Correlation functions in the prethermalized regime after a quantum quench of a spin chain

Aditi Mitra
Department of Physics, New York University, 4 Washington Place, New York, New York 10003, USA
(Received 12 February 2013; revised manuscript received 22 April 2013; published 9 May 2013)

Results are presented for a two-point correlation function of a spin chain after a quantum quench for an intermediate time regime where inelastic effects are weak. A Callan-Symanzik-like equation for the correlation function is explicitly constructed which is used to show the appearance of three distinct scaling regimes. One is for spatial separations within a light cone, the second is for spatial separations on the light cone, and the third is for spatial separations outside the light cone. In these three regimes, the correlation function is found to decay with power laws with nonequilibrium exponents that differ from those in equilibrium, as well as from those obtained from quenches in a quadratic Luttinger liquid theory. A detailed discussion is presented on how the existence of scaling depends on the properties of the initial state before the quench.

DOI: 10.1103/PhysRevB.87.205109 PACS number(s): 75.10.Jm, 05.70.Ln, 67.85.−d, 71.10.Pm

I. INTRODUCTION

Motivated by experiments in cold atomic gases[1] and ultrafast spectroscopy of strongly correlated materials,[2,3] the nonequilibrium dynamics of interacting quantum systems has become a topic at the forefront of research. In this context, dynamics arising due to a quantum quench, where a system is prepared in the ground state of an initial Hamiltonian H_i and then time-evolved with respect to a final Hamiltonian H_f, is of particular interest because of its potential for addressing several fundamental questions.[4] Some of these are the mechanisms and time scales for thermalization,[5–11] the possibility of an intermediate time prethermalized regime,[12–15] dynamical phase transitions associated with nonanalytic behavior during the time evolution[16,17], dynamics of integrable models,[18–25] and the possibility of describing their steady state in terms of a generalized Gibbs ensemble (GGE).[26] Yet another important question, which is related to the topic of this paper, is the appearance of universal behavior in the dynamics with the possibility of capturing such a behavior in a renormalization group (RG) approach, even though the system is out of equilibrium.

An RG approach has been actively used to study nonequilibrium time evolution after a quench in classical field theories.[27,28] The aim of this paper is to develop an RG approach to study nonequilibrium time evolution of correlation functions in interacting quantum field theories. So far the establishment of universal scaling functions in quantum systems that are out of equilibrium either due to a sudden quench or a "slow" quench that involves changing parameters in a prescribed time-dependent way has been mainly explored for exactly solvable or mean-field theories,[29,30] or in numerical studies of interacting field theories.[31–33] In the present paper on the other hand we present an analytical RG approach to study how a correlation function evolves after a quantum quench in an interacting field theory. In doing so a Callan-Symanzik (CS)-like equation for a two-point correlation function is derived which is used to explicitly show under what conditions scaling holds out of equilibrium, and is used to identify intrinsically nonequilibrium scaling regimes with new exponents.

We study a quantum quench in a generic one-dimensional spin chain $H = \sum_i [S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z] + \text{NNN}$ where NNN denotes additional next-nearest-neighbor couplings. We study the dynamics in a continuum field theory described by the quantum sine-Gordon model with the cosine potential representing the underlying commensurate lattice or periodic potential.[34] In particular we study the time evolution of the two-point correlation function of the staggered spin component $R(r,T_m) = (-1)^r \langle \psi_i | e^{iH_f T_m} S_0^z S_r^z e^{-iH_f T_m} | \psi_i \rangle$ where $|\psi_i\rangle$ is the ground state of the Hamiltonian H_i before the quench, H_f is the Hamiltonian after the quench, T_m is the time after the quench, and r is the spatial separation between spin operators. We find that when the quench involves the sudden switching on of the cosine potential, then in the vicinity of the critical point where the cosine potential is a marginal perturbation in equilibrium,[35–37] out of equilibrium three distinct scaling regimes appear for macroscopic distance and time scales $r, T_m \gg 1$ [we have set the sound velocity $u = 1$, and the distances and times are measured in units of an ultraviolet (UV) cutoff]. One of the scaling regimes is for spatial separations outside the light cone ($r \gg 2T_m$) where we find $R(r,T_m, r \gg 2T_m) \sim \frac{\sqrt{\ln T_m}}{r}$. The second scaling regime is for spatial separations inside the light cone $r \ll 2T_m$ where we find $R(r,T_m, r \ll 2T_m) \sim \frac{\sqrt{\ln r}}{r}$. The third scaling regime is for spatial separations on the light cone $r = 2T_m$ where we find $R(r,T_m, r = 2T_m) \sim \frac{\ln r}{r}$.

For more complicated quenches which involve not only a sudden switching on of the cosine potential, but also a change in the Luttinger interaction parameter from $K_0 \to K$, we find that the scaling within the light cone survives, where the correlation function is found to be $R(r,T_m, r \ll 2T_m) \sim \frac{(\ln r)^\theta}{r}$ where θ is a universal number that approaches $1/2$ as $K_0 \to K$. In addition we also find that the existence of scaling in the other two regimes, one being outside the light cone ($r \gg 2T_m$), and the second being on the light cone $r = 2T_m$, depends on the initial wave function. In particular if the initial Luttinger parameter K_0 is such that the cosine potential is a relevant or marginal perturbation, scaling holds on the light cone, whereas scaling on the light cone is violated when the cosine potential is an irrelevant perturbation for the initial state.

This paper is organized as follows. In Sec. II we introduce the model, establish notation, and also briefly summarize the results. The rest of the paper goes into more detail of how these results are obtained. In Sec. III we briefly present results for an interaction quench in the quadratic theory (the Luttinger liquid) in the language of Keldysh Green's functions. These results will be useful for later sections when we perform perturbation theory in the cosine potential. In Sec. IV, we perform perturbation theory in the cosine potential and derive the β function to two loop. In Sec. V we present results for the correlation function within perturbation theory to leading order in the cosine potential. These results then set the stage for doing renormalization improved perturbation theory which will be explicitly carried out in Sec. VI where a CS-like differential equation for the correlation function is derived. Results of the solution of the CS equation are presented in Sec. VII for the case where only the cosine or lattice potential is suddenly switched on, while results for the correlation function for a simultaneous lattice and interaction quench are presented in Sec. VIII. Finally in Sec. IX we present our conclusions.

II. MODEL AND A BRIEF DISCUSSION OF RESULTS

We study a quantum quench in a generic one-dimensional (1D) spin chain,

$$H = \sum_i \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z\right] + \mathrm{NNN}, \tag{1}$$

where NNN denotes additional next-nearest-neighbor couplings. If the NNN couplings are weak in comparison to the NN couplings, the spin chain has two phases: a gapless phase with linearly dispersing spin waves at long wavelengths, and a gapped antiferromagnetic Ising phase. In equilibrium and zero temperature the properties of the spin chain in its gapless phase are captured very well by a continuum theory that retains only the relevant operators, namely the Luttinger liquid.[34] In contrast, the effect of irrelevant operators can be important both at finite temperature[38–43] as well as out of equilibrium following a quench.[8,44] In this paper we study the dynamics of the spin chain in a continuum theory by retaining the effect of the leading irrelevant operator. For the spin chain in its gapless phase, the leading irrelevant operator is a commensurate periodic or lattice potential which gives rise to umpklapp or backscattering. In this paper we study the effect of this term on the dynamics; we expect its effect will dominate over those of other irrelevant terms such as band curvature.

Specifically we study a quench where initially the system is in the ground state of a Luttinger liquid,

$$H_i = \frac{u_0}{2\pi}\int dx \left\{K_0\,[\pi\Pi(x)]^2 + \frac{1}{K_0}[\partial_x\phi(x)]^2\right\}. \tag{2}$$

$-\partial_x\phi/\pi$ represents the density, Π is the variable canonically conjugate to ϕ, K_0 is the dimensionless interaction parameter, and u_0 is the velocity of the sound modes. The system is driven out of equilibrium via an interaction quench at $t = 0$ from $K_0 \to K$, with the leading irrelevant operator corresponding to a commensurate lattice or periodic potential V_{sg} also switched on suddenly, at the same time as the interaction quench. This triggers nontrivial time evolution from $t > 0$ due to the quantum sine-Gordon model,

$$H_f = H_{f0} + V_{\mathrm{sg}}, \tag{3}$$

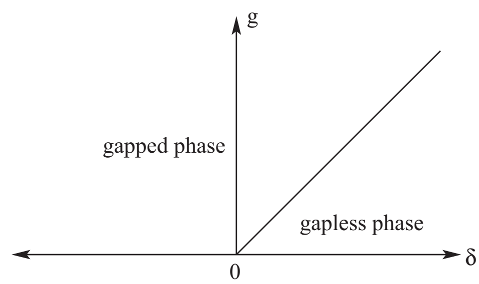

FIG. 1. Ground-state phase diagram of the quantum sine-Gordon model. A critical line separates a gapless phase where the cosine term is irrelevant from a gapped phase where the cosine term is relevant. The critical line is located at $\delta = 2\pi g$, where $\delta = K_{\mathrm{eq}} - 2$, with $K_{\mathrm{eq}} = \frac{\gamma^2 K}{4}$.

where

$$H_{f0} = \frac{u}{2\pi}\int dx \left\{K[\pi\Pi(x)]^2 + \frac{1}{K}[\partial_x\phi(x)]^2\right\} \tag{4}$$

$$V_{\mathrm{sg}} = -\frac{gu}{\alpha^2}\int dx\,\cos(\gamma\phi). \tag{5}$$

Above,

$$\Lambda = \frac{u}{\alpha} \tag{6}$$

is a short-distance UV cutoff, g is the strength of the commensurate periodic potential. Note that both for the NN spin chain as well as the NNN spin chain, the low-energy theory is represented by H_f with $\gamma = 4$. However, the precise values of the Luttinger parameter K and the strength of the cosine potential g depends on the microscopic details. For example, the point $K = 1, g = 0$ corresponds to the exactly solvable NN XX chain. Switching on NNN interactions can result in parameters where $K \neq 1$ but $g = 0$.[45]

The versatility of H_f is that it equally well applies to interacting bosons in a commensurate periodic potential.[46] For this case, $\gamma = 2$, while the point $K = 1$ corresponds to NN hard-core bosons or the Tonks-Girardeau gas. In this paper, in all our analytic results, we keep γ general so that the obtained results may be applied both to the spin chain as well as to bosons in a commensurate periodic potential.

The ground-state phase diagram of the quantum sine-Gordon model is shown in Fig. 1. A critical line defined by $\delta = 2\pi g$, where $\delta = K_{\mathrm{eq}} - 2$, with $K_{\mathrm{eq}} = \frac{\gamma^2 K}{4}$, separates a gapped phase where V_{sg} is a relevant perturbation, from a gapless phase, where V_{sg} is an irrelevant perturbation. For the spin chain the gapped phase corresponds to the Ising phase, while for interacting bosons in a lattice, the gapped phase is the Mott insulator.

In this paper we study the time evolution of the equal time two-point correlation function of the staggered spin component $R(r,T_m) = (-1)^r\langle S_0^z(T_m)S_r^z(T_m)\rangle$ where T_m is the time after the quench. In the continuum, this correlator is given by[47]

$$R(x_1T_m,x_2T_m) = 4\left\langle \cos\left(\frac{\gamma\phi(x_1T_m)}{2}\right)\cos\left(\frac{\gamma\phi(x_2T_m)}{2}\right)\right\rangle. \tag{7}$$

In equilibrium, and in the gapless phase, but in the vicinity of the critical line where V_{sg} is a marginal perturbation, logarithmic corrections arise. In particular R near the equilibrium critical point behaves as follows:[35–37]

$$R_{eq}(r) = \frac{\sqrt{\ln r}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \tag{8}$$

where r is the magnitude of the spatial separation $x_1 - x_2$. The aim of this paper is to determine how the correlator R evolves after a quantum quench. We will study R in the regime where V_{sg} is irrelevant or marginally irrelevant, where the meaning of these terms in a nonequilibrium situation will be clarified below.

We now briefly outline how R is calculated for the nonequilibrium problem. Denoting $\hat{O}(xt) = 2\cos(\frac{\gamma\phi(xt)}{2})$, R may be written as a Keldysh path integral representing the time evolution from the initial pure state $|\psi_i\rangle$ (hence an initial density matrix $\rho = |\psi_i\rangle\langle\psi_i|$) corresponding to the ground state of H_i,

$$R(x_1 T_m, x_2 T_m) = Tr[\rho(T_m)O(x_1 0)O(x_2 0)] \tag{9}$$

which may be written as

$$\begin{aligned} R(x_1 T_m, x_2 T_m) &= \mathrm{Tr}[e^{-iH_f T_m}|\psi_i\rangle\langle\psi_i|e^{iH_f T_m}\hat{O}(x_1)\hat{O}(x_2)] \\ &= \int \mathcal{D}[\phi_{cl},\phi_q]e^{i(S_0+S_{sg})}\hat{O}_I(x_1T_m)\hat{O}_I(x_2T_m), \end{aligned} \tag{10}$$

where $\hat{O}_I$ is the operator in the interaction representation of H_{f0}, S_0 describes the nonequilibrium Luttinger liquid ($g = 0$),

$$\begin{aligned} S_0 = \frac{1}{2}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\int_0^{T_m}dt_1\int_0^{T_m}dt_2(\phi_{cl}(1)\phi_q(1)) \\ \times\begin{pmatrix} 0 & G_A^{-1}(1,2) \\ G_R^{-1}(1,2) & -\left[G_R^{-1}G_K G_A^{-1}\right](1,2)\end{pmatrix}\begin{pmatrix}\phi_{cl}(2)\\ \phi_q(2)\end{pmatrix} \end{aligned} \tag{11}$$

and

$$S_{sg} = \frac{gu}{\alpha^2}\int_{-\infty}^{\infty}dx_1\int_0^{T_m}dt_1[\cos\{\gamma\phi_-(1)\} - \cos\{\gamma\phi_+(1)\}] \tag{12}$$

Above, $1(2) = (x_{1(2)},t_{1(2)})$, $\phi_{cl,q} = \frac{\phi_-\pm\phi_+}{\sqrt{2}}$ with $-/+$ representing fields that are time/antitime ordered on the Keldysh contour.[48]

We derive a CS-like differential equation for R by splitting the fields ϕ into slow and fast fields $\phi = \phi_< + \phi_>$ where the fast fields have a large weight at short wavelengths, and therefore oscillate rapidly in time. We integrate out the fast fields, and rescale the cutoff, position, and time. Such a procedure within the real-time Keldysh approach has been employed for the quantum sine-Gordon model both for steady-state[8,9,33] and transient behavior,[17] where in each case the β function was derived. Here we generalize this approach to the study of a two-point correlation function by performing a microscopic derivation of a CS-like differential equation for the correlation function. This approach reveals the conditions under which scaling holds after a quantum quench, and identifies different scaling regimes. So far such a treatment has only been employed for quenches in classical field theories where intermediate time nonequilibrium scaling regimes with new exponents have been identified.[27,28] Here we will show that similar new nonequilibrium scaling regimes can arise for quantum quenches of 1D systems. While interaction quenches in Luttinger liquids and related quadratic theories have been studied extensively,[18,20,22,49–53] and predict new nonequilibrium exponents as well, in this paper we show that these exponents are further modified by the presence of the commensurate periodic potential V_{sg}. For example when V_{sg} is marginal, as briefly stated in the introduction, V_{sg} gives rise to logarithmic corrections.

In a global quench like the one we study, the system is translationally invariant in space, so that $R(x_1T_m,x_2T_m) = R(r = x_1 - x_2, T_m)$. We define the following three exponents which play an important role in the dynamics,

$$\begin{aligned} K_{eq} &= \frac{\gamma^2 K}{4}; \quad K_{neq} = \frac{\gamma^2}{8}K_0\left(1+\frac{K^2}{K_0^2}\right), \\ K_{tr} &= \frac{\gamma^2}{8}K_0\left(1-\frac{K^2}{K_0^2}\right). \end{aligned} \tag{13}$$

K_{eq} governs the power-law decay of the correlator R in the ground state of H_{f0} (i.e., the Luttinger liquid with interaction parameter K_0 and $V_{sg} = 0$), K_{neq} determines the power-law decay of R at long times after an interaction quench in the Luttinger liquid ($V_{sg} = 0$),[8,49] K_{tr} determines the crossover from a short-time behavior determined primarily by the initial Luttinger parameter $K_{neq} + K_{tr} = \frac{\gamma^2 K_0}{4}$, to the long-time behavior determined by K_{neq}.[17]

One of the results of our study is that the dynamics of equal-time correlation functions after a quench has qualitatively different features in the three regimes shown in Fig. 2. One is the region outside the light cone where the spatial separation r is much larger than the time after the quench ($r \gg 2T_m$ setting the velocity $u = 1$); here the behavior of the correlator is primarily determined by the initial wave function. The second is an intrinsically nonequilibrium regime where the separation r lies on the light cone ($r = 2T_m$) and where we will identify universal behavior of the correlator R with new exponents. The third is a nonequilibrium steady-state regime where the spatial separation lies within the light cone

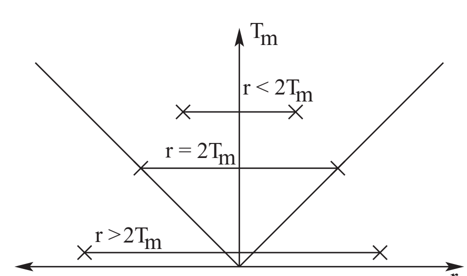

FIG. 2. The equal-time correlator shows distinctly different behavior for the following three cases: spatial separations outside the light cone ($r > 2T_m$), spatial separations on the light cone ($r = 2T_m$), and spatial separations within the light cone ($r < 2T_m$). The present paper gives results for correlation functions in these three regimes under the additional constraint that $T_m < 1/\eta$ where η is an inelastic scattering rate.

($r \ll 2T_m$). Here R is independent of time and shows scaling behavior in position with new nonequilibrium exponents which differ from those in the first two regimes just discussed. This qualitative change in the behavior of the correlators at $r = 2T_m$ is known as the "horizon effect"[54] where at time $T_m = r/2$ left- and right-moving excitations originating from the same spatial region reach the two local observables, thus maximally entangling them.

The presence of these three regimes is already apparent in the behavior of R for an interaction quench in a Luttinger liquid ($V_{sg} = 0$). Here (writing r,T_m in units of Λ) for $r,T_m \gg 1$ and far from the light cone ($|r \pm 2T_m| \gg 1$) one finds[17,49]

$$R^{(0)}(r,T_m,g=0) = \left[\left(\frac{1}{r^2}\right)^{\gamma^2K_0/4}\left(\left|\frac{r^2-(2T_m)^2}{r^2(2T_m)^2}\right|\right)^{-K_{tr}}\right]^{1/4}. \tag{14}$$

R shows the "horizon effect" where outside the light cone $r \gg 2T_m$, the correlator depends only on the initial wave function (and hence the initial Luttinger parameter K_0) albeit with a time-dependent prefactor $T_m^{K_{tr}/2}$. However, within the light cone $r \ll 2T_m$, the correlator reaches a steady state characterized by the nonequilibrium exponent K_{neq}. Moreover, on the light cone the correlator is found to decay as $R^{(0)}(r = 2T_m) \sim r^{-(\gamma^2K_0/8)+3(K_{tr}/4)}$. When $V_{sg} \neq 0$, using RG improved perturbation theory, we will show that this distinct behavior of R outside, on, and inside the light cone survives with exponents that differ from the ones above. It is interesting to contrast this behavior with that of a correlator in a quench from an initially gapped state.[24,54] For the latter the horizon effect is more pronounced as R is exponentially suppressed in position outside the light cone. In contrast for our case, where the quench is from an initially gapless state, the change in the behavior of R from outside the light cone to inside is less dramatic as the correlators both outside the light cone and inside decay as power laws in position.

Before we give results for the correlator, we first discuss the β function obtained from gradually lowering the cutoff from the bare value of Λ_0 to Λ_0/l. The actual derivation is presented in Sec. IV. Integrating out the fast modes generates corrections not only to the Luttinger liquid parameters of the Hamiltonian after the quench (H_{f0}), but also generates new terms whose physical meaning is inelastic scattering. Moreover the β function depends in general on the time after the quench because the strength of the corrections arising due to the commensurate potential together with the effect of the $K_0 \rightarrow K$ quench depend on this time. Expressing time in units of the cutoff, the β function is found to be[17]

$$\frac{dg}{d\ln l} = g\left[2 - \left(K_{neq} + \frac{K_{tr}}{1+4T_m^2}\right)\right], \tag{15}$$

$$\frac{dK^{-1}}{d\ln l} = \frac{\pi g^2\gamma^2}{4} I_K(T_m), \tag{16}$$

$$\frac{d\eta}{d\ln l} = \eta + \frac{\pi g^2\gamma^2 K}{2} I_\eta(T_m), \tag{17}$$

$$\frac{d(\eta T_{eff})}{d\ln l} = 2\eta T_{eff} + \frac{\pi g^2\gamma^2 K}{4} I_{T_{eff}}(T_m), \tag{18}$$

$$\frac{dT_m}{d\ln l} = -T_m. \tag{19}$$

Above T_m is the rescaled time after the quench where the rescaling with the changing cutoff in Eq. (19) is similar to the rescaling of position or frequency, the latter applicable for a system in steady state. Moreover Eq. (19) implies that T_m is related to the bare physical time T_{m0} as $T_m = \frac{T_{m0}}{l}$. This relation suggests that the time after the quench qualitatively acts as an inverse UV cutoff so that the larger is the time after the quench, the more important is the role of the long-wavelength modes on the dynamics.[55]

Equation (16) represents the usual corrections to the Luttinger parameter that arise even in the equilibrium theory. Equation (15) shows that there is a crossover from a short-time regime where the scaling dimension of V_{sg} is $K_{neq} + K_{tr} - 2 = \frac{\gamma^2K_0}{4} - 2$ and therefore depends on the initial Luttinger parameter K_0, to a long-time steady-state regime where the periodic potential V_{sg} has a nonequilibrium scaling dimension $K_{neq} - 2$. In the ground state of H_f, the scaling dimension of V_{sg} is $K_{eq} - 2$, and since $K_{eq} < K_{neq}$, the periodic potential at long times is always more irrelevant for the nonequilibrium problem. Thus the location of the critical point at steady state ($T_m = \infty$) shifts from $K_{eq} = 2$ to $K_{neq} = 2$. Physically this is because the interaction quench $K_0 \rightarrow K$ gives rise to a highly excited state of bosons which cannot be localized by the periodic potential as easily as when they are in the zero-temperature ground state. A consequence of the change in the scaling dimension with time from $K_{neq} + K_{tr} - 2$ to $K_{neq} - 2$ is a qualitative change in the two-point correlation function from outside the light cone to inside the light cone, an effect which will be discussed in detail later.

Equations (17) and (18) show that the notion of relevance or irrelevance should not be taken literally out of equilibrium as even when the cosine potential is irrelevant, it can cause the generation of new terms after a quantum quench. These terms are an inelastic scattering rate η which can be identified by the generation of quadratic corrections to the Luttinger liquid theory of the form $\eta(T_m)\phi_q\partial_{T_m}\phi_{cl}(T_m)$. The second new term is a noise for the long-wavelength modes of strength $\eta(T_m)T_{eff}(T_m)$ which corresponds to the generation of corrections to the Luttinger liquid theory of the form $\eta(T_m)T_{eff}(T_m)\phi_q^2$. Note that for $T_m = 0$, all the corrections $I_{K,\eta,T_{eff}}$ vanish as the effect of V_{sg} vanishes, while for $T_m \gg 1$, $I_{K,\eta,T_{eff}}$ take steady-state values implying that the dissipation and noise reach steady-state values. Thus at long times, the low-energy effective theory is a classical theory characterized by an effective-temperature T_{eff}, and a classical fluctuation-dissipation theorem is obeyed with a dissipation strength of[8] η.

An interaction quench in a Luttinger liquid ($V_{sg} = 0$) generates a highly nonequilibrium occupation of the bosonic modes which does not relax. However, in the presence of V_{sg}, the occupation probability of these bosonic modes is no longer conserved, and a nonzero η represents the rate at which the occupation probability of the long-wavelength modes relax. Here by η we imply the dissipation strength at long times ($T_m \gg 1$). Thus $1/\eta$ is a natural time scale associated with the quench, which is the time after which inelastic scattering events become strong and cause a significant deviation of the bosonic occupation probabilities. Thus the β function implies that the dynamics after a quench has three regimes shown

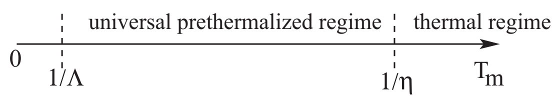

FIG. 3. The dynamics after a quench is characterized by three different regimes: a short-time regime that depends on microscopic parameters ($T_m \ll 1/\Lambda$), an intermediate-time regime ($1/\Lambda \ll T_m \ll 1/\eta$) where inelastic effects are weak and the correlator shows universal scaling behavior, and a long-time thermal regime ($T_m \gg 1/\eta$) where inelastic-scattering effects are strong and cause the system to thermalize.

in Fig. 3. A short-time regime $T_m \ll 1$ where the dynamics depends on microscopic details and can be easily treated within perturbation theory. The second is an intermediate-time prethermalized regime where the time is long as compared to microscopic time scales, while short as compared to the dissipation rate $1 \ll T_m \ll 1/\eta$. In this regime inelastic effects are weak. Using RG improved perturbation theory, we will show that the two-point correlation function R shows universal behavior in this intermediate-time regime, where the precise universal behavior also depends on the magnitude of the spatial separation relative to the time after the quench (horizon effect). Finally there is a third regime which we label the thermal regime $T_m \gg 1/\eta$ where inelastic effects are strong and lead to eventual thermalization. For small quenches ($|K_0 - K| \ll 1$), $\eta \sim g^2(K_0 - K)^4 \ll 1$,[9] the intermediate-time prethermalized regime can be quite large. In this paper we will give results for the correlation function in this intermediate-time regime where universal dynamics characterized by a CS-like differential equation will emerge. The dynamics in the thermal regime is also interesting to explore, and will be discussed elsewhere.

Note that in the nonequilibrium problem, the term irrelevant simply implies that the strength of the perturbation decreases under RG transformations so that perturbation theory in g is valid. In addition, the meaning of the leading irrelevant operator in the nonequilibrium problem is the same as in equilibrium in that it is the coupling constant that decreases under RG the slowest. For example, under RG (and at long times), the coupling strength g for the potential $\cos(\gamma\phi)$, according to Eq. (15), decreases as $l^{2-K_{\rm neq}}$. This is a slower decrease than that for the perturbation of the form $\cos(2\gamma\phi)$ which decreases as $l^{2-4K_{\rm neq}}$. Under RG transformation, the coupling constant for the band curvature $(\partial_x\phi)^3$ will also decrease faster than that of $V_{\rm sg}$ for the values of $K_{\rm neq}$ that we are concerned with in this paper. Thus while these other irrelevant terms will also give rise to additional inelastic scattering, these will only be small corrections to the inelastic-scattering rate already produced by $V_{\rm sg}$.

Equation (15) shows that in the prethermalized regime, there is a crossover from an intermediate time dynamics where the physics is determined by the initial wave function (and hence the initial Luttinger parameter K_0) and a long-time time dynamics determined by $K_{\rm neq}$. This can result in a situation where perturbation theory in g is violated at intermediate times when $\frac{\gamma^2 K_0}{4} < 2$, i.e., when g is a relevant perturbation in the initial state. In Secs. V and VIII we show how this happens and what it implies. Throughout this paper we will present results close to the critical point defined by

$$K_{\rm neq} = 2 + \delta, \quad \forall\ 0 < \delta \ll 1, \tag{20}$$

where $V_{\rm sg}$ is marginal and can give rise to logarithmic corrections to scaling.

We now outline how the CS-like differential equation is derived; this is a summary of the more detailed calculations presented in Sec. VI. We derive the CS-like differential equation for R by integrating out fast modes gradually and in the process lowering the cutoff from $\Lambda \to \Lambda/l = \Lambda - d\Lambda$. This leads to a relation of the form

$$R = R_< \left[1 - \frac{d\Lambda}{\Lambda}(\cdots)\right], \tag{21}$$

where $R_<$ is the correlator for the slow modes while R is the correlator for all the modes. To get an idea for what to expect, let us carry out this exercise for the quadratic theory after the quench ($g = 0$). Here for time T_m after the quench the following relation between the correlator for the full and the slow modes emerges:

$$R^{(0)}(r,T_m) = R^{(0)}_<(r,T_m)\left[1 - \frac{d\Lambda}{\Lambda}\gamma_{an,0}(r,T_m)\right]. \tag{22}$$

The above expression implies the following differential equation: $[\frac{\partial}{\partial \ln l} - \gamma_{an,0}(r,T_m)]R^{(0)}(\frac{r\Lambda_0}{l},\frac{T_{m0}\Lambda_0}{l}) = 0$, where (in units of Λ)

$$\gamma_{an,0}(r,T_m) = \frac{1}{2}\left[K_{\rm neq}\frac{r^2}{1+r^2} + \frac{K_{\rm tr}}{1+(2T_m)^2} - \frac{K_{\rm tr}}{2}\left\{\frac{1}{1+(2T_m+r)^2} + \frac{1}{1+(2T_m-r)^2}\right\}\right]. \tag{23}$$

Equation (23) shows that there are three scaling limits where $\gamma_{an,0}$ becomes independent of r,T_m. One is within the light cone $2T_m \gg r \gg 1$ where $\gamma_{an,0} = K_{\rm neq}/2$. The second is outside the light cone $r \gg 2T_m \gg 1$ where also $\gamma_{an,0} = K_{\rm neq}/2$, while the third is on the light cone $r = 2T_m, r \gg 1$ where $\gamma_{an,0} = (K_{\rm neq} - K_{\rm tr}/2)/2$. We now discuss the correction to R to next order in g where logarithmic corrections arise in the vicinity of the critical point.

At next order in the cosine potential [at $\mathcal{O}(g)$], in terms of slow and fast fields, we find

$$\begin{aligned} R^{(1)}(r,T_m) &= R^{(1)}_<(r,T_m)\left[1 + 2\frac{d\Lambda}{\Lambda} - 2K_{\rm neq}\frac{d\Lambda}{\Lambda} + \gamma_{an,0}(r,T_m)\frac{d\Lambda}{\Lambda}\right] \\ &\quad + 2g\pi I_C(r,T_m)\frac{d\Lambda}{\Lambda}R^0_<(r,t), \end{aligned} \tag{24}$$

where I_C is discussed below. To quadratic order, the correction $R^{(2)}$ leads to the β function which has been discussed above. Equations (22), (24), and the β function imply the following CS-like differential equation for R:

$$\left[\frac{\partial}{\partial \ln l} + \beta(g_i)\frac{\partial}{\partial g_i} - \gamma_{an,0} + 2\pi g I_C\right] \times R\left[\frac{r\Lambda_0}{l}, \frac{\Lambda_0 T_{m0}}{l}, g_i(l)\right] = 0; \tag{25}$$

above T_{m0} is the time after the quench, while r is the spatial separation between the local operators. $g_i = g,\delta,\eta,T_{\rm eff}$, are coupling constants while $\gamma_{an,0} - 2\pi g I_C$ is the anomalous scaling dimension of the correlator. For macroscopic lengths and times where $\frac{r\Lambda_0}{l} \gg 1, \frac{T_{m0}\Lambda_0}{l} \gg 1$, $I_C,\gamma_{an,0}$ are constants independent of position and time.

We make a simplifying assumption of being in the prethermalized regime $1 \ll T_{m0} \ll 1/\eta$. Here the new coupling constants related to dissipation and noise may be neglected and the β function becomes much simpler. On integrating Eq. (25) up to $l^* = \Lambda_0 \min(r,T_{m0})$, one may relate the correlator at long times and distances to the correlator at short times and distances and a renormalized coupling $g_i(l^*)$, where since $g(l^*) \ll 1$, the latter may be evaluated readily within perturbation theory. The anomalous scaling dimension $\gamma_{an,0} - 2\pi g I_C$ takes different values in the three regimes shown in Fig. 2, and is responsible for the distinctly different scaling behavior outside, on, and inside the light cone. We now present results for the correlation function for two cases: one is for the pure-lattice quench ($K_0 = K, g \neq 0$), and the second is a simultaneous lattice and interaction quench ($K_0 \neq K, g \neq 0$).

Pure lattice quench. This corresponds to $K_0 = K$ or $K_{\rm neq} = K_{\rm eq}$, but a periodic potential of strength g switched on suddenly at $T_m = 0$. We are interested in the physics in the vicinity of the critical point where $K_{\rm eq} = K_{\rm neq} = 2 + \delta\ \forall\ 0 < \delta \ll 1$. Here we find

$$I_C(r,T_m) = \frac{r^2+1}{4+r^2} - \left(\frac{1+r^2}{1+T_m^2}\right) \times \left[\frac{r^2 - 4T_m^2 + 4 - 8T_m^2}{\{r^2 - 4T_m^2 + 4\}^2 + 64T_m^2}\right]. \quad (26)$$

Note that $I_C(T_m = 0) = 0$ as the lattice has not had time to affect the correlator. Equation (26) shows the appearance of scaling in three cases: one is outside the light cone, where

$$I_C(r,T_m \gg 1, 2T_m \ll r) = 1 + \mathcal{O}\left(\frac{1}{r^2},\frac{1}{T_m^2}\right); \quad (27)$$

the second is within the light cone,

$$I_C(r,T_m \gg 1, 2T_m \gg r) = 1 + \mathcal{O}\left(\frac{1}{r^2},\frac{r^2}{T_m^4}\right); \quad (28)$$

and the third is on the light cone,

$$I_C(r,T_m \gg 1, 2T_m = r) = \frac{3}{2} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (29)$$

Equations (23) and (26) also show that scaling is valid until (restoring units) $l \sim \min[\Lambda_0 r, \Lambda_0 T_m]$. In the scaling limit, I_K in Eq. (16) is $I_K(T_m \gg 1) = \pi[1 - \frac{7}{8T_m^2} + \cdots]$. Thus the solution of the CS equation (25) in the three scaling limits where $I_C,\gamma_{an,0},I_K$ are constants in time and position is

$$R(\Lambda_0 r,\Lambda_0 T_{m0},g_0) = e^{-\int_{g_0}^{g(l)} dg'[\gamma_{an}(g')/\beta(g')]} R\left(\frac{r\Lambda_0}{l},\frac{T_{m0}\Lambda_0}{l},g(l)\right), \quad (30)$$

where $\gamma_{an} = 1 + \frac{\delta}{2} - 2\pi g I_C$ and the β function is $\frac{dg}{d\ln l} = -g\delta, \frac{d\delta}{d\ln l} = -(2\pi g)^2$. Note that the critical point corresponding to the $S = 1/2$ Heisenberg chain corresponds to $\delta = 2\pi g$. Equation (30) is one of the main results of this paper. It shows the existence of a scaling regime where the correlator at large times or distances is related to the correlator at shorter scales $(r/l, T_{m0}/l)$ and renormalized couplings $g(l)$, where since $g(l \gg 1) \ll 1$, the latter may be readily evaluated within perturbation theory. We first discuss the behavior of the correlator R at the critical point $\delta = 2\pi g$, and then discuss its behavior for slight deviations from this critical point such that $\delta > 2\pi g$.

The correlation function outside the light cone is determined by setting $l = \Lambda_0 T_{m0}$ in Eq. (30), and using $R(\frac{r}{2T_{m0}} \gg 1, l = \Lambda_0 T_{m0}, g = 0) \sim \frac{T_{m0}}{r}$ (see Sec. III). At the critical point $\delta = 2\pi g$ this gives

$$R\,(r \gg 2T_{m0}) \sim \frac{\sqrt{\ln T_{m0}}}{r}. \quad (31)$$

The correlation function within and inside the light cone is obtained by setting $l = \Lambda_0 r$ in Eq. (30), Moreover noting that $R\left(l = r\Lambda_0, \frac{T_{m0}}{r} \gg 1, g = 0\right) \sim \mathcal{O}(1)$ this gives the following correlator inside the light cone at the critical point:

$$R(r, 2T_{m0} \gg r) \sim \frac{\sqrt{\ln r}}{r}. \quad (32)$$

In a similar way, using $R(l = r\Lambda_0, \frac{2T_{m0}}{r} = 1, g = 0) \sim \mathcal{O}(1)$, the correlator on the light cone is found to be

$$R(r = 2T_{m0}) = \frac{\ln r}{r}. \quad (33)$$

Equations (31)–(33) are the main results for the correlator for a pure lattice quench at the critical point $\delta = 2\pi g$. Thus in the prethermalized regime the result Eq. (32) within the light cone for a pure lattice quench is the same as in the ground state of the Heisenberg chain.[35–37] In contrast, the quench leads to qualitatively new scaling behavior for spatial separations outside the light cone [Eq. (31)] and on the light cone [Eq. (33)]. Note that for a pure lattice quench, no dissipative effects are generated to $\mathcal{O}(g^2)$,[9] extending the regime of validity of the prethermalized regime.

For slight deviations $\delta > 2\pi g$ from the critical point, we obtain the following correlator outside the light cone:

$$R\,(\Lambda_0 r,\Lambda_0 T_{m0},g_0,r \gg 2T_{m0}) \sim \left(\frac{1}{r}\right)^{1+\delta/2} (T_{m0})^{\delta/2-(\sqrt{\delta^2-(2\pi g)^2})/2}. \quad (34)$$

Thus the correlator outside the light cone is primarily the one in the initial state with a time-dependent prefactor which depends on the strength g of the periodic potential. Eventually this time dependence drops off at very long times, with the correlator taking the following steady-state value inside the light cone:

$$R\,(\Lambda_0 r,\Lambda_0 T_{m0},g_0,r \ll 2T_{m0}) \sim \frac{1}{r^{1+(1/2)\sqrt{\delta^2-(2\pi g)^2}}}. \quad (35)$$

Unlike the case of the dynamics on the critical point, for slight deviations from the critical point $\delta > 2\pi g$, the leading asymptote for the correlator on and inside the light cone is the same. The expressions for the correlators in Eqs. (34) and (35) are valid for $(\sqrt{\delta^2 - (2\pi g)^2}) \ln(\min[T_{m0},r]) \gg 1$, whereas the universal logarithmic corrections discussed before this in Eqs. (31)–(33) are valid for the opposite case of $(\sqrt{\delta^2 - (2\pi g)^2}) \ln(\min[T_{m0},r]) \ll 1$.

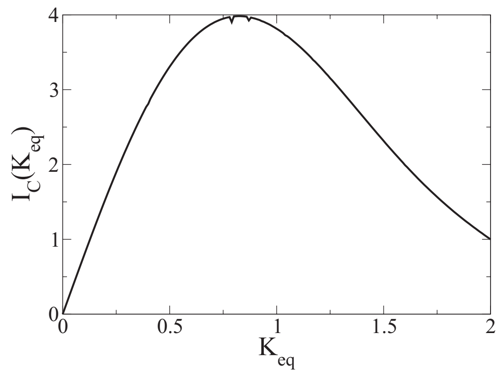

FIG. 4. Plot of I_C for a simultaneous lattice and interaction quench for points within the light cone and near the critical point $K_{\rm neq} = 2$ where $K_{\rm eq} = \frac{\gamma^2 K_0}{4}\sqrt{\frac{16}{\gamma^2 K_0} - 1}$.

Simultaneous lattice and interaction quench. Now we turn to the case where both the Luttinger interaction parameter and the cosine potential are simultaneously quenched at $T_m = 0$. Moreover we are interested in the physics close to the nonequilibrium critical point defined by $K_{\rm neq} = 2 + \delta, \forall\ 0 < \delta \ll 1$. This quench corresponds to a final Hamiltonian whose ground state can be in the gapped phase. However, due to the quench, since $K_{\rm neq} > K_{\rm eq}$, the periodic potential is more irrelevant (though it can give rise to inelastic scattering). Thus perturbation theory in the cosine potential may be valid for the nonequilibrium problem, even though it may not hold for determining the properties of the ground state.

For this general quench we find that I_K in Eq. (16) is $I_K(T_m, K_{\rm neq} = 2) = \pi\,\{c_1 - c_2 \sin[\pi(K_{\rm eq} - K_{\rm neq})]\ \ln(\min[T_m, 1/\eta])\}$ where $c_{1,2}$ are $\mathcal{O}(1)$ and depend on the initial Luttinger parameter K_0. When $K_0 = K$, $c_1 = 1$. As before, we are interested in the prethermalized regime where $1 \ll T_m < 1/\eta$. We will also assume a small quench where $|K_{\rm eq} - K_{\rm neq}| \ln T_m \ll 1$. In this case, $I_K \simeq \pi c_1$ is a constant in time. The β function in the vicinity of $K_{\rm neq} = 2 + \delta$ becomes $\frac{dg}{d\ln l} = -g\delta, \frac{d\delta}{d\ln l} = -g^2 B^2$, where $B = \pi\sqrt{c_1 \frac{\gamma^2 K_0}{4}}(\frac{16}{\gamma^2 K_0} - 1)^{3/4}$. In addition, I_C in Eq. (25) becomes a universal function of the initial Luttinger parameter K_0 and is plotted in Fig. 4.

Using Eq. (31) the correlation function inside the light cone is found to be

$$R\,(\Lambda_0 r, r \ll 2T_{m0} \ll 1/\eta, g_0) \simeq \frac{1}{(\Lambda_0 r)^{1+A/2}} \frac{1}{\sqrt{1-(r\Lambda_0)^{-2A}}}\left[\frac{1-(\Lambda_0 r)^{-A}}{1+(\Lambda_0 r)^{-A}}\right]^{2\pi I_C/B}, \qquad (36)$$

where $A = \sqrt{\delta^2 - g^2 B^2}$. Thus for an interaction and lattice quench, and for $A \ln r \gg 1$, the correlators decay as a power law with exponent $(1 + A/2)$. This exponent is not the same as in an interaction quench in a quadratic Luttinger liquid theory, which would have been $1 + \delta/2$. Thus the lattice, even though irrelevant, modifies the decay exponent. In the vicinity of the nonequilibrium critical point $A \to 0$, logarithmic corrections are obtained for $A \ll A \ln r \ll 1$, where

$$R(2T_{m0} \gg r) \sim \frac{1}{r}(\ln r)^{2\pi I_C/B - 1/2}. \qquad (37)$$

This steady-state behavior in the vicinity of the nonequilibrium critical point is significantly different from that near the equilibrium critical point, which is[35–37] $\sqrt{\ln r}/r$.

Let us briefly discuss scaling in the other two regimes: one is on the light cone and the other is outside the lightcone. For the former we find

$$I_C(r = 2T_{m0}) \sim r^{K_{\rm tr}/2}, \qquad (38)$$

where $K_{\rm tr} = \frac{\gamma^2 K_0}{4} - 2$. Thus scaling is recovered only if $K_{\rm tr} \leqslant 0$ where $I_C \to 0$ or to a constant at large distances, whereas for $K_{\rm tr} > 0$ scaling is lost as I_C grows with distance, only to be cut off at $r \sim 1/\eta$ where we expect the correlators to begin decaying in a thermal manner.

In contrast to the above, outside the light cone we find

$$I_C(r \gg 2T_{m0}) \sim \left(\frac{2T_{m0}}{r}\right)^{K_{\rm tr}}. \qquad (39)$$

Here when $K_{\rm tr} \geqslant 0$, scaling holds outside the light cone as I_C is either a constant or decays to zero for sufficiently large distances. On the other hand when $K_{\rm tr} < 0$, I_C grows with distance. This behavior is opposite to what one finds on the light cone, and is consistent with the fact that the behavior of the correlator outside the light cone is primarily determined by the initial wave function. Thus if the cosine potential is a relevant perturbation in the initial state $K_{\rm tr} < 0$, then the perturbative corrections are large indicating that perturbation theory may not be valid at large distances outside the light cone, even though it may be valid inside the light cone. This behavior is also consistent with the crossover in time of the scaling dimension of $V_{\rm sg}$ discussed earlier in this section.

The remaining part of the paper outlines how the above results were obtained. In Sec. III we reintroduce the model and briefly present results for an interaction quench in the quadratic theory (the Luttinger liquid) in the language of Keldysh Green's functions. These results will be used in later sections when we do perturbation theory in the cosine potential. In Sec. IV, we do perturbation theory in the cosine potential and derive the β function to two loops. In Sec. V we present results for the correlation function within perturbation theory to leading order in the cosine potential. These results set the stage for doing renormalization improved perturbation theory which will be explicitly carried out in Sec. VI where the CS equation for the correlation function is derived. Results of solution of the CS equation are presented in Sec. VII for the case where only the lattice potential is quenched, while results for the correlation function for a simultaneous lattice and interaction quench are presented in Sec. VIII. Finally in Sec. IX we summarize our results and discuss open questions.

III. MODEL AND GREEN'S FUNCTIONS FOR THE QUADRATIC THEORY

In order to study quench dynamics of the spin chain, we employ a bosonization prescription where

$$\phi(x) = -(N_R + N_L)\frac{\pi x}{L} - \frac{i\pi}{L}\sum_{p\neq 0}\left(\frac{L|p|}{2\pi}\right)^{1/2} \times \frac{1}{p}e^{-\alpha|p|/2 - ipx}(b_p^\dagger + b_{-p}), \qquad (40)$$

$$\theta(x) = (N_R - N_L)\frac{\pi x}{L} + \frac{i\pi}{L}\sum_{p\neq 0}\left(\frac{L|p|}{2\pi}\right)^{1/2} \times \frac{1}{|p|}e^{-\alpha|p|/2-ipx}(b_p^\dagger - b_{-p}). \quad (41)$$

We choose the initial Hamiltonian for $t \leqslant 0$ to be a Luttinger liquid,

$$H_i = \frac{u_0}{2\pi}\int dx\left[K_0\{\pi\Pi(x)\}^2 + \frac{1}{K_0}\{\partial_x\phi(x)\}^2\right] = \sum_{p\neq 0} u_0|p|\eta_p^\dagger\eta_p, \quad (42)$$

while the time evolution from $t > 0$ is due to the quantum sine-Gordon model,

$$H_f = H_{f0} + V_{\rm sg}, \quad (43)$$

$$H_{f0} = \frac{u}{2\pi}\int dx\left[K\{\pi\Pi(x)\}^2 + \frac{1}{K}\{\partial_x\phi(x)\}^2\right] = \sum_{p\neq 0} u|p|\gamma_p^\dagger\gamma_p, \quad (44)$$

$$V_{\rm sg} = -\frac{gu}{\alpha^2}\int dx\cos(\gamma\phi). \quad (45)$$

We will make the assumption that the quench connects the same zero-mode sectors of the initial and final Hamiltonian. In this case, the zero modes corresponding to the first terms in Eqs. (40) and (41) will not play a role in the dynamics.

We study a quench that preserves Galilean invariance, i.e., $u = v_F/K, u_0 = v_F/K_0$. At the microscopic level, this corresponds to a quench in the Luttinger model where the g_2 and g_4 interactions equal each other for both the initial and final Hamiltonians ($g_{2i} = g_{4i}, g_{2f} = g_{4f}$).[34] While this simplifies the algebra, relaxing this requirement is straightforward, and does not change the results in a qualitative way. The three bosonic operators b,η,γ are related by a linear Bogoliubov transformation,

$$\begin{pmatrix} b_p \\ b_{-p}^\dagger\end{pmatrix} = \begin{pmatrix}\cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta\end{pmatrix}\begin{pmatrix}\gamma_p \\ \gamma_{-p}^\dagger\end{pmatrix}, \quad (46)$$

$$\begin{pmatrix} b_p \\ b_{-p}^\dagger\end{pmatrix} = \begin{pmatrix}\cosh\beta_0 & -\sinh\beta_0 \\ -\sinh\beta_0 & \cosh\beta_0\end{pmatrix}\begin{pmatrix}\eta_p \\ \eta_{-p}^\dagger\end{pmatrix}, \quad (47)$$

where $e^{-2\beta_0} = K_0$, $e^{-2\beta} = K$.

Let us define the functions

$$f(pt) = \cos(u|p|t)\cosh\beta_0 - i\sin(u|p|t)\cosh(2\beta - \beta_0), \quad (48)$$

$$g(pt) = \cos(u|p|t)\sinh\beta_0 + i\sin(u|p|t)\sinh(2\beta - \beta_0), \quad (49)$$

which determine the time evolution after the quench ($t > 0$) for the quadratic theory ($g = 0$),

$$b_p^\dagger(t) + b_{-p}(t) = (f^*(pt) - g(pt))\eta_p^\dagger(0) + (f(pt) - g^*(pt))\eta_{-p}(0), \quad (50)$$

$$b_p^\dagger(t) - b_{-p}(t) = (f^*(pt) + g(pt))\eta_p^\dagger(0) - (f(pt) + g^*(pt))\eta_{-p}(0). \quad (51)$$

Since the system is out of equilibrium, it is convenient to study the problem using the Keldysh formalism. The Keldysh action is

$$Z_K = \mathrm{Tr}[\rho(t)] = \mathrm{Tr}[e^{-iH_f t}|\psi_i\rangle\langle\psi_i|e^{iH_f t}] \quad (52)$$

$$= \int \mathcal{D}[\phi_{cl},\phi_q]e^{i(S_0+S_{\rm sg})}, \quad (53)$$

where S_0 is the quadratic part which describes the physics in the absence of the periodic potential which corresponds to an interaction quench in a Luttinger liquid. In particular at a time t after the quench (note that the fields ϕ are real),

$$S_0 = \frac{1}{2}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\int_0^t dt_1\int_0^t dt_2(\phi_{cl}(1)\phi_q(1)) \times \begin{pmatrix} 0 & G_A^{-1}(1,2) \\ G_R^{-1}(1,2) & -\left[G_R^{-1}G_K G_A^{-1}\right](1,2)\end{pmatrix}\begin{pmatrix}\phi_{cl}(2) \\ \phi_q(2)\end{pmatrix}, \quad (54)$$

where $1 = (x_1,t_1)$, $2 = (x_2,t_2)$, and

$$\phi_{cl,q} = \frac{\phi_- \pm \phi_+}{\sqrt{2}}, \quad (55)$$

where $G_{R,A,K}$ are the retarded, advanced, and Keldysh Green's functions, with

$$[G_{R,A}(1,2)]^{-1} = -\delta(x_1 - x_2)\delta(t_1 - t_2)\frac{1}{\pi K u}\left[\partial^2_{t_1\pm i\delta} - u^2\partial^2_{x_1}\right] \quad (56)$$

and

$$-i\left\langle\begin{pmatrix}\phi_{cl}(1) \\ \phi_q(1)\end{pmatrix}(\phi_{cl}(2)\phi_q(2))\right\rangle = \begin{pmatrix} G_K(1,2) & G_R(1,2) \\ G_A(1,2) & 0\end{pmatrix}, \quad (57)$$

whereas

$$S_{\rm sg} = \frac{gu}{\alpha^2}\int_{-\infty}^{\infty}dx\int_0^t dt_1 \times [\cos\{\gamma\phi_-(1)\} - \cos\{\gamma\phi_+(1)\}]. \quad (58)$$

For the quadratic theory after the quench,

$$G_R(x_1t_1,x_2t_2) = -i\theta(t_1 - t_2)\langle[\phi(x_1t_1),\phi(x_2t_2)]\rangle = -\frac{K}{2}\theta(t_1 - t_2)\sum_{\epsilon=\pm}\tan^{-1}\left(\frac{u(t_1 - t_2) + \epsilon(x_1 - x_2)}{\alpha}\right), \quad (59)$$

$$G_A(x_1t_1,x_2t_2) = i\theta(t_1 - t_2)\langle[\phi(x_1t_1),\phi(x_2t_2)] = \frac{K}{2}\theta(t_2 - t_1)\sum_{\epsilon=\pm}\tan^{-1}\left(\frac{u(t_1 - t_2) + \epsilon(x_1 - x_2)}{\alpha}\right), \quad (60)$$

and

$$G_K(x_1t_1,x_2t_2) = -i\langle\{\phi(x_1t_1),\phi(x_2t_2)\}\rangle \tag{61}$$

$$= -i\frac{K_0}{4}\left(1+\frac{K^2}{K_0^2}\right)\int_0^\infty \frac{dp}{p}e^{-\alpha p} \times \sum_{\epsilon=\pm}\cos[up(t_1-t_2)+\epsilon p(x_1-x_2)] - i\frac{K_0}{4}\left(1-\frac{K^2}{K_0^2}\right)\int_0^\infty \frac{dp}{p}e^{-\alpha p} \times \sum_{\epsilon=\pm}\cos[up(t_1+t_2)+\epsilon p(x_1-x_2)]. \tag{62}$$

Note that G_K is logarithmically divergent, but in all physical quantities it is always the combination $G_K(1,2) - \frac{1}{2}G_K(1,1) - \frac{1}{2}G_K(2,2)$ that appears, which is finite.

Let us define

$$C_{ab,m}(x_1t_1,x_2t_2) = \langle e^{im\gamma\phi_a(x_1t_1)}e^{-im\gamma\phi_b(x_2t_2)}\rangle = e^{-(\gamma^2m^2/2)[iG_K(11)/2+iG_K(22)/2-iG_K(12)+iaG_A(1,2)+ibG_R(12)]}, \tag{63}$$

where

$$-\frac{\gamma^2}{2}\left[\frac{iG_K(1,1)}{2}+\frac{iG_K(2,2)}{2}-iG_K(1,2)\right] = -K_{\rm neq}\sum_{\epsilon=\pm}\left[\ln\frac{\sqrt{\alpha^2+\{u(t_1-t_2)+\epsilon(x_1-x_2)\}^2}}{\alpha}\right] - K_{\rm tr}\left[\ln\sqrt{\frac{\alpha^2+\{u(t_1+t_2)+(x_1-x_2)\}^2}{\alpha^2+(2ut_1)^2}} + \ln\sqrt{\frac{\alpha^2+\{u(t_1+t_2)-(x_1-x_2)\}^2}{\alpha^2+(2ut_2)^2}}\right], \tag{64}$$

with the coefficients $K_{\rm eq},K_{\rm neq},K_{\rm tr}$ defined in Eq. (13). The above implies that the equal-time correlator is given by (setting $u=1$ and $\Lambda = 1/\alpha$)

$$C_{m=1}(rt,0t) = \frac{1}{(\sqrt{1+\Lambda^2r^2})^{2K_{\rm neq}}} \times \left[\frac{\sqrt{1+\Lambda^2(2t+r)^2}\sqrt{1+\Lambda^2(2t-r)^2}}{1+(2\Lambda t)^2}\right]^{-K_{\rm tr}}. \tag{65}$$

At equal times $C_{ab,m}$ does not depend on the ab indices since $G_{R,A}(t,t) = 0$. Therefore we have dropped the ab indices.

At positions and times large as compared to the cutoff and far from the light cone $|r \pm 2t| \gg 1$ (with r,t measured in units of Λ)

$$C_{m=1}(rt,0t) = \left[\frac{1}{r^2}\right]^{\gamma^2K_0/4}\left[\left|\frac{r^2-(2t)^2}{r^2(2t)^2}\right|\right]^{-K_{\rm tr}}. \tag{66}$$

This agrees with Ref. 49 where it was pointed out that outside the light cone ($2t \ll r$) the correlator decays in position in the same way as in the initial state, but with a time-dependent prefactor which goes as $t^{2K_{\rm tr}}$. Moreover within the light cone $r \ll 2t$, the correlator reaches a steady-state value where it decays in position with the new nonequilibrium exponent $K_{\rm neq}$, $C_{m=1}(r \ll 2t) \sim \frac{1}{r^{2K_{\rm neq}}}$. Exactly on the light cone $C_{m=1}(r = 2t) \simeq [\frac{1}{r^2}]^{\gamma^2K_0/4}[\frac{1}{r^3}]^{-K_{\rm tr}}$.

In this paper we aim to calculate the correlator $R_{ab} = \langle e^{i\gamma\phi_a(1)/2}e^{-i\gamma\phi_b(2)/2}\rangle = C_{ab,m=1/2}$ at equal times and unequal positions in the presence of the cosine potential. In particular we will derive a CS-like differential equation that will allow us to relate the correlator at large scales of the bare theory to the correlator at short scales and renormalized couplings, where the latter is well approximated by the correlator of the free theory. Three useful results for the free correlator at short scales that will be used later are for points within the light cone $r \ll 2t, \Lambda r \sim 1$, points outside the light cone $r \gg 2t, \Lambda t \sim 1$, and points on the light cone $r = 2t, \Lambda r \sim 1$. For these three cases using Eq. (65) we find

$$C_{m=1/2}(r \ll 2t, \Lambda r = 1) \sim \mathcal{O}(1), \tag{67}$$

$$C_{m=1/2}(r = 2t, \Lambda r = 1) \sim \mathcal{O}(1), \tag{68}$$

$$C_{m=1/2}(r \gg 2t, \Lambda t = 1) \sim \left(\frac{t}{r}\right)^{K_{\rm neq}/2}\left(\frac{r}{t}\right)^{-K_{\rm tr}/2}. \tag{69}$$

IV. DERIVATION OF THE β FUNCTION FROM THE ACTION

In this section we discuss how the β function is derived from the Keldysh action. We split the fields into slow ($\phi^<$) and fast ($\phi^>$) components where the fast components have a large weight at high momentum, and therefore oscillate rapidly in time,[8,9,17,33]

$$\phi_\pm = \phi_\pm^< + \phi_\pm^>. \tag{70}$$

We integrate out the fast fields perturbatively in the cosine potential. In doing so to $\mathcal{O}(g^2)$, we obtain

$$S = S_0^< + \delta S^<, \tag{71}$$

where $S_0^<$ is the quadratic action for the slow fields,

$$S_0^< = \int_{-\infty}^\infty dR\int_0^{ut/\sqrt{2}} d(uT_m)\frac{1}{2\pi K}\Big[\phi_q^<\left(\partial_R^2-\partial_{uT_m}^2\right)\phi_{cl}^< + \phi_{cl}^<\left(\partial_R^2-\partial_{uT_m}^2\right)\phi_q^< + \frac{\delta u}{u}\phi_q^<\left(\partial_R^2+\partial_{uT_m}^2\right)\phi_{cl}^< + \frac{\delta u}{u}\phi_{cl}^<\left(\partial_R^2+\partial_{uT_m}^2\right)\phi_q^< - 2\frac{\eta}{u}\phi_q^<\partial_{uT_m}\phi_{cl}^< + i\frac{4\eta T_{\rm eff}}{u^2}(\phi_q^<)^2\Big]. \tag{72}$$

Above δu represents corrections to the velocity, η represents dissipation of the long-wavelength modes, and $\eta T_{\rm eff}$ denotes the strength of the noise on the long-wavelength modes. Initially, $\delta u = T_{\rm eff} = \eta = 0$, but in the first step of the RG, as we show below, the cosine potential generates corrections (contained in $\delta S^<$) that not only renormalize the interaction parameter K and the velocity u, but also generates the dissipative term ($\phi_q\partial_{T_m}\phi_{cl}$) and noise term ($\phi_q^2$).

The corrections arising from integrating the fast fields are

$$\delta S^< = \frac{g\Lambda^2}{u}\int_{-\infty}^{\infty} dx \int_0^t dt_1 \times [\cos\gamma\phi_-^<(1) - \cos\gamma\phi_+^<(1)]e^{-\frac{\gamma^2}{4}\langle(\phi_{cl}^>(1))^2\rangle} \quad (73)$$

$$+\frac{ig^2\Lambda^4}{2u^2}\int_{-\infty}^{\infty} dx_1 \int_0^t dt_1 \int_{-\infty}^{\infty} dx_2 \int_0^t dt_2 \theta(t_1-t_2) \times :\cos[\gamma\phi_-^<(1) - \gamma\phi_-^<(2)]: e^{-(\gamma^2/2)\langle[\phi_-(1)-\phi_-(2)]^2\rangle} \times \left[1 - e^{-\gamma^2\langle\phi_-^>(1)\phi_-^>(2)\rangle}\right] \quad (74)$$

$$+\frac{ig^2\Lambda^4}{2u^2}\int_{-\infty}^{\infty} dx_1 \int_0^t dt_1 \int_{-\infty}^{\infty} dx_2 \int_0^t dt_2 \times \theta(t_2-t_1){:}\cos[\gamma\phi_+^<(1) - \gamma\phi_+^<(2)] \times {:}e^{-(\gamma^2/2)\langle[\phi_+(1)-\phi_+(2)]^2\rangle}\left[1 - e^{-\gamma^2\langle\phi_+^>(1)\phi_+^>(2)\rangle}\right] \quad (75)$$

$$-\frac{ig^2\Lambda^4}{2u^2}\int_{-\infty}^{\infty} dx_1 \int_0^t dt_1 \int_{-\infty}^{\infty} dx_2 \int_0^t dt_2 \times \{\theta(t_1-t_2)+\theta(t_2-t_1)\}{:}\cos[\gamma\phi_+^<(1) - \gamma\phi_-^<(2)] \times {:}e^{-(\gamma^2/2)\langle[\phi_+(1)-\phi_-(2)]^2\rangle} \times \left[1 - e^{-\gamma^2\langle\phi_+^>(1)\phi_-^>(2)\rangle}\right]. \quad (76)$$

Above we have used that $\cos(a) =: \cos(a) : e^{-\langle a^2\rangle/2}$ where the operators inside the symbol :: are normal ordered. Moreover, all expectation values $\langle\cdots\rangle$ are with respect to the initial state, and therefore depend on the quench. Since the correlators for the full fields are related to the correlator for the slow and fast fields as follows $G = G^< + G^>$, the correlator for the fast fields may be related in a simple way to derivatives of the full correlators,[56]

$$G^> = d\Lambda \frac{dG}{d\Lambda}. \quad (77)$$

Explicit expressions for the fast correlators at equal time are

$$\langle[\phi_{cl}^>(t)]^2\rangle = \frac{d\Lambda}{\Lambda}\left[\frac{K_0}{2}\left(1+\frac{K^2}{K_0^2}\right) + \frac{K_0}{2}\left(1-\frac{K^2}{K_0^2}\right)\left\{\frac{1}{1+(2t\Lambda)^2}\right\}\right] \quad (78)$$

$$\xrightarrow{t\ll 1/\Lambda} \frac{d\Lambda}{\Lambda}K_0 \quad (79)$$

$$\xrightarrow{t\gg 1/\Lambda} \frac{d\Lambda}{\Lambda}\frac{K_0}{2}\left(1+\frac{K^2}{K_0^2}\right). \quad (80)$$

The short- and long-time limits of $\langle[\phi_{cl}^>(t)]^2\rangle$ reflect the fact that at short times after the quench, it is the initial wave function and hence the initial Luttinger parameter K_0 that determines the behavior of the correlators, while at long times, a new nonequilibrium exponent related to K_{neq} determines the behavior.

For the nonlocal fast correlators, we have

$$\langle\phi_{cl}^>(1)\phi_{cl}^>(2)\rangle = \frac{d\Lambda}{\Lambda}\sum_{\epsilon=\pm}\left[\frac{K_0}{4}\left(1+\frac{K^2}{K_0^2}\right)\frac{1}{1+\Lambda^2[(t_1-t_2)+\epsilon(x_1-x_2)/u]^2} + \frac{K_0}{4}\left(1-\frac{K^2}{K_0^2}\right)\frac{1}{1+\Lambda^2[(t_1+t_2)+\epsilon(x_1-x_2)/u]^2}\right] \quad (81)$$

$$\langle\phi_{cl}^>(1)\phi_q^>(2)\rangle = -i\frac{K}{2}\frac{d\Lambda}{\Lambda}\theta(t_1-t_2) \times \sum_{\epsilon=\pm}\left[\frac{\Lambda[(t_1-t_2)+\epsilon(x_1-x_2)/u]}{1+\Lambda^2[(t_1-t_2)+\epsilon(x_1-x_2)/u]^2}\right] \quad (82)$$

$$\langle\phi_q^>(1)\phi_{cl}^>(2)\rangle = i\frac{K}{2}\frac{d\Lambda}{\Lambda}\theta(t_2-t_1) \times \sum_{\epsilon=\pm}\left[\frac{\Lambda[(t_1-t_2)+\epsilon(x_1-x_2)/u]}{1+\Lambda^2[(t_1-t_2)+\epsilon(x_1-x_2)/u]^2}\right]. \quad (83)$$

In the next step we define new variables corresponding to center of mass (R,T_m) and relative coordinates (r,τ),

$$R = \frac{x_1+x_2}{2}, \quad T_m = \frac{t_1+t_2}{2}, \quad (84)$$

$$r = x_1 - x_2, \quad \tau = t_1 - t_2. \quad (85)$$

Thus

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 = \int_{-\infty}^{\infty} dR \int_{-\infty}^{\infty} dr, \quad (86)$$

$$\int_0^t dt_1 \int_0^t dt_2 = \int_0^{t/\sqrt{2}} dT_m \int_{-2T_m}^{2T_m} d\tau. \quad (87)$$

Since quantities have a slower variation with respect to the center-of-mass coordinates as compared to the relative coordinates, we perform a gradient expansion in R,T_m and obtain

$$\delta S^< = \delta S_g^< + \delta S_0^< + \delta S_{T_{\text{eff}}}^< + \delta S_\eta^<, \quad (88)$$

where

$$\delta S_g^< = \frac{gu}{\alpha^2}\int_{-\infty}^{\infty} dx_1 \int_0^t dt_1[\cos\gamma\phi_-^<(1) - \cos\gamma\phi_+^<(1)] \times e^{-(\gamma^2/4)\langle[\phi_{cl}^>(1)]^2\rangle} \quad (89)$$

and

$$\delta S_0^< = \frac{g^2\Lambda^4\gamma^2}{2u^2}\frac{d\Lambda}{\Lambda}\int_{-\infty}^{\infty} dR \int_{-\infty}^{\infty} dr \int_0^{t/\sqrt{2}} dT_m \int_{-2T_m}^{2T_m} d\tau \times \theta(\tau)[(r\partial_R\phi_{cl}^<)(r\partial_R\phi_q^<) + (\tau\partial_{T_m}\phi_{cl}^<)(\tau\partial_{T_m}\phi_q^<)] \times \text{Im}\left[e^{-(\gamma^2/2)\langle[\phi_+(R+r/2,T_m+\tau/2)-\phi_-(R-r/2,T_m-\tau/2)]^2\rangle} \times F(r,T_m,\tau)\right] \quad (90)$$

while

$$\delta S_{T_{\text{eff}}}^< = \frac{ig^2\Lambda^4\gamma^2}{2u^2}\frac{d\Lambda}{\Lambda}\int_{-\infty}^{\infty} dR \int_{-\infty}^{\infty} dr \times \int_0^{t/\sqrt{2}} dT_m \int_{-2T_m}^{2T_m} d\tau[\phi_q^<(R,T_m)]^2 \times \text{Re}\left[e^{-(\gamma^2/2)\langle[\phi_+(R+r/2,T_m+\tau/2)-\phi_-(R-r/2,T_m-\tau/2)]^2\rangle} \times F(r,T_m,\tau)\right] \quad (91)$$

and

$$\delta S_\eta^< = \frac{g^2\Lambda^4\gamma^2}{2u^2}\frac{d\Lambda}{\Lambda}\int_{-\infty}^{\infty} dR \int_{-\infty}^{\infty} dr \int_0^{t/\sqrt{2}} dT_m \int_{-2T_m}^{2T_m} d\tau \times \phi_q^<(R,T_m)\tau\partial_{T_m}\phi_{cl}^<(R,T_m) \times \mathrm{Im}\big[e^{-(\gamma^2/2)\langle[\phi_+(R+r/2,T_m+\tau/2)-\phi_-(R-r/2,T_m-\tau/2)]^2\rangle} \times F(r,T_m,\tau)\big], \tag{92}$$

where

$$F(r,T_m,\tau) = K_{\rm neq}\left[\frac{1}{1+\Lambda^2(\tau+r/u)^2}+\frac{1}{1+\Lambda^2(\tau-r/u)^2}\right] + K_{\rm tr}\left[\frac{1}{1+\Lambda^2(2T_m+r/u)^2}+\frac{1}{1+\Lambda^2(2T_m-r/u)^2}\right] - iK_{\rm eq}\left[\frac{\Lambda(\tau+r/u)}{1+\Lambda^2(\tau+r/u)^2}+\frac{\Lambda(\tau-r/u)}{1+\Lambda^2(\tau-r/u)^2}\right] \tag{93}$$

whereas

$$e^{-(\gamma^2/2)\langle[\phi_+(R+r/2,T_m+\tau/2)-\phi_-(R-r/2,T_m-\tau/2)]^2\rangle} = \left[\frac{1}{\sqrt{1+\Lambda^2(\tau+r/u)^2}}\frac{1}{\sqrt{1+\Lambda^2(\tau-r/u)^2}}\right]^{K_{\rm neq}} \times \left[\frac{\sqrt{1+\Lambda^2\{2(T_m+\tau/2)\}^2}}{\sqrt{1+\Lambda^2(2T_m+r/u)^2}}\frac{\sqrt{1+\Lambda^2\{2(T_m-\tau/2)\}^2}}{\sqrt{1+\Lambda^2(2T_m-r/u)^2}}\right]^{K_{\rm tr}} \times e^{-iK_{\rm eq}[\tan^{-1}[\Lambda(\tau+r/u)]+\tan^{-1}[\Lambda(\tau-r/u)]]} \tag{94}$$

and $\mathrm{Re}[A]=(A+A^*)/2, \mathrm{Im}[A]=(A-A^*)/(2i)$.

Collecting all terms we find

$$\delta S_g^< = g\Lambda^2\int_{-\infty}^{\infty} d(R/u)\int_0^t dT_m[\cos\gamma\phi_-^<(R,T_m) - \cos\gamma\phi_+^<(R,T_m)]e^{-(\gamma^2/4)\langle[\phi_{cl}^>(T_m)]^2\rangle}, \tag{95}$$

$$\delta S_0^< = \frac{g^2\gamma^2}{2}\frac{d\Lambda}{\Lambda}\int_{-\infty}^{\infty} d(R/u)\int_0^{t/\sqrt{2}} dT_m \times \big[-I_R(T_m)(\partial_{R/u}\phi_{cl}^<)(\partial_{R/u}\phi_q^<) - I_{T_m}(T_m)\left(\partial_{T_m}\phi_{cl}^<\right)\left(\partial_{T_m}\phi_q^<\right)\big], \tag{96}$$

$$\delta S_{T_{\rm eff}}^< = \frac{ig^2\gamma^2\Lambda^2}{2}\frac{d\Lambda}{\Lambda}\int_{-\infty}^{\infty} d(R/u)\int_0^{t/\sqrt{2}} dT_m \times (\phi_q^<)^2 I_{T_{\rm eff}}(T_m), \tag{97}$$

$$\delta S_\eta^< = -\frac{g^2\gamma^2\Lambda}{2}\frac{d\Lambda}{\Lambda}\int_{-\infty}^{\infty} d(R/u)\int_0^{t/\sqrt{2}} dT_m \times \phi_q^<[\partial_{T_m}\phi_{cl}^<]I_\eta(T_m), \tag{98}$$

where

$$I_R(T_m) = -\Lambda^4\int_{-\infty}^{\infty} d(r/u)\int_0^{2T_m} d\tau (r/u)^2 \times \mathrm{Im}\big[e^{-(\gamma^2/2)\langle[\phi_+(R+r/2,T_m+\tau/2)-\phi_-(R-r/2,T_m-\tau/2)]^2\rangle} \times F(r,T_m,\tau)\big], \tag{99}$$

$$I_{T_m}(T_m) = -\Lambda^4\int_{-\infty}^{\infty} d(r/u)\int_0^{2T_m} d\tau\tau^2 \times \mathrm{Im}\big[e^{-(\gamma^2/2)\langle[\phi_+(R+r/2,T_m+\tau/2)-\phi_-(R-r/2,T_m-\tau/2)]^2\rangle} \times F(r,T_m,\tau)\big], \tag{100}$$

$$I_{T_{\rm eff}}(T_m) = \Lambda^2\int_{-\infty}^{\infty} d(r/u)\int_{-2T_m}^{2T_m} d\tau \times \mathrm{Re}\big[e^{-(\gamma^2/2)\langle[\phi_+(R+r/2,T_m+\tau/2)-\phi_-(R-r/2,T_m-\tau/2)]^2\rangle} \times F(r,T_m,\tau)\big], \tag{101}$$

$$I_\eta(T_m) = -\Lambda^3\int_{-\infty}^{\infty} d(r/u)\int_{-2T_m}^{2T_m} d\tau \tag{102}$$

$$\tau\,\mathrm{Im}\big[e^{-(\gamma^2/2)\langle[\phi_+(R+r/2,T_m+\tau/2)-\phi_-(R-r/2,T_m-\tau/2)]^2\rangle}F(r,T_m,\tau)\big], \tag{103}$$

$$I_K = I_R - I_{T_m}, \tag{104}$$

$$I_u = I_R + I_{T_m}. \tag{105}$$

At the next step we rescale the cutoff back to the original value of Λ, and in the process rescale position and time to $R,T_m \to \frac{\Lambda}{\Lambda'}(R,T_m)$, where $\Lambda' = \Lambda - d\Lambda$. This rescaling is not necessary in expressions for $\delta S^<_{0,T_{\rm eff},\eta}$ as they are already of $\mathcal{O}(\frac{d\Lambda}{\Lambda})$. We also express everything in dimensionless units of $\bar{R} = \Lambda R/u, \bar{T} = T\Lambda$. Thus, to summarize, one obtains

$$S_0^< = \int_{-\infty}^{\infty} d\bar{R}\int_0^{(t/\sqrt{2})\Lambda(\frac{\Lambda'}{\Lambda})} d\bar{T}_m \frac{1}{2\pi K}\bigg[\phi_q^<\left(\partial_{\bar{R}}^2-\partial_{\bar{T}_m}^2\right)\phi_{cl}^< + \phi_{cl}^<\left(\partial_{\bar{R}}^2-\partial_{\bar{T}_m}^2\right)\phi_q^< + \frac{\delta u}{u}\phi_q^<\left(\partial_{\bar{R}}^2+\partial_{\bar{T}_m}^2\right)\phi_{cl}^< + \frac{\delta u}{u}\phi_{cl}^<\left(\partial_{\bar{R}}^2+\partial_{\bar{T}_m}^2\right)\phi_q^< - 2\eta\left(\frac{\Lambda}{\Lambda'}\right)\phi_q^<\partial_{\bar{T}_m}\phi_{cl}^< + i4\eta T_{\rm eff}\left(\frac{\Lambda}{\Lambda'}\right)^2(\phi_q^<)^2\bigg] \tag{106}$$

and

$$\delta S_g^< = g\left(\frac{\Lambda}{\Lambda'}\right)^2\int_{-\infty}^{\infty} d\bar{R}\int_0^{t\Lambda(\Lambda'/\Lambda)} d\bar{T}_m[\cos\gamma\phi_-^<(R,T_m) - \cos\gamma\phi_+^<(R,T_m)]e^{-(\gamma^2/4)\langle[\phi_{cl}^>(T_m)]^2\rangle}, \tag{107}$$

$$\delta S_0^< = \frac{g^2\gamma^2}{2}\frac{d\Lambda}{\Lambda}\int_{-\infty}^{\infty} d\bar{R}\int_0^{t\Lambda/\sqrt{2}} d\bar{T}_m \times \big[-I_R(T_m)(\partial_{\bar{R}}\phi_{cl}^<)(\partial_{\bar{R}}\phi_q^<) - I_{T_m}(T_m)\left(\partial_{\bar{T}_m}\phi_{cl}^<\right)\left(\partial_{\bar{T}_m}\phi_q^<\right)\big], \tag{108}$$

$$\delta S_{T_{\rm eff}}^< = \frac{ig^2\gamma^2}{2}\frac{d\Lambda}{\Lambda}\int_{-\infty}^{\infty} d\bar{R}\int_0^{t\Lambda/\sqrt{2}} d\bar{T}_m(\phi_q^<)^2 I_{T_{\rm eff}}(T_m), \tag{109}$$

$$\delta S_\eta^< = -\frac{g^2\gamma^2}{2}\frac{d\Lambda}{\Lambda}\int_{-\infty}^{\infty} d\bar{R}\int_0^{t\Lambda/\sqrt{2}} d\bar{T}_m\phi_q^<\left[\partial_{\bar{T}_m}\phi_{cl}^<\right]I_\eta(T_m). \tag{110}$$

δS_η represents dissipation of the long-wavelength modes due to the integrated out high momentum modes, and appears as a term proportional to $\phi_q \partial_{T_m} \phi_{cl}$, with the strength of the dissipation $\eta(T_m)$ depending on the time after the quench. $\delta S_{T_{\rm eff}}$ represents terms that are proportional to ϕ_q^2 and represents the noise on the long-wavelength modes due to the integrated out modes. We denote the strength of this noise as $\eta(T_m)T_{\rm eff}(T_m)$ because in classical systems, the ratio of the noise and the dissipation strength gives a temperature. Here too, this allows us to define a time-dependent temperature. Defining $\ln l$ such that

$$\frac{\Lambda}{\Lambda'} = e^{d\ln(l)}; \quad \left|\frac{d\Lambda}{\Lambda}\right| = \frac{\Lambda - \Lambda'}{\Lambda} = d\ln(l). \tag{111}$$

Therefore the RG equations in terms of dimensionless variables such as $T_m \to T_m\Lambda, \eta \to \eta/\Lambda, T_{\rm eff} \to T_{\rm eff}/\Lambda$ are[17]

$$\frac{dg}{d\ln l} = g\left[2 - \left(K_{\rm neq} + \frac{K_{\rm tr}}{1+4T_m^2}\right)\right], \tag{112}$$

$$\frac{dK^{-1}}{d\ln l} = \frac{\pi g^2\gamma^2}{4} I_K(T_m), \tag{113}$$

$$\frac{1}{Ku}\frac{du}{d\ln l} = \frac{\pi g^2\gamma^2}{4} I_u(T_m), \tag{114}$$

$$\frac{d\eta}{d\ln l} = \eta + \frac{\pi g^2\gamma^2 K}{2} I_\eta(T_m), \tag{115}$$

$$\frac{d(\eta T_{\rm eff})}{d\ln l} = 2\eta T_{\rm eff} + \frac{\pi g^2\gamma^2 K}{4} I_{T_{\rm eff}}(T_m), \tag{116}$$

$$\frac{dT_m}{d\ln l} = -T_m. \tag{117}$$

Note that the renormalization of the velocity in Eq. (114) is a minor effect which will be neglected for the rest of the discussion. In the Appendix, the expression for $I_{K,u,\eta,T_{\rm eff}}$ is presented in dimensionless units. The physical meaning of the various terms of the β function has been discussed in detail in Sec. II.

V. PERTURBATIVE EVALUATION OF EQUAL TIME CORRELATION FUNCTION

We now turn to a perturbative evaluation of the correlator R defined in Eq. (7) to $\mathcal{O}(g)$. This will set the stage for the remaining sections where RG will be used to improve on this result revealing interesting nonequilibrium scaling regimes. Equation (112) shows that there is a crossover from an intermediate time dynamics where the physics is determined by the initial wave function (and hence the initial Luttinger parameter K_0) and a long-time dynamics determined by $K_{\rm neq}$. This can result in a situation where perturbation theory in g is violated at intermediate times when $\frac{\gamma^2 K_0}{4} < 2$ i.e., when $V_{\rm sg}$ is a relevant perturbation in the initial state. We show below what this implies for R.

Denoting

$$R(x_1t,x_2t) = R^{(0)} + R^{(1)} + \cdots, \tag{118}$$

where $R^{(i)}$ is the correlator to $\mathcal{O}(g^i)$, and using results from the previous section, we find

$$\begin{aligned} R^{(0)}&(x_1t,x_2t) \\ &= 2e^{-(\gamma^2/8)[iG_K(11)/2+iG_K(22)/2-iG_K(12)]} \\ &= 2\left[\left(\frac{1}{\sqrt{1+\frac{(x_1-x_2)^2}{\alpha^2}}}\right)^{2K_{\rm neq}}\right. \\ &\quad\times\left.\left(\frac{\sqrt{1+\frac{(2ut+x_1-x_2)^2}{\alpha^2}}\sqrt{1+\frac{(2ut-x_1+x_2)^2}{\alpha^2}}}{1+\frac{(2ut)^2}{\alpha^2}}\right)^{-K_{\rm tr}}\right]^{1/4}. \end{aligned} \tag{119}$$

The above gives Eq. (14) in the scaling limit (and setting $t = T_m$).

To next order the equal time correlator is

$$\begin{aligned} R^{(1)}&(x_1t,x_2t) \\ &= \frac{4igu}{\alpha^2}\int_{-\infty}^{\infty}dx'\int_0^t dt' \sum_{c=\pm}\mathrm{sgn}(-c) \\ &\quad\times\left\langle \cos\left[\gamma\phi_c(x't')\right]\cos\left(\frac{\gamma}{2}\phi(x_1t)\right)\cos\left(\frac{\gamma}{2}\phi(x_2t)\right)\right\rangle \\ &= \frac{igu}{\alpha^2}\int_{-\infty}^{\infty}dx'\int_0^t dt' \sum_{c=\pm}\mathrm{sgn}(-c) \\ &\quad\times\left\langle \cos\left(\gamma\phi_c(x't') - \frac{\gamma}{2}\phi(x_1t) - \frac{\gamma}{2}\phi(x_2t)\right)\right\rangle. \end{aligned} \tag{120}$$

On evaluating the expectation value one finds

$$\begin{aligned} R^{(1)}(x_1t,x_2t) &= \left(\frac{2gu}{\alpha^2}\right)e^{(\gamma^2/8)[iG_K(11)/2+iG_K(22)/2-iG_K(12)]}\int_{-\infty}^{\infty}dx'\int_0^t dt' \\ &\quad\times\sin\left[\frac{\gamma^2 K}{8}\sum_{\epsilon=\pm}\left\{\tan^{-1}\left(\frac{u(t-t')+\epsilon(x_1-x')}{\alpha}\right)+\tan^{-1}\left(\frac{u(t-t')+\epsilon(x_2-x')}{\alpha}\right)\right\}\right] \\ &\quad\times e^{-(\gamma^2/4)[iG_K(1'1')/2+iG_K(11)/2-iG_K(1'1)+iG_K(1'1')/2+iG_K(22)/2-iG_K(1'2)]}. \end{aligned} \tag{121}$$

The above expression shows that at microscopically short times $R^{(1)}(\Lambda t \ll 1) \propto \Lambda t^2$.

The behavior of $R^{(1)}$ for several different quench protocols is shown in Fig. 5. Since the initial state is gapless, the correlator is nonzero outside the light cone ($T_m < r/2$). Moreover, for a quench where the initial state is such that the cosine potential is relevant ($\frac{\gamma^2 K_0}{4} < 2$; dashed line in Fig. 5), R can be parametrically large at these initial times outside the light cone indicating that at these initial times perturbation theory in g is not valid. When the initial state is one where the potential is irrelevant or marginally irrelevant ($\frac{\gamma^2 K_0}{4} \geqslant 2$, solid and dotted line in Fig. 5), the correlator is most enhanced on

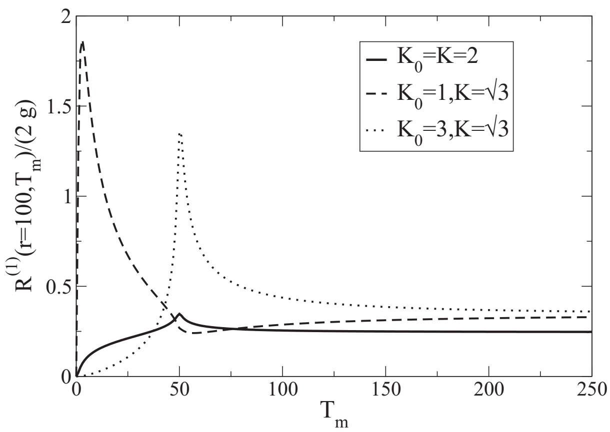

FIG. 5. The first-order correction to the equal-time correlation function $R^{(1)}(r=100,T_m)$ as a function of the time T_m after the quench. Length and time are measured in units of the cutoff Λ. Three different quenches are considered: a pure lattice quench where $K_0 = K = 2$ (solid line), lattice and interaction quench corresponding to $K_0 = 1, K = \sqrt{3}$ (dashed line), and $K_0 = 3, K = \sqrt{3}$ (dotted line). The light cone is at $T_m = r/2 = 50$. We choose $\gamma = 2$ and all quenches are to the critical point $K_{\rm neq} = 2$.

the light cone ($r = 2T_m$). For all these cases, within the light cone ($T_m > r/2$) the correlator reaches a steady state.

We now turn to an RG treatment for the correlator where a CS-like equation will be derived.

VI. DERIVATION OF A CALLAN-SYMANZIK-LIKE DIFFERENTIAL EQUATION OUT OF EQUILIBRIUM

In this section we carry out RG improved perturbation theory and derive a CS-like differential equation for the equal-time correlation function R. We split the fields ϕ into slow and fast fields, and integrate over the fast fields. In doing so, the leading-order ($g = 0$) correlator $R^{(0)}$ is found to be

$$R^{(0)}(x_1t,x_2t) = R^0_<(x_1t,x_2t)e^{-(d\Lambda/\Lambda)\gamma_{an,0}(x_1t,x_2t)}, \tag{122}$$

where $R_<$ is the correlator for the slow fields and R is the correlator for all the fields, and

$$\gamma_{an,0}(x_1t,x_2t) = \frac{1}{2}\left[K_{\rm neq}\frac{(x_1-x_2)^2\Lambda^2/u^2}{1+(x_1-x_2)^2\Lambda^2/u^2} + \frac{K_{\rm tr}}{1+(2t\Lambda)^2} - \frac{K_{\rm tr}}{2}\left\{\sum_{\epsilon=\pm}\frac{1}{1+\Lambda^2[2t+\epsilon(x_1-x_2)/u]^2}\right\}\right]. \tag{123}$$

Then we rescale the cutoff, and also rescale position and time. These transformations do not change the above expression. Expanding to $\mathcal{O}(\frac{d\Lambda}{\Lambda})$, and noting that

$$\Lambda\frac{R^{(0)}(\Lambda) - R^{(0)}_<(\Lambda - d\Lambda)}{d\Lambda} = \Lambda\frac{\partial R^{(0)}}{\partial\Lambda} = -\frac{\partial R^{(0)}}{\partial \ln l}, \tag{124}$$

we obtain the following CS-like equation to leading order:

$$\left[\frac{\partial}{\partial \ln l} - \gamma_{an,0}(x_1t,x_2t)\right]R^{(0)}\left(\frac{\Lambda_0}{l}\right) = 0, \tag{125}$$

where Λ_0 is the bare cutoff.

Some limiting expressions for $\gamma_{an,0}$ are as follows: At long distances and microscopically short times,

$$\gamma_{an,0}(|x_1-x_2|\Lambda \gg 1, t\Lambda \ll 1) = \frac{1}{2}(K_{\rm neq}+K_{\rm tr}) = \frac{\gamma^2K_0}{8}. \tag{126}$$

At long distances and times and inside the light cone,

$$\gamma_{an,0}(|x_1-x_2|\Lambda \gg 1, t\Lambda \gg 1, 2t \gg |x_1-x_2|) = \frac{K_{\rm neq}}{2}. \tag{127}$$

At long distances and times, and outside the light cone,

$$\gamma_{an,0}(|x_1-x_2|\Lambda \gg 1, t\Lambda \gg 1, 2t \ll |x_1-x_2|) = \frac{K_{\rm neq}}{2}. \tag{128}$$

At long distances and times, and on the light cone,

$$\gamma_{an,0}(|x_1-x_2|\Lambda \gg 1, t\Lambda \gg 1, 2t = |x_1-x_2|) = \frac{1}{2}\left(K_{\rm neq} - \frac{K_{\rm tr}}{2}\right). \tag{129}$$

The correlator at next order, $R^{(1)}$, may also be evaluated by splitting it into slow and fast fields as follows:

$$\begin{aligned}R^{(1)}&(x_1t,x_2t)\\ &= 4{\rm Tr}\Bigg[e^{i(S_0^<+S_0^>)}iS_{\rm sg}(\phi_<+\phi_>)\\ &\quad\times\cos\left(\frac{\gamma\phi^<(x_1t)+\gamma\phi^>(x_1t)}{2}\right)\\ &\quad\times\cos\left(\frac{\gamma\phi^<(x_2t)+\gamma\phi^>(x_2t)}{2}\right)\Bigg]\\ &= \frac{igu}{\alpha^2}\int_{-\infty}^{\infty}dx'\int_0^t dt'\sum_{c=\pm}{\rm sgn}(-c)\\ &\quad\times\left\langle\cos\left(\gamma\phi_c^<(x't')-\frac{\gamma}{2}\phi^<(x_1t)-\frac{\gamma}{2}\phi^<(x_2t)\right)\right\rangle\\ &\quad\times\left\langle\cos\left(\gamma\phi_c^>(x't')-\frac{\gamma}{2}\phi^>(x_1t)-\frac{\gamma}{2}\phi^>(x_2t)\right)\right\rangle.\end{aligned} \tag{130}$$

On integrating out the fast fields, we obtain

$$\begin{aligned}R^{(1)}&(x_1t,x_2t)\\ &= \frac{igu}{\alpha^2}e^{(d\Lambda/\Lambda)\gamma_{an,0}(x_1,x_2,t)}\int_{-\infty}^{\infty}dx'\int_0^t dt'\\ &\quad\times\sum_{c=\pm}{\rm sgn}(-c)e^{-(d\Lambda/\Lambda)\delta(x_1,x_2,t,x',t')}\\ &\quad\times\left\langle\cos\left(\gamma\phi_c^<(x't')-\frac{\gamma}{2}\phi^<(x_1t)-\frac{\gamma}{2}\phi^<(x_2t)\right)\right\rangle,\end{aligned} \tag{131}$$

where

$$
\begin{aligned}
&\delta(x_1,x_2,t,x',t')\\
&\quad=\Bigg[2K_{\rm neq}+\frac{K_{\rm tr}}{1+(2t'\Lambda)^2}+\frac{K_{\rm tr}}{1+(2t\Lambda)^2}\\
&\qquad-\frac{K_{\rm neq}}{2}\sum_{\epsilon=\pm}\left\{\frac{1}{1+\Lambda^2[t-t'+\epsilon(x_1-x')/u]^2}\right\}\\
&\qquad-\frac{K_{\rm tr}}{2}\sum_{\epsilon=\pm}\left\{\frac{1}{1+\Lambda^2[t+t'+\epsilon(x_1-x')/u]^2}\right\}\\
&\qquad-\frac{K_{\rm neq}}{2}\sum_{\epsilon=\pm}\left\{\frac{1}{1+\Lambda^2[t-t'+\epsilon(x_2-x')/u]^2}\right\}\\
&\qquad-\frac{K_{\rm tr}}{2}\sum_{\epsilon=\pm}\left\{\frac{1}{1+\Lambda^2[t+t'+\epsilon(x_2-x')/u]^2}\right\}\\
&\qquad-ic\frac{K_{\rm eq}}{2}\sum_{\epsilon=\pm}\left\{\frac{\Lambda[t-t'+\epsilon(x_1-x')/u]}{1+\Lambda^2[t-t'+\epsilon(x_1-x')/u]^2}\right\}\\
&\qquad-ic\frac{K_{\rm eq}}{2}\sum_{\epsilon=\pm}\left\{\frac{\Lambda[t-t'+\epsilon(x_2-x')/u]}{1+\Lambda^2[t-t'+\epsilon(x_2-x')/u]^2}\right\}\Bigg].
\end{aligned}
\tag{132}
$$

Rescaling the cutoff and correspondingly the position and time, we obtain the following differential equation:

$$
\begin{aligned}
R^{(1)}(x_1-x_2,t)&=R^{(1)}_<(x_1-x_2,t)\Bigg[1+2\frac{d\Lambda}{\Lambda}-2K_{\rm neq}\frac{d\Lambda}{\Lambda}\\
&\quad+\gamma_{an,0}(x_1-x_2,t)\frac{d\Lambda}{\Lambda}\Bigg]+2g\pi I_C(x_1,x_2,t)\\
&\quad\times\frac{d\Lambda}{\Lambda}R^0_<(x_1-x_2,t).
\end{aligned}
\tag{133}
$$

Rewriting the zero-order term Eq. (122) for convenience,

$$
\begin{aligned}
&R^{(0)}(x_1-x_2,t)\\
&\quad=R^{(0)}_<(x_1-x_2,t)\left[1-\frac{d\Lambda}{\Lambda}\gamma_{an,0}(x_1-x_2,t)\right],
\end{aligned}
\tag{134}
$$

and combining Eqs. (133) and (134), we get

$$
\begin{aligned}
R&=R^{(0)}_<\Bigg[1-\frac{d\Lambda}{\Lambda}\gamma_{an,0}(x_1-x_2,t)\\
&\quad+\frac{d\Lambda}{\Lambda}2g\pi I_C(x_1-x_2,t)\Bigg]+R^{(1)}_<\Bigg[1+(2-K_{\rm neq})\frac{d\Lambda}{\Lambda}\\
&\quad+\{-K_{\rm neq}+\gamma_{an,0}(x_1-x_2,t)\}\frac{d\Lambda}{\Lambda}\Bigg].
\end{aligned}
\tag{135}
$$

The term in the first square brackets $(\gamma_{an,0}-2\pi gI_C)$ is the anomalous scaling dimension of the correlator R. The $(2-K_{\rm neq})d\Lambda/\Lambda$ term in the second square brackets is simply the renormalization of g which has been discussed before in the context of the β function.

To next order in the cosine potential, we obtain the $\mathcal{O}(g^2)$ terms of the β function whose derivation is already discussed in Sec. IV. Thus to $\mathcal{O}(g^2)$ the following CS differential-equation for R is obtained:

$$
\begin{aligned}
&\left[\frac{\partial}{\partial\ln l}+\beta(g_i)\frac{\partial}{\partial g_i}-\gamma_{an,0}+2\pi gI_C\right]\\
&\quad\times R\left[\frac{r\Lambda_0}{l},\frac{\Lambda_0T_{m0}}{l},g_i(l)\right]=0,
\end{aligned}
\tag{136}
$$

where $g_i=g,K,\eta,T_{\rm eff}$, and Λ_0,T_{m0},g_{i0} denote bare values. Moreover I_C is given by

$$
\begin{aligned}
&I_C(r=x_1-x_2,t,K_{\rm neq},K_{\rm eq})\\
&=-\left(\frac{1}{2\pi}\right)e^{K_{\rm neq}\ln\sqrt{1+(x_1-x_2)^2}}\int_{-\infty}^{\infty}dx'\int_0^t dt'\sin\left[\frac{K_{\rm eq}}{2}\sum_{\epsilon=\pm}\left\{\tan^{-1}[t'+\epsilon(x'-x_1)]+\tan^{-1}[t'+\epsilon(x'-x_2)]\right\}\right]\\
&\times e^{-(K_{\rm neq}/2)\sum_{\epsilon=\pm}[\ln\sqrt{1+[t'+\epsilon(x'-x_1)]^2}+\ln\sqrt{1+[t'+\epsilon(x'-x_2)]^2}]}\\
&\times e^{-(K_{\rm tr}/2)[\ln\sqrt{[1+(2t-t'+x'-x_1)^2]/(1+4t^2)}+\ln\sqrt{\{1+[2t-t'-(x'-x_1)]^2\}/[1+4(t-t')^2]}+\ln\sqrt{[1+(2t-t'+x'-x_2)^2]/(1+4t^2)}+\ln\sqrt{\{1+[2t-t'-(x'-x_2)]^2\}[1+4(t-t')^2]}]}\\
&\times\Bigg[\frac{K_{\rm tr}}{1+4t^2}+\frac{K_{\rm tr}}{1+4(t-t')^2}-\frac{K_{\rm neq}}{2}\sum_{\epsilon=\pm}\left\{\frac{1}{1+[t'+\epsilon(x'-x_1)]^2}+\frac{1}{1+[t'+\epsilon(x'-x_2)]^2}\right\}\\
&-\frac{K_{\rm tr}}{2}\sum_{\epsilon=\pm}\left\{\frac{1}{1+[2t-t'+\epsilon(x'-x_1)]^2}+\frac{1}{1+[2t-t'+\epsilon(x'-x_2)]^2}\right\}\Bigg]
\end{aligned}
\tag{137}
$$

$$
\begin{aligned}
&+\frac{1}{2\pi}e^{K_{\rm neq}\ln\sqrt{1+(x_1-x_2)^2}}\int_{-\infty}^{\infty}dx'\int_0^t dt'\cos\left[\frac{K_{\rm eq}}{2}\sum_{\epsilon=\pm}\{\tan^{-1}[t'+\epsilon(x'-x_1)]+\tan^{-1}[t'+\epsilon(x'-x_2)]\}\right]\\
&\times e^{-(K_{\rm neq}/2)\sum_{\epsilon=\pm}[\ln\sqrt{1+[t'+\epsilon(x'-x_1)]^2}+\ln\sqrt{1+[t'+\epsilon(x'-x_2)]^2}]}\\
&\times e^{-(K_{\rm tr}/2)[\ln\sqrt{[1+(2t-t'+x'-x_1)^2]/(1+4t^2)}+\ln\sqrt{\{1+[2t-t'-(x'-x_1)]^2\}/[1+4(t-t')^2]}+\ln\sqrt{[1+(2t-t'+x'-x_2)^2]/(1+4t^2)}+\ln\sqrt{\{1+[2t-t'-(x'-x_2)]^2\}/[1+4(t-t')^2]}]}\\
&\times\frac{K_{\rm eq}}{2}\sum_{\epsilon=\pm}\left[\frac{[t'+\epsilon(x'-x_1)]}{1+[t'+\epsilon(x'-x_1)]^2}+\frac{[t'+\epsilon(x'-x_2)]}{1+[t'+\epsilon(x'-x_2)]^2}\right].
\end{aligned}
\tag{138}
$$

The usual CS equation encountered in equilibrium for the Heisenberg chain[35–37] may be obtained from above by setting the time after the quench $t=\infty$, and by setting $K_0 = K$ and by also being at the critical point $K_{\rm eq}=2$. Here, noting that $-{\rm Im}[e^{-i\tan^{-1}x}\frac{1}{1+ix}] = \frac{\sin(\tan^{-1}x)}{1+x^2} + \frac{x}{1+x^2}\cos(\tan^{-1}x)$, one may write

$$I_C = -\frac{[1+(x_1-x_2)^2]}{2\pi}{\rm Im}\bigg[\int_{-\infty}^{\infty}dx'\int_0^{\infty}dt' \times e^{-\sum_{\epsilon=\pm}\ln\{1+i[t'+\epsilon(x'-x_1)]\}+\ln\{1+i[t'+\epsilon(x'-x_2)]\}} \times \sum_{\epsilon=\pm}\bigg\{\frac{1}{1+i[t'+\epsilon(x'-x_1)]}+\frac{1}{1+i[t'+\epsilon(x'-x_2)]}\bigg\}\bigg]. \tag{139}$$

The above integral may be evaluated to give

$$I_C = \frac{(x_1-x_2)^2+1}{4+(x_1-x_2)^2}. \tag{140}$$

Thus for $|x_1-x_2|\gg 1$,

$$I_C\left(K_0=K, K_{\rm eq}=2, t=\infty\right) = 1. \tag{141}$$

In the next two sections we will solve the CS equation (136) for two cases: one is when the Luttinger liquid interaction parameter is held fixed, but the cosine or lattice potential is suddenly switched on, and the second is when the Luttinger parameter is changed at the same time as when the lattice potential is switched on.

VII. CORRELATION FUNCTION FOR THE PURE LATTICE QUENCH

In the previous section we showed that in order to determine the correlation function at spatial separation r and a time T_{m0} after a quench, we need to solve

$$\left[\frac{\partial}{\partial\ln l}+\beta(g_i)\frac{\partial}{\partial g_i}-\gamma_{an,0}+2\pi g I_C\right] \times R\left[\frac{r\Lambda_0}{l},\frac{\Lambda_0 T_{m0}}{l},g_i(l)\right]=0, \tag{142}$$

where $g_i = g, K, \eta, T_{\rm eff}$, and Λ_0, g_{i0} denote bare values. In this section we will solve Eq. (142) for the lattice quench $K_0 = K$ and near the critical point $K_{\rm eq} = 2+\delta, \forall\ 0<\delta\ll 1$. For this case, $\gamma_{an,0}$ has the following limiting forms:

$$\gamma_{an,0}(r\Lambda\gg 1, T_m\Lambda\ll 1) = \frac{1}{2}(K_{\rm neq}+K_{\rm tr}) = 1+\frac{\delta}{2}, \tag{143}$$

$$\gamma_{an,0}(r\Lambda\gg 1, T_m\Lambda\gg 1, 2T_m\gg r) = \frac{K_{\rm neq}}{2} = 1+\frac{\delta}{2}, \tag{144}$$

$$\gamma_{an,0}(r\Lambda\gg 1, T_m\Lambda\gg 1, 2T_m\ll r) = \frac{K_{\rm neq}}{2} = 1+\frac{\delta}{2}, \tag{145}$$

$$\gamma_{an,0}(r\Lambda\gg 1, T_m\Lambda\gg 1, 2T_m = r) = \frac{1}{2}\left(K_{\rm neq}-\frac{K_{\rm tr}}{2}\right) = 1+\frac{\delta}{2}. \tag{146}$$

Moreover for $K_0 = K$ and in the vicinity of the critical point, the full expression for I_C in Eqs. (137) and (138) reduces to

$$I_C(r,T_m) = \frac{r^2+1}{4+r^2} - \frac{r^2+1}{2\pi}{\rm Im}\left[i\int_{-\infty}^{\infty}dx\left\{\frac{1}{(1+iT_m)^2+x^2} \times \frac{1}{(1+iT_m)^2+(x-r)^2}\right\}\right] = \frac{r^2+1}{4+r^2} - \frac{1+r^2}{1+T_m^2}\left[\frac{r^2-4T_m^2+4-8T_m^2}{\left\{r^2-4T_m^2+4\right\}^2+64T_m^2}\right]. \tag{147}$$

I_C has the following different limits already discussed in Sec. II:

$$I_C(r\gg 1, T_m\gg 1, 2T_m\gg r) = 1+\mathcal{O}\left(\frac{1}{r^2},\frac{r^2}{T_m^4}\right), \tag{148}$$

$$I_C(r\gg 1, T_m\gg 1, 2T_m\ll r) = 1+\mathcal{O}\left(\frac{1}{T_m^2},\frac{1}{r^2}\right), \tag{149}$$

$$I_C(r\gg 1, T_m\gg 1, 2T_m = r) = \frac{3}{2}+\mathcal{O}\left(\frac{1}{r^2}\right). \tag{150}$$

In addition the β function for a pure lattice quench and using $K_{\rm neq} = K_{\rm eq} = 2+\delta$ is (neglecting velocity renormalization)

$$\frac{dg}{d\ln l} = -g\delta, \tag{151}$$

$$\frac{d\delta}{d\ln l} = -4\pi g^2 I_K(T_m), \tag{152}$$

$$\frac{d\eta}{d\ln l} = \eta + 4\pi g^2 I_\eta(T_m), \tag{153}$$

$$\frac{d(\eta T_{\rm eff})}{d\ln l} = 2\eta T_{\rm eff} + 2\pi g^2 I_{T_{\rm eff}}(T_m), \tag{154}$$

$$\frac{dT_m}{d\ln l} = -T_m, \tag{155}$$

where

$$I_K(T_m, K_{\rm neq} = K_{\rm eq} = 2) = \pi - \frac{\pi}{2}\left[\frac{2+20T_m^2+7\times 16T_m^4}{\left(1+4T_m^2\right)^3}\right]. \tag{156}$$

The above shows that at short times, $I_K(T_m\ll 1) = 2\pi T_m^2+\cdots$, while at long times I_K reaches a steady state as follows: $I_K(T_m\gg 1) = \pi[1-\frac{7}{8T_m^2}+\cdots]$. Moreover,

$$I_\eta(T_m, K_{\rm neq} = K_{\rm eq} = 2) = 4\pi\frac{(2T_m)^3}{[1+(2T_m)^2]^3}, \tag{157}$$

$$I_{T_{\rm eff}}(T_m, K_{\rm neq} = K_{\rm eq} = 2) = 6\pi T_m\left[\frac{1-\frac{4}{3}T_m^2}{\left(1+4T_m^2\right)^3}\right]. \tag{158}$$

Note that the above expressions show that at long times ($T_m\to\infty$) a pure lattice quench does not generate any dissipation and noise, at least to $\mathcal{O}(g^2)$. This extends the regime of validity of the prethermalized regime which we defined as the regime where time is larger than microscopic time scales, but smaller than the steady-state inelastic scattering rate (Fig. 3).

Thus neglecting dissipative and thermal effects, and at macroscopically long times, the β function simplifies considerably to

$$\frac{dg}{d\ln l} = -g\delta; \quad \frac{d\delta}{d\ln l} = -(2\pi g)^2 \tag{159}$$

with the anomalous dimension given by

$$\gamma_{an} = \gamma_{an,0} - 2\pi g I_C = 1 + \frac{\delta}{2} - 2\pi g I_C. \tag{160}$$

As discussed earlier in this section, $I_C = 1$ inside $(2T_{m0} \gg r)$ and outside $(2T_{m0} \ll r)$ the light cone, whereas $I_C = 3/2$ for points on the light cone $r = 2T_{m0}$. We will derive expressions for the correlator for two cases separately. The first case is when the final Hamiltonian is on the critical line $\delta = 2\pi g$, while the second is when the final Hamiltonian is slightly away from the critical line $\delta > 2\pi g$.

For the first case, i.e., on the critical line $\delta = 2\pi g$, the β function further simplifies to

$$\frac{d\delta}{d\ln l} = -\delta^2 \tag{161}$$

and the anomalous dimension of the correlator becomes

$$\gamma_{an} = 1 + \frac{\delta}{2} - \delta I_C. \tag{162}$$

The explicit forms of I_C and $\gamma_{an,0}$ in Eqs. (23) and (26) show that scaling stops when $l^* = \min(\Lambda_0 r, \Lambda_0 T_{m0})$. Thus there are three interesting cases to consider: one for spatial separations outside the light cone, $r \gg 2T_{m0}$, the second is for spatial separations on the light cone, $r = 2T_{m0}$, and the third is for spatial separations within a light cone, $r \ll 2T_{m0}$. These three cases are shown pictorially in Fig. 2, and the corresponding correlators are derived next.

Integrating the CS equation up to l^* we obtain

$$R(\Lambda_0 r, \Lambda_0 T_{m0}, g_0) = e^{-\int_{g_0}^{g(l^*)} dg'[\gamma_{an}(g')/\beta(g')]} R\left(\frac{r\Lambda_0}{l^*}, \frac{T_{m0}\Lambda_0}{l^*}, g(l^*)\right). \tag{163}$$

Integrating up to l^*, the solution of Eq. (161) is

$$\frac{1}{\delta(l^*)} - \frac{1}{\delta_0} = \ln(l^*), \tag{164}$$

whereas for evaluating the correlator we need

$$\int_{\delta_0}^{\delta(l^*)} d\delta \frac{\gamma_{an}}{\beta} = \ln(l^*) + \left(I_C - \frac{1}{2}\right)\ln\left[\frac{\delta(l^*)}{\delta_0}\right]. \tag{165}$$

For points outside the light cone, setting $l^* = 2\Lambda_0 T_{m0}$, the correlation function is found to be

$$R(\Lambda_0 r, \Lambda_0 T_{m0}, g_0, 2T_{m0} \ll r) \sim \frac{\sqrt{\ln T_{m0}}}{T_{m0}} R\left(\frac{r}{2T_{m0}} \gg 1, T_{m0}\Lambda_0 = l^*, g(l^*)\right). \tag{166}$$

Since $l^* \gg 1$, $g(l^*) \ll g_0$, so that the short-distance correlator may be evaluated perturbatively in the cosine potential. Using the results from Eq. (69), where $R(\frac{r}{2T_{m0}}, T_{m0}\Lambda_0 = l^*, g = 0) \sim T_{m0}/r$, we find that the correlator outside the light cone behaves as

$$R(\Lambda_0 r, g_0, 2T_{m0} \ll r) \sim \frac{\sqrt{\ln T_{m0}}}{r}. \tag{167}$$

The second interesting case is the behavior of the correlator inside the light cone where $2T_{m0} \gg r$. Here integrating up to $l^* = \Lambda_0 r$, we get

$$R(\Lambda_0 r, 2T_{m0} \gg r, g_0) = e^{-\int_{g_0}^{g(\Lambda_0 r)} dg'[\gamma_{an}(g')/\beta(g')]} R\left(\frac{r\Lambda_0}{l^*} = 1, 2T_{m0} \gg r, g(l^*)\right). \tag{168}$$

From Eq. (67), $R(l^* = \Lambda_0 r, 2T_{m0} \gg r, g = 0) = \mathcal{O}(1)$, while using the result that $I_C = 1$ within the light cone we obtain

$$R(\Lambda_0 r, 2T_{m0} \gg r, g_0) \sim \frac{\sqrt{\ln r}}{r}. \tag{169}$$

At these long times, the explicit dependence on the time after the quench drops out and the correlator reaches a steady-state value. Moreover this result is the same as in the ground state of the final Hamiltonian. Later when we study an interaction and lattice quench, we will see that within the light cone, a steady-state behavior again arises, however the logarithmic corrections are different than those obtained for a system which is in equilibrium, and near the critical point.

The third interesting case is for spatial separations on the light cone where $r = 2T_{m0}$. Here integrating up to $l^* = \Lambda_0 r$

$$R(\Lambda_0 r, 2T_{m0} = r, g_0) = e^{-\int_{g_0}^{g(\Lambda_0 r)} dg'[\gamma_{an}(g')/\beta(g')]} R\left(\frac{r\Lambda_0}{l^*} = 1, 2T_{m0} = l^*, g(l^*)\right). \tag{170}$$

Using Eq. (68) and the fact that $I_C = 3/2$ on the light cone we find

$$R(\Lambda_0 r, 2T_{m0} = r, g_0) \sim \frac{\ln r}{r}. \tag{171}$$

On the light cone $(r = 2T_{m0})$ the correlator decays somewhat slower than within and outside the light cone.

We now turn to the case where $\delta > 2\pi g$. Here the solution of Eq. (159) on integrating up to a scale l^* [defining $A_1 = \sqrt{\delta^2 - (2\pi g)^2}$] is

$$\int_{g_0}^{g(l^*)} \frac{\gamma(g')}{\beta(g')} dg' = \ln l^* + \frac{1}{2}\ln\left[\frac{\sinh\left(A_1 \ln l^* + \tanh^{-1}\frac{A_1}{\delta_0}\right)}{\sinh\left(\tanh^{-1}\frac{A_1}{\delta_0}\right)}\right] - I_C A_1 \ln\left[\frac{\tanh\left(\frac{A_1}{2}\ln l^* + \frac{1}{2}\tanh^{-1}\frac{A_1}{\delta_0}\right)}{\tanh\left(\frac{1}{2}\tanh^{-1}\frac{A_1}{\delta_0}\right)}\right]. \tag{172}$$

The correlation function therefore is found to be

$$R(\Lambda_0 r, \Lambda_0 T_{m0}, g_0) \simeq \frac{1}{(l^*)^{1+A_1/2}} \sqrt{\frac{1}{1-(l^*)^{-2A_1}}} \left[\frac{1-(l^*)^{-A_1}}{1+(l^*)^{-A_1}}\right]^{I_C}. \tag{173}$$

For points outside the light cone we set $l^* = \Lambda_0 T_{m0}$ in Eq. (173), and using Eq. (69) we obtain

$$R(\Lambda_0 r, \Lambda_0 T_{m0}, g_0, r \gg 2T_{m0}) \sim \left(\frac{1}{r}\right)^{1+\delta/2} (T_{m0})^{\delta/2-(\sqrt{\delta^2-(2\pi g)^2})/2}. \tag{174}$$

For points on the light cone, or inside the light cone we set $l^* = \Lambda_0 r$ in Eq. (173) to obtain

$$R(\Lambda_0 r, \Lambda_0 T_{m0}, g_0, r \leqslant 2T_{m0}) \sim \frac{1}{r^{1+A_1/2}}. \quad (175)$$

Note that for small deviations away from the critical point such that $A_1 \ln r \gg 1$, the correlator on the light cone and inside the light cone have the same leading behavior in position. This is however not the case for quenches to the critical point where $A_1 \ln r \ll 1$. Here Eqs. (169) and (171) show that differences arise even in the leading asymptote.

VIII. CORRELATION FUNCTION FOR THE LATTICE AND INTERACTION QUENCH

The β function shows that an interaction quench $K_0 \neq K$ changes the location of the critical point to $K_{\rm neq} = 2$. Since $K_{\rm neq} > K_{\rm eq}$, this implies that the ground state of the final Hamiltonian can be in the gapped phase, however the quench results in a highly excited and more delocalized state of bosons. In this section we are interested in evaluating the correlator R in the vicinity of this new nonequilibrium critical point. Setting $K_{\rm neq} = 2$ implies the following relation between K_0 and K:

$$K = K_0\sqrt{\frac{16}{\gamma^2 K_0} - 1}. \quad (176)$$

This implies the following:

$$K_{\rm eq} = \frac{\gamma^2 K_0}{4}\sqrt{\frac{16}{\gamma^2 K_0} - 1}, \quad (177)$$

$$K_{\rm tr} = \frac{\gamma^2 K_0}{4} - 2. \quad (178)$$

When $K_0 \neq K$, dissipative effects are generated. However in the prethermalized regime these effects are still weak and may be neglected. Writing $K_{\rm neq} = 2 + \delta$, the RG equations in the prethermalized regime ($T_m < 1/\eta$) are the following:

$$\frac{dg}{d\ln l} = -g\delta, \quad (179)$$

$$\frac{d\delta}{d\ln l} = -\pi g^2 \left(\frac{\gamma^2 K_0}{4}\right)^2 \left(\frac{16}{\gamma^2 K_0} - 1\right)^{3/2} I_K(T_m), \quad (180)$$

$$\frac{dT_m}{d\ln l} = -T_m; \quad (181)$$

above we have used that since $K_{\rm neq} = 2 + \delta$, $dK = \frac{4}{\gamma^2}\frac{K_0}{K} d\delta$, and $K_{\rm eq}$ is given by Eq. (177). Moreover,

$$\begin{aligned} I_K(T_m = \infty, K_{\rm neq} = 2) &= -4\int_0^\infty du \frac{u}{(1+u^2)^2}\cos(K_{\rm eq}\tan^{-1}u) \\ &\quad \times \underbrace{\int_0^u dv \frac{v}{1+v^2}\sin(K_{\rm eq}\tan^{-1}v)}_{i_1(u)} \quad (182) \\ &\quad -4\int_0^\infty du \frac{u}{1+u^2}\cos(K_{\rm eq}\tan^{-1}u) \\ &\quad \times \underbrace{\int_0^u dv \frac{v}{(1+v^2)^2}\sin(K_{\rm eq}\tan^{-1}v)}_{i_2(u)} \quad (183) \\ &\quad -2K_{\rm eq}\int_0^\infty du \frac{u}{1+u^2}\cos(K_{\rm eq}\tan^{-1}u) \\ &\quad \times \underbrace{\int_0^u dv \frac{v^2}{(1+v^2)^2}\cos(K_{\rm eq}\tan^{-1}v)}_{i_3(u)} \quad (184) \\ &\quad +2K_{\rm eq}\int_0^\infty du \frac{u^2}{(1+u^2)^2}\sin(K_{\rm eq}\tan^{-1}u) \\ &\quad \times \underbrace{\int_0^u dv \frac{v}{1+v^2}\sin(K_{eq}\tan^{-1}v)}_{i_1(u)}. \quad (185) \end{aligned}$$

Note that Eqs. (183) and (184) are logarithmically divergent (such a divergence does not arise for a pure lattice quench). The reason for this divergence is because we have used leading-order perturbation theory to evaluate the correlators that go into I_K, whereas we should have used the correlators from the full theory which correspond to a nonzero dissipation η. Taking this into account, the logarithmic divergence is cut off by $\min(T_m, 1/\eta)$. Denoting the well-behaved terms in Eqs. (182) and (185) as πc_1, and using $i_2(\infty) = -\frac{1}{K_{\rm eq}^2-4}\sin(\frac{\pi K_{\rm eq}}{2})$ and $i_3(\infty) = \frac{(K_{\rm eq}^2-2)}{K_{\rm eq}(K_{\rm eq}^2-4)}\sin(\frac{\pi K_{\rm eq}}{2})$, we may write

$$\begin{aligned} I_K(T_m, K_{\rm neq} = 2) &= \pi\left[c_1 - 2c_2'\sin\left(\frac{\pi K_{\rm eq}}{2}\right)\right. \\ &\quad \left.\times \int_0^{\min(T_m,1/\eta)} du \frac{u}{1+u^2}\cos\left(K_{\rm eq}\tan^{-1}u\right)\right] \quad (186) \\ &\simeq \pi\{c_1 - c_2\sin(\pi K_{\rm eq})\ln[\min(T_m, 1/\eta)]\}, \quad (187) \end{aligned}$$

where $c_{1,2}$ are $\mathcal{O}(1)$ and depend on $K_{\rm eq}$ and hence K_0. When $K_0 = K$, $c_1 = 1$.

In what follows we will consider a prethermalized regime where $T_m < 1/\eta$, and also weak quenches such that $|K_{\rm eq} - K_{\rm neq}|\ln T_m \ll 1$. In this case, $I_K \simeq \pi c_1$ and is independent of time. We will solve the RG equations in a regime where the periodic potential is marginally irrelevant, $g_{\rm eff} = gB < \delta$, where

$$B = \pi\sqrt{c_1}\left(\frac{\gamma^2 K_0}{4}\right)\left(\frac{16}{\gamma^2 K_0} - 1\right)^{3/4}. \quad (188)$$

The RG flow equations $\frac{dg_{\rm eff}}{d\ln l} = -g_{\rm eff}\delta$, $\frac{d\delta}{d\ln l} = -g_{\rm eff}^2$ are characterized by the constant of motion,

$$A = \sqrt{\delta^2 - g_{\rm eff}^2}, \quad (189)$$

in terms of which the solution of the RG equations are

$$\delta(l) = A\coth\left[A\ln l + \tanh^{-1}\left(\frac{A}{\delta_0}\right)\right], \quad (190)$$

$$g_{\rm eff}(l) = \frac{A}{\sinh\left[A\ln l + \tanh^{-1}\left(\frac{A}{\delta_0}\right)\right]}. \quad (191)$$

In the vicinity of the nonequilibrium critical point $K_{\rm neq} = 2 + \delta, \forall\ \delta \ll 1$, and inside the light cone, I_C is found to be

$$\begin{aligned} I_C(r \gg 1, T_m \gg 1, 2T_m \gg r) \\ = \frac{K_{\rm eq}}{2} + \left[1 - \left(\frac{K_{\rm eq}}{2}\right)^2\right] L(K_{\rm eq}), \end{aligned} \quad (192)$$

where

$$L(K_{\rm eq}) = \lim_{|r|\gg 1}\left(\frac{r^2}{4\pi}\int_0^\infty du\int_{-u}^{u} dv \sin\left[\frac{K_{\rm eq}}{2}\{\tan^{-1}u+\tan^{-1}v+\tan^{-1}(u+r)+\tan^{-1}(v-r)\}\right]\frac{1}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+v^2}}\right.$$
$$\left.\times\frac{1}{\sqrt{1+(u+r)^2}}\frac{1}{\sqrt{1+(v-r)^2}}\left[\frac{1}{1+u^2}+\frac{1}{1+v^2}+\frac{1}{1+(u+r)^2}+\frac{1}{1+(v-r)^2}\right]\right)+(r\to -r). \quad (193)$$

$L(K_{\rm eq})$ is a universal function of $K_{\rm eq}$ and hence the initial Luttinger parameter. This results in a universal expression for I_C which is plotted in Fig. 4 for different initial states.

The solution of the CS equation is

$$R\left(\Lambda_0 r,\Lambda_0 T_{m0},g_0\right) = e^{-\int_{g_0}^{g(l)} dg'[\gamma_{an}(g')/\beta(g')]}\quad R\left(\frac{r\Lambda_0}{l},\frac{T_{m0}\Lambda_0}{l},g(l)\right). \quad (194)$$

We will solve this first for points within the light cone $1/\eta \gg T_{m0} \gg r$. For this case, dissipative effects are neglected and the RG equations are integrated up to $l=\Lambda_0 r$ to give

$$R\left(\Lambda_0 r,r\ll T_{m0}\ll 1/\eta,g_0\right) = e^{-\int_{g_0}^{g(\Lambda_0 r)} dg'[\gamma_{an}(g')/\beta(g')]}R\left(\frac{r\Lambda_0}{l}=1,\frac{T_{m0}}{r}\gg 1,g(l)\right). \quad (195)$$

On the right-hand side, R being a short-distance correlator, is $\mathcal{O}(1)$. Note that

$$\gamma_{an}(r\ll 2T_{m0}) = 1+\frac{\delta}{2}-2\pi g I_C. \quad (196)$$

Then,

$$\int_{g_0}^{g(l)}\frac{\gamma(g')}{\beta(g')}dg' = \ln l+\frac{1}{2}\ln\left[\frac{\sinh\left(A\ln l+\tanh^{-1}\frac{A}{\delta_0}\right)}{\sinh\left(\tanh^{-1}\frac{A}{\delta_0}\right)}\right]-\frac{2\pi I_C A}{B}\ln\left[\frac{\tanh\left(\frac{A}{2}\ln l+\frac{1}{2}\tanh^{-1}\frac{A}{\delta_0}\right)}{\tanh\left(\frac{1}{2}\tanh^{-1}\frac{A}{\delta_0}\right)}\right], \quad (197)$$

where $B=\pi\sqrt{c_1}(\frac{\gamma^2 K_0}{4})(\frac{16}{\gamma^2 K_0}-1)^{3/4}$.

The correlation function within the light cone is given by setting $l=\Lambda_0 r$ which gives

$$R\left(\Lambda_0 r,r\ll 2T_{m0}\ll 1/\eta,g_0\right)\simeq\frac{1}{(\Lambda_0 r)^{1+A/2}}\sqrt{\frac{1}{1-(\Lambda_0 r)^{-2A}}}\left[\frac{1-(\Lambda_0 r)^{-A}}{1+(\Lambda_0 r)^{-A}}\right]^{2\pi I_C/B}. \quad (198)$$

For $A\ln r\ll 1$, we obtain the following logarithmic correction to scaling:

$$R\sim\frac{1}{r}(\ln r)^{2\pi I_C/B-1/2}. \quad (199)$$

In the above expression the exponent $\theta=\frac{2\pi I_C}{B}-\frac{1}{2}$ approaches $1/2$ as K_0 approaches K.

Let us now discuss how I_C behaves for spatial separations on the light cone. Simplifying the full expression for I_C in Eqs. (137) and (138) by noting that for $r\gg 1,t\gg 1,r=2t$,

$$\sqrt{\frac{1+(2t-t'+x')^2}{1+4t^2}}\simeq 1,\quad \sqrt{\frac{1+(2t-t'-x')^2}{1+4(t-t')^2}}\simeq 1,$$

$$\sqrt{\frac{1+(2t-t'+x'-r)^2}{1+4t^2}}=\sqrt{\frac{1+(t'-x')^2}{1+4t^2}},\quad \sqrt{\frac{1+(2t-t'-x'+r)^2}{1+4(t-t')^2}}\simeq 2,$$

one may write

$$I_C(r=2t,r\gg 1,K_{\rm neq}=2) = 2^{-K_{\rm tr}/2}{\rm Lt}_{r\to\infty}\frac{r^{2+K_{\rm tr}/2}}{2\pi}\int_{-\infty}^{\infty}dx'\int_0^{r/2}dt'$$
$$\times\sin\left[\frac{K_{\rm eq}}{2}\{\tan^{-1}(t'+x')+\tan^{-1}(t'-x')+\tan^{-1}(t'+x'-r)+\tan^{-1}(t'-x'+r)\}\right]$$
$$\times\frac{1}{\sqrt{1+(t'+x')^2}}\left(\frac{1}{\sqrt{1+(t'-x')^2}}\right)^{1+K_{\rm tr}/2}\frac{1}{\sqrt{1+(t'+x'-r)^2}}\frac{1}{\sqrt{1+(t'-x'+r)^2}}$$
$$\times\left\{\frac{1}{1+(t'+x')^2}+\frac{\left(1+\frac{K_{\rm tr}}{2}\right)}{1+(t'-x')^2}+\frac{1}{1+(t'+x'-r)^2}+\frac{1}{1+(t'-x'+r)^2}\right\}$$

$$
\begin{aligned}
&+2^{-K_{\rm tr}/2}{\rm Lt}_{r\to\infty}\frac{r^{2+K_{\rm tr}/2}}{2\pi}\left(\frac{K_{\rm eq}}{2}\right)\int_{-\infty}^{\infty}dx'\int_0^{r/2}dt'\\
&\times\cos\left[\frac{K_{\rm eq}}{2}\{\tan^{-1}(t'+x')+\tan^{-1}(t'-x')+\tan^{-1}(t'+x'-r)+\tan^{-1}(t'-x'+r)\}\right]\\
&\times\frac{1}{\sqrt{1+(t'+x')^2}}\left(\frac{1}{\sqrt{1+(t'-x')^2}}\right)^{1+K_{\rm tr}/2}\frac{1}{\sqrt{1+(t'+x'-r)^2}}\frac{1}{\sqrt{1+(t'-x'+r)^2}}\\
&\times\left[\frac{(t'+x')}{1+(t'+x')^2}+\frac{(t'-x')}{1+(t'-x')^2}+\frac{(t'+x'-r)}{1+(t'+x'-r)^2}+\frac{(t'-x'+r)}{1+(t'-x'+r)^2}\right].
\end{aligned}\tag{200}
$$

The above implies that

$$I_C(r=2t,r\gg 1)\sim r^{K_{\rm tr}/2}.\tag{201}$$

Thus the scaling for spatial separations on the light cone which we found for a pure lattice quench is lost for a simultaneous lattice and interaction quench when $K_{\rm tr}>0$ (i.e., $\frac{\gamma^2K_0}{4}>2$). This corresponds to a situation where the cosine potential is an irrelevant perturbation in the initial state. However, scaling holds when $K_{\rm tr}\leqslant 0$ where I_C either approaches zero at large distances or is a constant. This case of $K_{\rm tr}\leqslant 0$ corresponds to the cosine potential being a marginal or a relevant perturbation in the initial state.

We now turn to the behavior of the correlator for points outside the light cone. In the previous section we found that for a pure lattice quench, scaling holds outside the light cone. We would like to explore whether this continues to be the case for a simultaneous lattice and interaction quench. We make the following approximations in Eqs. (137) and (138) by noting that for $r\gg 1,t\gg 1,r\gg 2t$,

$$
\sqrt{\frac{1+(2t-t'+x')^2}{1+4t^2}}\simeq 1,\quad \sqrt{\frac{1+(2t-t'-x')^2}{1+4(t-t')^2}}\simeq 1,
$$
$$
\sqrt{\frac{1+(2t-t'+x'-r)^2}{1+4t^2}}\simeq\frac{r}{2t},\quad \sqrt{\frac{1+(2t-t'-x'+r)^2}{1+4(t-t')^2}}\simeq\frac{r}{2t}
$$

to obtain

$$
\begin{aligned}
I_C(r,t\gg 1;r\gg 2t,K_{\rm neq}=2)\simeq{}&\frac{r^2}{2\pi}\left(\frac{2t}{r}\right)^{K_{\rm tr}}\int_{-\infty}^{\infty}dx'\int_0^{t}dt'\\
&\times\sin\left[\frac{K_{\rm eq}}{2}\{\tan^{-1}(t'+x')+\tan^{-1}(t'-x')+\tan^{-1}(t'+x'-r)+\tan^{-1}(t'-x'+r)\}\right]\\
&\times\frac{1}{\sqrt{1+(t'+x')^2}}\frac{1}{\sqrt{1+(t'-x')^2}}\frac{1}{\sqrt{1+(t'+x'-r)^2}}\frac{1}{\sqrt{1+(t'-x'+r)^2}}\\
&\times\left\{\frac{1}{1+(t'+x')^2}+\frac{1}{1+(t'-x')^2}+\frac{1}{1+(t'+x'-r)^2}+\frac{1}{1+(t'-x'+r)^2}\right\}\\
&+\frac{r^2}{2\pi}\left(\frac{2t}{r}\right)^{K_{\rm tr}}\left(\frac{K_{\rm eq}}{2}\right)\int_{-\infty}^{\infty}dx'\int_0^{t}dt'\\
&\times\cos\left[\frac{K_{\rm eq}}{2}\{\tan^{-1}(t'+x')+\tan^{-1}(t'-x')+\tan^{-1}(t'+x'-r)+\tan^{-1}(t'-x'+r)\}\right]\\
&\times\frac{1}{\sqrt{1+(t'+x')^2}}\frac{1}{\sqrt{1+(t'-x')^2}}\frac{1}{\sqrt{1+(t'+x'-r)^2}}\frac{1}{\sqrt{1+(t'-x'+r)^2}}\\
&\times\left[\frac{(t'+x')}{1+(t'+x')^2}+\frac{(t'-x')}{1+(t'-x')^2}+\frac{(t'+x'-r)}{1+(t'+x'-r)^2}+\frac{(t'-x'+r)}{1+(t'-x'+r)^2}\right].
\end{aligned}\tag{202}
$$

The above implies that outside the light cone,

$$I_C\,(r\gg 2t,r\gg 1,t\gg 1)\sim\left(\frac{2t}{r}\right)^{K_{\rm tr}}.\tag{203}$$

Thus outside the light cone, we find that if $K_{\rm tr}>0$, i.e., the cosine potential is an irrelevant perturbation in the initial state, then I_C goes to zero with distance as a power law. However, if $K_{\rm tr}<0$, then I_C grows with distance, and the perturbative correction to the correlation function becomes large. This result is expected as the cosine potential for $K_{\rm tr}<0$ is a relevant perturbation in the initial state. Thus perturbation theory at initial times outside the light cone is violated. An example of this was also discussed in Sec. V and shown in Fig. 5.

IX. SUMMARY AND CONCLUSIONS

We have studied quench dynamics in a generic strongly correlated one-dimensional system which is represented by the quantum sine-Gordon model. We develop a time-dependent renormalization-group approach which reveals that the dynamics after a quantum quench can be quite rich by being characterized by several time scales (Fig. 3). One is a perturbatively accessible short-time scale ($T_m \ll 1/\Lambda$), the second is an intermediate-time prethermalized regime where inelastic effects are small ($1 \ll T_m \ll 1/\eta$) and the system can show universal scaling behavior, and the third is a longer time scale ($T_m \gg 1/\eta$) where inelastic scattering is strong, leading to thermal behavior. In this paper we explicitly derived a CS-like differential equation (25) for a two-point correlation function, and solved it in the prethermalized regime. This CS equation shows that even in the universal prethermalized regime, three distinctly different scaling regimes can exist that are summarized in Fig. 2. One is for spatial separations of the local operators outside the light cone, the other is for spatial separations on the light cone and the third is for spatial separations inside the light cone.

When only the cosine potential is quenched, and for a final Hamiltonian H_f which is at the critical point, the results for the correlator in the three scaling regimes are given in Eqs. (31)–(33). Universal logarithmic corrections due to the marginal cosine potential are found and, consistent with the horizon effect, the correlator is most enhanced right on the light cone. For a final Hamiltonian with parameters that correspond to slight deviations away from the critical point, the result for the correlation function outside the light cone is given in Eq. (34), and inside the light cone is given in Eq. (35).

For more complicated quenches where the initial Luttinger parameter is quenched at the same time as the cosine potential is switched on, scaling holds within the light cone, and the results are given in Eqs. (36) and (37). Whether scaling holds on the light cone and outside it depends on the initial state (or the initial Luttinger parameter). In particular when the cosine potential is a relevant or marginal perturbation in the initial state, scaling holds on the light cone, otherwise it is violated. In contrast outside the light cone, if the cosine potential is a relevant perturbation in the initial state, the perturbative corrections grow with distance. The latter behavior is consistent with our understanding that outside the light cone the behavior is primarily determined by the initial state. Thus even though the cosine potential may be a marginal or irrelevant perturbation for the final Hamiltonian (or the wave function at long times after the quench), if it is a relevant perturbation for the initial Hamiltonian, perturbation theory will be violated outside the light cone.

There are many interesting open questions. One is to generalize the results of this paper to unequal time correlation functions with the aim of studying issues such as aging.[28,30] The current paper focuses on the prethermalized regime where the main assumption is that inelastic effects being weak, the nonequilibrium boson distribution function generated due to the quench hardly changes in time. The dynamics in the thermal regime, and in particular how the boson distribution function evolves in time due to strong inelastic scattering is also very interesting to study. If the time scale $1/\eta$ is very short (i.e., the quench amplitude is large), so that the prethermalized regime is almost absent, a quantum kinetic equation may be employed to study how the boson distribution function evolves in time.[57] When $1/\eta$ is large resulting in a long prethermalized regime, understanding the difficult problem of how the crossover in time from the prethermalized to the thermalized regime occurs, and observing this in numerical studies[58] is also an important open question. It is also interesting to study the regime where the cosine potential is a relevant perturbation either in the initial or final state or both. When the cosine potential is irrelevant in the initial state, but relevant in the final state after the quench, RG may be used to identify a critical time after the quench when perturbation theory breaks down.[17] Studying the time evolution of two-point correlation functions and also quantities such as the Loschmidt echo when the cosine potential is relevant are important open questions. Finally an interesting direction to pursue is to employ the approach of this paper to study quench dynamics in quantum field theories in higher spatial dimensions.

ACKNOWLEDGMENTS

The author is deeply indebted to F. Essler for many helpful discussions and for a critical reading of the manuscript. This research was supported by the National Science Foundation under Grants No. PHY11-25915 and No. DMR-1004589.

APPENDIX: EXPRESSIONS FOR $I_{u,K,\eta,T_{\rm eff}}$

$$I_{T_{\rm eff}}(T_m\Lambda) = \int_{-\infty}^{\infty} d\bar{r}\int_{-2T_m\Lambda}^{2T_m\Lambda} d\bar{\tau}\,{\rm Re}\,[B(\bar{r},T_m\Lambda,\bar{\tau})]\,, \quad (A1)$$

$$I_{\eta}(T_m\Lambda) = -\int_{-\infty}^{\infty} d\bar{r}\int_{-2T_m\Lambda}^{2T_m\Lambda} d\bar{\tau}\,\bar{\tau}\,{\rm Im}\,[B(\bar{r},T_m\Lambda,\bar{\tau})]\,, \quad (A2)$$

$$I_u(T_m\Lambda) = -\int_{-\infty}^{\infty} d\bar{r}\int_{0}^{2T_m\Lambda} d\bar{\tau}(\bar{r}^2+\bar{\tau}^2) \times {\rm Im}[B(\bar{r},T_m\Lambda,\bar{\tau})], \quad (A3)$$

$$I_K(T_m\Lambda) = -\int_{-\infty}^{\infty} d\bar{r}\int_{0}^{2T_m\Lambda} d\bar{\tau}(\bar{r}^2-\bar{\tau}^2) \times {\rm Im}[B(\bar{r},T_m\Lambda,\bar{\tau})], \quad (A4)$$

where ${\rm Re}[B]=(B+B^*)/2$, ${\rm Im}[B]=(B-B^*)/(2i)$, and

$$B(\bar{r},T_m\Lambda,\bar{\tau}) = C_{+-,1}(\bar{r},T_m\Lambda,\bar{\tau})F(\bar{r},T_m\Lambda,\bar{\tau}) \quad (A5)$$

with $C_{+-,1}(\bar{r},T_m\Lambda,\bar{\tau}) = \langle e^{i\phi_+(\bar{r},\bar{\tau}+T_m\Lambda/2)}e^{-i\phi_-(0,\bar{\tau}-T_m\Lambda/2)}\rangle$. This quantity within leading order in perturbation theory is

$$\begin{aligned} &C_{+-,1}(\bar{r},T_m\Lambda,\bar{\tau}) \\ &= \left[\frac{1}{\sqrt{1+(\bar{\tau}+\bar{r})^2}}\frac{1}{\sqrt{1+(\bar{\tau}-\bar{r})^2}}\right]^{K_{\rm neq}} \\ &\quad\times\left[\frac{\sqrt{1+\{2(T_m\Lambda+\bar{\tau}/2)\}^2}}{\sqrt{1+(2T_m\Lambda+\bar{r})^2}}\frac{\sqrt{1+\{2(T_m\Lambda-\bar{\tau}/2)\}^2}}{\sqrt{1+(2T_m\Lambda-\bar{r})^2}}\right]^{K_{\rm tr}} \\ &\quad\times e^{-iK_{\rm eq}[\tan^{-1}(\bar{\tau}+\bar{r})+\tan^{-1}(\bar{\tau}-\bar{r})]} \end{aligned} \quad (A6)$$

while F is given by

$$F(\bar{r},T_m\Lambda,\bar{\tau}) = K_{\rm neq}\left[\frac{1}{1+(\bar{\tau}+\bar{r})^2}+\frac{1}{1+(\bar{\tau}-\bar{r})^2}\right] + K_{\rm tr}\left[\frac{1}{1+(2T_m\Lambda+\bar{r})^2}+\frac{1}{1+(2T_m\Lambda-\bar{r})^2}\right] - i\,K_{\rm eq}\left[\frac{\bar{\tau}+\bar{r}}{1+(\bar{\tau}+\bar{r})^2}+\frac{\bar{\tau}-\bar{r}}{1+(\bar{\tau}-\bar{r})^2}\right]. \tag{A7}$$

[1] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
[2] D. Fausti, R. I. Tobey, N. Dean, S. Kaiser, A. Dienst, M. C. Hoffmann, S. Pyon, T. Takayama, H. Takagi, and A. Cavalleri, Science **331**, 189 (2011).
[3] C. L. Smallwood, J. P. Hinton, C. Jozwiak, W. Zhang, J. D. Koralek, H. Eisaki, D.-H. Lee, J. Orenstein, and A. Lanzara, Science **336**, 1137 (2012).
[4] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
[5] M. Rigol, Phys. Rev. A **80**, 053607 (2009).
[6] L. F. Santos and M. Rigol, Phys. Rev. E **81**, 036206 (2010).
[7] G. Biroli, C. Kollath, and A. M. Läuchli, Phys. Rev. Lett. **105**, 250401 (2010).
[8] A. Mitra and T. Giamarchi, Phys. Rev. Lett. **107**, 150602 (2011).
[9] A. Mitra and T. Giamarchi, Phys. Rev. B **85**, 075117 (2012).
[10] A. Riera, C. Gogolin, and J. Eisert, Phys. Rev. Lett. **108**, 080402 (2012).
[11] J. Sirker, N. Konstantinidis, and N. Sedlmayr, arXiv:1303.3064.
[12] J. Berges, S. Borsányi, and C. Wetterich, Phys. Rev. Lett. **93**, 142002 (2004).
[13] M. Moeckel and S. Kehrein, Phys. Rev. Lett. **100**, 175702 (2008).
[14] M. Eckstein and M. Kollar, Phys. Rev. Lett. **100**, 120404 (2008).
[15] J. Sabio and S. Kehrein, New J. Phys. **12**, 055008 (2010).
[16] M. Heyl, A. Polkovnikov, and S. Kehrein, Phys. Rev. Lett. **110**, 135704 (2013).
[17] A. Mitra, Phys. Rev. Lett. **109**, 260601 (2012).
[18] M. A. Cazalilla, Phys. Rev. Lett. **97**, 156403 (2006).
[19] T. Barthel and U. Schollwöck, Phys. Rev. Lett. **100**, 100601 (2008).
[20] J. Lancaster and A. Mitra, Phys. Rev. E **81**, 061134 (2010).
[21] J.-S. Caux and R. M. Konik, Phys. Rev. Lett. **109**, 175301 (2012).
[22] J. Rentrop, D. Schuricht, and V. Meden, New J. Phys. **14**, 075001 (2012).
[23] F. H. L. Essler, S. Evangelisti, and M. Fagotti, Phys. Rev. Lett. **109**, 247206 (2012).
[24] P. Calabrese, F. H. L. Essler, and M. Fagotti, Phys. Rev. Lett. **106**, 227203 (2011).
[25] D. Iyer and N. Andrei, Phys. Rev. Lett. **109**, 115304 (2012).
[26] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. **98**, 050405 (2007).
[27] H. Janssen, B. Schaub, and B. Schmittmann, Z. Phys. B **73**, 539 (1989).
[28] P. Calabrese and A. Gambassi, J. Phys. A: Math. Gen. **38**, R133 (2005).
[29] A. Chandran, A. Erez, S. S. Gubser, and S. L. Sondhi, Phys. Rev. B **86**, 064304 (2012).
[30] B. Sciolla and G. Biroli, arXiv:1211.2572.
[31] C. De Grandi, A. Polkovnikov, and A. W. Sandvik, Phys. Rev. B **84**, 224303 (2011).
[32] M. Kolodrubetz, B. K. Clark, and D. A. Huse, Phys. Rev. Lett. **109**, 015701 (2012).
[33] E. G. Dalla Torre, E. Demler, T. Giamarchi, and E. Altman, Phys. Rev. B **85**, 184302 (2012).
[34] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, Oxford, 2004).
[35] R. R. P. Singh, M. E. Fisher, and R. Shankar, Phys. Rev. B **39**, 2562 (1989).
[36] I. Affleck, J. Phys. A: Math. Gen. **31**, 4573 (1998).
[37] V. Barzykin and I. Affleck, J. Phys. A: Math. Gen. **32**, 867 (1999).
[38] S. Sachdev, Phys. Rev. B **50**, 13006 (1994).
[39] H. Castella, X. Zotos, and P. Prelovšek, Phys. Rev. Lett. **74**, 972 (1995).
[40] X. Zotos, Phys. Rev. Lett. **82**, 1764 (1999).
[41] A. Rosch and N. Andrei, Phys. Rev. Lett. **85**, 1092 (2000).
[42] B. Altshuler, R. Konik, and A. Tsvelik, Nucl. Phys. B **739**, 311 (2006).
[43] J. Sirker, R. G. Pereira, and I. Affleck, Phys. Rev. Lett. **103**, 216602 (2009).
[44] J. Lancaster, T. Giamarchi, and A. Mitra, Phys. Rev. B **84**, 075143 (2011).
[45] F. D. M. Haldane, Phys. Rev. B **25**, 4925 (1982).
[46] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, Rev. Mod. Phys. **83**, 1405 (2011).
[47] Here we have dropped the $\langle\nabla\phi(1)\nabla\phi(2)\rangle$ term as it decays faster in position, and is therefore less important than $\langle\cos\{\gamma\phi(1)\}\cos\{\gamma\phi(2)\}\rangle$ near the critical point where $V_{\rm sg}$ is marginal. This holds true both in and out of equilibrium.
[48] A. Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, Cambridge, England, 2011).
[49] A. Iucci and M. A. Cazalilla, Phys. Rev. A **80**, 063619 (2009).
[50] D. M. Kennes and V. Meden, Phys. Rev. B **82**, 085109 (2010).
[51] E. Perfetto, Phys. Rev. B **74**, 205123 (2006).
[52] B. Dóra, M. Haque, and G. Zaránd, Phys. Rev. Lett. **106**, 156406 (2011).
[53] E. Perfetto and G. Stefanucci, Europhys. Lett. **95**, 10006 (2011).
[54] P. Calabrese and J. Cardy, Phys. Rev. Lett. **96**, 136801 (2006).
[55] L. Mathey and A. Polkovnikov, Phys. Rev. A **80**, 041601 (2009).
[56] P. Nozieres and F. Gallet, J. Phys. (Paris) **48**, 353 (1987).
[57] M. Tavora and A. Mitra (unpublished).
[58] C. Karrasch, J. Rentrop, D. Schuricht, and V. Meden, Phys. Rev. Lett. **109**, 126406 (2012).