Interplay between interaction and chiral anomaly: Anisotropy in the electrical resistivity of interacting Weyl metals

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We predict that long-range interactions give rise to anisotropy in the electrical resistivity of Weyl metals at low temperatures, where the electrical resistivity becomes much reduced when electric fields are applied to the direction of the momentum vector to connect two paired Weyl points. Performing the renormalization group analysis, we find that the distance between two Weyl points becomes enhanced logarithmically at low temperatures although the coupling constant of such interactions vanishes inverse-logarithmically. Considering the Adler-Bell-Jackiw anomaly, scattering between these two Weyl points becomes suppressed to increase electrical conductivity in the "longitudinal" direction, counter intuitive in the respect that interactions are expected to reduce metallicity. We also propose that the anomalous contribution in the Hall effect shows the logarithmic enhancement as a function of temperature, originating from the fact that the anomalous Hall coefficient turns out to be proportional to the distance between two paired Weyl points. Correlations with topological constraints allow unexpected and exotic transport properties.

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I. INTRODUCTION

Nontrivial global structures of ground states sometimes violate classically respected conservation laws at quantum levels, referred to as (quantum) anomalies.[1] When anomalies are associated with breaking of local (gauge) symmetries, it means that their corresponding quantum theories are not consistent and such anomalies should be canceled, introducing meaningful quantum fields. Indeed, the standard model and string theories are constructed consistently, canceling gauge and gravitational (also conformal) anomalies, respectively.[2] On the other hand, when such anomalies are related with breaking of global symmetries, they give rise to various interesting physical properties. In particular, various types of topological terms associated with quantum anomalies arise to play essential roles in quantum criticality of quantum matter.[3] In addition, they turn out to be responsible for quantum number fractionalization, given by Goldstone-Wilczek currents.[4] Actually, an emergent non-Abelian chiral anomaly has been proposed to cause so-called deconfined quantum criticality in low dimensional spin systems.[5,6] Furthermore, such topological terms sometimes give rise to anomalous (quantized) electrical or thermal Hall effects.[7,8] Quantum anomalies govern quantum criticality, quantum number fractionalization, and anomalous transport phenomena.[9]

In this study we focus on the role of the Adler-Bell-Jackiw anomaly or chiral anomaly[1] in anomalous transport phenomena. This anomaly means that classically conserved chiral currents, that is, currents of right-handed (Weyl) fermions minus those of left-handed fermions, are not preserved at quantum levels due to nontrivial global configurations of gauge (or electric and magnetic) fields. Applying magnetic fields to gapless semiconductors, a Dirac point described by the four-component Dirac spinor splits into two Weyl points governed by the two-component Weyl spinors with opposite chiralities, where the distance between two Weyl points is proportional to the applied magnetic field.[10] The chiral anomaly gives rise to a topological constraint in dynamics of Weyl fermions, where right-handed Weyl fermions at one Weyl point should scatter into left-handed Weyl fermions at the other Weyl point, when currents are driven to the same direction as the momentum to connect these two Weyl points.[11] Even if short-range scatterers are taken into account, scattering between these two Weyl points becomes suppressed due to the finite distance in the momentum space. As a result, the longitudinal ($E \parallel B$) magnetoconductivity is enhanced, which turns out to be proportional to the square of the applied magnetic field or the distance of two Weyl points.[11,12]

In this paper we investigate effects of interactions on the "longitudinal" "magneto"-transport in Weyl metals. Here, " " will be clarified later. It is almost trivial to observe that local four-fermion interactions are irrelevant at low energies in a perturbative sense since the density of states vanishes at zero energy. Long-range Coulomb interactions have been investigated both extensively and intensively for transport phenomena in graphene.[13] In addition, transverse gauge interactions have also been discussed in Weyl- or Dirac-type systems.[14] Recently, the chiral anomaly has been calculated in the Weyl system.[15] However, the interplay between long-range interactions and the chiral anomaly has not been investigated clearly. In particular, it remains mysterious how this combination gives rise to anomalous "longitudinal" "magneto"-transport phenomena.

Performing the renormalization group analysis, we reveal that the distance between two Weyl points becomes enhanced logarithmically at low temperatures although the coupling constant for transverse long-range interactions vanishes inverse-logarithmically (expected in three dimensions). This is in contrast with "conventional" Weyl metals without interactions,[10–12] where the distance between two corresponding Weyl points remains finite. As a result, scattering between two Weyl points becomes suppressed much more than the case of noninteracting Weyl metals, which increases electrical conductivity in the direction to connect the momentum vector between two Weyl points. We predict that anisotropic metallicity arises where the electrical resistivity becomes much reduced for the longitudinal direction while normal metallic behaviors result for other directions. Furthermore, we propose that the anomalous contribution in the Hall effect becomes enhanced as a function of temperature, originating from the fact that the anomalous Hall coefficient turns out to be proportional to the distance between two paired Weyl points.[16] We discuss this interaction-enhanced anisotropy in the longitudinal resistivity and the increase of the anomalous contribution in the Hall effect based on the quantum Boltzmann equation approach in the presence of both long-range transverse interactions and the chiral anomaly.

II. INTERPLAY BETWEEN LONG-RANGE TRANSVERSE INTERACTIONS AND THE CHIRAL ANOMALY

We start from quantum electrodynamics with a topological θ term in three spatial dimensions ($\theta-\mathrm{QED}_4$)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi + \frac{e^2\theta}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{1}$$

where ψ is a four-component Dirac spinor to take both chirality (associated with either orbital or sublattice indices) and spin quantum numbers, and A_μ is an electromagnetic vector potential regarded as a quantum field. $D_\mu = \partial_\mu + ieA_\mu$ in $\not{D} = \gamma^\mu D_\mu$ is a covariant derivative with an electric charge e, where γ^μ is the Dirac gamma matrix satisfying the Clifford algebra with $\mu = 0,1,2,3$. $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is a magnetic dual tensor of the electromagnetic field strength tensor. One may consider that this field theory results from certain lattice models with spin-orbit interactions for topological insulators except for long-range transverse interactions.[17] However, we would like to emphasize that even gauge fluctuations can emerge in some lattice models, supporting so-called topological spin liquids.[18] The θ term is the fingerprint of the topological insulator in three dimensions, the source of the (longitudinal) magnetoelectric effect or equivalently, the half-quantized Hall conductance on its surface.[8]

The Adler-Bell-Jackiw anomaly states that the classically conserved chiral current is not conserved at quantum levels,[1] given by

$$\partial_\mu J^{5\mu} = -\frac{e^2}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{2}$$

with the chiral current $J^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi = J^R_\mu - J^L_\mu$ mentioned before, where the γ_5 matrix is a Dirac matrix that anticommutes with other Dirac matrices. In other words, when electric fields are applied in parallel with magnetic fields, the chiral current is not conserved. It is important to realize that the θ term is a boundary term, implying that the coefficient θ cannot be renormalized by interactions. However, if inhomogeneous magnetic fields can be applied to this topological insulating state, the θ coefficient depends on position.[17,19] As a result, this term is not a boundary term anymore, which can be renormalized by interactions.

Resorting to this anomaly equation, we rewrite the effective field theory as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - e\not{A} + \not{c}\,\gamma_5)\psi, \tag{3}$$

where c_μ with the γ_5 Dirac matrix is a chiral gauge field, given by $\partial_\mu\theta$. See Appendix A1 for the derivation from Eq. (3) to Eq. (1) with Eq. (2). It is interesting to observe that the Dirac point splits into two Weyl points at $\boldsymbol{K} = \pm\boldsymbol{c}$. In this respect our problem is to investigate the nature of the quantum critical point between a topological insulator and a band insulator in the presence of inhomogeneous magnetic fields. In other words, we study how both the interaction parameter e and the distance between two Weyl points c_μ are renormalized to evolve at low temperatures, and reveal how these renormalization effects modify transport properties, compared with the case in the absence of interactions. Such inhomogeneous magnetic fields may be created by either ferrimagnetism[17] or some ferromagnetic clusters, given by randomly distributed magnetic ions.[20]

Introducing counter terms, we rewrite this effective field theory as

$$\mathcal{L} = -\frac{Z_A}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(Z_\psi i\not{\partial} + Z_c\not{c}\gamma_5 - Z_e e\not{A})\psi, \tag{4}$$

where Z_ψ, Z_A, Z_c, and Z_e are field renormalization constants of ψ, A_μ, c_μ, and the vertex or coupling (e) renormalization constant, respectively. See Appendix A1 for details. We emphasize that the renormalization factor Z_c has never been introduced as far as we know, thus regarded as an essential aspect of this study. We alert that c_μ is a background gauge field, not dynamical in the present study.

In order to perform the renormalization group analysis, we resort to the dimensional regularization. A subtle point arises due to the presence of the γ_5 matrix, which needs some care for its treatment, because its existence depends on dimensionality.[1] We point out that the presence of the γ_5 matrix makes our calculations much more complicated and laborious. One nontrivial check for the validity of our calculations is that the Ward identity is respected in the one-loop level. All details are presented in Appendix A2. As a result, we obtain our coupled renormalization group equations for e and c_μ,

$$\beta_e(\mu) = \mu\frac{de}{d\mu} = \frac{e^3}{12\pi^2}, \tag{5}$$

$$\beta_c(\mu) = \mu\frac{dc_\nu}{d\mu} = -\frac{e^2}{4\pi^2}c_\nu. \tag{6}$$

See Appendix A3 for the derivation of Eqs. (5) and (6).

The first equation is nothing but the conventional renormalization group equation of the coupling constant, which tells us that the electric charge renormalizes to vanish at zero temperature. Even if the coupling constant vanishes, the background chiral gauge field flows to go to infinity. Inserting the solution of the first equation into the second equation, we find

$$e^2(T) = \frac{e_D^2}{1 + \frac{e_D^2}{4\pi^2}\ln\left(\frac{D}{T}\right)}, \tag{7}$$

$$c_\nu(T) = c_\nu^D\left|1 + \frac{e_D^2}{4\pi^2}\ln\frac{D}{T}\right|, \tag{8}$$

where c_μ^D and e_D are the chiral gauge field and coupling constant at the energy scale of the bandwidth or cutoff. See

Appendix A4. The chiral gauge field increases in a logarithmic way. This indicates that the distance between two Weyl points becomes "infinite" at zero temperature, implying that their scattering events are suppressed "completely." This renormalization effect should be observed in transport coefficients.

One may criticize that the renormalization group equation for the chiral gauge field should not be trusted at low temperatures because the chiral gauge field increases beyond its "critical" value to define the criterion of the renormalization group analysis. However, this statement is not correct because the chiral gauge field has been taken into account nonperturbatively. Here, "nonperturbatively" means that the chiral gauge field was introduced into the electron Green's function from the start. In other words, we used the Green's function of a Weyl electron instead of that of a Dirac fermion. In this respect the renormalization group equation for the chiral gauge field is valid even if the value of the chiral gauge field becomes large. More accurately, the validity of the renormalization group equation for the chiral gauge field is preserved as long as the one-loop renormalization group analysis for the gauge coupling e^2 is justified, actually protected in the large-N limit, where N is the number of flavors or spin degeneracy of fermions.

Although we did not check out what happens if we start from the Green's function of a Dirac fermion and take into account the chiral gauge field perturbatively, we doubt the resulting renormalization group equation for the chiral gauge field in this treatment. We note that the symmetry of the Weyl metallic state differs from that of the Dirac metal phase, which implies that these two phases will not be connected adiabatically.

III. ANISOTROPY IN THE LONGITUDINAL ELECTRICAL TRANSPORT AND ENHANCEMENT OF THE ANOMALOUS CONTRIBUTION IN THE HALL EFFECT

Our framework for anomalous transport is the "semiclassical" quantum Boltzmann equation approach. Here, the term "semiclassical" means that the role of both Berry curvature and chiral anomaly or the topological θ term is introduced from coupled semiclassical equations of motion based on the wave-packet picture in solids.[7] Benchmarking a recent transport study based on the classical Boltzmann equation,[12] we incorporate this information into the quantum Boltzmann equation, which has been applied to transport dynamics in strongly correlated electrons.[21] As a result, inelastic scattering events can be taken into account naturally in the presence of the topological θ term. We consider the case of a finite chemical potential, more generic than the case with two Weyl points. In principle, one can derive the quantum Boltzmann equation in a matrix form, regarded as a full quantum transport theory.[22,23] However, its derivation is much more complicated and not easy to perform. We would like to emphasize that our phenomenological "quantum" transport theory with the introduction of the topological θ term recovers the known result for the longitudinal transport coefficient in Weyl metals, the so-called "negative magnetoresistance" proposed in Ref. 11.

We start from the quantum Boltzmann equation for a steady state[24]

$$\begin{aligned}\dot{\boldsymbol{p}}\cdot\frac{\partial G^<(\boldsymbol{p},\omega)}{\partial \boldsymbol{p}} &+ \dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{p}}\frac{\partial G^<(\boldsymbol{p},\omega)}{\partial\omega} \\ &- \dot{\boldsymbol{p}}\cdot\left\{\frac{\partial\Sigma^<(\boldsymbol{p},\omega)}{\partial\omega}\frac{\partial\Re G_{ret}(\boldsymbol{p},\omega)}{\partial\boldsymbol{p}}\right. \\ &\left.-\frac{\partial\Re G_{ret}(\boldsymbol{p},\omega)}{\partial\omega}\frac{\partial\Sigma^<(\boldsymbol{p},\omega)}{\partial\boldsymbol{p}}\right\} \\ &= -2\Gamma(\boldsymbol{p},\omega)G^<(\boldsymbol{p},\omega) + \Sigma^<(\boldsymbol{p},\omega)A(\boldsymbol{p},\omega). \qquad (9)\end{aligned}$$

$G^<(\boldsymbol{p},\omega)$ is the lesser Green's function, regarded as a quantum distribution function, where $\boldsymbol{p}$ and ω represent momentum and frequency for relative coordinates, respectively. $\dot{\mathcal{O}}$ denotes the derivative with respect to time t. $\Sigma^<(\boldsymbol{p},\omega)$ and $G_{ret}(\boldsymbol{p},\omega)$ indicate the lesser self-energy and the retarded Green's function, respectively, where $\Re$ is their real part. The right-hand side introduces collision terms, where $\Gamma(\boldsymbol{p},\omega)$ and $A(\boldsymbol{p},\omega)$ indicate the scattering rate and the spectral function.

$\boldsymbol{r}$ and $\boldsymbol{p}$ are governed by semiclassical equations of motion,[7] given by

$$\dot{\boldsymbol{r}} = \frac{\partial\epsilon_{\boldsymbol{p}}}{\partial\boldsymbol{p}} + \dot{\boldsymbol{p}}\times\boldsymbol{\Omega}_{\boldsymbol{p}}, \quad \dot{\boldsymbol{p}} = e\boldsymbol{E} + \frac{e}{c}\dot{\boldsymbol{r}}\times\boldsymbol{B}, \qquad (10)$$

where $\boldsymbol{\Omega}_{\boldsymbol{p}}$ represents the Berry curvature of the momentum space. Solving these equations, one obtains[12]

$$\begin{aligned}\dot{\boldsymbol{r}} &= \left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}\left\{\boldsymbol{v}_{\boldsymbol{p}} + e\boldsymbol{E}\times\boldsymbol{\Omega}_{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{\Omega}_{\boldsymbol{p}}\cdot\boldsymbol{v}_{\boldsymbol{p}}\boldsymbol{B}\right\}, \\ \dot{\boldsymbol{p}} &= \left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}\left\{e\boldsymbol{E} + \frac{e}{c}\boldsymbol{v}_{\boldsymbol{p}}\times\boldsymbol{B} + \frac{e^2}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_{\boldsymbol{p}}\right\}. \qquad (11)\end{aligned}$$

An essential point is the presence of the $\boldsymbol{E}\cdot\boldsymbol{B}$ term in the second equation, imposing the Adler-Bell-Jackiw anomaly.

Inserting Eq. (11) into Eq. (9) and performing straightforward but rather tedious algebra, we reach the following expression for the longitudinal conductivity

$$\sigma_L(T) \longrightarrow (1+\mathcal{K}[c(T)]^2)\sigma_n(T), \qquad (12)$$

where $\mathcal{K}$ is a positive numerical constant and the normal conductivity $\sigma_n(T)$ is determined by (gauge-interaction induced) intrascattering events at one Weyl point. All details are shown in Appendix B. An essential point is in the $c(T)$ term, where $c(T)$ is the distance between two paired Weyl points, given by our renormalization group analysis. Here, the vector index μ is fixed and omitted for simplicity. Actually, this expression is to replace the applied magnetic field in noninteracting Weyl metals with the distance between Weyl points in interacting Weyl metals, when electric fields are applied in parallel with "magnetic fields" or the momentum vector to connect such Weyl points,[25] implying the reason why we call "longitudinal" in front of the conductivity. We emphasize that this expression recovers that of the original proposal,[11] where the magnetoconductivity is proportional to the square of the distance between two Weyl points. Although the distance does not renormalize in the noninteracting case, the presence of transverse long-range interactions gives rise to the logarithmic enhancement at zero chemical potential.

One aspect should be pointed out carefully. We proved that the distance between two paired Weyl points increases logarithmically at low temperatures, originating from long-range transverse interactions. It should be noticed that this result appears at zero chemical potential. On the other hand, we found the longitudinal conductivity [Eq. (9)] for a finite chemical potential. Can we expect the similar enhancement of the distance between two paired Weyl points in the case of a finite chemical potential? The renormalization group analysis for the Weyl metallic state with a finite chemical potential turns out to be more complex and technically involved, originating from the treatment of four by four matrices and angular integrals along Fermi surfaces. Besides such technical difficulties, we reach the conclusion that an exotic phenomenon may appear. It is natural to expect that the Weyl metallic state with two corresponding Fermi surfaces will not be stable at low temperatures when there exist interactions. This instability originates from their perfect nesting. As a result, some types of charge or spin density waves are expected to arise at low temperatures. However, the main result of the previous section in the case of zero chemical potential is that the distance between two paired Weyl points increases to diverge, implying that scattering between these two Weyl points is suppressed. Then, we expect that the competition between two kinds of divergences, one of which comes from the perfect nesting while the other originates from the chiral anomaly with long-range transverse interactions, may allow an exotic balance, which gives rise to an interacting fixed point, identified with a novel non-Fermi liquid metallic state. In this respect we believe that the study in the case of a finite chemical potential should be performed more carefully and would like to leave it as a future work.

Although we cannot determine the temperature dependence of the distance between two Weyl points in the presence of Fermi surfaces, the presence of the prefactor $(1 + \mathcal{K}[c(T)]^2)$ in the longitudinal resistivity guarantees an anisotropic metallic behavior because such a prefactor does not exist in the transverse resistivity. Unfortunately, we cannot quantify the degree of anisotropy at present.

We propose another fingerprint of the interacting or critical Weyl metallic state with long-range transverse interactions, that is, peculiar temperature dependencies for the anomalous contribution of the Hall coefficient, depending on the chemical potential. Recently, the anomalous contribution for the Hall effect has been evaluated,[15,16] given by

$$\sigma_{\mu\nu} = \frac{e^2}{2\pi^2}\epsilon_{\mu\nu\gamma}\boldsymbol{c}_\gamma \tag{13}$$

in the absence of Fermi surfaces, where $\boldsymbol{c}_\gamma$ is the distance between two paired Weyl points. Based on our renormalization group analysis [Eq. (8)], we find

$$\sigma_{xy}(T) = \frac{e^2}{2\pi^2}c_z(D)\left|1 + \frac{e_D^2}{4\pi^2}\ln\frac{D}{T}\right|, \tag{14}$$

where $c_z(D)$ is the distance of two paired Weyl points at $T = D$. This expression is rather unexpected because the anomalous Hall coefficient diverges, dominating over the normal Hall effect. Here, the term "divergence" should be regarded more carefully. Since our long-wavelength effective description is valid only below the momentum cutoff, at most within the first Brillouin zone, the term "divergence" is more accurate to be replaced with enhancement at low temperatures.

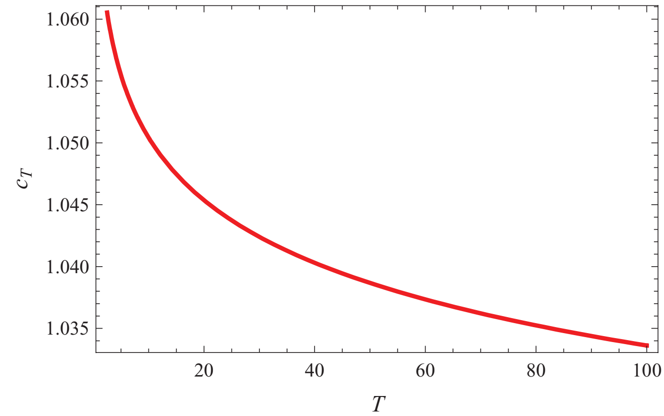

FIG. 1. (Color online) Temperature dependence of the anomalous contribution for the Hall effect in an interacting Weyl metal with zero chemical potential. We plot a dimensionless anomalous Hall coefficient $\sigma_{xy}(T)/\sigma_{xy}(D) = |1 + \frac{e_D^2}{4\pi^2}\ln\frac{D}{T}|$ as a function of temperature T of a unit of K, using $D = 10^4$ K and $\frac{e_D^2}{4\pi^2} = 1/137$. An essential feature is the logarithmic enhancement of the anomalous Hall coefficient as a result of the interplay between transverse long-range interactions and the chiral anomaly.

We show $\sigma_{xy}(T)/\sigma_{xy}(D)$ with $\sigma_{xy}(D) = \frac{e^2}{2\pi^2}c_z(D)$ in Fig. 1 for clarity of physics. Interestingly, it has been also shown that Eq. (13) is not modified even in the presence of a finite chemical potential, based on the Kubo formula.[15] It will be quite interesting to reveal the temperature dependence for the anomalous Hall coefficient in the presence of Fermi surfaces near two paired Weyl points.

IV. CONCLUSION AND PERSPECTIVES

In summary, we investigated the quantum critical point of the topological phase transition from a topological insulator to a band insulator in the presence of inhomogeneous ferromagnetism or under nonuniform magnetic fields, where the topological θ term gives rise to a topological constraint in dynamics of bulk fermions, referred to as the Adler-Bell-Jackiw anomaly. Such inhomogeneous magnetic fields serve background chiral gauge fields, splitting the Dirac point into two Weyl points. Introducing long-range transverse interactions and performing the renormalization group analysis, we uncovered that the distance between these two Weyl points becomes enhanced logarithmically at low temperatures although the coupling constant vanishes as expected. Resorting to the semiclassical quantum Boltzmann equation approach, we claimed that the enhancement of the distance strengthens metallic properties at low temperatures when electric fields are applied to the same direction as the momentum to connect these Weyl points because scattering between the Weyl points are suppressed due to their huge distance in the momentum space. Besides this emergent enhanced anisotropy in electrical resistivity, we predicted the logarithmically "divergent" temperature dependence for the anomalous contribution of the Hall effect. These two anomalous transport properties are proposed to be fingerprints of Weyl metals with transverse long-range interactions.

There remain three important problems in our direction. The first question is to perform the renormalization group analysis in the case of a finite chemical potential, as discussed before. Since the competition between the enhancement of the distance between two paired Weyl points and the presence of perfect nesting between two paired Fermi surfaces is expected to cause a delicate balance, we are expecting an interacting fixed point, which can be identified with a novel non-Fermi liquid metal. The second question is what happens if we take into account chiral gauge fields quantum mechanically. This situation arises when ferromagnetic phase transitions occur near the topological phase transition.[17] Is it possible to obtain a novel interacting fixed point, too? The third question is more practical thus experimentally verified. If we introduce weak antilocalization corrections in the transport theory, how is the longitudinal transport coefficient modified? This question is still meaningful even without interactions because this transport signature can be measured actually.[26]

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APPENDIX A: RENORMALIZATION GROUP ANALYSIS IN THE PRESENCE OF CHIRAL ANOMALY

1. Introduction of counter terms

We start from the following Lagrangian

$$\mathcal{L}_B = i\bar{\psi}_B\gamma^\mu\partial_\mu\psi_B - e_B A_{B\mu}\bar{\psi}_B\gamma^\mu\psi_B - \frac{1}{4}F_{B\mu\nu}F_B^{\mu\nu} + \frac{e_B^2\theta_B}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{B\mu\nu}F_{B\rho\sigma}, \quad \text{(A1)}$$

where ψ_B, $A_{B\mu}$, e_B, θ_B are bare quantities to be renormalized. Introducing renormalization factors of $\psi_B = Z_\psi^{1/2}\psi$, $A_{B\mu} = Z_A^{1/2}A_\mu$, $e_B Z_A^{1/2} Z_\psi = Z_e e$, $\theta_B = Z_c\theta$, one can rewrite the above bare Lagrangian in terms of its renormalized part and counter-term part:

$$\begin{aligned}\mathcal{L}_B &= \mathcal{L}_r + \mathcal{L}_{c.t.}\\ \mathcal{L}_r &= i\bar{\psi}\gamma^\mu\partial_\mu\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2\theta}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\\ \mathcal{L}_{c.t.} &= \delta_\psi i\bar{\psi}\gamma^\mu\partial_\mu\psi - \delta_e eA_\mu\bar{\psi}\gamma^\mu\psi - \frac{\delta_A}{4}F_{\mu\nu}F^{\mu\nu} + \delta_c\frac{e^2\theta}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},\end{aligned} \quad \text{(A2)}$$

where $\delta_\psi = Z_\psi - 1$, $\delta_e = Z_e - 1$, $\delta_A = Z_A - 1, \delta_c = Z_c - 1$.

Incorporating the anomaly equation [Eq. (2)] for renormalized fields into the above expression, we obtain Eq. (4) for our renormalization group analysis. This procedure can be performed in a more formal way. This Lagrangian functional is invariant under the chiral transformation of $\psi \to e^{i\alpha(x)\gamma_5}\psi$ as long as fermions remain massless. Consider the following replacement:

$$\begin{aligned}Z &= \int DAD\psi D\bar{\psi}\,\mathbf{exp}\left\{i\int d^4x\mathcal{L}_r + \mathcal{L}_{c.t.}\right\}\\ &\to Z = \int DAD\psi D\bar{\psi}\,\mathbf{exp}\left\{i\int d^4x\mathcal{L}_r + \mathcal{L}_{c.t.}\right.\\ &\quad \left. + \alpha(x)\left(\partial_\mu J^{\gamma_5\mu} + \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)\right\},\end{aligned} \quad \text{(A3)}$$

where gauge fixing is assumed. $\alpha(x)$ is an arbitrary and infinitesimal local parameter. Taking $\alpha(x) = -\theta(x)$, we see that the $\theta F\tilde{F}$ term is replaced with the chiral gauge-field term in Eq. (4). As a result, we obtain the following expression:

$$\begin{aligned}Z &= \int DAD\psi D\bar{\psi}\,\mathbf{exp}\left\{i\int d^4x\mathcal{L}_1 + \mathcal{L}_{1.c.t.}\right\}\\ \mathcal{L}_1 &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\partial_\mu - ic_\mu\gamma_5)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi\\ \mathcal{L}_{1.c.t.} &= -\frac{\delta_A}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu(\delta_\psi\partial_\mu - i\delta_c c_\mu\gamma_5)\psi\\ &\quad - \delta_e eA_\mu\bar{\psi}\gamma^\mu\psi,\end{aligned} \quad \text{(A4)}$$

where gauge fixing is also assumed. It is important to notice that the chiral anomaly equation is satisfied for renormalized fields, not bare fields.

2. One-loop structure of QED_4 with a background chiral gauge field

We perform one-loop renormalization group analysis in the presence of the background chiral gauge field, i.e., c_μ = constant. We obtain renormalization constants from $\Sigma_1(\not{p},c)$, $\Gamma_1^\mu(p,p',c)$, $\Pi_1^{\mu\nu}(q,c)$, corresponding to one-loop fermion self-energy, one-loop vertex correction, and one-loop gauge-boson self-energy, respectively.

It is important to notice that the background chiral gauge field is taken into account nonperturbatively. In other words, our vacuum state is a state that chiral currents are flowing. Thus, the fermion propagator is modified to be

$$\frac{1}{\not{p}} \to \frac{1}{\not{p} + \not{c}\gamma_5} = \frac{(p^2+c^2)\not{p} + 2(p\cdot c)\not{p}\gamma_5 - (p^2+c^2)\not{c}\gamma_5 - 2(p\cdot c)\not{c}}{(p-c)^2(p+c)^2}. \quad \text{(A5)}$$

In this respect the key point is how self-energies of fermions and gauge bosons and vertex corrections are modified in this novel vacuum state. A subtle point arises due to the presence of the γ_5 matrix in the regularization procedure.[1] When dimensional regularization is used to regularize loop integrals including γ_5, some anomalous terms appear. They originate from components perpendicular to physical four dimensions. We separate out these perpendicular momentum components of $l_\perp^\mu = l^\mu - l_\parallel^\mu$ explicitly in our dimensional regularization. However, it turns out that they do not result in divergent contributions.

First, we calculate the fermion self-energy

$$-i\Sigma_1(\not p,c) = (-ie)^2 \int \frac{d^dk}{(2\pi)^d}\gamma^\mu i\frac{(k^2+c^2)\not k - 2(k\cdot c)\not c + 2(k\cdot c)\not k\gamma_5 - (k^2+c^2)\not c\gamma_5}{(k+c)^2(k-c)^2}\gamma^\nu \frac{-i\eta_{\mu\nu}}{(p-k)^2}$$
$$= -e^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{(p-k)^2(k-c)^2(k+c)^2}$$
$$\times\{(k^2+c^2)\gamma^\mu\not k\gamma_\mu - 2(k\cdot c)\gamma^\mu\not c\gamma_\mu + 2(k\cdot c)\gamma^\mu\not k\gamma_5\gamma_\mu - (k^2+c^2)\gamma^\mu\not c\gamma_5\gamma_\mu\}. \quad \text{(A6)}$$

Replacing the denominator with Feynman parameters, we rewrite the above expression as

$$-i\Sigma_1(\not p,c) = -2e^2\int_0^1 dxdydz\,\delta(x+y+z-1)\int\frac{d^dl}{(2\pi)^d}\frac{1}{[l^2-\Delta]^3}\{-2((l+a)^2+c^2)(\not l+\not a) + 4(l\cdot c + a\cdot c)\not c$$
$$+4(l\cdot c+a\cdot c)(\not l+\not a)\gamma_5 - 2((l+a)^2+c^2)\not c\gamma_5\}, \quad \text{(A7)}$$

where $l_\mu = k_\mu - a_\mu$, $a_\mu = xp_\mu + (y-z)c_\mu$, $\Delta = (x^2-x)p^2 + (y^2-y+z^2-z-2yz)c^2 + (2xy-2xz)(p\cdot c)$. Note that the degree of divergence in each term depends on only the power of redefined loop momenta l. Since we need to calculate only divergent terms, we consider

$$-i\Sigma_1(\not p,c) = -2e^2\int_0^1 dxdydz\,\delta(x+y+z-1)\int\frac{d^dl}{(2\pi)^d}\frac{1}{[l^2-\Delta]^3}\{-2l^2\not a - 4l^\lambda l^\kappa a_\lambda\gamma_\kappa + 4l^\lambda l^\kappa c_\lambda\gamma_\kappa\gamma_5 - 2l^2\not c\gamma_5\} + \textit{finite}.$$
$$= -2e^2\int_0^1 dxdydz\,\delta(x+y+z-1)\left\{-2(\not a+\not c\gamma_5)\left(\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{[l^2-\Delta]^3}\right)\right.$$
$$\left.-4(a_\lambda\gamma_\kappa - c_\lambda\gamma_\kappa\gamma_5)\left(\int\frac{d^dl}{(2\pi)^d}\frac{l^\lambda l^\kappa}{[l^2-\Delta]^3}\right)\right\} + \textit{finite}. \quad \text{(A8)}$$

Integrating over the loop momenta l, we obtain

$$-i\Sigma_1(\not p,c) = \frac{e^2 i}{\pi^2\epsilon}\int_0^1 dxdydz\,\delta(x+y+z-1)\left\{\frac{3}{4}\not a + \frac{1}{4}\not c\gamma_5\right\} + \textit{finite} = \frac{e^2 i}{8\pi^2\epsilon}(\not p + \not c\gamma_5) + \textit{finite}, \quad \text{(A9)}$$

with $\epsilon = 4-d$.

Second, we calculate the vertex correction in the same way as above:

$$\Gamma_1^\mu(p,p',c) = (-ie)^2\int\frac{d^dk}{(2\pi)^d}\left\{\gamma^\nu i\frac{(k'^2+c^2)\ \not k' - 2(k'\cdot c)\not c + 2(k'\cdot c)\ \not k'\gamma_5 - (k'^2+c^2)\not c\gamma_5}{(k'-c)^2(k'+c)^2}\right.$$
$$\left.\times\gamma^\mu i\frac{(k^2+c^2)\ \not k - 2(k\cdot c)\not c + 2(k\cdot c)\ \not k\gamma_5 - (k^2+c^2)\not c\gamma_5}{(k-c)^2(k+c)^2}\gamma^\rho\right\}\frac{-i\eta_{\nu\rho}}{(k-p)^2}$$
$$= -ie^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k-p)^2(k'-c)^2(k'+c)^2(k-c)^2(k+c)^2}\gamma^\nu\{(k'^2+c^2)\ \not k' - 2(k'\cdot c)\not c + 2(k'\cdot c)\ \not k'\gamma_5$$
$$-(k'^2+c^2)\not c\gamma_5\}\gamma^\mu\{(k^2+c^2)\not k - 2(k\cdot c)\not c + 2(k\cdot c)\not k\gamma_5 - (k^2+c^2)\not c\gamma_5\}\gamma_\nu, \quad \text{(A10)}$$

where $p^2 = p'^2 = m_f^2 = 0$, $k_\mu + q_\mu = k'_\mu$, $p_\mu + q_\mu = p'_\mu$. Similarly, replacing the denominator with Feynman parameters and taking only divergent terms, we obtain

$$\Gamma_1^\mu(p,p',c) = -24ie^2\int_0^1 dxdydzdudv\,\delta(x+y+z+u+v-1)\int\frac{d^dk}{(2\pi)^d}\frac{1}{[l^2-\Delta]^5}\gamma^\nu\{((l+a+q)^2+c^2)(\not l+\not a+\not q)$$
$$-2(l\cdot c+a\cdot c+q\cdot c)\not c + 2(l\cdot c+a\cdot c+q\cdot c)(\not l+\not a+\not q)\gamma_5 - ((l+a+q)^2+c^2)\not c\gamma_5\}$$
$$\times\gamma^\mu\{((l+a)^2+c^2)(\not l+\not a) - 2(l\cdot c+a\cdot c)\not c + 2(l\cdot c+a\cdot c)(\not l+\not a)\gamma_5 - ((l+a)^2+c^2)\not c\gamma_5\}\gamma_\nu$$
$$= 48ie^2\int_0^1 dxdydzdudv\,\delta(x+y+z+u+v-1)\left\{\int\frac{d^dk}{(2\pi)^d}\frac{(l^2)^2 l_\lambda l_\rho}{[l^2-\Delta]^5}\right\}\{\gamma^\lambda\gamma^\mu\gamma^\rho\} + \textit{finite}.$$
$$= \frac{e^2}{8\pi^2\epsilon}\gamma^\mu + \textit{finite}, \quad \text{(A11)}$$

where $l_\mu = k_\mu - a_\mu$, $a_\mu = xp_\mu - (y+z)q_\mu + (y-z+u-v)c_\mu$.

Third, we evaluate the vacuum polarization tensor. Although the procedure is essentially the same as before, this calculation is much more complicated, given by

$$\begin{aligned} i\Pi_1^{\mu\nu}(q,c) &= -(-ie)^2\int\frac{d^dk}{(2\pi)^d}Tr\bigg[\gamma^\mu i\frac{(k^2+c^2)\not{k}-2(k\cdot c)\not{c}+2(k\cdot c)\not{k}\gamma_5-(k^2+c^2)\not{c}\gamma_5}{(k-c)^2(k+c)^2} \\ &\quad\times\gamma^\nu i\frac{((k+q)^2+c^2)(\not{k}+\not{q})-2(k\cdot c+q\cdot c)\not{c}+2(k\cdot c+q\cdot c)(\not{k}+\not{q})\gamma_5-((k+q)^2+c^2)\not{c}\gamma_5}{(k+q-c)^2(k+q+c)^2}\bigg] \\ &= -e^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k-c)^2(k+c)^2(k+q-c)^2(k+q+c)^2}Tr[\gamma^\mu\{(k^2+c^2)\not{k}-2(k\cdot c)\not{c}+2(k\cdot c)\not{k}\gamma_5 \\ &\quad -(k^2+c^2)\not{c}\gamma_5\}\gamma^\nu\{((k+q)^2+c^2)(\not{k}+\not{q})-2(k\cdot c+q\cdot c)\not{c}+2(k\cdot c+q\cdot c)(\not{k}+\not{q})\gamma_5-((k+q)^2+c^2)\not{c}\gamma_5\}]. \end{aligned} \tag{A12}$$

Notice that the extra (-1) factor comes from the fermion loop in the diagram. Now, we have to change the loop momenta as before. Straightforward but rather tedious algebras give us the following expression:

$$\begin{aligned} i\Pi_1^{\mu\nu}(q,c) &= -6e^2\int_0^1 dxdydz\,\delta(x+y+z-1)\int\frac{d^dl}{(2\pi)^d}\frac{1}{[l^2-\Delta]^4}[(l^2)^3\{-4\eta^{\mu\nu}\}+8(l^2)^2l^\mu l^\nu \\ &\quad +(l^2)^2\{4a^\mu b^\nu+4a^\nu b^\mu-4\eta^{\mu\nu}(a\cdot b)-4i\epsilon^{\mu\nu\rho\sigma}c_\rho q_\sigma+8c^\mu c^\nu-12\eta^{\mu\nu}c^2-4\eta^{\mu\nu}(a^2+b^2)\} \\ &\quad +l^2l^\mu l^\lambda\{8h_\lambda h^\nu-32c_\lambda c^\nu\}+l^2l^\nu l^\lambda\{8h_\lambda h^\mu-32c_\lambda c^\mu\} \\ &\quad +l^2l^\lambda l^\kappa\{-8h_\lambda h_\kappa\eta^{\mu\nu}-16a_\lambda b_\kappa\eta^{\mu\nu}+16c_\lambda c_\kappa\eta^{\mu\nu}+16ic_\lambda b_\sigma\eta_{\kappa\rho}\epsilon^{\mu\nu\rho\sigma}+16ic_\lambda a_\rho\eta_{\kappa\sigma}\epsilon^{\mu\nu\rho\sigma}\} \\ &\quad +l^2l^\mu l^\nu\{8a^2+8b^2+16c^2\}+l^\mu l^\nu l^\lambda l^\kappa\{32a_\lambda b_\kappa+32c_\kappa c_\lambda\}]+\textit{finite}, \end{aligned} \tag{A13}$$

where

$$\begin{aligned} l_\mu &= k_\mu-a_\mu,\quad a_\mu=Ac_\mu+Bq_\mu=(x-y+z-u)c_\mu+(u+z)q_\mu, \\ b_\mu &= a_\mu+q_\mu=Ac_\mu+(B+1)q_\mu,\quad h_\mu=a_\mu+b_\mu=2Ac_\mu+(2B+1)q_\mu, \\ \Delta &= l^2-[x(k-c)^2+y(k+c)^2+z(k+q-c)^2+u(k+q+c)^2] \\ &= 2(c\cdot q)\{u^2-u-xu+yu+z-xz+zy-z^2\}+q^2\{u^2-u+z^2-z+2uz\} \\ &\quad +c^2\{x^2+y^2+z^2+u^2-x-y-z-u-2xy+2xz-2yz-2xu+2yu-2zu\}. \end{aligned} \tag{A14}$$

Integrating over the loop momenta l, we get

$$\begin{aligned} i\Pi_1^{\mu\nu}(q,c) &= -\frac{6e^2i}{\pi^2\epsilon}\int_0^1 dxdydz\,\delta(x+y+z-1)\bigg\{-\eta^{\mu\nu}\Delta+c^\mu c^\nu\left(\frac{10}{3}A^2-\frac{2}{3}\right)+\eta^{\mu\nu}c^2\left(-\frac{7}{3}A^2-\frac{1}{3}\right) \\ &\quad +q^\mu q^\nu\left(\frac{10}{3}B^2+\frac{10}{3}B+\frac{1}{2}\right)+\eta^{\mu\nu}q^2\left(-\frac{7}{3}B^2-\frac{7}{3}B-\frac{1}{2}\right)\bigg\}+\textit{finite} \\ &= -\frac{e^2i}{6\pi^2\epsilon}(\eta^{\mu\nu}q^2-q^\mu q^\nu)+\textit{finite}. \end{aligned} \tag{A15}$$

It is interesting to observe that the divergent part is not modified by the background chiral gauge field although it makes the expression much more complicated in the intermediate stage. Contributions from the background chiral gauge field turn out to be canceled exactly in the polarization contribution. It is quite laborious to check this cancellation.

We summarize leading divergent contributions of fermion one-loop self-energy, one-loop gauge-fermion vertex, and gauge-boson one-loop self-energy as follows:

$$-i\Sigma_1(\not{p},c)=\frac{e^2i}{8\pi^2\epsilon}\not{p}+\frac{e^2i}{8\pi^2\epsilon}\not{c}\gamma_5+\textit{finite}, \tag{A16}$$

$$\Gamma_1^\mu(p,p',c)=\frac{e^2}{8\pi^2\epsilon}\gamma^\mu+\textit{finite}, \tag{A17}$$

$$i\Pi_1^{\mu\nu}(q,c)=-\frac{e^2i}{6\pi^2\epsilon}(\eta^{\mu\nu}q^2-q^\mu q^\nu)+\textit{finite}. \tag{A18}$$

As a result, we obtain

$$\delta_\psi=-\frac{e^2}{8\pi^2\epsilon}+\textit{finite}, \tag{A19}$$

$$\delta_c=-\frac{e^2}{8\pi^2\epsilon}+\textit{finite}, \tag{A20}$$

$$\delta_e=-\frac{e^2}{8\pi^2\epsilon}+\textit{finite}, \tag{A21}$$

$$\delta_A=-\frac{e^2}{6\pi^2\epsilon}+\textit{finite}. \tag{A22}$$

3. Renormalization group equations

We would like to point out that the Ward identity of $Z_\psi=Z_e$ is satisfied, guaranteeing the gauge invariance. We emphasize that this serves quite a nontrivial check for our renormalization group analysis, where complex dependencies for chiral gauge fields are all canceled to give rise to the Ward identity. Recalling our perturbative analysis in the one-loop level, the satisfaction of the Ward identity implies that the renormalization group equation for the coupling constant does not change, compared with the case in the absence of the background chiral gauge field. On the other hand, the renormalization group equation for the chiral gauge field is an essential point of our study.

The beta function for the chiral gauge field is given by

$$\beta_{c_\nu}(\mu) = \mu \frac{dc_\nu}{d\mu}. \tag{A23}$$

Considering that the bare quantity $c_{B\nu} = Z_c c_\nu$ is independent of the scale parameter of μ, we obtain the renormalization group equation for the chiral gauge field

$$0 = \frac{d}{d\ln\mu}\ln c_{B\nu} = \frac{dM_c}{de}\frac{de}{d\ln\mu} + \frac{d}{d\ln\mu}\ln c_\nu, \tag{A24}$$

where

$$M_c = \ln Z_c = \sum_{n=1}^{\infty}\frac{m_n(e,c)}{\epsilon^n} = \frac{-\frac{e^2}{8\pi^2} + O(e^4)}{\epsilon} + O\left(\frac{1}{\epsilon^2}\right). \tag{A25}$$

In the one-loop level we obtain $m_1(e,c) = -\frac{e^2}{8\pi^2}$. Inserting $\frac{de}{d\ln\mu} = \beta_e(\mu) - \epsilon e$ into the above expression, we reach the following formula:

$$0 = \left(\left(-\frac{e}{4\pi^2} + O(e^3)\right)\frac{1}{\epsilon} + O\left(\frac{1}{\epsilon^2}\right)\right)(\beta_e(\mu) - \epsilon e) + \frac{1}{c_\nu}\beta_c(\mu). \tag{A26}$$

Renormalizability guarantees the cancellation in higher negative orders. As a result, we obtain

$$\beta_{c_\nu}(\mu) = \mu\frac{dc_\nu}{d\mu} = -\frac{e^2}{4\pi^2}c_\nu + O(e^4). \tag{A27}$$

4. Low temperature behaviors for background chiral gauge fields

Solving the renormalization group equation for the coupling constant

$$\beta_e(\mu) = \frac{de}{d\ln\mu} = \frac{e^3}{12\pi^2}, \tag{A28}$$

we obtain

$$e^2(\mu) = \frac{e_D^2}{1 - \frac{e_D^2}{4\pi^2}\ln\left(\frac{\mu}{D}\right)}. \tag{A29}$$

Substituting this solution into the renormalization group equation for the chiral gauge field

$$\beta_{c_\nu}(\mu) = \frac{dc_\nu}{d\ln\mu} = -\frac{e^2}{4\pi^2}c_\nu, \tag{A30}$$

we obtain

$$\begin{aligned}\ln\left(\frac{c_\nu(\mu)}{c_\nu(D)}\right) &= -\frac{1}{4\pi^2}\int_D^\mu d(\ln\mu)\frac{e_D^2}{1 - \frac{e_D^2}{4\pi^2}\ln\left(\frac{\mu}{D}\right)} \\ &= -\int_{\ln D}^{\ln\mu} dx \frac{1}{\left[\alpha_D^{-1} + \ln D\right] - x} \\ &= \int_0^{\ln\frac{\mu}{D}} dx\frac{1}{x - \alpha_D^{-1}} = \ln\left|\frac{\ln\frac{\mu}{D} - \alpha_D^{-1}}{\alpha_D^{-1}}\right|,\end{aligned} \tag{A31}$$

where $\alpha_D = \frac{e_D^2}{4\pi^2}$ is the fine structure constant at the cutoff scale. As a result, we find

$$c_\nu(\mu) = c_\nu(D)\,\alpha_D\left|\ln\frac{\mu}{D} - \alpha_D^{-1}\right|. \tag{A32}$$

APPENDIX B: QUANTUM BOLTZMANN EQUATION APPROACH IN THE PRESENCE OF BOTH CHIRAL ANOMALY AND GAUGE INTERACTION

1. A formal development of the quantum Boltzmann equation in the presence of the topological $\boldsymbol{E}\cdot\boldsymbol{B}$ term

Inserting the solutions [Eq. (11)] of semiclassical equations [Eq. (10)] into the quantum Boltzmann equation [Eq. (9)] and performing some algebra, we obtain the following expression:

$$\begin{aligned}&\left(1 + \frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}\frac{e}{c}\boldsymbol{v}_{\boldsymbol{p}}\cdot\left(\boldsymbol{B}\times\frac{\partial G^<}{\partial\boldsymbol{p}}\right) + \left(1 + \frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-2}\left(\frac{e}{c}\boldsymbol{v}_{\boldsymbol{p}}\times\boldsymbol{B}\right)\cdot(e\boldsymbol{E}\times\boldsymbol{\Omega}_{\boldsymbol{p}})\frac{\partial G^<}{\partial\omega} \\ &- [A(\boldsymbol{p},\omega)]^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)\left(1 + \frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-2}\left\{e\boldsymbol{E} + \frac{e^2}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_{\boldsymbol{p}}\right\}\cdot\left\{\boldsymbol{v}_{\boldsymbol{p}} + \frac{e}{c}(\boldsymbol{\Omega}_{\boldsymbol{p}}\cdot\boldsymbol{v}_{\boldsymbol{p}})\boldsymbol{B}\right\}\Gamma = -i[2\Gamma G^< - \Sigma^< A],\end{aligned} \tag{B1}$$

where the argument of $(\boldsymbol{p},\omega)$ is omitted for simplicity.

The lesser self-energy is given by[21]

$$\Sigma^<(\boldsymbol{p},\omega) = \sum_{\boldsymbol{q}}\int_0^\infty d\nu\left|\frac{\boldsymbol{p}\times\hat{\boldsymbol{q}}}{m}\right|^2\Im D_a(\boldsymbol{q},\nu)\{[n(\nu)+1]G^<(\boldsymbol{p}+\boldsymbol{q},\omega+\nu) + n(\nu)G^<(\boldsymbol{p}+\boldsymbol{q},\omega-\nu)\}. \tag{B2}$$

Here, we consider gauge interactions for example. Thus, $D_a(\boldsymbol{q},\nu)$ represents the Green function of gauge fluctuations. $n(\nu)$ is the Bose-Einstein distribution function. One can replace the gauge-boson propagator with some other types of fluctuations such as phonons, spin fluctuations, etc. One may consider the diffusion-mode propagator for weak antilocalization, where the form of its vertex should be changed, of course.

We write down the lesser Green's function in the following way:[21]

$$G^{<}(\boldsymbol{p},\omega) = if(\omega)A(\boldsymbol{p},\omega) + i\left(-\frac{\partial f(\omega)}{\partial\omega}\right)A(\boldsymbol{p},\omega)\boldsymbol{v}_{\boldsymbol{p}}\cdot\Lambda(\boldsymbol{p},\omega), \tag{B3}$$

which consists of the equilibrium part (the first term) and its correction term (the second term). We call $\Lambda(\boldsymbol{p},\omega)$ "vertex distribution function" although it sounds somewhat confusing. $f(\omega)$ is the Fermi-Dirac distribution function.

Inserting this ansatz into the quantum Boltzmann equation with the expression of the lesser self-energy and performing some straightforward algebra, we obtain

$$\begin{aligned}
&i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}\frac{e}{mc}\boldsymbol{v}_{\boldsymbol{p}}\cdot\left(\boldsymbol{B}\times\frac{\partial \boldsymbol{p}_{\alpha}}{\partial\boldsymbol{p}}\right)\Lambda_{\alpha}(\boldsymbol{p},\omega) - i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-2}\left(\frac{e}{c}\boldsymbol{v}_{\boldsymbol{p}}\times\boldsymbol{B}\right)\cdot(e\boldsymbol{E}\times\boldsymbol{\Omega}_{\boldsymbol{p}})\\
&\quad - A(\boldsymbol{p},\omega)\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-2}\left\{e\boldsymbol{E}+\frac{e^2}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_{\boldsymbol{p}}\right\}\cdot\left\{\boldsymbol{v}_{\boldsymbol{p}}+\frac{e}{c}(\boldsymbol{\Omega}_{\boldsymbol{p}}\cdot\boldsymbol{v}_{\boldsymbol{p}})\boldsymbol{B}\right\}\Gamma(\boldsymbol{p},\omega)\\
&= 2\Gamma(\boldsymbol{p},\omega)\boldsymbol{v}_{\boldsymbol{p}}\cdot\Lambda(\boldsymbol{p},\omega) - \sum_{\boldsymbol{q}}\int_0^{\infty}d\nu\left|\frac{\boldsymbol{p}\times\hat{\boldsymbol{q}}}{m}\right|^2\Im D_a(\boldsymbol{q},\nu)\{[n(\nu)+f(\omega+\nu)]A(\boldsymbol{p}+\boldsymbol{q},\omega+\nu)\boldsymbol{v}_{\boldsymbol{p}+\boldsymbol{q}}\cdot\Lambda(\boldsymbol{p}+\boldsymbol{q},\omega+\nu)\\
&\quad - [n(-\nu)+f(\omega-\nu)]A(\boldsymbol{p}+\boldsymbol{q},\omega-\nu)\boldsymbol{v}_{\boldsymbol{p}+\boldsymbol{q}}\cdot\Lambda(\boldsymbol{p}+\boldsymbol{q},\omega-\nu)\},
\end{aligned} \tag{B4}$$

where we have used the following relation:

$$2\Gamma(\boldsymbol{p},\omega) = \sum_{\boldsymbol{q}}\int_0^{\infty}d\nu\left|\frac{\boldsymbol{p}\times\hat{\boldsymbol{q}}}{m}\right|^2\Im D_a(\boldsymbol{q},\nu)\{[n(\nu)+f(\omega+\nu)]A(\boldsymbol{p}+\boldsymbol{q},\omega+\nu) - [n(-\nu)+f(\omega-\nu)]A(\boldsymbol{p}+\boldsymbol{q},\omega-\nu)\}. \tag{B5}$$

Writing down the quantum Boltzmann equation in terms of components and focusing on dynamics near the Fermi surface, we reach the following expression for each component:

$$\begin{aligned}
&\frac{\Lambda_F^x(\omega)}{\tau_{tr}(\omega)} + i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-1}\frac{eB_z}{mc}\Lambda_F^y(\omega) - i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-1}\frac{eB_y}{mc}\Lambda_F^z(\omega)\\
&\quad = -i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-2}\left\{\frac{e}{c}B_y(e\boldsymbol{E}\times\boldsymbol{\Omega}_F)_z - \frac{e}{c}B_z(e\boldsymbol{E}\times\boldsymbol{\Omega}_F)_y\right\}\\
&\qquad - A(\boldsymbol{p}_F,\omega)\Gamma(\boldsymbol{p}_F,\omega)\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-2}\left\{eE_x + \frac{e^2}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_F^x + \frac{e^2}{c}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_F^x\right\},
\end{aligned} \tag{B6}$$

$$\begin{aligned}
&\frac{\Lambda_F^y(\omega)}{\tau_{tr}(\omega)} - i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-1}\frac{eB_z}{mc}\Lambda_F^x(\omega) + i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-1}\frac{eB_x}{mc}\Lambda_F^z(\omega)\\
&\quad = -i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-2}\left\{-\frac{e}{c}B_x(e\boldsymbol{E}\times\boldsymbol{\Omega}_F)_z + \frac{e}{c}B_z(e\boldsymbol{E}\times\boldsymbol{\Omega}_F)_x\right\}\\
&\qquad - A(\boldsymbol{p}_F,\omega)\Gamma(\boldsymbol{p}_F,\omega)\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-2}\left\{eE_y + \frac{e^2}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_F^y + \frac{e^2}{c}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_F^y\right\},
\end{aligned} \tag{B7}$$

and

$$\begin{aligned}
&\frac{\Lambda_F^z(\omega)}{\tau_{tr}(\omega)} + i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-1}\frac{eB_y}{mc}\Lambda_F^x(\omega) - i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-1}\frac{eB_x}{mc}\Lambda_F^y(\omega)\\
&\quad = -i\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-2}\left\{-\frac{e}{c}B_y(e\boldsymbol{E}\times\boldsymbol{\Omega}_F)_x + \frac{e}{c}B_x(e\boldsymbol{E}\times\boldsymbol{\Omega}_F)_y\right\}\\
&\qquad - A(\boldsymbol{p}_F,\omega)\Gamma(\boldsymbol{p}_F,\omega)\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)^{-2}\left\{eE_z + \frac{e^2}{c}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_F^z + \frac{e^2}{c}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_F\right)(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{\Omega}_F^z\right\},
\end{aligned} \tag{B8}$$

where the transport time is given by

$$\begin{aligned}
\frac{1}{\tau_{tr}(\omega)} = \sum_{\boldsymbol{q}}\int_0^{\infty}d\nu\left|\frac{\boldsymbol{p}_F\times\hat{\boldsymbol{q}}}{m}\right|^2\Im D_a(\boldsymbol{q},\nu)(1-\cos\theta)\{[n(\nu)+f(\omega+\nu)]A(\boldsymbol{p}_F+\boldsymbol{q},\omega+\nu)\\
- [n(-\nu)+f(\omega-\nu)]A(\boldsymbol{p}_F+\boldsymbol{q},\omega-\nu)\}.
\end{aligned} \tag{B9}$$

We note the $1-\cos\theta$ factor in this expression, which extracts out backscattering contributions.

2. Current formulation

It is natural to define a current in the following way:[12]

$$\boldsymbol{J} = -e\frac{1}{\beta}\sum_{i\omega}\int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}\left\{\boldsymbol{v}_{\boldsymbol{p}}+e\boldsymbol{E}\times\boldsymbol{\Omega}_{\boldsymbol{p}}+\frac{e}{c}(\boldsymbol{\Omega}_{\boldsymbol{p}}\cdot\boldsymbol{v}_{\boldsymbol{p}})\boldsymbol{B}\right\}[-iG^{<}(\boldsymbol{p},i\omega)]. \tag{B10}$$

We note the $\dot{\boldsymbol{r}}$ term in the integral expression.

Inserting the ansatz for the lesser Green's function into the above expression, we obtain

$$\begin{aligned}\boldsymbol{J} =& -e^2\frac{1}{\beta}\sum_{i\omega}\int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}(\boldsymbol{E}\times\boldsymbol{\Omega}_{\boldsymbol{p}})f(\omega)A(\boldsymbol{p},\omega)\\ &-e\frac{1}{\beta}\sum_{i\omega}\int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}\left\{\boldsymbol{v}_{\boldsymbol{p}}+\frac{e}{c}(\boldsymbol{\Omega}_{\boldsymbol{p}}\cdot\boldsymbol{v}_{\boldsymbol{p}})\boldsymbol{B}\right\}\left(-\frac{\partial f(\omega)}{\partial\omega}\right)A(\boldsymbol{p},\omega)\boldsymbol{v}_{\boldsymbol{p}}\cdot\Lambda(\boldsymbol{p},\omega).\end{aligned} \tag{B11}$$

Then, each component is given by

$$\begin{aligned}J_x =& -e^2\frac{1}{\beta}\sum_{i\omega}\int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}(E_y\boldsymbol{\Omega}_{\boldsymbol{p}}^z-E_z\boldsymbol{\Omega}_{\boldsymbol{p}}^y)f(\omega)A(\boldsymbol{p},\omega)\\ &-e\frac{1}{\beta}\sum_{i\omega}\int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}(v_F^x)^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)A(\boldsymbol{p},\omega)\Lambda_x(\boldsymbol{p},\omega)\\ &-e\frac{1}{\beta}\sum_{i\omega}\int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}\frac{e}{c}B_x\left\{\left(v_{\boldsymbol{p}}^x\right)^2\boldsymbol{\Omega}_{\boldsymbol{p}}^x\Lambda_x(\boldsymbol{p},\omega)+\left(v_{\boldsymbol{p}}^y\right)^2\boldsymbol{\Omega}_{\boldsymbol{p}}^y\Lambda_y(\boldsymbol{p},\omega)+\left(v_{\boldsymbol{p}}^z\right)^2\boldsymbol{\Omega}_{\boldsymbol{p}}^z\Lambda_z(\boldsymbol{p},\omega)\right\}\\ &\times\left(-\frac{\partial f(\omega)}{\partial\omega}\right)A(\boldsymbol{p},\omega),\end{aligned} \tag{B12}$$

and

$$\begin{aligned}J_y =& e^2\frac{1}{\beta}\sum_{i\omega}\int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}(E_z\boldsymbol{\Omega}_{\boldsymbol{p}}^x-E_x\boldsymbol{\Omega}_{\boldsymbol{p}}^z)f(\omega)A(\boldsymbol{p},\omega)\\ &+e\frac{1}{\beta}\sum_{i\omega}\int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}(v_F^y)^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)A(\boldsymbol{p},\omega)\Lambda_y(\boldsymbol{p},\omega)\\ &+e\frac{1}{\beta}\sum_{i\omega}\int \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\left(1+\frac{e}{c}\boldsymbol{B}\cdot\boldsymbol{\Omega}_{\boldsymbol{p}}\right)^{-1}\frac{e}{c}B_y\left\{\left(v_{\boldsymbol{p}}^x\right)^2\boldsymbol{\Omega}_{\boldsymbol{p}}^x\Lambda_x(\boldsymbol{p},\omega)+\left(v_{\boldsymbol{p}}^y\right)^2\boldsymbol{\Omega}_{\boldsymbol{p}}^y\Lambda_y(\boldsymbol{p},\omega)+\left(v_{\boldsymbol{p}}^z\right)^2\boldsymbol{\Omega}_{\boldsymbol{p}}^z\Lambda_z(\boldsymbol{p},\omega)\right\}\\ &\times\left(-\frac{\partial f(\omega)}{\partial\omega}\right)A(\boldsymbol{p},\omega).\end{aligned} \tag{B13}$$

3. Longitudinal magnetotransport

Solving the quantum Boltzmann equation in the unconventional setup of $\boldsymbol{B}=B_x\hat{\boldsymbol{x}}$ and $\boldsymbol{E}=E_x\hat{\boldsymbol{x}}$, we find

$$\Lambda_F^x(\omega) = -eA(\boldsymbol{p}_F,\omega)\frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)}E_x, \tag{B14}$$

$$\begin{aligned}\Lambda_F^y(\omega) =& me\frac{\omega_c^x\tau_{tr}(\omega)}{\left(1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x\right)^2+\left[\omega_c^x\tau_{tr}(\omega)\right]^2}\left(-\boldsymbol{\Omega}_F^z+\boldsymbol{\Omega}_F^y\frac{\omega_c^x\tau_{tr}(\omega)}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}\right)E_x\\ &-A(\boldsymbol{p}_F,\omega)\frac{\frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)}}{\left(1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x\right)^2+\left[\omega_c^x\tau_{tr}(\omega)\right]^2}\left\{\frac{e^2}{c}+\frac{e^2}{c}\left(1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x\right)\right\}\left(\boldsymbol{\Omega}_F^y+\boldsymbol{\Omega}_F^z\frac{\omega_c^x\tau_{tr}(\omega)}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}\right)E_xB_x,\end{aligned} \tag{B15}$$

and

$$\begin{aligned}\Lambda_F^z(\omega) =& me\frac{\omega_c^x\tau_{tr}(\omega)}{\left(1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x\right)^2+\left[\omega_c^x\tau_{tr}(\omega)\right]^2}\left(\boldsymbol{\Omega}_F^y+\boldsymbol{\Omega}_F^z\frac{\omega_c^x\tau_{tr}(\omega)}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}\right)E_x\\ &-A(\boldsymbol{p}_F,\omega)\frac{\frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)}}{\left(1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x\right)^2+\left[\omega_c^x\tau_{tr}(\omega)\right]^2}\left\{\frac{e^2}{c}+\frac{e^2}{c}\left(1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x\right)\right\}\left(\boldsymbol{\Omega}_F^z-\boldsymbol{\Omega}_F^y\frac{\omega_c^x\tau_{tr}(\omega)}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}\right)E_xB_x,\end{aligned} \tag{B16}$$

where the relaxation rate $1/\tau_{sc}(\omega)$ via intranode scattering (node = Weyl point) is defined as $1/\tau_{sc}(\omega) = \Gamma(\boldsymbol{p}_F,\omega)$ given by Eq. (B5), and $\omega_c^x = \frac{eB_x}{mc}$ is the "cyclotron" frequency associated with the B_x field. We notice that there are $\boldsymbol{E}\cdot\boldsymbol{B} = E_xB_x$ terms, which are topological in their origin.

Inserting these vertex distribution functions into the current formula, we obtain a rather complicated expression for the x component of the current,

$$\begin{aligned}
J_x &= e^2\frac{1}{\beta}\sum_{i\omega}\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\frac{1}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}(v_F^x)^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)[A(\boldsymbol{p}_F,\omega)]^2\frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)}E_x \\
&+\frac{e^3}{c}\frac{1}{\beta}\sum_{i\omega}\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\frac{1}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}(v_{\boldsymbol{p}}^x)^2\boldsymbol{\Omega}_F^x\left(-\frac{\partial f(\omega)}{\partial\omega}\right)[A(\boldsymbol{p}_F,\omega)]^2\frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)}B_xE_x \\
&-\frac{me^3}{c}\frac{1}{\beta}\sum_{i\omega}\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\frac{1}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}(v_F^y)^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)A(\boldsymbol{p}_F,\omega) \\
&\times\frac{\omega_c^x\tau_{tr}(\omega)}{\left(1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x\right)^2+\left[\omega_c^x\tau_{tr}(\omega)\right]^2}\left(-\boldsymbol{\Omega}_F^y\boldsymbol{\Omega}_F^z+(\boldsymbol{\Omega}_F^y)^2\frac{\omega_c^x\tau_{tr}(\omega)}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}\right)B_xE_x \\
&+\frac{e^4}{c^2}\frac{1}{\beta}\sum_{i\omega}\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\frac{2+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}(v_F^y)^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)[A(\boldsymbol{p}_F,\omega)]^2\frac{\frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)}\left(\left(\boldsymbol{\Omega}_F^y\right)^2+\boldsymbol{\Omega}_F^y\boldsymbol{\Omega}_F^z\frac{\omega_c^x\tau_{tr}(\omega)}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}\right)}{\left(1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x\right)^2+\left[\omega_c^x\tau_{tr}(\omega)\right]^2}E_xB_x^2 \\
&-\frac{me^3}{c}\frac{1}{\beta}\sum_{i\omega}\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\frac{1}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}(v_F^z)^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)A(\boldsymbol{p}_F,\omega) \\
&\times\frac{\omega_c^x\tau_{tr}(\omega)}{\left(1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x\right)^2+\left[\omega_c^x\tau_{tr}(\omega)\right]^2}\left(\boldsymbol{\Omega}_F^y\boldsymbol{\Omega}_F^z+(\boldsymbol{\Omega}_F^z)^2\frac{\omega_c^x\tau_{tr}(\omega)}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}\right)B_xE_x \\
&+\frac{e^4}{c^2}\frac{1}{\beta}\sum_{i\omega}\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}\frac{2+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}(v_F^z)^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)[A(\boldsymbol{p}_F,\omega)]^2\frac{\frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)}\left(\left(\boldsymbol{\Omega}_F^z\right)^2-\boldsymbol{\Omega}_F^y\boldsymbol{\Omega}_F^z\frac{\omega_c^x\tau_{tr}(\omega)}{1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x}\right)}{\left(1+\frac{e}{c}B_x\boldsymbol{\Omega}_F^x\right)^2+\left[\omega_c^x\tau_{tr}(\omega)\right]^2}E_xB_x^2. \qquad \text{(B17)}
\end{aligned}$$

Expanding the above expression up to the second order for the Berry curvature and keeping only even-power contributions,[12] we obtain

$$\begin{aligned}
J_x &\approx e^2\frac{1}{\beta}\sum_{i\omega}\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}(v_F^x)^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)[A(\boldsymbol{p}_F,\omega)]^2\frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)}E_x \\
&-2\frac{me^3}{c}\frac{1}{\beta}\sum_{i\omega}\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}(v_F^y)^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)A(\boldsymbol{p}_F,\omega)\frac{(\boldsymbol{\Omega}_F^y)^2\omega_c^x\tau_{tr}(\omega)}{1+\left[\omega_c^x\tau_{tr}(\omega)\right]^2}\left[\omega_c^x\tau_{tr}(\omega)\right]B_xE_x \\
&+2\frac{e^4}{c^2}\frac{1}{\beta}\sum_{i\omega}\int\frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3}(v_F^y)^2\left(-\frac{\partial f(\omega)}{\partial\omega}\right)[A(\boldsymbol{p}_F,\omega)]^2\frac{\frac{\tau_{tr}(\omega)}{\tau_{sc}(\omega)}(\boldsymbol{\Omega}_F^y)^2}{1+\left[\omega_c^x\tau_{tr}(\omega)\right]^2}E_xB_x^2 \\
&= \mathcal{C}N_Fe^2v_F^2\tau_{tr}(T)E_x+2\mathcal{C}'\frac{e^4}{c^2}N_Fv_F^2\frac{\tau_{tr}(T)}{1+\left[\omega_c^x\tau_{tr}(T)\right]^2}B_x^2E_x-2\mathcal{C}''\frac{me^3}{c}N_Fv_F^2\frac{\tau_{sc}(T)\left[\omega_c^x\tau_{tr}(T)\right]^2}{1+\left[\omega_c^x\tau_{tr}(T)\right]^2}B_xE_x. \qquad \text{(B18)}
\end{aligned}$$

The first term is also the conventional contribution near the Fermi surface, but there is no dependence for magnetic fields. This is certainly expected because the magnetic field is in the same direction as the electric field. On the other hand, the second contribution originates from the topological $\boldsymbol{E}\cdot\boldsymbol{B}$ term. The third term is also anomalous, which results from the Berry curvature but not from the $\boldsymbol{E}\cdot\boldsymbol{B}$ term.

4. Discussion

The longitudinal magnetoconductivity is

$$\begin{aligned}
\sigma_L(B_x,T) &= \mathcal{C}N_Fe^2v_F^2\tau_{tr}(T) \\
&+2\mathcal{C}'\frac{e^4}{c^2}N_Fv_F^2\frac{\tau_{tr}(T)}{1+\left[\omega_c^x\tau_{tr}(T)\right]^2}B_x^2 \\
&-2\mathcal{C}''\frac{me^3}{c}N_Fv_F^2\frac{\tau_{sc}(T)\left[\omega_c^x\tau_{tr}(T)\right]^2}{1+\left[\omega_c^x\tau_{tr}(T)\right]^2}B_x. \qquad \text{(B19)}
\end{aligned}$$

If we limit our discussion to low magnetic fields, we are allowed to neglect the last contribution. Then, the above expression can be rewritten as follows:

$$\sigma_L(B_x,T) = \left(1+\mathcal{C}_WB_x^2\right)\sigma_n(T), \qquad \text{(B20)}$$

where $\sigma_n(T) = \mathcal{C}N_Fe^2v_F^2\tau_{tr}(T)$ is the normal conductivity and $\mathcal{C}_W = 2(\mathcal{C}'/\mathcal{C})(e^2/c^2)$ is a positive constant.

Our proposal is to replace the B_x^2 term with $[c(T)]^2$, where $c(T)$ represents the distance between two Weyl points. Then, the final expression for the "longitudinal" conductivity becomes

$$\sigma_L(T) \longrightarrow (1 + \mathcal{K}[c(T)]^2)\sigma_n(T), \qquad \text{(B21)}$$

where $\mathcal{K}$ is a positive numerical constant and the normal conductivity is determined by intrascattering events at one Weyl point.

A cautious person may point out the physical procedure for this replacement and how to determine the value of $\mathcal{K}$. Applying magnetic fields into a Dirac metal phase, a Weyl metallic state appears, where such magnetic fields are introduced into the Dirac Lagrangian as a form of the Zeeman energy. Comparing this Dirac Lagrangian with a Zeeman term to our effective Lagrangian with a chiral gauge field [Eq. (3)], one finds that the chiral gauge field is identified with $g\boldsymbol{B}$, where g is the Lande g factor and $\boldsymbol{B}$ is the applied magnetic field. This is the identification between the applied magnetic field and the chiral gauge field.

However, we would like to confess one uncertainty in this identification. Considering our derivation for the longitudinal conductivity, an essential term of the quantum Boltzmann equation approach is the topological $\boldsymbol{E} \cdot \boldsymbol{B}$ term, introduced from the help of the semiclassical equations of motion. In the situation where magnetic fields are applied, the distance between two paired Weyl points is given by the applied magnetic field, the same $\boldsymbol{B}$ as that of the topological $\boldsymbol{E} \cdot \boldsymbol{B}$ term. On the other hand, the chiral gauge field is given by the spatial gradient of the angle θ in the axion term, as explicitly shown in Appendix A1, but this differs from the magnetic field in the $\boldsymbol{E} \cdot \boldsymbol{B}$ term. If one compares this chiral gauge field with the applied magnetic field in the level of the Lagrangian, they can be identified clearly. However, in the level of the quantum Boltzmann equation approach, this identification is not clarified because we did not verify how the introduction of the chiral gauge field can be consistently made with the semiclassical equation-of-motion approach with the topological $\boldsymbol{E} \cdot \boldsymbol{B}$ term. In order to resolve this uncertain point, we need to derive the quantum Boltzmann equation from the Weyl Lagrangian with the chiral gauge field, based on the Schwinger-Keldysh formalism.[24] This work is beyond the scope of the present study, which deserves to be investigated more sincerely in the future. This is the reason why we use the term "replacement." In this respect it is not fully clarified how to determine the value of $\mathcal{K}$ although we suspect that it would be identified with $\mathcal{C}_W = 2(\mathcal{C}'/\mathcal{C})(e^2/c^2)$ as Eq. (B20) except for the Lande g factor. In order to make the value of $\mathcal{K}$ definite, it seems necessary to derive the quantum Boltzmann equation from the Weyl Lagrangian via the Schwinger-Keldysh formalism.

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