities are distributed symmetrically with respect to the line (a, b) connecting the contacts, then $\sum_v \sin 2k_n x_v = 0$ and, consequently, $\mathcal{T}(1/2) = 0$. On the language of trajectories, this can be explained by cancellation of the contributions of the mirrored paths just as in the case of the clean ring. The width of the dip is on the order of $|r| \sqrt{N}$. If

impurities are distributed symmetrically with respect to the line perpendicular to (a, b) and crossing the center of the ring, then $\sum_v \cos 2k_n x_v = 0$ and the amplitude of the dip in $\mathcal{T}(\phi)$ goes to zero with increasing r for $\sqrt{N}|r| \gg \gamma$. Finally, if impurities are distributed symmetrically with respect to the ring center, one gets $\sum_v \exp(2ik_n x_v) = 0$ and the dependence $\mathcal{T}(\phi)$ becomes the same as in the clean ring.

IV. THE RING WITH IMPURITIES AND SPIN-ORBIT INTERACTION

In this section, we discuss the effect of impurities on the conductance of a ring with spin-orbit and Zeeman interactions. The case of a clean ring was studied in detail in Ref. 53.

The Hamiltonian of a clean ring with SO interaction induced by an axially symmetric built-in field is given by

$$\hat{H} = \hat{H}_{\text{kin}} + \hat{H}_Z + \hat{H}_{\text{SO}}, \quad (49)$$

where

$$\hat{H}_{\text{kin}} = -\frac{\hbar^2}{2m} D_x^2 \quad (50)$$

is the kinetic energy, $D_x = \partial/\partial x + 2\pi i\phi/L$,

$$\hat{H}_Z = \frac{1}{2}\hbar\omega_Z\hat{\sigma}_z \quad (51)$$

is the Zeeman term ($\hbar\omega_Z$ is the Zeeman splitting energy in the external magnetic field parallel to the z axis) and \hat{H}_{SO} describes the SO coupling:

$$\hat{H}_{\text{SO}} = -i\xi \frac{\hbar^2}{2m} \left\{ \begin{bmatrix} -\cos\theta & \sin\theta e^{-2\pi ix/L} \\ \sin\theta e^{2\pi ix/L} & \cos\theta \end{bmatrix}, D_x \right\}. \quad (52)$$

Here, θ is the angle between the effective SO-induced magnetic field and the z axis, ξ is the dimensionless parameter characterizing the strength of SO interaction, and $\{\dots\}$ stands for the anticommutator.

The problem is studied in the quasiclassical case ($k_F L \gg 1, \xi \ll k_F L$) in which the effect of the SO interaction is described by the rotation of the electron spin in an effective magnetic field, which varies along the electron trajectory. The stationary wave functions and energies in this case read

$$\begin{aligned} \psi_{n,\pm}^{(1)}(x) &= e^{\pm i2\pi nx/L} \begin{bmatrix} \cos\vartheta_{\pm}/2 \\ -\sin\vartheta_{\pm}/2 e^{2\pi ix/L} \end{bmatrix}, \\ \psi_{n,\pm}^{(2)}(x) &= e^{\pm i2\pi nx/L} \begin{bmatrix} \sin\vartheta_{\pm}/2 e^{-2\pi ix/L} \\ \cos\vartheta_{\pm}/2 \end{bmatrix}, \\ \epsilon_{n,+}^{(1)} &= \Delta(n + \phi - \delta_+), \quad \epsilon_{n,-}^{(1)} = \Delta(n - \phi + \delta_-), \\ \epsilon_{n,+}^{(2)} &= \Delta(n + \phi + \delta_+), \quad \epsilon_{n,-}^{(2)} = \Delta(n - \phi - \delta_-). \end{aligned} \quad (53)$$

Here, we introduced the notations

$$\begin{aligned} \delta_{\pm} &= |\varkappa_{\pm}| - \frac{1}{2}, \quad e^{i\vartheta_{\pm}} = \frac{\varkappa_{\pm}}{|\varkappa_{\pm}|}, \\ \varkappa_{\pm} &= \frac{1}{2} + \xi e^{i\theta} \mp \Omega_Z, \quad \Omega_Z = \omega_Z L / 4\pi v_F. \end{aligned} \quad (54)$$

As seen from Eq. (53), the degeneracy of the levels occurs for the following eight values of magnetic flux: $\phi = \pm\delta, \phi = \pm\delta', \phi = 1/2 \pm \delta$, and $\phi = 1/2 \pm \delta'$, where $\delta = (\delta_+ + \delta_-)/2, \delta' = (\delta_+ - \delta_-)/2 + 1/2$. However, at four of

these values, $\phi = \pm\delta$ and $\phi = \pm\delta'$, the resonances are absent because of the backscattering on the contacts (see Appendix B in Ref. 53). At four other points, the backscattering is negligible, and there appear the antiresonances with the width γ . The amplitudes of the antiresonances at $\phi = 1/2 \pm \delta$ and $\phi = 1/2 \pm \delta'$ are γc^2 and γs^2 , respectively, where $c = \cos(\vartheta_+ - \vartheta_-)/2, s = \sin(\vartheta_+ - \vartheta_-)/2$.

At $\phi = 1/2 + \delta$, the degeneracy occurs between the states $\psi_{n,+}^{(1)}$ and $\psi_{n+1,-}^{(1)}$; at $\phi = 1/2 + \delta$, between $\psi_{n,+}^{(2)}$ and $\psi_{n+1,-}^{(2)}$; at $\phi = \delta' - 1/2$, between $\psi_{n,+}^{(1)}$ and $\psi_{n,-}^{(2)}$; and, finally, at $\phi = -\delta' + 1/2$, between $\psi_{n,+}^{(2)}$ and $\psi_{n,-}^{(1)}$. We note that the amplitudes of the antiresonances (γc^2 and γs^2) are determined by the scalar products of the corresponding spinors: $c^2 = |\langle \psi_{n,+}^{(1)} | \psi_{n+1,-}^{(1)} \rangle|^2 = |\langle \psi_{n,+}^{(2)} | \psi_{n+1,-}^{(2)} \rangle|^2$ and $s^2 = |\langle \psi_{n,+}^{(2)} | \psi_{n,-}^{(1)} \rangle|^2 = |\langle \psi_{n,+}^{(1)} | \psi_{n,-}^{(2)} \rangle|^2$.

The relation between the transmission coefficient and the stationary states of the closed ring, derived in Appendix B, allows us to easily find out the influence of the impurities on the four resonances, described above. As in the previous section, we use the two-level approximation (we assume that the distance between the antiresonances is much larger than $\sqrt{N}|r|$). The matrix elements of impurity potential read

$$\begin{aligned} \langle \psi_{n,+}^{(1)} | \hat{V} | \psi_{m,-}^{(1)} \rangle &= \langle \psi_{n,+}^{(2)} | \hat{V} | \psi_{m,-}^{(2)} \rangle = cV_{nm}, \\ \langle \psi_{n,+}^{(2)} | \hat{V} | \psi_{m,-}^{(1)} \rangle &= -\langle \psi_{n,+}^{(1)} | \hat{V} | \psi_{m,-}^{(2)} \rangle = sV_{nm}, \end{aligned} \quad (55)$$

where $V_{nm} = i\Delta r \sum_v \exp(i(n+m)2\pi x_v/L)/2\pi$ is the matrix element appearing in the spinless problem.

Using Eqs. (53), (55), and (14) (the latter equation was modified for the spinful case), we obtain the following result for the transmission coefficient:

$$\begin{aligned} \mathcal{T}_{\text{SO}}(\phi) &= \frac{c^2}{2} [T(\phi + \delta; rc) + T(\phi - \delta; rc)] \\ &+ \frac{s^2}{2} [T(\phi + \delta'; rs) + T(\phi - \delta'; rs)], \end{aligned} \quad (56)$$

where $\mathcal{T}(\phi; \rho)$ is the transmission coefficient in the spinless problem [given by Eq. (33)] with the reflection amplitude r substituted by ρ . We see that the effect of the impurity scattering in the spinless and in the spinful case is essentially the same: the antiresonances are broadened and their amplitudes become smaller (for strong enough impurities the amplitudes

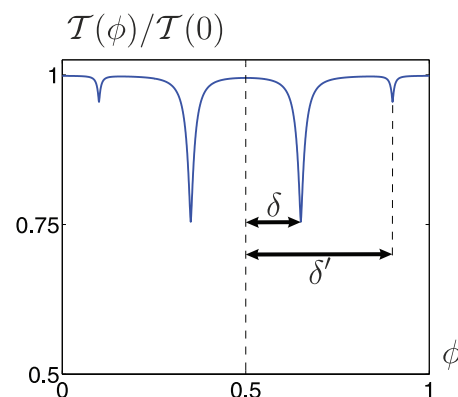


FIG. 7. (Color online) Transmission coefficient in the ring with impurities. Both spin-orbit and Zeeman interactions are present.

are two times smaller than in a clean ring). The only difference is the appearance of “effective” reflection amplitudes r_c and r_s for the antiresonances at $\phi = 1/2 \pm \delta$ and $\phi = 1/2 \pm \delta'$, respectively.

The expression for transmission coefficient is especially simple in the absence of Zeeman interaction (in this case $s = 0, c = 1$):

$$\mathcal{T}_{\text{SO}}(\phi) = \frac{1}{2}[\mathcal{T}(\phi + \delta; r) + \mathcal{T}(\phi - \delta; r)]. \quad (57)$$

This equation shows two types of periodic oscillations: AB oscillations with magnetic flux and AC oscillations with δ . The obtained results are illustrated in Fig. 7.

V. SUMMARY

In this work, we have studied the effect of disorder on the Aharonov-Bohm interferometer made of a single-channel quantum ring tunnel-coupled to the leads. We focused on the case of large temperature (compared to the level spacing) and the relatively weak disorder, when scattering time τ is large compared to L/v_F . In this case, backscattering events occur after many rotations around the ring, so that the system is in the ballistic regime (as opposed to regime of the strong localization which develops for $v_F\tau \ll L$).

In a clean ring, the thermally averaged transmission coefficient $\mathcal{T}(\phi)$ exhibits sharp dips at half-integer values of the magnetic flux [see Eq. (12) and Fig. 2]. The conductance turns to zero at the bottoms of the dips, which is due to cancellation of the amplitudes of mirrored paths, as explained in Sec. II. The width of the dips is determined by the transparency of the contacts. We considered two types of disorder: the potential created by N short-range impurities randomly distributed along the ring, and a smooth potential slowly varying on the scale of the Fermi wavelength. The short-range potential was shown to broaden the dips in the conductance. Remarkably, when the dips are significantly broadened, their depth is independent of the impurity scattering rate and is two times smaller compared to the case of a clean ring. We derived an analytical expression for $\mathcal{T}(\phi)$ for arbitrary $N > 2$ [see Eq. (37)] and demonstrated that this expression simplifies for $N \gg 1$ [see Eq. (38)]. The analytical expressions for $\mathcal{T}(\phi)$ in the special cases $N = 1$ and $N = 2$ were also obtained [see Eqs. (34), and (35), respectively]. We found that the shape of the dips is non-Lorentzian for $N > 1$, [see Eqs. (35)–(38) and Fig. 6]. Some partially correlated impurity distributions for which Eqs. (35)–(38) become invalid were also discussed [see Sec. III B3].

We also gave a physically transparent explanation of obtained results using the picture of the electron transitions through the quasistationary states in the ring. In this picture, the origins of the dips are essentially different in the cases of a clean ring and a ring with impurities. In the clean ring, the effect is due to the interference of processes of transitions through pairs of close energy levels. On the other hand, in a ring with sufficiently strong scattering on impurities this interference is suppressed due to repulsion of energy levels. In this case, the dips in conductance result from a strong dependence of the wave functions on the magnetic flux at $\phi \approx n + 1/2$ [see the discussion in the two paragraphs after Eq. (45)]. The long-range smooth disorder leads to the appearance of additional negative resonant peaks at integer

values of the flux ($\phi = n$). The amplitudes and the widths of the peaks (both at $\phi = n$ and at $\phi = n + 1/2$) oscillate with a disorder-induced phase difference between the shortest clockwise and counterclockwise interfering trajectories [see Eqs. (25) and (26)].

The results have been generalized to account for the spin-orbit and Zeeman coupling. Such interactions lead to the appearance of additional peaks in $\mathcal{T}(\phi)$. In the general case when both the SO and Zeeman interactions are present, there are four negative peaks per unit interval of ϕ [see Eq. (56) and Fig. 7]. We found that the effect of impurities in the spinful case is similar to the one in the spinless case. Namely, all four dips are broadened and their amplitudes decrease, becoming two times smaller compared to the case of a clean ring. Just as in the case of the ring without disorder, the amplitudes and the positions of the peaks depend on the strength of the SO and Zeeman couplings. Notably, in contrast to the clean case, the widths of the dips also depend on the strength of both couplings in the absence of Zeeman interaction, two of the four dips disappear and the analytical expression for $\mathcal{T}(\phi)$ simplifies [see Eq. (57)]. The distance between dips depends on the strength of the SO coupling, while the width and amplitudes of the peaks are independent of this coupling and depend only on the strength of disorder. Physically, this means that the tunneling conductance reveals co-existence of two types of disorder-dependent periodic oscillations: the Aharonov-Bohm oscillations with magnetic flux and the Aharonov-Casher oscillations with the strength of the SO interaction.

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APPENDIX A

In this Appendix, we calculate the sum

$$\sum_{n=0}^{\infty} |(\boldsymbol{\alpha}, \hat{A}^n \boldsymbol{\beta})|^2, \quad (A1)$$

where \hat{A} is an arbitrary 2×2 matrix, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are arbitrary two-component vectors. First, we note that

$$\hat{A}^n = \Sigma_n + \Delta_n(\hat{A} - \text{Tr}\hat{A}/2), \quad (A2)$$

where $\Sigma_n = (\lambda_1^n + \lambda_2^n)/2$ and $\Delta_n = (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)$. To prove Eq. (A2), one can make a similarity transformation that reduces \hat{A} to a diagonal (or Jordan) form. Using Eq. (A2), we can rewrite Eq. (A1) as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} |(\boldsymbol{\alpha}, \hat{A}^n \boldsymbol{\beta})|^2 &= |a|^2 \sum_{n=0}^{\infty} |\Sigma_n|^2 + |b - a\text{Tr}\hat{A}/2|^2 \sum_{n=0}^{\infty} |\Delta_n|^2 \\ &+ 2\text{Re} \left[a^*(b - a\text{Tr}\hat{A}/2) \sum_{n=0}^{\infty} \Sigma_n^* \Delta_n \right], \end{aligned} \quad (A3)$$

where $a = (\boldsymbol{\alpha}, \boldsymbol{\beta}), b = (\boldsymbol{\alpha}, \hat{A}\boldsymbol{\beta})$. The sums entering Eq. (A3) are easily calculated:

$$\begin{aligned}\sum_{n=0}^{\infty} |\Sigma_n|^2 &= \frac{1}{4} \left(\frac{1}{1 - |\lambda_1|^2} + \frac{1}{1 - |\lambda_2|^2} + 2\text{Re} \frac{1}{1 - \lambda_1^* \lambda_2} \right), \\ \sum_{n=0}^{\infty} |\Delta_n|^2 &= \frac{1}{|\lambda_1 - \lambda_2|^2} \left(\frac{1}{1 - |\lambda_1|^2} + \frac{1}{1 - |\lambda_2|^2} - 2\text{Re} \frac{1}{1 - \lambda_1^* \lambda_2} \right), \\ \sum_{n=0}^{\infty} \Sigma_n^* \Delta_n &= \frac{1}{2(\lambda_1 - \lambda_2)} \left(\frac{1}{1 - |\lambda_1|^2} - \frac{1}{1 - |\lambda_2|^2} + 2i\text{Im} \frac{1}{1 - \lambda_2^* \lambda_1} \right).\end{aligned}$$

These expressions can be rewritten in terms of $D = \det \hat{A}, S = \text{Tr} \hat{A}$. After some algebra, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} |\Sigma_n|^2 &= \frac{1}{4} Z^{-1} [4(1 - |D|^2) - |S|^2(|D|^2 + 3) + 2\text{Re} D^* S^2] \\ \sum_{n=0}^{\infty} |\Delta_n|^2 &= Z^{-1}(1 - |D|^2), \quad \sum_{n=0}^{\infty} \Sigma_n^* \Delta_n = \frac{1}{2} Z^{-1} [S^*(1 + |D|^2) - 2D^* S],\end{aligned}\tag{A4}$$

where $Z = (1 - |D|^2)^2 - (1 + |D|^2)|S|^2 + 2\text{Re} D^* S^2$. Finally, we get

$$\sum_{n=0}^{\infty} |(\boldsymbol{\alpha}, \hat{A}^n \boldsymbol{\beta})|^2 = \frac{|a|^2(1 - |D|^2 - |S|^2 - |DS|^2 + 2\text{Re} D^* S^2) + |b|^2(1 - |D|^2) + 2\text{Re}[a^* b(S^* |D|^2 - D^* S)]}{(1 - |D|^2)^2 - (1 + |D|^2)|S|^2 + 2\text{Re} D^* S^2}.\tag{A5}$$

In the case of real D and S , this expression is simplified:

$$\sum_{n=0}^{\infty} |(\boldsymbol{\alpha}, \hat{A}^n \boldsymbol{\beta})|^2 = \frac{|a|^2(1 + D - S^2 + DS^2) + |b|^2(1 + D) - 2DS\text{Re}(a^* b)}{(1 - D)(1 + D)^2 - S^2}.\tag{A6}$$

APPENDIX B

In this Appendix, we show that the transmission coefficient may be easily expressed in terms of stationary levels of the closed ring. We start from discussion of the clean ring and then generalize obtained results for the ring with short-range disorder.

1. Clean ring

Using Eqs. (2), (6), and (10), one can write the transmission coefficient in the following way:

$$T(\phi) = \left\langle \left| \left(\mathbf{e}, \frac{1}{1 - e^{ikL} \hat{A}} \boldsymbol{\beta}_0 \right) \right|_E^2 \right\rangle.\tag{B1}$$

Neglecting the backscattering by the contacts [see Eq. (27)], we get

$$\begin{aligned}T(\phi) &= t_{\text{out}}^2 \left\langle \left| \frac{e^{-i\pi\phi}}{1 - (1 - 2\gamma)e^{i(kL - 2\pi\phi)}} \right. \right. \\ &\quad \left. \left. + \frac{e^{i\pi\phi}}{1 - (1 - 2\gamma)e^{i(kL + 2\pi\phi)}} \right|_E^2 \right\rangle.\end{aligned}\tag{B2}$$

For $\gamma = 0$, the integrand of Eq. (B1) has poles on the real axis, $k_n^\pm = 2\pi(n \pm \phi)/L$. These poles are related to the energy levels of the closed ring in following way [see Eq. (18)]: $\epsilon_n^\pm = \hbar v_F k_n^\pm + c$ (it is notable that k_n^\pm do not coincide with the eigenvalues of momentum operator).

Expanding denominators of these fractions near k_n^\pm , after simple algebra we find that Eq. (B2) approximately coincides

with Eq. (14) of the main text where G_E is found from Eq. (15). In this case, the wave functions entering Eq. (15) are simply given by $\psi_n^\pm(0) = 1/\sqrt{L}$ and $\psi_n^\pm(L/2) = (-1)^n/\sqrt{L}$. Below, we demonstrate that Eq. (14) also holds for dirty ring where wave functions strongly depend on realization of disorder.

2. Ring with short-range disorder

The transmission coefficient of disordered ring is also given by Eq. (B1) where both \hat{A} and $\boldsymbol{\beta}_0$ depend on disorder. As we discussed in Sec. III B, the matrix \hat{A} can be approximately written as $\hat{A} = (1 - 2\gamma)\hat{A}_0$, where the unitary matrix \hat{A}_0 is given by Eq. (30).

Let us also introduce matrix \hat{U} , such that $e^{ikL/2}\hat{U}$ is a transfer matrix from contact a to contact b . Then, one can express $\boldsymbol{\beta}_0$ in terms of this matrix: $\boldsymbol{\beta}_0 = e^{ikL/2} t_{\text{out}} \hat{U} \mathbf{e}$. Denoting the eigenvalues of $e^{ikL} \hat{A}_0$ as $e^{iQ_1(k)L}$ and $e^{iQ_2(k)L}$ and the corresponding eigenvectors as $\boldsymbol{\chi}_1(k)$ and $\boldsymbol{\chi}_2(k)$, we rewrite Eq. (B1) as follows:

$$T(\phi) = t_{\text{out}}^2 \left\langle \left| \sum_{\alpha=1,2} \frac{(\mathbf{e}, \boldsymbol{\chi}_\alpha)(\boldsymbol{\chi}_\alpha, \hat{U} \mathbf{e})}{1 - (1 - 2\gamma)e^{iQ_\alpha(k)L}} \right|_E^2 \right\rangle.\tag{B3}$$

Equation (B3) is a generalization of Eq. (B2) for a disordered ring. [In the clean ring, $Q_1(k) = k - 2\pi\phi/L$, $Q_2(k) = k + 2\pi\phi/L$, $\boldsymbol{\chi}_1 = (1, 0)$, $\boldsymbol{\chi}_2 = (0, 1)$, and \hat{U} is a diagonal matrix with the elements $\exp(-i\pi\phi)$ and $\exp(i\pi\phi)$.]

As was pointed out in Sec. III B, the matrix \hat{A} relates the amplitudes C_\pm, D_\pm of the waves in the vicinity of the contact

b to each other [see Fig. 5 and Eq. (29)]. In the closed ring ($\gamma = 0$), the stationary states can be found from the conditions $C_+ = D_+$ and $C_- = D_-$. This allows us to establish a relation between the matrix \hat{A}_0 and the stationary states of the closed ring. Specifically, the wave vectors $k = k_{1n}$ and $k = k_{2n}$, for which one of the eigenvalues of the matrix $e^{ikL}\hat{A}_0$ equals to unity, correspond to the energy levels.

These wave vectors are found from the equation

$$Q_\alpha(k_{\alpha n}) = 2\pi n/L.$$

Each of the corresponding eigenvectors with components (C_+, C_-) describes the stationary wave function in the vicinity of the point b (x close to $L/2$):

$$\psi(x) = \frac{C_+ e^{i(k_{\alpha n} - 2\pi\phi/L)(x-L/2)} + C_- e^{-i(k_{\alpha n} + 2\pi\phi/L)(x-L/2)}}{\sqrt{L}}.$$

As expected, for $\gamma = 0$, Eq. (B3) has poles as a function of k for $k = k_{1n}$ and $k = k_{2n}$. For an almost closed ring, the poles of the right-hand side of Eq. (B3) slightly shift away from the axis of real k .

Just as in the clean ring, the main contribution to the integral over E comes from the poles of the fractions $[1 - (1 - 2\gamma)e^{iQ_\alpha(k)L}]^{-1}$, while the terms $(\mathbf{e}, \chi_\alpha)(\chi_\alpha, \hat{U}\mathbf{e})$ as well as function $\partial f/\partial E$ can be taken at the poles for $\gamma = 0$. Next, we notice that the vectors χ_α and the matrix \hat{U} are defined in such a way that

$$(\mathbf{e}, \chi_\alpha)|_{k=k_{\alpha n}} = \sqrt{L}\psi_{\alpha n}(L/2), \quad (\chi_\alpha, \hat{U}\mathbf{e})|_{k=k_{\alpha n}} = \sqrt{L}\psi_{\alpha n}^*(0),$$

where $\psi_{\alpha n}(x)$ are the stationary wave functions of the ring with disorder. Using these equations, expanding denominators in Eq. (B3) near the poles, and neglecting $dQ_\alpha/dk - 1 \sim \sqrt{N}|r| \ll 1$ with respect to unity, we arrive at Eqs. (14) and (15).

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- ⁵⁴In Ref. 22, it was argued that at sufficiently high T , the temperature averaging is equivalent to the averaging over phases acquired while passing two shoulders of interferometer. We briefly discuss this assumption in Sec. III A.
- ⁵⁵The AB conductance through the single-channel ring with a scattering potential barrier in one of the arms was discussed in Ref. 25 for the case of almost transparent contacts ($\gamma \simeq 1$). It was shown that the interference part of the conductance is not entirely suppressed for $T \gg \Delta$.