

Boson topological insulators: A window into highly entangled quantum phases

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We study several aspects of the realization of global symmetries in highly entangled phases of quantum matter. Examples include gapped topological ordered phases, gapless quantum spin liquids, and non-Fermi liquid phases. An insightful window into such phases is provided by recent developments in the theory of *short-ranged entangled* symmetry protected topological (SPT) phases. First, they generate useful no-go constraints on how global symmetry may be implemented in a highly entangled phase. Possible symmetry implementation in gapped topological phases and some proposed gapless spin/Bose liquids are examined in this light. We show that some previously proposed spin liquid states for 2D quantum magnets do not in fact have consistent symmetry implementation unless they occur as the surface of a 3D SPT phase. A second SPT-based insight into highly entangled states is the development of a view point of such states as SPT phases of one of the emergent excitations. We describe this in the specific context of time-reversal symmetric 3D $U(1)$ quantum spin liquids with an emergent photon. Different such spin liquids are shown to be equivalent to different SPT insulating phases of the emergent monopole excitation of such phases. The highly entangled states also in turn enrich our understanding of SPT phases. We use the insights obtained from our results to provide an explicit construction of bosonic SPT phases in 3D in a system of coupled layers. This includes construction of a time-reversal symmetric SPT state that is not currently part of the cohomology classification of such states.

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I. INTRODUCTION

A focus of modern quantum condensed matter physics is the study of phases of matter whose characterization is not captured by the concepts of broken symmetry and associated Landau order parameters. Striking examples are gapped phases of matter with topological order. These are characterized by emergent excitations with unusual quantum statistics and ground-state degeneracies that depend on the topology of the underlying manifold.¹ Other examples are gapless phases of matter where the gaplessness is protected but not by a broken symmetry. The most familiar example of such a phase is a Fermi liquid but gapless spin liquids and various non-Fermi liquid phases provide other examples. A common characterization of these different phases is the presence of nonlocal many-body quantum entanglement in their ground-state wave function. Such phases have come to be known as “highly entangled phases” of matter.

A simpler example of phases of matter that are not captured by notions of broken symmetry are the celebrated electronic topological insulators.² These phases are gapped in the bulk and are *short-ranged entangled* but nevertheless are distinguished from trivial electronic band insulators. Symmetry plays a key role in maintaining the distinction between these different short-ranged entangled phases. The electronic topological insulators are well described by free-fermion models that have nontrivial surface states protected by global symmetries. They clearly do not have bulk topological order or fractionalization. The free fermion topological insulator is a member of a more general class of phases—dubbed symmetry protected topological (SPT) phases—which all have no bulk topological order but have protected surface states. A well-known example is the Haldane spin chain in one dimension. A general formal classification of such phases in diverse dimension has been proposed.^{3,4} Very recently, their physical properties in both two^{5–7} and three dimensions^{8,9} have been elaborated.

The story of the electronic and bosonic topological insulators raises the important question of the role that symmetry plays in the characterization of highly entangled phases. In the specific context of gapped highly entangled phases, the interplay of symmetry and topological order has recently attracted renewed interest. For gapless phases and critical points, global symmetry obviously plays a much more important role, which is very poorly understood. In this paper, we will study several aspects of the realization of symmetry in these exotic phases—both gapped and gapless, and in both two and three space dimensions. Of particular importance to us are the results of Ref. 8 on the protected surface states of three-dimensional bosonic SPT phases. The surface phase diagram was argued to admit a phase with surface topological order though the bulk itself has no such order. Furthermore, this surface topological order implements the defining global symmetry in a manner not allowed in strictly two-dimensional systems.

We are thus led to consider in detail a consistent implementation of global symmetries in several highly entangled quantum phases. First, we obtain several new results and insights into both gapped and gapless phases that are allowed to exist in strictly 2D systems. These results have immediate application to theories of quantum spin liquid insulators and of non-Fermi liquid metals. Along the way, we also obtain an explicit construction of the various 3D symmetry protected topological insulators of bosons studied recently in Ref. 8. In particular, we construct a time-reversal symmetric 3D SPT phase that was suggested to exist in Ref. 8 but is not currently part of the cohomology classification of Ref. 3.

Second, we study symmetry realization in three-dimensional gapless quantum spin liquids with an emergent photon. Focusing on time-reversal symmetry and on phases that can exist in strictly 3D systems, we show that different such spin liquids may be distinguished by whether the emergent electric charge excitation is a Kramers singlet or doublet and its statistics. We show that this distinction is nicely

captured by viewing these phases as different SPT insulators of the dual “magnetic” particle (the monopole).

Reference 10 proposed a formal classification of the two-dimensional topological order described by a deconfined Z_2 gauge theory in the presence of global symmetries. The topological quasiparticles can, in principle, carry fractional quantum numbers of the global symmetry. More formally, this means that they are allowed to transform projectively under the global symmetry group. The approach of Ref. 10 involves finding all consistent ways of assigning projective representations to the different topological quasiparticles. A different classification has also appeared¹¹ that considers topological order with unitary symmetry but restricts to phases where one of the bosonic quasiparticles has trivial global quantum numbers. Earlier, Refs. 12 and 13 classified all two-dimensional time-reversal invariant gapped Abelian insulators using a Chern-Simons/ K -matrix approach. It is expected that all such insulators can always be described by a multicomponent Chern-Simons theory. The key idea of Refs. 12 and 13 is that the bulk two-dimensional theory can be completely characterized by studying the $1+1$ -dimensional edge theory at the interface with the vacuum (or equivalently a topologically trivial gapped insulator). This is a multicomponent Luttinger liquid theory in which operators corresponding to various bulk quasiparticles can be easily identified. In particular, constraints coming from global symmetries can be straightforwardly implemented. This approach has recently been used to study other symmetry enriched 2D topological order in Refs. 14 and 15.

How does the interplay of symmetry and topological order that is only allowed at the surface of 3D systems fit in with the emerging results on the classification of 2D topological order with symmetry? In the first part of this paper, we address this question in details for a few examples. Specifically, we restrict attention to boson systems with a few simple internal global symmetries. We also restrict to topological order described by a deconfined Z_2 gauge theory. We first use the procedure of Ref. 10 to obtain all distinct allowed implementation of the global internal symmetry. Some of these can be realized at the surface of 3D SPT phases. Then we use elementary arguments and the results of Ref. 13 to determine which ones of the phases are allowed in strictly 2D systems. Interestingly, the remaining phases are *all* shown to be realized at the surface of 3D SPT phases.

Why does the Chern-Simons/edge theory approach select out only those phases that can exist in strict two dimensions, while the approach of Ref. 10 does not? The key point is that the former approach assumes that the state in question can have a physical edge with the vacuum (equivalently, a topologically trivial gapped insulator) while preserving the symmetry. For topological order realized at the surface of a 3D SPT phase, this possibility simply does not exist. A trivial gapped symmetry preserving state to which the surface topological ordered state can have an interface is forbidden at the SPT surface. In contrast, the methods of Ref. 10 only worry about consistent assignment of symmetries to the various topological quasiparticles. The requirement that the state allow a physical edge to the vacuum is not part of the considerations of this method.

We will denote states that allow a physical edge to the vacuum as “*edgible*.” The topologically ordered states at the

surface of a 3D SPT phase are not edgible, while those allowed in strictly 2D are edgible. The concept of edgibility will prove to be a powerful criterion in deciding which topological ordered states with symmetry are allowed in strictly 2D systems and which not.

Below, we will flesh all this out in several concrete examples. We first study 2D gapped Z_2 topological order with a few different symmetries in Secs. II–IV. Next, we use the insights from these results to provide an explicit construction of SPT phases with the same symmetries in a system of coupled layers in Sec. V. We provide a brief discussion of the relationship between the possible surface topological order in a 3D SPT and its bulk topological field theory in Sec. VI. We turn our attention then to highly entangled gapless phases. In Sec. VII, we argue that a previously proposed gapless vortex fluid (dubbed the “algebraic vortex liquid”) cannot exist with time-reversal symmetry in strictly 2D systems. Section VIII contains our results on 3D $U(1)$ quantum liquids. We conclude in Sec. IX with a discussion. The two appendixes contain important details. In particular, in Appendix B, we describe the surface Landau-Ginzburg theories for the 3D SPT phases of interest in terms of dual vortices with nontrivial structure and discuss the surface phase structure. We also provide an explicit derivation of this dual surface vortex theory.

II. TOPOLOGICAL ORDERED BOSON INSULATORS: SYMMETRY $U(1) \times Z_2^T$

We begin by considering a system of bosons with a global $U(1)$ symmetry and time reversal (Z_2^T). The bosons are taken to have charge 1 under the global $U(1)$ symmetry. In this section, we assume that the boson destruction operator $b \rightarrow b$ under Z_2^T . This means that the global symmetry group is $U(1) \times Z_2^T$. We will assume that the topological order in question has 2 nontrivial bosonic particles (dubbed e and m) and a fermion (dubbed ϵ). Any two of these are mutual semions. Further, any one of these may be thought of as a bound state of the other two. This corresponds precisely to the excitation structure of a deconfined Z_2 gauge theory in two space dimensions. What are the allowed topological phases with Z_2 gauge structure according to the analysis of Ref. 10? The time-reversal operation \mathcal{T} when it acts on physical states of the bosons must satisfy $\mathcal{T}^2 = 1$. Let us denote by $\mathcal{T}_{e,m}$ the action of time reversal on the e and m particles. The only restriction on these is that they satisfy

$$\mathcal{T}_e^2 = \mu_e, \quad (1)$$

$$\mathcal{T}_m^2 = \mu_m \quad (2)$$

with $\mu_{e,m} = \pm 1$. A value -1 of either of these means that the corresponding particle forms a Kramers doublet. What about symmetry under global $U(1)$ rotations? Here, the distinct possibilities correspond to whether the (e,m) particles carry integer or fractional charge. In the latter case, their charge must be shifted from an integer by $\frac{1}{2}$. These possibilities are nicely distinguished by asking about the action of a 2π global $U(1)$ rotation $R_{2\pi}$. On physical states, $R_{2\pi} = 1$. Let us again denote by $R_{2\pi}^{e,m}$ the action on the e and m sectors. We then have

$$R_{2\pi}^e = \sigma_e, \quad (3)$$

$$R_{2\pi}^m = \sigma_m \quad (4)$$

TABLE I. Symmetry action $U(1) \times Z_2^T$ for Z_2 topological ordered states. The first five are allowed in strict 2D, while the last five can only be realized at surface of 3D SPT phases (or derived from them).

Phase	σ_e	σ_m	μ_e	μ_m	Comments
$e0m0$	1	1	1	1	No fractionalization
eT	1	1	-1	1	No fractional charge but Kramers
eC	-1	1	1	1	$b = \Phi^2$
eCT	-1	1	-1	1	$b = \epsilon_{\alpha\beta} f_\alpha f_\beta$; f_α in trivial band
$eCmT$	-1	1	1	-1	$b = \epsilon_{\alpha\beta} f_\alpha f_\beta$; f_α in topological band
$eTmT$	1	1	-1	-1	3D SPT surface
$eCmC$	-1	-1	1	1	3D SPT surface
$eCTmC$	-1	-1	-1	1	$eCmC \oplus eCmT$
$eCTmT$	-1	1	-1	-1	$eTmT \oplus eCmT$
$eCTmCT$	-1	-1	-1	-1	$eCTmC \oplus eCTmT$

with $\sigma_{e,m} = \pm 1$. The realization of the symmetry in this topologically ordered state is thus described by the numbers $(\sigma_e, \mu_e, \sigma_m, \mu_m)$. Naively, this gives 16 phases, but we must remember that interchanging e and m does not produce a new phase. This removes six possibilities so we are left with a total of 10 phases for this symmetry.

In Table I, we display the quantum numbers of the e and m excitations of these ten phases. We label these phases by the excitations that carry nontrivial charge (C) or time-reversal (T) quantum numbers. Thus $e0m0$ means both the e and m particles carry trivial quantum numbers, while mT refers to a phase where the m particle is Kramers doublet and neither e nor m carry half-integer charge, etc.

Note that the discussion above made no reference to the edgability of the state. To consider only edgable states, let us now switch gears and consider the possibilities for Z_2 topological order with the same symmetry as above but within the Chern-Simons/edge theory approach of Ref. 13. This naturally selects out edgable states and hence will enable us to decide which of the ten states in Table I can be realized in strictly 2D systems. We will show that only the first five of these are captured in the Chern-Simons approach.

In the Chern-Simons description of an Abelian two-dimensional insulator, the effective Lagrangian is given by a multicomponent Chern-Simons term:

$$L = \frac{K_{IJ}}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu^I \partial_\nu a_\lambda^J + \frac{1}{2\pi} \tau_I \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda^I, \quad (5)$$

where the current density of quasiparticle I is given by $j_\mu^I = \frac{\epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda^I}{2\pi}$. The K matrix gives the topological information of the system, while the charge vector τ_I is an integer valued charge of each quasiparticle through coupling with the external gauge field A_μ . The allowed quasiparticles carry integer charge under the different gauge fields a^I , which can be expressed in terms of an integer valued vector l . The mutual statistics of two quasiparticles labeled by l and l' is $\theta_{ll'} = 2\pi l^T K^{-1} l'$, while the self-statistics of a quasiparticle is $\theta_l = \pi l^T K^{-1} l$. To describe Z_2 topological order, we begin with a 2×2 K matrix

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad (6)$$

which captures the statistics of the e and m particles. We will determine the distinct ways in which the $U(1) \times Z_2^T$ symmetry

can be realized. First of all, note that the electrical Hall conductivity is given by $\sigma_{xy} = \tau^T K^{-1} \tau = \tau_1 \tau_2$. Thus time-reversal invariant states must necessarily have at most one of $\tau_{1,2} \neq 0$. Henceforth, without loss of generality, we will therefore set $\tau_1 = 0$ and $\tau_2 = t$. Next, the physical charge of a quasiparticle labeled by l is given by $q_l = l^T K^{-1} \tau = \frac{lt}{2}$. Since we only want to distinguish a half-integer physical charge from an integer one, the distinct possibilities correspond to $t = 0, 1$. Let us now demand time-reversal invariance of the Chern-Simons Lagrangian. The symmetry realizations classified by the first approach above assume that the symmetry transformation does not interchange e and m particles. Therefore we restrict attention to that subclass here. For the first term to be a time-reversal invariant, it must be that the spatial components a_{1i}, a_{2i} transform oppositely under time reversal. Further, if $\tau_2 = t$ is nonzero, then $\epsilon_{ij} \partial_i a_{2j}$ must be even under time reversal. Thus we choose the action of time reversal on the a_i^I to be $a_i^I \rightarrow T_{IJ} a_j^J$ with

$$T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (7)$$

As described in Ref. 13, we also need to describe the transformation of the quasiparticle creation operators. This is conveniently accomplished by using the standard edge theory that corresponds to the bulk Chern-Simons Lagrangian:

$$\mathcal{L} = \frac{1}{4\pi} (K_{IJ} \partial_x \phi_I \partial_t \phi_J + \dots) + \frac{1}{2\pi} \epsilon_{\mu\nu} \tau_I \partial_\mu \phi_I A_\nu. \quad (8)$$

Quasiparticle creation operators corresponding to $l = (1, 0)$ and $l = (0, 1)$ are $e^{i\phi_1}$ and $e^{i\phi_2}$, respectively. The time-reversal transformation of a_{Ii} fixes the transformation of ϕ_I up to an overall phase. Thus we write

$$e^{i\phi_1} \rightarrow e^{i(\phi_1 + \alpha_1)}, \quad (9)$$

$$e^{i\phi_2} \rightarrow e^{-i(\phi_2 + \alpha_2)}. \quad (10)$$

However, by a shift of ϕ_1 , we can always set $\alpha_1 = 0$. This is not possible for α_2 . A further constraint comes from requiring that all physical operators transform such that $\mathcal{T}^2 = 1$. In particular, \mathcal{T}^2 should take $e^{2i\phi_2} \rightarrow e^{2i\phi_2}$. This imposes the restriction that $\alpha_2 = \frac{\pi x}{2}$ with $x = 0, 1$. If $x = 0$, then the particle created by $e^{i\phi_2}$ is a Kramers singlet. If $x = 1$, however, \mathcal{T}^2 takes $e^{i\phi_2} \rightarrow -e^{i\phi_2}$ so that the particle is a Kramers doublet.

Thus within this 2×2 K matrix, we have four possible states corresponding to the four possible values of the pair t, x . In terms of Table I, these correspond to the four phases $e0m0, eT, eC, eCmT$. Actually, a fifth phase eCT is also allowed in strict 2D but requires a 4×4 K matrix. To see why this is so, it is useful to better understand the physics of the four states described so far.

First, note that with the K matrix in Eq. (6), the edge phase fields ϕ_1, ϕ_2 satisfy commutation relations such that the fields $f_\pm = e^{i(\phi_1 \pm \phi_2)}$ satisfy fermion anticommutation relations. Indeed, these correspond to $l = (1, 1), l = (1, -1)$ and describe the bulk fermionic ϵ particle. f_\pm are the right- and left-moving fermions of the one-dimensional edge Luttinger liquid theory. Under a global $U(1)$ symmetry rotation U_θ by

angle θ and time reversal, the f_{\pm} transform as

$$U_{\theta}^{\dagger} f_{\pm} U_{\theta} = e^{i\frac{\theta}{2}} f_{\pm}, \quad (11)$$

$$\mathcal{T}^{-1} f_{\pm} \mathcal{T} = e^{\mp i\frac{\pi x}{2}} f_{\mp}. \quad (12)$$

Note that as the ϵ particle may be regarded as a bound state of e and m , it has quantum numbers $\sigma_{\epsilon} = \sigma_e \sigma_f$ and $\mu_f = \mu_e \mu_f$. For the four cases described above in terms of edge Lagrangians, this is consistent with the symmetry transformation of the edge fermion fields.

Further insight is obtained by understanding how the four phases corresponding to the four choices of (t, x) are obtained within a slave particle (parton) construction in the bulk. Consider a slave particle (parton) construction obtained by writing the boson operator $b_r = a_r s_r$ at each site r of the lattice. Here, a_r destroys a bosonic parton with charge 1 while s_r is an Ising parton (with charge 0). We may take them to belong to the e sector. Under time reversal, a_r, s_r remain invariant. Further, as the a_r, s_r carry only an integer charge, $\sigma_e = \mu_e = 1$. First, we take the a_r to form a simple bosonic Mott insulator and s_r to form a simple Ising paramagnet. Then the vison of the Z_2 gauge field associated with the slave particle construction will have trivial quantum numbers so that $\sigma_m = \mu_m = 1$. This corresponds to phase $e0m0$ in Table I.

Phase eT can be likewise constructed if we start with two species of physical bosons $b_{1,2}$ and require $b_1 \leftrightarrow b_2$ under time reversal (e.g., spin-half bosons). Then we write the boson operators as $b_{1,2} = a_{1,2} s_{1,2}$ and put the system into a state such that time reversal is implemented through $(a_1, a_2) \rightarrow (-a_2, a_1)$ and $(s_1, s_2) \rightarrow (-s_2, s_1)$. The e particles in this phase ($a_{1,2}$ and $s_{1,2}$) are Kramer's doublets, while the m particle (the vison) transforms trivially under time reversal. Nothing carries fractional charge in this phase.

Phase eC is a familiar one and can be obtained in a slave particle construction by writing $b_r = \Phi_r^2$. The Φ_r destroys a charge-1/2 bosonic parton (denoted "chargon" in the literature). Under time reversal, Φ_r is invariant. Explicit microscopic models for the corresponding phase were studied in Refs. 16–18 with the standard implementation of the time-reversal symmetry for bosons ($b \rightarrow b$).

Phase eCT is also a familiar one. It can be obtained through a parton construction by writing the boson operator as $b_r = \epsilon_{\alpha\beta} f_{r\alpha} f_{r\beta}$ with $f_{r\alpha}$ a fermion. We will refer to $\alpha = 1, 2$ as a pseudospin index. The fermions carry charge-1/2. Time reversal is implemented through $f_{r\alpha} \rightarrow i(\sigma_y)_{\alpha\beta} f_{r\beta}$. Now consider a mean-field ansatz where the fermion $f_{r\alpha}$ forms a (topologically trivial) band insulator that preserves time reversal but does not conserve any component of the fermion pseudospin. The result is a Z_2 topologically ordered state with symmetry implemented as defined for phase 4.

Phase $eCmT$ is obtained from the same parton construction as for phase eCT but when the $f_{r\alpha}$ band structure is topologically nontrivial, i.e., the fermions form a 2D topological insulator. Then a π flux seen by the fermions (which we may take to be the m particle) is known to bind a Kramer's doublet.¹⁹ Indeed, in the edge theory above, if we choose $t = 1$ and $x = 1$, the edge Lagrangian becomes identical to that of a fermionic topological insulator formed by the ϵ particle. Thus this parton construction has the symmetries

of phase 5. Three-dimensional analogs of these phases were studied in Refs. 20 and 21.

It is clear now that the phase eCT can exist in strictly 2D systems but is not captured by a Chern-Simons/edge theory description with a 2×2 K matrix. This can also be seen by noting that since the physical charge is invariant under time reversal, one cannot have a particle that's nontrivial under both $U(1)$ and \mathcal{T} symmetries within this 2×2 K -matrix formulation.

Note that for $eCmT$ the edge theory is gapless so long as the global symmetry is preserved. In contrast, for the phases $e0m0, eT, eC$, the edge theory can be gapped by adding symmetry-allowed perturbations. Similarly, from the parton construction, we know that though the ϵ particle carries the same quantum numbers for both $eCmT$ and for eCT , the edge theory for eCT can be gapped. From the theory of the fermion topological insulator, it follows that trivial band structure for the ϵ can be built up from the topological band structure by taking two copies and allowing all symmetry allowed perturbations. This suggests that the minimal description of eCT uses a $4 \times 4K$ matrix. Specifically consider

$$K = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (13)$$

with the charge vector $\tau = (0, 1, 0, 0)$. Time reversal is implemented on the edge boson fields ϕ_I through $\phi_I \rightarrow T_{IJ}(\phi_J + \alpha_J)$ with $\alpha = (0, 0, \pi, 0)$. It is readily seen that this describes the eCT phase.

In passing, we note that we can easily generate other 2D Z_2 topological phases with this symmetry by simply adding a layer of the 2D SPT phase allowed with $U(1) \times Z_2^T$ symmetry to one of the five examples discussed above. This obvious extension does not affect our subsequent discussion and we will not consider it further.

In Appendix A, we explain in detail why all the other phases are not possible within the K -matrix formulation. In the next section, we argue that, independent of the K -matrix formulation, the existence of those phases on SPT surfaces implies their nonexistence in strict 2D systems.

For now, we make a few comments. Note that in the first four phases, the m particle has trivial quantum numbers. It is only natural that such states, where one of the e or m particles have trivial quantum numbers, can always be realized in strictly 2D systems. From such states, we can always destroy the Z_2 topological order by condensing the m particle to produce a trivial symmetry preserving insulator. This will not be possible for states that can only be realized at the surface of 3D SPT phases. In phase $eCmT$, both the e and m carry nontrivial quantum numbers. Despite this, as we have seen, it can be realized in strict 2D.

Now lets move to the last five phases of Table I. Reference 8 showed that phases $eTmT$ and $eCmC$ both arise at the surface of 3D SPT phases. To discuss the other phases, we first define the concept of "surface equivalence" of topologically ordered phases.

A. Surface equivalence

We say that two topologically ordered states at the surface of a 3D SPT phase are “surface equivalent” if one can be obtained from the other by combining with a strictly 2D states with the same symmetry. The notion of combining two states will be described in detail below. Consider two Z_2 topologically ordered states—say states A and B—with distinct realizations of the global symmetry. This means that at least one of the e, m particles transform differently under the global symmetry for the two states. Assume now that A and B have the same symmetry for the e particle or—in obvious notation—that $(\sigma_{eA}, \mu_{eA}) = (\sigma_{eB}, \mu_{eB}) \equiv (\sigma_e, \mu_e)$. Then we must have $(\sigma_{mA}, \mu_{mA}) \neq (\sigma_{mB}, \mu_{mB})$.

Now consider the composite system $A + B$. We allow A and B to couple through all symmetry allowed short-ranged interactions. For weak interaction strengths, the two states will be decoupled, and the combined system will have deconfined $Z_2 \times Z_2$ topological order. However, for stronger interactions, e_A can mix with e_B as they have the same symmetry. This partially confines the $Z_2 \times Z_2$ topological order to a simpler topological ordered state with just a single deconfined Z_2 gauge structure. We will denote this new phase $A \oplus B$. In this new state, the m particles of A and B will be confined together to produce a new particle $m_{A \oplus B} \sim m_{AmB}$. Thus $A \oplus B$ has the quantum numbers $(\sigma_e, \sigma_{mA} \sigma_{mB}, \mu_e, \mu_{mA} \mu_{mB})$.

This concept of combining phases enables us to see several equivalences in Table I. For instance, it is clear that phase eCT can be obtained as $eC \oplus eT$ (by letting the m particles mix). Let us now consider surface equivalence. Phases $eCmC$ and $eCmT$ share the same quantum numbers for the e particle. Thus we may combine them to produce a new Z_2 phase $eCmC \oplus eCmT$, which, by inspection, has the same symmetries as phase $eCTmC$ (after a relabeling of e and m). This means that phases $eCmC$ and $eCTmC$ are surface equivalent. Specifically, consider the 3D SPT phase with phase $eCmC$ as its surface topological ordered state. We may then deposit a layer of phase $eCmT$ (which is allowed in strict 2D) at its surface, and then let the e particles mix. This mixing will induce a surface phase transition where the surface topological order becomes that of phase $eCTmC$. It follows that phase $eCTmC$ can also only be realized at the surface of the 3D SPT boson insulator.

Similarly, the m particle of phase $eTmT$ has the same quantum numbers as the m particle of phase $eCmT$. Letting them mix, we get phase $eCTmT$. Phase $eCTmCT$ is also readily seen to be $eCTmC \oplus eCTmT$.

Thus we see that the last five phases of Table I are all obtained at the surface of 3D SPT phases. All these five phases are obtained from two “root” phases (phases $eTmT$ and $eCmC$) by combining with phases that are allowed in strict 2D or with each other.

It is interesting to notice that the realization of the five phases at the SPT surfaces implies their absence in strict 2D systems, independent of K -matrix consideration. One can understand this as follows: if a surface state can also be realized in strict 2D, then one can deposit such a 2D system onto the surface. The quasiparticles in the two systems [let us call them (e_1, e_2) and (m_1, m_2)] will then have exactly the same symmetry properties, and the bound states of two particles of the same

kind in the two systems ($e_1 e_2$ and $m_1 m_2$) will be trivial under all symmetries. Moreover, $e_1 e_2$ and $m_1 m_2$ are mutual bosons to each other. Hence one can condense both $e_1 e_2$ and $m_1 m_2$ without breaking any symmetry. However, this will confine all the fractional quasiparticles since any one of them will have mutual π statistics with either $e_1 e_2$ or $m_1 m_2$, and the surface will become a trivial phase, i.e., symmetric, gapped, and confined. By definition, the corresponding bulk cannot be an SPT state. Hence the states at SPT surfaces must not be realizable in strict 2D. This will have interesting implications for 2D systems, and an example will be given toward the end of this paper.

It is also interesting to view this result from a different point of view, which inverts the logic followed above. Consider the problem of identifying 3D boson SPT states with this symmetry. The results of this section show that there are precisely two distinct “root” Z_2 topological orders that can only occur at the surface of SPT phases. This then gives us two “root” 3D SPT states with this symmetry. This is the same conclusion arrived at by direct consideration of surface theories in Ref. 8 and ties in nicely with the formal cohomology classification (which also gives two root states). Note, in particular, that of the two root states, $eTmT$ is simply inherited from the 3D SPT with Z_2^T symmetry alone. Thus the only nontrivial SPT state that is unique to the extra $U(1)$ symmetry is the one with surface topological order $eCmC$ as was suggested in Ref. 8.

III. TOPOLOGICAL Z_2 SPIN LIQUIDS

Here, we repeat the exercise above for symmetries appropriate to quantum spin systems. We consider two cases: symmetry $U(1) \times Z_2^T$ and symmetry Z_2^T . The former describes time-reversal symmetric quantum spin Hamiltonians with a conserved component of spin. In the latter, we only assume time-reversal symmetry. The consistent symmetry assignments for Z_2 topological order with bosonic e and m particles are given in Tables II and III.

Let us first consider $U(1) \times Z_2^T$ in which case the $U(1)$ charge goes to minus itself under time reversal. The analysis of Ref. 10 again gives the same ten phases as before and we will use the same labels. However, a difference appears in the

TABLE II. Symmetry action $U(1) \times Z_2^T$ for Z_2 topological ordered states. The first four are allowed in strict 2D, while the last six can only be realized at surface of 3D SPT phases (or derived from them).

Phase	σ_e	σ_m	μ_e	μ_m	Comments
$e0m0$	1	1	1	1	No fractionalization
eT	1	1	-1	1	No fractional charge but Kramers
eC	-1	1	1	1	$b = \Phi^2$
eCT	-1	1	-1	1	$b = \epsilon_{\alpha\beta} f_\alpha f_\beta$
$eTmT$	1	1	-1	-1	3D SPT surface
$eCTmT$	-1	1	-1	-1	3D SPT surface
$eCTmCT$	-1	-1	-1	-1	3D SPT surface
$eCmT$	-1	1	1	-1	$eTmT \oplus eCTmT$
$eCTmC$	-1	-1	-1	1	$eCTmT \oplus eCTmCT$
$eCmC$	-1	-1	1	1	$eCTmC \oplus eCmT$

TABLE III. Symmetry action Z_2^T for Z_2 topological ordered states. The first two are allowed in strict 2D, while the last one can only be realized at the surface of 3D SPT phases.

Phase	μ_e	μ_m	Comments
$e0m0$	1	1	No fractionalization
eT	-1	1	Kramers
$eTmT$	-1	-1	3D SPT surface

K -matrix classification. For this symmetry class, we will see that a 2×2 K matrix is enough to describe all the 2D states. We have again

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (14)$$

but now the charge vector must be taken to be $\tau = (t, 0)$ with $t = 0, 1$ due to the different transformation of the density of the global $U(1)$ charge under time reversal. Time reversal on the edge boson fields continues to be implemented as $\Phi_I \rightarrow T_{IJ}(\phi_I + \alpha_I)$ with $\alpha = (0, \frac{\pi x}{2})$ ($x = 0, 1$). With this symmetry implementation, we see that the edge field $e^{i\phi_2}$ creates a particle that can either carry $1/2$ charge, be a Kramers doublet, or both. The other edge field $e^{i\phi_1}$ creates a particle with trivial quantum numbers. This leads to four phases corresponding to $e0m0$, eT , eC , and eCT .

The standard slave boson or fermion construction of Z_2 spin liquids—as in the classic work of Refs. 22 and 23—give [when the spin symmetry is $U(1)$] the state eCT . The spinon in these constructions both carries spin $1/2$ and is a Kramers doublet. The easy axis kagome lattice spin model of Ref. 24 provides an explicit microscopic model for a Z_2 spin liquid with $U(1) \times Z_2^T$ symmetry. In the standard interpretation, the spin S_z of that model labels the two members of a Kramers doublet states of a microscopic Ising spin. Time reversal is then implemented in terms of the spin operators as usual through $\vec{S} \rightarrow -\vec{S}$. In that case, in the Z_2 spin liquid phase, the spinons are readily seen to both carry $S_z = \pm \frac{1}{2}$ and be Kramers doublets to realize the eCT class. There is a different implementation of time-reversal symmetry in this easy-axis kagome spin model. If S_z labels two members of a microscopic non-Kramers doublet, then we must interpret \vec{S} as a “pseudospin” $1/2$ operator that acts in this two-dimensional Hilbert space at each site. Time reversal takes $S_z \rightarrow -S_z, S^+ \rightarrow S^-$. In that case, the spinons in the Z_2 spin liquid phase will have spin $S_z = \pm \frac{1}{2}$ but will be Kramers singlets. Thus we have a realization of the eC phase in the model of Ref. 24.

Phase $eCmT$, which was allowed earlier in Sec. II, now does not appear. Physically, this is because the “topological band” in the $U(1) \times Z_2^T$ symmetry becomes trivial in the $U(1) \times Z_2^T$ case. The easiest way to see this is to consider the edge theory, which has two counterpropagating fermions. With $U(1) \times Z_2^T$ symmetry, one can mix the two fermions (hence gap out the edge) without breaking any symmetry, even if the fermions form Kramer’s pairs.

The absence of the $eCmT$ phase in strict 2D modifies the equivalence relation established in the last section. In particular, the last three phases in Table I will not be equivalent to either $eTmT$ or $eCmC$. Actually, in Ref. 8, three distinct

“root” SPT phases were discussed corresponding to those with surface topological orders $eTmT$, $eCTmT$, and $eCTmCT$. The last three phases in Table II are thus surface topological orders corresponding to other SPT phases that may be obtained by combining these root phases. This is in perfect agreement with the results of Ref. 8.

Next, we consider Z_2^T symmetry alone, which is much simpler. It is straightforward to see that the phases $e0m0$ and $eTm0$ can be realized in strict 2D, while $eTmT$ can only appear on an SPT surface. The corresponding table is simply a subset of the previous two.

IV. ALL-FERMION Z_2 LIQUIDS

We now extend our analysis to a very interesting topological order where there are three distinct topological quasiparticles, all of which are fermions $f_{1,2,3}$, and there is a mutual π statistics between any two of them. This can be viewed as a variant of the usual Z_2 liquid in which both the e and m particles become fermions. Since they have a mutual π statistics, the bound state $\epsilon = em$ is still a fermion and has π statistics with both e and m .

The statistics of this phase is perfectly compatible with time-reversal symmetry, but the realization in strict 2D turns out to be always chiral and hence breaks time reversal. One way to understand this is to start from a conventional Z_2 topologically ordered liquid with bosonic e and m particles. Then we put the fermionic ϵ particle into a band structure such that the vison also becomes a fermion. This may be fruitfully discussed in terms of the edge Lagrangian for the ϵ field. The vison operator appears as a “twist” field that creates a π phase shift for ϵ . For a single branch of chiral (complex) fermion on the edge $e^{i\phi_{L,R}}$, the twist operator is $e^{i\phi_{L,R}/2}$. This has conformal spin $\pm 1/8$ so that in this case, the vison is an anyon with fractional statistics. Very generally, take a theory with n_R right-moving and n_L left-moving fermions all of which correspond to the same bulk ϵ particle that sees a single common vison. This acts as a common twist field for all the edge fermions and hence has conformal spin $\frac{n_R - n_L}{8}$. Therefore to make the vison fermionic, one needs $n_L - n_R = 4 \pmod 8$. One such realization is given by the 4×4 K matrix:

$$K = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}, \quad (15)$$

which has chiral central charge 4. Since the chiral central charge is nonzero, this phase clearly cannot arise in a time-reversal invariant strictly 2D system.

However, Ref. 8 suggested that such an all fermion Z_2 topological order can arise at the surface of a 3D SPT phase with time-reversal symmetry. In this state, if the surface is gapped by breaking time-reversal symmetry, then there is a quantized thermal Hall conductivity $\kappa_{xy} = \pm 4$. However, if time-reversal symmetry is present and the surface is gapped, there will be surface topological order. Reference 8 proposed that this is a Z_2 topological order, which is a time-reversal symmetric all fermion state. To understand why this is reasonable, consider starting from the all fermion surface topological ordered state.

TABLE IV. Symmetry action Z_2^T for all-fermion Z_2 states. Both states can only be realized at surfaces of 3D SPT phases.

Phase	μ_e	μ_m	Comments
$e^f 0 m^f 0$	1	1	All fermions, singlets
$e^f T m^f T$	-1	-1	$e^f 0 m^f 0 \oplus e T m T$

What should we do to confine all the fermion excitations in the surface? It is clear from the discussion above that if we take one of the fermions and put it in a Chern band such that the surface $\kappa_{xy} = \pm 4$, then the other two topological quasiparticles will become bosons. These bosons can now be condensed to get a confined surface state. However, this clearly requires broken time-reversal symmetry and will give a $\kappa_{xy} = \pm 4$, which is, indeed, the right Z_2^T broken surface state for this proposed SPT. This kind of 3+1-D SPT phase with Z_2^T symmetry is not present in the cohomology table of Ref. 3. Including this 3D SPT surface state and using the language in the last few sections, we have a new table (Table IV).

The second phase in the table is obtained from the first by adding a usual Z_2 liquid in the $e T m T$ phase, then condense the bound state of the $\epsilon^f = e^f m^f$ in the fermionic liquid and the $\epsilon = em$ in the $e T m T$ liquid. Since the $e T m T$ phase cannot be realized in strict 2D, the two phases $e^f 0 m^f 0$ and $e^f T m^f T$ should be viewed as inequivalent, hence they give rise to two distinct SPT phases with time-reversal symmetry in addition to the one with the $e T m T$ surface Z_2 topological order. Thus, in total with Z_2^T symmetry, we actually have three nontrivial SPT phases corresponding to a Z_2^2 classification. (The cohomology classification of Ref. 3 gives instead a Z_2 classification.)

V. CONSTRUCTING SPT WITH COUPLED LAYERS OF Z_2 LIQUIDS

From the considerations in the previous sections, it is clear that to construct a 3+1-D SPT state, we only need to construct the corresponding topological order on the surface but have a confined bulk with gapped excitations. In this section, we give one such explicit construction using coupled layers of 2D Z_2 liquids. Specifically, we consider a system of stacked layers where each layer realizes a Z_2 topological order that is allowed in strictly 2D systems. Then we couple the different layers together in such a way that the bulk is confined and gapped, but we show that the surface layer is left unconfined and further corresponds to the surface Z_2 topological order of an SPT phase. A similar coupled-layer construction of the free fermion topological insulator was proposed²⁵ to obtain the single Dirac cone on the surface. We first illustrate this by constructing the $e T m T$ with Z_2^T symmetry, and it will be clear later that this can be generalized to all the SPT states mentioned in this paper.

Consider stacking N layers of Z_2 liquids in the $e T$ state, which is allowed in strictly 2D. Now turn on an interlayer coupling to make the composite particles $e_i m_{i+1} e_{i+2}$ condensed, where i is the layer index running from 1 to $N-2$. Note that the $e_i m_{i+1} e_{i+2}$ all have the bosonic self and mutual statistics so that they may be simultaneously condensed. As illustrated in Fig. 1, this procedure confines all the nontrivial quasiparticles

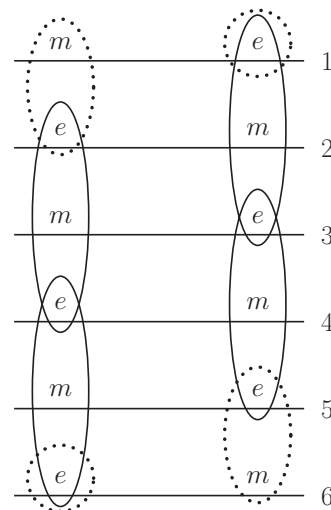


FIG. 1. Coupled-layer construction of SPT states. The particle composites in the ellipses are condensed, and only the four surface particles in the dotted ellipses survived as deconfined topological quasiparticles.

in the bulk. However, four particles on the surfaces survive as the only deconfined objects: $e_1, m_1 e_2, e_N, m_N e_{N-1}$. Notice that e_1 and $m_1 e_2$ are mutual semions and have self-boson statistics. Thus they form a Z_2 liquid at the top surface. Similarly, e_N and $m_N e_{N-1}$ are self-bosons, have mutual semion statistics, and form a Z_2 liquid at the bottom surface. The key point, however, is that all these particles have $\mathcal{T}^2 = -1$. Thus either surface is in the $e T m T$ state though the bulk has no exotic excitations. By the analysis above, we identify this with the 3D SPT state with Z_2^T symmetry.

This construction can be immediately generalized to other SPT states. For example, to get the $e C m C$ (or $e C T m C T$) state with $U(1) \times Z_2^T$ or $U(1) \times Z_2^T$ symmetry, just stack layers of $e C$ (or $e C T$) states and condense $e_i m_{i+1} e_{i+2}$.

Most interestingly, the all-fermion Z_2 surface topological state with global Z_2^T symmetry, which is quite hard to construct using other methods, can also be constructed in this way: simply start with stacked 2D Z_2 liquids where all particles e, m, ϵ are invariant under T reversal. Such a Z_2 state is obviously allowed in strict 2D. Now condense $\epsilon_i m_{i+1} \epsilon_{i+2}$ instead of $e_i m_{i+1} e_{i+2}$ in the above constructions, where $\epsilon_i = e_i m_i$ is the fermion in the 2D Z_2 gauge theory. Again, the $\epsilon_i m_{i+1} \epsilon_{i+2}$ have both self- and mutual boson statistics so that they may be simultaneously condensed. This confines all bulk topological quasiparticles. The surviving surface quasiparticles will be $\epsilon_1, m_1 \epsilon_2$ at the top surface and $\epsilon_N, m_N \epsilon_{N-1}$ at the bottom surface. These particles are all fermions, and the two particles at either surface have mutual semion statistics. It follows that either surface realizes the all-fermion Z_2 topological order but now in the presence of Z_2^T symmetry. We have thus explicitly constructed the SPT phase discussed in Sec. IV.

This coupled layer construction gives very strong support to the results of Ref. 8 on the various SPT phases. In particular, it removes any concerns on the legitimacy of the state of Sec. IV with Z_2^T symmetry not currently present in the cohomology classification.

VI. RELATION WITH BULK TOPOLOGICAL FIELD THEORIES

Here, we provide an understanding of the results obtained above from topological field theories in the bulk. It was shown in Ref. 8 that bosonic topological insulators in 3D with $[U(1)]^N$ symmetry have a bulk response to external “probe” gauge fields A^I characterized by a θ term with $\theta = \pi$:

$$L_\theta = \frac{\theta}{8\pi^2} K_{IJ} \epsilon^{ijkl} \partial_i A_j^I \partial_k A_l^J. \quad (16)$$

If under symmetry transformations (e.g., time reversal) the θ -angle transforms as $\theta \rightarrow -\theta$, then the $\theta = \pi$ term is symmetric in the bulk, but on the boundary, it reduces to a (mutual) Chern-Simons term with symmetry violating responses. This was a familiar issue in the noninteracting fermionic topological insulator, where a single Dirac cone was introduced on the boundary to cancel the time-reversal violating response through parity anomaly.

In our cases, let us understand how this works out when the surface is in a symmetry preserving gapped topological ordered phase of the kind studied in this paper. We will show that the symmetries of the topological order on the boundary are such as to cancel the Chern-Simons response arising from the θ term. To illustrate the idea, we take $K_{IJ} = \sigma_x$, which applies to a large class of SPT phases in 3D. This will give a mutual Chern-Simons term on the boundary:

$$L_{CS} = \frac{1}{4\pi} \epsilon^{ijk} A_{1,i} \partial_j A_{2,k}. \quad (17)$$

This term alone would give a response that breaks time-reversal symmetry. To cure it, we put a Z_2 topological liquid on the boundary, with the e and m particles coupling to $A_{1,2}$, respectively. The Lagrangian is given by

$$L_{Z_2} = \frac{1}{\pi} \epsilon^{ijk} a_{1,i} \partial_j a_{2,k} + \frac{1}{2\pi} (\epsilon^{ijk} A_{1,i} \partial_j a_{1,k} + \epsilon^{ijk} A_{2,i} \partial_j a_{2,k}). \quad (18)$$

Integrating out $a_{1,2}$ induces a mutual Chern-Simons term for $A_{1,2}$, which exactly cancels what arose from the bulk θ term and hence restores time-reversal symmetry.

The topological ordered states with symmetry that are forbidden in strict 2D realize the symmetry in an “anomalous” way. The corresponding topological field theories cannot be given consistent lattice regularizations, which implement the symmetry in a local manner. The discussion in this section illustrates how these theories can nevertheless be given a higher dimensional regularization as the boundary of a nonanomalous field theory. This has the same essence with the anomaly cancellation in free fermion TI. It will be interesting for the future to have criteria to directly identify such anomalous symmetry in a topological field theory.

VII. CONSTRAINTS ON GAPLESS 2D QUANTUM SPIN LIQUIDS: ABSENCE OF ALGEBRAIC VORTEX LIQUIDS

We now turn to gapless quantum liquids in two space dimensions. Examples are gapless quantum spin liquid phases of frustrated quantum magnets or non-Fermi liquid phases of

itinerant fermions or bosons. Symmetry plays a very crucial role in the stability of these phases. The example of the topologically ordered gapped states considered in previous sections lead us to pose the question of what kinds of putative gapless phases and critical points are allowed to exist with a certain symmetry in strictly 2D systems. First of all, we note that in contrast with gapped topologically ordered phases, global symmetries typically play a much more important role in protecting the gaplessness of a phase. The symmetry may forbid a relevant perturbation to the low-energy renormalization group fixed point that, if present, may lead to a flow to a gapped fixed point. Here, however, we are interested in a more general question. We wish to consider gapless fixed points that can be obtained by tuning any finite number of relevant perturbations. This includes not just bulk 2D phases but also critical or even finitely multicritical quantum systems. We are particularly interested in such gapless 2D fixed points with symmetry that cannot exist in strict 2D but may only exist at the surface of a 3D insulator (SPT or otherwise).

To set the stage, consider a simple and familiar example in a free fermion system. The surface of the celebrated time-reversal symmetric electron topological insulator (symmetry $U(1) \rtimes Z_2^T$ has an odd number of Dirac cones. Such a gapless state cannot exist in strict 2D fermion systems with the same symmetry even as a multicritical point. However, if we give up time-reversal symmetry, this state is allowed as a critical point in strict 2D. An example is provided by a 2D free fermion model poised right at the integer quantum Hall transition. Thus symmetry provides a strong restriction on what gapless fixed points are allowed in strict 2D.

We focus now on a very interesting gapless state proposed²⁶ to exist in strict 2D in frustrated XY quantum magnets [symmetry $U(1) \times Z_2^T$] or in boson systems [symmetry $U(1) \rtimes Z_2^T$]. This state—dubbed an algebraic vortex liquid (AVL)—was obtained in a dual vortex description by fermionizing the vortices and allowing them to be massless. A suggestive approximation was then used to derive a low energy effective field theory consisting of an even number of massless two-component Dirac fermions (the vortices) coupled to a non-compact $U(1)$ gauge field. The AVL state has been proposed to describe quantum spin liquid states on the Kagome and triangular lattices. In terms of development of the theory of gapless spin liquids and non-Fermi liquids, the AVL proposal is extremely important. To date, the only known theoretical route to accessing such exotic gapless phases of matter (in $d > 1$) is through a slave particle construction where the spin, electron operator is split into a product of other operators. If some of the resulting slave particles are fermions, they can be gapless. In contrast, the AVL presents a new paradigm for a gapless highly entangled state that is likely beyond the standard slave particle approach. It is therefore crucial to explore and understand it thoroughly.

We now argue that the AVL state cannot exist in strictly 2D models with either $U(1) \rtimes Z_2^T$ or $U(1) \times Z_2^T$ symmetry. This is already hinted at by several observations. First, it has never been clear how to implement time reversal in a consistent way in the AVL theory. The AVL is obtained from the usual bosonic dual vortex theory through a flux attachment procedure to fermionize the vortices. This leads to an additional Chern-Simons gauge field that couples to the

fermionized vortices. However, this new gauge field can be absorbed into the usual dual gauge field to leave behind a three derivative term for a residual gauge field. It was argued that this three derivative interaction is formally irrelevant in the low-energy effective theory. This argument is delicate though. In the simplest context²⁷ where such an approximation was made, an alternate description in terms of a sigma model revealed the presence of a topological θ term at $\theta = \pi$. The topological term also has three derivatives, but its coefficient is protected by time-reversal symmetry and does not flow under the RG, its presence presumably crucially alters the physics of the model. Thus one may worry about the legitimacy of the approximations invoked to justify the AVL phase.

Note that the fundamental issue that needs to be addressed with the AVL phase is whether it realizes symmetry in a manner that is allowed in 2D spin/boson systems. This is of course the kind of question that is the essence of this paper. A final hint that the AVL phase may not exist in strict 2D comes from recent work⁸ showing that gapless quantum vortex liquids with fermionic vortices can actually arise at the surface of time-reversal symmetric 3D SPT phases. This strongly suggests that such phases cannot arise in strict 2D with the same symmetries. Below we will sharpen these arguments.

Consider the proposed effective field theory for the AVL phase:

$$\mathcal{L} = \overline{\psi}_\alpha (i\partial_\mu - i\phi_\mu) \psi_\alpha + \frac{1}{2e^2} (\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda)^2 + \frac{1}{2\pi} a_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda^{\text{ext}}. \quad (19)$$

Here, ψ_α ($\alpha = 1, \dots, 2N$) are the fermionized vortices, a_μ is a noncompact U(1) gauge field whose curl is 2π times the global U(1) current, and A_μ^{ext} is an external probe U(1) gauge field. Note that the vortices themselves do not carry physical U(1) charge. As mentioned above the realization of time reversal in terms of these fermionized vortex fields has always been a tricky issue for the AVL theory. To sharpen the issue, we now consider a phase that is accessed from the AVL phase by pairing and condensing the ψ_α fields. In the original AVL literature,²⁶ a number of phases proximate to the AVL were studied by assuming that four fermion interactions were strong enough to give a mass to the fermions. A number of different such mass terms leading to various symmetry breaking orders were examined. Here instead, we imagine a mass term that corresponds to vortex pairing that preserves the global internal symmetry.

The vortex pair condensation will gap out the gauge field a_μ and will give an insulator. However, as the ψ_α fields are vortices of the original boson, this is a phase with Z_2 topological order. The fermionized vortices survive as “unpaired” gapped quasiparticles in this topological phase. We may identify them with the ϵ particle, which in this case, has zero global U(1) charge. In the notation of previous sections, $\sigma_\epsilon = 1$. Furthermore, the pair condensation will quantize the gauge flux $\vec{\nabla} \times \vec{a}$ in units of π so that one of the topological quasiparticles (which we take to be the e particle) has 1/2 charge, i.e., $\sigma_e = -1$.

Now it is clear from Tables I and II that in strict 2D such a state can exist only if the ϵ particle also carries 1/2 charge, i.e., $\sigma_\epsilon = -1$. However, we just argued that the Z_2 topological

ordered state realized from the AVL state has $\sigma_\epsilon = 1$, i.e., it carries zero global U(1) charge. It follows that such a Z_2 topological ordered state cannot exist in strict 2D. Note that we did not explicitly rely on time-reversal symmetry in our analysis (though it is implicit in deciding which Z_2 states are allowed in 2D).

Thus the Z_2 topological ordered states that descends from the AVL is not allowed to exist in strict 2D. This then implies that the AVL itself cannot exist in strictly 2D systems so long as both global U(1) and Z_2^T symmetries are present.

Can gapless quantum vortex liquids ever exist in strictly 2D? One option is to break time-reversal symmetry. Then our arguments do not prohibit the formation of fermionic vortices, which can then be in a gapless fluid state. Indeed, such a gapless magnetic-field induced vortex metal state was proposed to exist in 2D superconducting films in Ref. 29. A different option—which we will elaborate elsewhere³⁰—that preserves internal symmetries is obtained by fractionalizing the vortices into fermionic partons, which can then be gapless.

VIII. TIME REVERSAL SYMMETRIC U(1) QUANTUM LIQUIDS IN 3 + 1 DIMENSIONS

We now turn our attention to three dimensional highly entangled states with time-reversal symmetry. In three dimensions, interesting gapless quantum liquids with an emergent gapless U(1) gauge field are possible.³¹ Explicit lattice models for such phases were constructed and their physics studied in Refs. 17 and 32–37. Interest in such phases has been revived following a recent proposed realization³⁸ in quantum spin ice materials on three-dimensional pyrochlore lattices. It is thus timely to understand the possibilities for the realization of symmetry in such phases with emergent photons. Here, we will restrict attention to time-reversal symmetry in keeping with the theme of the rest of the paper.

The excitation spectrum of the U(1) quantum liquid consists, in addition to the gapless photon, point “electric” charges (the e particle) and point “magnetic” charges (the m particle or monopole). We will only consider the situation in which both the e and m particles are gapped, and will focus on phases that can be realized in strictly 3D systems (as opposed to U(1) phases allowed at the boundary of 4 + 1-dimensional SPT phases). Following the discussion of previous sections, a simple restriction that ensures this is to assume that one of the e or m particles has trivial global quantum numbers and is a boson. Without loss of generality, we will assume that it is the m particle.

The low-energy long wavelength physics of the U(1) liquid state is described by Maxwell’s equations. As usual, they imply that the emergent electric and magnetic fields transform oppositely under time reversal. We will distinguish two cases depending on whether the electric field is even or odd under time reversal.

A. Even electric field

First, we consider the case $\vec{E} \rightarrow \vec{E}, \vec{B} \rightarrow -\vec{B}$ under time reversal. This is what happens in the usual slave particle constructions of U(1) spin liquids through Schwinger bosons or fermions. The electric field on a bond gets related to

the bond energy, which is clearly even under time reversal. Consistent with this, the magnetic field gets identified with the scalar spin chirality which is odd under time reversal. Then the electric charge $q_e \rightarrow q_e$ and magnetic charge $q_m \rightarrow -q_m$. Let us introduce creation operators e^\dagger, m^\dagger for the e and m particles. With the assumption that m^\dagger has trivial global quantum numbers and is a boson, it must transform under time reversal as

$$\mathcal{T}^{-1} m^\dagger \mathcal{T} = e^{i\alpha_m} m. \quad (20)$$

However, the phase α_m has no physical significance. It can be removed by combining \mathcal{T} with a (dual) U(1) gauge transformation that rotates the phase of m (more detail follows in Sec. VIII C). So we may simply set $\alpha_m = 0$. Let us consider now the e particle. If there is just a single species of e particle, then we must have

$$\mathcal{T}^{-1} e^\dagger \mathcal{T} = e^{i\alpha_e} e^\dagger. \quad (21)$$

Now the phase α_e can be absorbed by redefining the e operator and so we set $\alpha_e = 0$. The e particle transforms trivially under Z_2^T . There are nevertheless two distinct phases depending on whether e is a boson or a fermion. More phases are obtained by considering a 2-component e field: $e = (e_1, e_2)$. The new nontrivial possibility is that this two-component e field transforms as a Kramers doublet under Z_2^T :

$$\mathcal{T}^{-1} e^\dagger \mathcal{T} = i\sigma_y e^\dagger. \quad (22)$$

Clearly, we have $\mathcal{T}^{-2} e^\dagger \mathcal{T}^2 = -e^\dagger$ but the action of \mathcal{T}^2 on physical (*gauge invariant* local) operators gives 1. For instance, $e_1^\dagger e_2$ is a physical operator and we clearly have $\mathcal{T}^{-2} e_1^\dagger e_2 \mathcal{T}^2 = e_1^\dagger e_2$. When e is a Kramers doublet, it can again be either a boson or a fermion. The former is obtained in the standard Schwinger boson construction and the latter in the Schwinger fermion construction. Thus we have a total of four possible phases corresponding to e being a Kramers singlet or doublet with Bose or Fermi statistics and a boson monopole with trivial global quantum numbers.

B. U(1) quantum liquids as monopole topological insulators

The four U(1) quantum liquids described above were distinguished by the symmetry and statistics of the e particle. We now develop a very interesting alternate view point where we understand these four states as different SPT insulators of the bosonic monopole with trivial global quantum numbers. As the magnetic charge is odd under time reversal, the monopole transforms under $U_g(1) \times Z_2^T$ where $U_g(1)$ is the gauge transformation generated by the monopole charge. It is useful (though not necessary) to perform an electric-magnetic duality transformation; this exchanges the e and m labels:

$$e \leftrightarrow m_d, \quad (23)$$

$$m \leftrightarrow e_d. \quad (24)$$

We included a subscript d on the right side to indicate that these are the dual labels. Now e_d is a gapped boson that transforms under $U_g(1) \times Z_2^T$. Thus we may regard the U(1) quantum liquids as insulating phases of e_d obtained by gauging the U(1) part of a $U(1) \times Z_2^T$ symmetry. Note that e_d transforms under a linear (i.e., not projective) representation of $U_g(1) \times Z_2^T$. As

discussed in previous sections such bosons can be in a number of different SPT phases. We now study their fate when the U(1) symmetry is gauged.

1. Gauged bosonic SPT phases in 3D

In 2D, Ref. 39 studied the fate of bosonic SPT insulators with discrete global unitary symmetry when that symmetry is gauged. It was shown that the result was a topologically ordered gapped quantum liquid with long-range entanglement. A general abstract discussion of such gauged SPT phases for unitary symmetry groups (i.e., not involving time reversal) has also appeared.⁴⁰ Here, we are interested in 3D SPT phases with $U(1) \times Z_2^T$ symmetry. A gauged 3D SPT phase with $U(1) \times Z_2^T$ was also studied very recently in a beautiful paper.⁴¹ Using the known $\theta = 2\pi$ electromagnetic response,⁸ it was argued that the monopole of this gauged SPT is a fermion, and this was used as a conceptual starting point to discuss the surface of this SPT. Here, we will discuss the gauged SPT from a different and more general point of view that will enable us to also discuss SPT phases where the electromagnetic response has no θ term (necessary for the results in this subsection).

References 8 and 9 show that a key distinction between different SPT phases with the same symmetry is exposed by considering the end points of vortex lines of the boson at the interface with the vacuum. It will be convenient to label the SPT phases by their possible surface topological order (whether or not such order is actually present in any particular microscopic realization). For one simple example, the SPT phase (the one whose surface topological order is $eCmC$), these papers argued that a surface Landau-Ginzburg theory is obtained in a dual description in terms of fermionic vortices. In another SPT phase (labeled by surface topological order $eCmT$), the surface vortex is a boson but is a Kramers doublet. By stacking these two phases together we can get a third SPT phase where the surface vortex is a fermionic Kramers doublet. In contrast, for topologically trivial insulators, the surface vortex is a bosonic Kramers singlet. In Appendix B, we describe these surface dual Landau-Ginzburg theories and their implied surface phase structure. We also provide an explicit derivation that is complementary to the arguments of Ref. 8.

Closely related to this we can also consider external point sources for vortex lines directly in the bulk. In the Hilbert space of the microscopic boson model, the vortex lines do not have open ends in the bulk. So these external sources for vortex lines must be thought of as ‘‘probes’’ that locally modify the Hilbert space. These will behave similarly to the surface end points of vortices. For example, in the SPT labeled $eCmT$, Ref. 9 shows that the ground-state wave function is a loop gas of vortices where each vortex core is described as a Haldane spin chain. An externally imposed open end for a vortex string will terminate the core Haldane chain so that there is a Kramers doublet localized at this end point. In this case, the external vortex source is a bosonic Kramers doublet. In the other example, the SPT labelled $eCmC$, the vortices are ribbons with a phase factor (-1) associated with each self-linking of the ribbon. Open end points of such vortex strings are fermionic Kramers singlets. Obviously stacking these two phases together produces an SPT where bulk vortex sources are fermionic Kramers doublets. In contrast, in trivial

TABLE V. Phases of U(1) quantum liquids (Z_2^T symmetry and even emergent electric field), labeled by symmetry properties of the electric charge, and the corresponding type of monopole SPT, conveniently labeled by the possible surface topological order.

Electric particle	Monopole insulator
$\mathcal{T}^2 = 1$, boson	Trivial
$\mathcal{T}^2 = -1$, boson	SPT- $eCmT$
$\mathcal{T}^2 = 1$, fermion	SPT- $eCmC$
$\mathcal{T}^2 = -1$, fermion	SPT- $eCTmC = eCmT \oplus eCmC$

boson insulators, such bulk external vortex sources are bosons with trivial quantum numbers under global symmetries.

This understanding of the different SPT phases immediately determines what happens when the U(1) symmetry is gauged. As these phases are gapped insulators (at least in the bulk) there will now be a dynamical photon. More interesting for our purposes is the fate of the magnetic monopole m_d . The monopole serves as a source of 2π magnetic flux for the e_d particle. Thus it should precisely be identified with the source of vortex lines. It follows that m_d can therefore either be a Kramers singlet or doublet and have Bose or Fermi statistics.

Reversing the duality transformation we see that these are precisely the four distinct U(1) quantum liquids discussed in the previous subsection. We have thus established our promised claim that these different U(1) quantum liquids may be equivalently viewed as different bosonic monopole SPT insulators.

In Table V, we list all the distinct phases of the U(1) gauge theory with their monopole quantum numbers and the corresponding SPT states (labeled by the surface topological states) formed by the bosonic matter field. Notice that SPT states descended from that of Z_2^T symmetry (the $eTmT$ and the all-fermion states) did not appear in Table V. One can understand this by thinking of these states as combinations of trivial insulators and Z_2^T SPT states formed by charge-neutral bosons, hence the U(1) gauge field is decoupled from the SPT states and the vortex source (i.e., monopole $m_d = e$) remains trivial.

C. Odd electric field

We now consider U(1) liquid states where under time reversal the electric field is odd and the magnetic field is even. In the convention of Ref. 33, this includes the case of quantum spin ice. Again we restrict attention to U(1) liquids where the magnetic monopole m is bosonic and transforms trivially under Z_2^T . What then are the possibilities for the e particle?

Based on the insights of the previous section, let us first see what we can learn by considering different monopole SPT phases. Now the magnetic charge $q_m \rightarrow q_m$ under time reversal so that

$$\mathcal{T}^{-1}m^\dagger\mathcal{T} = m^\dagger. \quad (25)$$

Thus m (or equivalently e_d after the duality transformation) transforms under $U_g(1) \times Z_2^T$.

For bosons with global symmetry $U(1) \times Z_2^T$, there is one nontrivial SPT phase, which is, again, conveniently labeled by its surface topological order $eCmC$. Other SPT phases are

inherited from Z_2^T and hence are not pertinent to our present concerns (see the end of Sec. VIII B1). Thus we have two possible phases: the trivial insulator and the SPT insulator labeled by $eCmC$. In the former case, external probes where bulk vortex lines end are bosons, while in the latter, they are fermions. In both cases, the vortex sources are Kramers trivial.

Let us now follow the logic of the previous section and gauge the U(1) symmetry. The resulting monopole m_d will be identified with the vortex source and will therefore be a Kramers singlet, which can be either boson or fermion. Thus this reasoning suggests that for the odd electric field case there are only two possibilities for the e particle ($=m_d$)—it is a Kramers singlet that is either boson or fermion.

Let us understand the above claim directly from the gauge theory point of view, independent of the argument based on SPT. With odd electric field the electric charge at any site q_e is also odd under time reversal. This implies that the e particles transform under $U_g(1) \times Z_2^T$, where $U_g(1)$ is the gauge transformation generated by q_e . Notice that we have $U_\theta\mathcal{T} = \mathcal{T}U_\theta$ for $U(1) \times Z_2^T$ symmetry, where U_θ gives the U(1) rotation. Allowing for the possibility of a multicomponent field e_I , time reversal will be implemented by

$$\mathcal{T}^{-1}e_I\mathcal{T} = e^{-i\alpha_e}T_{IJ}e_J^\dagger. \quad (26)$$

We can always change the common phase α_e by defining a new time-reversal operator $\tilde{\mathcal{T}} = U(\theta)\mathcal{T}$. As $U(\theta)$ is a gauge transformation $\tilde{\mathcal{T}}$ and \mathcal{T} will have the same action on all physical operators. We therefore can set $\alpha_e = 0$ (or any other value for that matter). In particular, under this redefinition, \mathcal{T}^2 goes to $(U_\theta\mathcal{T})^2 = U_{2\theta}\mathcal{T}^2$, so that the over all phase in the action of \mathcal{T}^2 on e can be changed at will, and one can always choose $\mathcal{T}^2 = 1$. The algebraic structure of $U_g(1) \times Z_2^T$ still guarantees a degenerate doublet structure, but the degeneracy here is protected by $U_g(1) \times Z_2^T$ as a whole rather than by Z_2^T alone as in Kramer's theorem. In particular, one can lift the degeneracy by breaking the $U_g(1)$ symmetry but still preserving time-reversal invariance, which is in sharp contrast with the Kramer's case. It is appropriate to regard the electric charge $q_e = \pm 1$ as a non-Kramers doublet. Hence with $U_{\text{gauge}}(1) \times Z_2^T$ symmetry, any charged particle should always be viewed as time-reversal trivial. This implies that the e particle is always time reversal trivial for a U(1) gauge theory where the electric field is odd, in full agreement with what we obtained from the SPT point of view.

Before concluding this section, let us briefly discuss the putative U(1) spin liquid in quantum spin ice from this point of view. We have just argued that the “spinons” (in the notation of Ref. 33) are not Kramers doublets. If the quantum spin ice Hamiltonian has S_z conservation then the spinons will generically carry fractional S_z . However, the realistic Hamiltonians currently proposed³⁸ for quantum spin ice do not have conservation of any component of spin. Thus the “spinons” of the possible U(1) spin liquid in quantum spin ice do not carry any quantum numbers associated with internal symmetries. Their non-Kramers doublet structure is independent of whether or not the microscopic Ising spin is itself Kramers or not. Further microscopically there are two species of electric charge e_1, e_2 (associated with two sub lattices of the diamond lattice formed by the centers of

pyrochlore tetrahedra). Time reversal should be implemented by letting $e_1 \rightarrow e_1^\dagger, e_2 \rightarrow \pm e_2^\dagger$, so that the physical spin operator $e_1^\dagger e_2$ transforms as appropriate with $-$ sign for a microscopic Kramers doublet spin and the $+$ sign for a non-Kramers doublet.

Finally, we note that current theoretical work treats the spinons in quantum spin ice as bosons. This is reasonable as the electric strings connecting them are simply made up of the physical spins and do not have the ribbon structure and associated phase factors expected if they were fermions.

IX. DISCUSSION

In this paper, we studied many aspects of the realization of symmetry in highly entangled quantum phases of matter. We relied heavily on insights obtained from recent work on *short-range entangled* symmetry protected topological phases. Despite their short-range entanglement, the SPT phases provide a remarkable window into the properties of the much more nontrivial highly entangled phases. In turn, the connections to the highly entangled phases enhances our understanding of SPT phases themselves. Below we briefly reiterate some of our results and their implications.

The very existence of SPT phases emphasizes the role of symmetry in maintaining distinctions between phases of matter even in the absence of any symmetry breaking. For highly entangled states, this leads to the question of whether symmetry is realized consistently in the low energy theory of such a state. We addressed this for the example of 2D gapped topological phases described by a Z_2 gauge theory with time-reversal symmetry [and possibly a global $U(1)$ symmetry]. By combining the methods of two different recent approaches^{10,13} to assigning symmetry to the topological quasiparticles, we showed that there are consistent symmetry realizations that nevertheless are not possible in strictly 2D systems. Such states were, however, shown to occur at the surface of 3D bosonic SPT phases. Conversely, we provided simple arguments that if a Z_2 topological order can occur at the surface of a 3D SPT, then it is not allowed to occur in strictly 2D systems. Crucial to our discussion was the concept of edgability. A topologically ordered state with (internal) symmetry is allowed in strictly 2D if and only if it is edgable, i.e., it must admit a physical edge with the vacuum. Topological ordered states that are only allowed at the surface of a 3D SPT phase are clearly not edgable.

Thus illustrates how the study of SPT surfaces can provide a very useful “no-go” constraint on what kinds of phases are acceptable in strictly d -dimensional systems. If a phase occurs at the surface of a $(d + 1)$ -dimensional SPT phase (for $d > 1$) then it cannot occur with the same realization of symmetry in strictly d dimensions. A nice application of this kind of no-go constraint is to the possibility of gapless vortex fluid phases proposed to exist²⁶ in two space dimensions in boson spin systems with both time-reversal and global $U(1)$ symmetries. Such phases were argued to exist at the surface of 3D SPT phases in Ref. 8, thereby strongly suggesting that they cannot exist in strictly 2D. We sharpened this conclusion by considering a descendant Z_2 topological ordered phase that is obtained by pairing and condensing vortices of this putative

vortex fluid. We showed that the result was a phase that cannot exist in strict 2D but can of course exist at the surface of 3D SPT.

The study of SPT surfaces thus gives us valuable guidance in writing down legal theories of strictly d -dimensional systems. It thus becomes an interesting exercise to study boundary states of SPT phases in $4 + 1$ dimensions as a quick route to obtaining some restriction on physically relevant effective field theories of strictly $(3 + 1)$ -dimensional systems. Quite generally, the issue of consistent symmetry realization is likely related to the existence of “quantum anomalies” in the continuum field theory. For instance, the surface field theory of free fermion topological insulator are “anomalous” and require the higher dimensional bulk for a consistent symmetry-preserving regularization. For other states such as, for example, topological quantum field theories it will be interesting if there is a useful direct diagnostic of whether the theory is anomalous or not.

A different aspect of our results is the development of a view point on some highly entangled states as phases in which one of the emergent excitations itself is in an SPT phase. A similar result was first established in $d = 2$ in the work of Levin and Gu.³⁹ We discussed how three-dimensional quantum phases with an emergent $U(1)$ gauge field may be viewed as SPT phases of the magnetic monopole excitations. In the context of $3 + 1$ -d $U(1)$ spin liquids with time-reversal symmetry, whether the electric charges are “spinons” (i.e., Kramers doublet under time reversal) and whether they are bosonic or fermionic is equivalent to the different possible SPT insulating phases of the dual magnetic monopole. This new view point may potentially be useful for future studies of these interesting gapless spin liquids and their phase transitions. As we demonstrated, there is a nice consistency between possible symmetry realizations in such spin liquids and the possible corresponding SPT phases.

Finally, a very interesting outcome of our results is the explicit construction of coupled layer models for 3D SPT phases. For all the symmetry classes discussed in Ref. 8, we provided such a construction. The strategy is to start from a layered 3D system where each layer is in a Z_2 topological ordered state that is allowed in strict 2D. We then coupled the layers together to confine all topological excitations in the bulk but left behind a deconfined Z_2 topological state at the surface. This surface topological order was shown to match the various possible such orders at SPT surfaces. In particular, this scheme provides an explicit construction of a 3D SPT state whose surface is a time-reversal symmetric gapped Z_2 topological ordered state with three fermionic excitations that are all mutual semions. This topological order is expected to occur at the surface of a bosonic SPT state with time-reversal symmetry proposed in Ref. 8 and is not currently part of the classification of Ref. 3.

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APPENDIX A: K MATRIX DESCRIPTIONS OF Z_2 TOPOLOGICAL ORDER

In this Appendix, we consider 2D states in detail. In most cases, a 2×2 K matrix is enough to describe the state because we can identify all particles with the same symmetry and topological properties through condensing appropriate combinations of them, and there remains only one species of e and m particles, respectively. For example, consider a Kramer’s doublet carrying spin-1/2 b_{\pm} , the combination $b_+ b_-$ is a singlet under time reversal and carries no spin, so we can condense it and identify $b_- \sim b_+^\dagger$, and time reversal could be realized through $b_+ \rightarrow i b_+^\dagger$ so that $T^2 = -1$.

The 2×2 K matrix was considered thoroughly in the main text, and it was straightforward to get all the possible states within the framework. It is also clear from the analysis above that 2×2 K matrix is enough to describe every state with Z_2^T and $U(1) \times Z_2^T$ (spin) symmetries. For $U(1) \times Z_2^T$ (charge) symmetry, the 2×2 K matrix describes most of the states, except when there is at least one particle that carries both half-charge and Kramer’s doublet in which case there is no particle bilinear that preserves all symmetries, and we should really consider two species of such particles. For these states, a 4×4 K matrix is needed.

The general form of such K matrices was given in Ref. 13, with slight modifications due to the bosonic nature of our systems here. There are three possible forms of K and T matrices. The simplest one of them

$$K = \begin{pmatrix} 0 & A_{2 \times 2} \\ A_{2 \times 2} & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -1_{2 \times 2} & 0 \\ 0 & 1_{2 \times 2} \end{pmatrix} \quad (\text{A1})$$

does not work because the T matrix does not allow a particle to carry both charge and Kramer’s doublet structure. The next possibility

$$K = \begin{pmatrix} K_{2 \times 2} & W_{2 \times 2} \\ W_{2 \times 2}^T & -K_{2 \times 2} \end{pmatrix}, \quad \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_1 \\ \tau_2 \end{pmatrix}, \quad (\text{A2})$$

$$T = \begin{pmatrix} 0 & 1_{2 \times 2} \\ 1_{2 \times 2} & 0 \end{pmatrix}$$

with $W_{2 \times 2}$ antisymmetric, does not work either. To see this, simply look at the charge carried by any particle $q_l = l_I K_{IJ}^{-1} \tau_J$. The entries of K_{IJ}^{-1} are either integers or half-integers. From the structure of the T -matrix and the assumption that time reversal does not interchange e and m particles, we find that the only half-integer entries of K^{-1} are $K_{12}^{-1} = K_{21}^{-1}, K_{14}^{-1} = K_{41}^{-1}, K_{23}^{-1} = K_{32}^{-1}, K_{34}^{-1} = K_{43}^{-1}$. Then from the structure of the τ vector it is easy to see that the charge $q_l = l_I K_{IJ}^{-1} \tau_J$ must be

an integer for any integer vector l , so there is no quasiparticle that carries half-charge.

The only possibility left is thus

$$K = \begin{pmatrix} 0 & A & B & B \\ A & 0 & C & -C \\ B & C & D & 0 \\ B & -C & 0 & -D \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 \\ \tau_2 \\ \tau_3 \\ \tau_3 \end{pmatrix}, \quad (\text{A3})$$

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

with $\det K = (AD - 2BC)^2 = 4$. The inverse of the K matrix is thus

$$K^{-1} = \frac{\text{sgn}(AD - 2BC)}{2} \begin{pmatrix} 0 & D & -C & -C \\ D & 0 & -B & B \\ -C & -B & A & 0 \\ -C & B & 0 & -A \end{pmatrix}. \quad (\text{A4})$$

Therefore to have the right self- and mutual statistics, we need $A = 4m, D = 2n$, and B, C odd, which makes particle 1 [$l = (1, 0, 0, 0)$] or 2 [$l = (0, 1, 0, 0)$] having π statistics with particle 3 [$l = (0, 0, 1, 0)$] or 4 [$l = (0, 0, 0, 1)$], and all the other mutual or self-statistics trivial.

It is clear from the T matrix that particle 2 is time-reversal trivial. Since the bound state of particles 1 and 2 [$l = (1, 1, 0, 0)$] has trivial statistics with any particle from the structure of K^{-1} , it must be physical hence time-reversal trivial, which implies that particle 1 should also be time-reversal trivial. Now consider the charge of these two particles. It is straightforward to see that with the given charge vector τ , the charge carried by particle 1 or 2 $q_{1,2} = \tau_I K_{IJ}^{-1} l_J, (J = 1, 2)$ can only be an integer. Hence particles 1 and 2 carry neither fractional charge nor Kramer’s doublet.

Recall that our purpose here is to describe phases with a particle that carries both charge-1/2 and Kramer’s doublet. Hence particles 3 and 4 must form a Kramer’s doublet and carry charge 1/2. So we want the charge vector that makes $q = \tau_I K_{IJ}^{-1} l_J$ half-integer when $l \in \{(0, 0, 1, 0), (0, 0, 0, 1)\}$. It is then straightforward to show that we need τ_2 to be odd and $\tau_3 = \tau_4$ to be any integer. What we have shown above is that if the e particle carries both charge-1/2 and Kramer’s doublet structure, the m particle must be trivial under both symmetry transforms, i.e., the phase has to be eCT .

APPENDIX B: DUAL LANDAU-GINZBURG THEORY OF SPT SURFACE

In this Appendix, we briefly describe the Landau-Ginzburg theory of the surface of 3D SPT states [symmetry $U(1) \times Z_2^T$ or $U(1) \times Z_2^T$] in terms of dual vortex fields. Reference 8 showed that the difference with a trivial surface is captured very simply in terms of the difference in the structure of the vortex (this should be understood as the point of penetration of the 3D vortex line with the surface). Here, we elaborate on this

dual theory and present an explicit derivation starting from the surface topological ordered phase.

Let us consider $U(1) \times Z_2^T$ and consider the phase labeled by surface topological order $eCmC$ [the symmetry $U(1) \times Z_2^T$ analysis is essentially the same]. According to Ref. 8, the boundary vortex is then a Kramers singlet fermion. The corresponding surface Landau-Ginzburg Lagrangian may be written schematically:

$$\mathcal{L}_d = \mathcal{L}[c, a_\mu] + \frac{1}{2\pi} A_\mu \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda. \quad (\text{B1})$$

The first term describes a (spineless) fermionic field c coupled minimally to the dual internal gauge field a_μ , and A_μ is an external probe gauge field. The field c describes the fermionic vortex. The global $U(1)$ current is as usual

$$j_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda. \quad (\text{B2})$$

If, instead, the field c were a boson, the Lagrangian above would be the standard dual Lagrangian for a system of strictly 2D bosons. Let us first describe the phase structure of this dual fermionic vortex theory. We will then provide an explicit derivation that is complementary to the general considerations of Ref. 8.

If the fermionic vortex c is gapped and in a trivial ‘‘band’’ insulator, then, as usual, we get a surface superfluid. Note that as c is a fermion it cannot condense. This precludes the usual mechanism of vortex condensation to obtaining a trivial boson insulator as expected for an SPT surface. The surface superfluid order can be killed if pairs of c condense, i.e., $\langle cc \rangle \neq 0$. This leads to a surface topological order described by a Z_2 gauge theory. There is the unpaired fermion that survives as a gapped excitation carrying zero global $U(1)$ charge. We identify this with the neutral ϵ particle. The pair condensation quantizes flux of a_μ in units of π . This carries global $U(1)$ charge $1/2$, and we identify this with the e particle. It follows that this is the $eCmC$ phase.

As described in Ref. 8, if we break time reversal at the surface we can get a gapped phase without topological order. This is obtained by simply letting the c -fermionic vortices completely fill a topological band with Chern number ± 1 , i.e., the Hall conductivity of the c fermion is $\sigma_{xy}^c = \pm 1$. To see that this indeed describes the correct T -broken surface state, we use a Chern-Simons description of this state. First, we rewrite the fermion current j^f in terms of a dual gauge field \tilde{a} :

$$j_\mu^f = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial_\nu \tilde{a}_\lambda. \quad (\text{B3})$$

When the fermion has Hall conductivity, say $+1$, the effective Chern-Simons Lagrangian in terms of (a, \tilde{a}) becomes

$$\mathcal{L} = \frac{1}{4\pi} \tilde{a} d\tilde{a} + \frac{1}{2\pi} a d\tilde{a} + \frac{1}{2\pi} A da. \quad (\text{B4})$$

(We have used the compact notation $da \equiv \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda$.) This is a two-component Chern-Simons theory with a K matrix

$$K = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{B5})$$

with charge vector $\tau = (0, 1)$. This state has electrical Hall conductivity $\sigma_{xy} = \tau^T K^{-1} \tau = -1$. Further, as this K matrix

has one positive and one negative eigenvalue, it is a nonchiral state with $\kappa_{xy} = 0$. Finally, there is no surface topological order as $|\det K| = 1$. These are exactly the right properties of the T -broken surface state without topological order at this SPT surface.

Thus the fermionic vortex Landau-Ginzburg theory correctly reproduces the surface phase structure of this SPT phase. We note that if the c -fermion has a band structure with an even number of gapless Dirac fermions, we get exactly the proposed Lagrangian for the AVL phase, consistent with the claim in Sec. VII that the AVL state can only occur at the surface of an SPT state (with T reversal) and not in strict 2D.

A different SPT state with $U(1) \times Z_2^T$ symmetry has a bosonic Kramers doublet vortex z_α , $\alpha = 1, 2$. The corresponding dual Landau-Ginzburg theory takes the form

$$\mathcal{L}_d = \mathcal{L}[z_\alpha, a_\mu] + \frac{1}{2\pi} A da. \quad (\text{B6})$$

Under time reversal, $z_\alpha \rightarrow i\sigma_{\alpha\beta}^y z_\beta$. Finally, by stacking the two SPT phases described above, we obtain a third SPT with a fermionic Kramers doublet vortex field c_α with Lagrangian

$$\mathcal{L}_d = \mathcal{L}[c_\alpha, a_\mu] + \frac{1}{2\pi} A da. \quad (\text{B7})$$

The surface phase structure of these other SPTs can be readily discussed in terms of these dual Landau-Ginzburg theories. In all these cases, there is a bulk-edge correspondence that relates the structure of the surface vortex to the properties of the bulk monopole when the global $U(1)$ symmetry is gauged. Including the trivial (i.e., non-SPT) insulator, we have four possible SPT phases with four distinct surface vortices (end points of bulk vortex lines). These correspond precisely to the four possible bulk monopoles of the gauged SPT as discussed in Sec. VIII B 1.

We now provide an explicit derivation of Eq. (B1) for the corresponding SPT phase. Let us begin with the surface topological order $eCmC$. Under time reversal, the e and m particles transform as

$$T^{-1} e T = e^\dagger, \quad (\text{B8})$$

$$T^{-1} m T = m^\dagger, \quad (\text{B9})$$

while the ϵ particle is left invariant. It is convenient for our purposes to focus on the e (described by a boson field b) and ϵ (described by a fermion f) particles with mutual semion interactions. We will implement this in a lattice model of the surface through two Ising gauge fields σ, μ with a mutual Ising Chern-Simons term.¹⁸ The mutual Chern-Simons term imposes a constraint relating the integer valued lattice 3-current \vec{j}_b to the Ising gauge flux of σ that the fermion sees:

$$(-1)^{\vec{j}_b} = \prod_P \sigma_{ij}. \quad (\text{B10})$$

Here, the plaquette product in the RHS is taken over the space-time plaquette pierced by the link of the dual lattice on which the boson current flows. The lattice space-time Lagrangian

may be taken to be

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f, \quad (\text{B11})$$

$$\mathcal{L}_b = \kappa \vec{j}_{b\alpha}^2 + i \vec{j}_{b\alpha} \frac{(\vec{A}_\alpha)}{2}, \quad (\text{B12})$$

$$\mathcal{L}_f = -\sigma_{ij} (t_{ij} f_i^\dagger f_j + \text{H.c.} + \dots). \quad (\text{B13})$$

The fermion Lagrangian will, in general, also include pairing terms $f_i f_j + \text{H.c.}$ As before, A is the external probe gauge field. We now implement a standard duality transformation on the b field by first writing $\vec{j}_b = \vec{\nabla} \times \vec{a}$ with α an integer. As this kind of duality has been explained in detail in Refs. 18 and 28, we will be very brief. The mutual semion constraint Eq. (B10) can be solved to write

$$(-1)^\alpha = \sigma. \quad (\text{B14})$$

This means that the integer $\alpha = 2\alpha' + \frac{\pi}{2}(1 - \sigma)$ with α' an integer. Imposing the integer constraint on α' softly leads to a term

$$-\lambda \cos(2\pi\alpha') = -\lambda \sigma_{ij} \cos(\pi\alpha_{ij}). \quad (\text{B15})$$

We may now define $\pi\alpha = a$ and extract a longitudinal piece $a_{ij} \rightarrow a_{ij} + \phi_i - \phi_j$ to obtain a dual vortex Lagrangian in Euclidean space-time:

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_f, \quad (\text{B16})$$

$$\mathcal{L}_\phi = -\lambda \sigma_{ij} \cos(\vec{\nabla}\phi + \vec{a}) + \frac{\kappa}{\pi^2} (\vec{\nabla} \times \vec{a})^2 \quad (\text{B17})$$

$$+ i \frac{1}{2\pi} \vec{A} \cdot \vec{\nabla} \times \vec{a}. \quad (\text{B18})$$

Finally, tracing over the Ising gauge fields σ gives a dual vortex theory in terms of two fermionic fields c_\pm defined through

$$c_\pm = f e^{i\pm\phi}. \quad (\text{B19})$$

However, generically due to pairing terms in the f Lagrangian, c_- can mix with c_+^\dagger so that there is a unique fermion field $c = c_+ \sim c_-^\dagger$. This then gives us the dual fermionic vortex Landau-Ginzburg theory in Eq. (B1). For $U(1) \times Z_2^T$ under time reversal, we must have $c \rightarrow c$. The dual vortex theory for the surface of the $U(1) \times Z_2^T$ SPT (with $eCmC$ surface topological order) can be derived identically and takes the same form except that under time reversal $c \rightarrow c^\dagger$.

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