

Low-temperature properties of ferromagnetic spin chains in a magnetic field

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The thermodynamic properties of ferromagnetic spin chains have been analyzed with a variety of microscopic methods over the years: Bethe ansatz, spin-wave theory, Schwinger-boson mean-field theory, Green functions, and renormalization group methods. Surprisingly, in all these different studies, the manifestation of the spin-wave interaction in the low-temperature series for the thermodynamic quantities, in the presence of a finite magnetic field, has been largely neglected. In the present work, we address this problem by following a different path, based on the systematic effective Lagrangian method. We evaluate the partition function up to two-loop order and derive the low-temperature expansion of the energy density, entropy density, heat capacity, magnetization, and susceptibility in the presence of a weak external magnetic field. Remarkably, the spin-wave interaction only manifests itself beyond two-loop order. In particular, there is no term of order T^2 in the low-temperature series of the free energy density. This is the analog of Dyson's statement that there is no term of order T^4 in the low-temperature series of the free energy density in the case of three-dimensional ideal ferromagnets. The range of validity of our series is critically examined in view of the nonperturbatively generated energy gap. We also compare our results with the condensed matter literature and point out that there are some misleading statements.

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I. INTRODUCTION

Ideal ferromagnets, i.e., ferromagnetic systems, which are governed by purely isotropic exchange coupling between the nearest neighbors and by the interaction with a weak external magnetic field, have been the subject of an impressive number of publications over the past few decades. In three spatial dimensions, the situation is well-known: after various unsuccessful attempts, the correct low-temperature series for the spontaneous magnetization was first given by Dyson in Ref. 1. Many authors after Dyson also discussed the low-temperature series for the three-dimensional ideal ferromagnet, based on other microscopic methods, such as spin-wave theory and Green functions. A simple and elegant method, according to Dyson,² is provided by Ref. 3. Recently, using the systematic effective Lagrangian technique, Dyson's series was rederived in Ref. 4 and extended to higher orders in Ref. 5.

Remarkably, regarding the low-temperature series describing two-dimensional ferromagnets, only a few papers are available, almost all of them dealing with noninteracting spin waves.⁶⁻¹⁹ Within the effective Lagrangian framework, the question of how the spin-wave interaction manifests itself in the low-temperature properties of two-dimensional ideal ferromagnets has been solved in Refs. 20 and 21.

In the present work, we apply the effective Lagrangian method to ferromagnetic spin chains: one-dimensional systems have never been studied within the systematic effective loop expansion. As we explain below, Lorentz- or pseudo-Lorentz-invariant systems, such as antiferromagnets, cannot be systematically analyzed within the framework of effective Lagrangians in one spatial dimension: the linear, i.e., relativistic, dispersion relation of the magnons in an antiferromagnet spoils the systematic loop expansion where the method is based upon. In this respect, ferromagnetic magnons, which display a quadratic dispersion relation, represent an interesting exception; for this nonrelativistic system, the loop expansion perfectly works in one spatial dimension, such that the

powerful method of effective Lagrangians can indeed be applied to ferromagnetic spin chains.

The effective Lagrangian method corresponds to an expansion of observables in powers of momentum or, equivalently, in powers of temperature. The systematic effective framework is based upon the fact that loops in Feynman diagrams are suppressed by some power n of momentum: otherwise the loop expansion does not converge and the effective field theory method fails. As we will see, the power n referring to the suppression of momentum depends on the spatial dimension of the system and on the dispersion relation. In (pseudo-)Lorentz-invariant effective field theories, which include the effective theories of quantum chromodynamics and antiferromagnets, the Goldstone bosons display a linear, i.e., relativistic, dispersion relation. Here, every loop in a Feynman diagram corresponds to a suppression of p^{d_s-1} powers of momentum. The effective expansion thus works in three and two spatial dimensions but is not applicable to one-dimensional (pseudo-)Lorentz-invariant systems. In this respect, the ferromagnet, its magnon displaying a quadratic dispersion relation, represents a peculiar case. Here, every loop in a Feynman diagram leads to a suppression of p^{d_s} powers of momentum, implying that the systematic effective Lagrangian method works in three, two, and one spatial dimension.

In the present study, we evaluate the partition function of ferromagnetic spin chains up to two-loop order in the presence of a weak external magnetic field. The low-temperature series for the free energy density, energy density, entropy density, heat capacity, magnetization, and susceptibility are given. It is pointed out that the spin-wave interaction does not yet manifest itself at this order of the effective expansion: it only enters at the three-loop level.

The range of validity of the low-temperature series is more restricted in one than in two spatial dimensions. This has to do with the fact that, unlike in two spatial dimensions, the nonperturbatively generated correlation length of ferromagnetic magnons is no longer exponentially large. We

carefully examine the domain of validity of the effective low-temperature series and, in particular, underline that it is conceptually inconsistent to switch off the magnetic field in these expressions.

While the thermodynamics of ferromagnetic spin chains has not been analyzed with effective Lagrangians so far, these systems have attracted a lot of attention over the past few decades and many methods have been used to derive their low-temperature properties. Early studies were based on the Bethe ansatz, amounting to numerically solving a system of coupled integral equations.^{22–30} Later on, modified spin-wave theory—a variant of conventional spin-wave theory, designed to cope with two- and one-dimensional systems—was used in Refs. 9, 10, and 31. Ferromagnetic spin chains were also addressed with Schwinger-boson mean-field theory,^{13,14} Green functions,^{16,32–38} spin-wave theory at constant order parameter,¹⁹ renormalization group and scaling methods,^{33,39–45} and by Monte Carlo simulations.^{37,46–52} Yet other approaches to ferromagnetic spin chains can be found in Refs. 53–58.

Most of these studies focus on the limit of a zero magnetic field. Our effective analysis, on the other hand, is valid in a different regime where the magnetic field is weak but not zero. Still, some of the above authors also consider the case of a nonzero magnetic field, such that their findings can be compared with our effective results. As it turns out, there are some misleading statements in the literature regarding conventional and modified spin-wave theories.

We would like to stress again that the manifestation of the spin-wave interaction in the low-temperature series describing ferromagnetic spin chains has not been considered explicitly in any of the above references, except for Ref. 19, which we discuss in detail. So it remains rather unclear whether the low-temperature series presented in these studies are indeed correct, i.e., complete up to the order considered, or whether they receive corrections due to the spin-wave interaction. This is one of the main problems we address in the present work.

The rest of the paper is organized as follows. In Sec. II, we provide a brief outline of the effective Lagrangian method with special attention to the loop counting in one spatial dimension. The partition function for ferromagnetic spin chains is evaluated perturbatively up to two-loop order and the low-temperature series for various thermodynamic quantities are derived in Sec. III. The range of validity of these series is critically examined in Sec. IV. The relevant condensed matter literature on ferromagnetic spin chains is reviewed and compared with our effective results in Sec. V. Finally, Sec. VI contains our conclusions.

At the end of this section, we would like to mention that the systematic effective Lagrangian method has been used to study a variety of condensed matter systems with a spontaneously broken internal spin symmetry. In three spatial dimensions, the low-energy properties of ferromagnets and antiferromagnets were analyzed in Refs. 59–66. Two-dimensional ferromagnets and antiferromagnets were the subject of Refs. 20, 21, and 67–70. Of particular interest are two-dimensional antiferromagnets, which turn into high-temperature superconductors upon doping with either holes or electrons. These systems have been analyzed within the effective field theory framework, both for underlying square and honeycomb lattices, in Refs. 71–81.

Finally, the consistency of the effective Lagrangian method with high-precision numerical simulations and microscopic models was demonstrated in Refs. 82–86.

II. EFFECTIVE LAGRANGIANS AND LOOP COUNTING

In this section, we focus on some essential aspects of the effective Lagrangian method at finite temperature. The interested reader may find a more detailed account on finite-temperature effective Lagrangians in Appendix A in Ref. 5 and in the various references given therein. In addition, for pedagogic introductions to the effective Lagrangian technique, we refer to Refs. 87–91. We would like to stress, however, that the present section is self-contained and that there is no need to read many references in parallel. The references above and those cited in the course of this section are provided in case the reader is interested in the foundations of the method, in the explicit derivation of the effective Lagrangian, or in other details, not really needed to understand the present calculation.

The basic degrees of freedom of the effective Lagrangian are the Goldstone bosons which are a consequence of the spontaneously broken continuous symmetry. At low energies or low temperatures, these particles dominate the physical behavior of the system. In the present case of ferromagnetic spin chains, we are dealing with magnons that are the Goldstone bosons of the spontaneously broken spin rotation symmetry; while the Heisenberg model is invariant under the group $O(3)$, the ground state—at zero temperature—is only invariant under $O(2)$.

The systematic construction of the terms in the effective Lagrangian is straightforward: the link between the underlying theory and the effective theory is provided by the symmetries.^{92–94} One first identifies all symmetries of the underlying theory. In our case, the Heisenberg model exhibits an $O(3)$ spin rotation symmetry as well as parity and time reversal symmetry. The effective Lagrangian, or more precisely, the effective action for the ferromagnetic spin chain,

$$\mathcal{S}_{\text{eff}} = \int d^2x \mathcal{L}_{\text{eff}}, \quad (1)$$

inherits all these symmetries of the underlying Heisenberg model.

Since we are interested in the low-energy—or low-temperature—dynamics, terms in the effective Lagrangian which contain only a few derivatives are the most significant ones, while terms with more derivatives are suppressed. This organization of terms represents the very basis for the systematic expansion of observables in powers of momentum p (or energy or temperature).

As derived in Ref. 59, the leading-order effective Lagrangian of the ideal ferromagnet in d_s spatial dimensions is of order p^2 and takes the form

$$\mathcal{L}_{\text{eff}}^2 = \Sigma \frac{\epsilon_{ab} \partial_0 U^a U^b}{1 + U^3} + \Sigma \mu H U^3 - \frac{1}{2} F^2 \partial_r U^i \partial_r U^i, \quad (2)$$

$$r = x_1, \dots, x_{d_s}.$$

The effective degrees of freedom are contained in the two real components of the magnon field, $U^a (a = 1, 2)$, which represent the first two components of the three-dimensional magnetization unit vector $U^i = (U^a, U^3)$. The magnetic field

points along the third direction, $\vec{H} = (0, 0, H)$, much like the magnetization. Our convention is $H = |\vec{H}| > 0$, i.e., the quantity H is always positive. While the structure of the above terms is an immediate consequence of the symmetries of the underlying theory, at this order, two *a priori* unknown low-energy constants emerge: the spontaneous magnetization at zero temperature Σ and F . These low-energy constants have to be determined by experiment, numerical simulation or by comparison with the underlying microscopic theory. In particular, the constant F can be extracted from comparison with microscopic results (see Sec. IV) and expressed as

$$F^2 = JS^2a, \quad (3)$$

where J is the exchange integral of the Heisenberg model and a is the distance between two spins. Note also that the sign of Σ is positive: taking the limit $\vec{H} \rightarrow 0$ at zero temperature, the spontaneous magnetization points into the same direction as the magnetic field, which is the positive 3-direction by our convention.

The Lagrangian (2) leads to a quadratic dispersion relation,

$$\omega(\vec{k}) = \gamma \vec{k}^2 + \mathcal{O}(|\vec{k}|^4), \quad \gamma \equiv \frac{F^2}{\Sigma}, \quad (4)$$

obeyed by ferromagnetic magnons. The low-energy constant F is thus related to the spin stiffness, or helicity modulus, γ . Note that this relation dictates how time and space derivatives in the systematic effective expansion have to be counted: one time derivative (∂_0) is on the same footing as two space derivatives (∂_r, ∂_r), i.e., two powers of momentum count as only one power of energy or temperature: $k^2 \propto \omega, T$.

As derived in Ref. 4, the next-to-leading terms in the effective Lagrangian are of order p^4 and contain four spatial derivatives. In two or three spatial dimensions, we have a total of three independent terms:

$$\mathcal{L}_{\text{eff}}^4 = l_1(\partial_r U^i \partial_r U^i)^2 + l_2(\partial_r U^i \partial_s U^i)^2 + l_3 \Delta U^i \Delta U^i \quad (d_s = 2, 3), \quad (5)$$

where Δ is the Laplace operator in d_s spatial dimensions. At next-to-leading order in the effective Lagrangian, we have three effective couplings: l_1, l_2 , and l_3 . In one spatial dimension, however, the first two terms coincide and we are left with only two independent terms of order p^4 ,

$$\mathcal{L}_{\text{eff}}^4 = l_1(\partial_{x_1} U^i \partial_{x_1} U^i)^2 + l_3 \partial_{x_1}^2 U^i \partial_{x_1}^2 U^i \quad (d_s = 1). \quad (6)$$

Higher-order pieces $\mathcal{L}_{\text{eff}}^6, \mathcal{L}_{\text{eff}}^8, \dots$ of the effective Lagrangian, as we discuss below, are irrelevant for the evaluation of the partition function considered in this work. Much like the leading-order constants F and Σ , these higher-order effective constants have to be determined by experiment, numerical simulation or by comparison with the underlying microscopic theory. The latter is performed in Secs. IV and V.

In conclusion, the effective Lagrangian for the ferromagnetic spin chain relevant for the present evaluation is

$$\mathcal{L}_{\text{eff}} = \Sigma \frac{\epsilon_{ab} \partial_0 U^a U^b}{1 + U^3} + \Sigma \mu H U^3 - \frac{1}{2} F^2 \partial_r U^i \partial_r U^i + l_1 (\partial_{x_1} U^i \partial_{x_1} U^i)^2 + l_3 \partial_{x_1}^2 U^i \partial_{x_1}^2 U^i. \quad (7)$$

The fundamental object in finite-temperature field theory is the partition function, represented as a Euclidean functional integral

$$\text{Tr}[\exp(-\mathcal{H}/T)] = \int [dU] \exp\left(-\int_T d^{d_s+1}x \mathcal{L}_{\text{eff}}\right). \quad (8)$$

Here, the integration extends over all magnon field configurations that are periodic in the Euclidean time direction $U(\vec{x}, x_4 + \beta) = U(\vec{x}, x_4)$, where $\beta \equiv 1/T$ is the inverse temperature. The quantity \mathcal{L}_{eff} on the right-hand side of the representation (8) is the Euclidean effective Lagrangian, consisting of a string of terms,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^2 + \mathcal{L}_{\text{eff}}^4 + \mathcal{O}(p^6), \quad (9)$$

which involve an increasing number of derivatives.

The essential point in the perturbative evaluation of Eq. (8) is that to a given order in the low-temperature expansion only a finite number of Feynman graphs and only a finite number of effective constants are relevant. More concretely, the low-temperature expansion of the partition function is obtained by analyzing the fluctuations of the spontaneous magnetization vector $\vec{U} = (U^1, U^2, U^3)$ around the ground state $\vec{U}_0 = (0, 0, 1)$, i.e., by expanding its third component, U^3 , in powers of the spin-wave fluctuations U^a ,

$$U^3 = \sqrt{1 - U^a U^a} = 1 - \frac{1}{2} U^a U^a - \frac{1}{8} U^a U^a U^b U^b - \dots \quad (10)$$

Inserting this expansion into Eq. (8), one then generates the set of Feynman diagrams depicted in Fig. 1. The leading contribution in the exponential on the RHS of Eq. (8) is of order p^2 and stems from $\mathcal{L}_{\text{eff}}^2$. It contains a term quadratic in the spin-wave field U^a —with the appropriate derivatives and the magnetic field given in Eq. (2)—and describes free magnons. The corresponding diagram for the partition function is the one-loop diagram 3 of Fig. 1.

The nonleading terms in the effective Lagrangian in the path integral representation of the partition function (8), i.e., $\mathcal{L}_{\text{eff}}^4 + \mathcal{L}_{\text{eff}}^6 + \dots$, are treated as perturbations. The Gaussian integrals are evaluated in the standard way (see Ref. 95, in

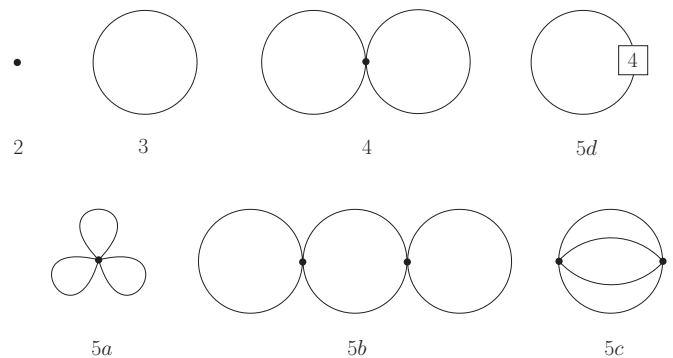


FIG. 1. Feynman graphs related to the low-temperature expansion of the partition function for the ferromagnetic spin chain up to order p^5 . The numbers attached to the vertices refer to the piece of the effective Lagrangian they come from. Vertices associated with the leading term $\mathcal{L}_{\text{eff}}^2$ are denoted by a filled circle. Note that ferromagnetic loops are suppressed by one power of momentum in one spatial dimension, $d_s=1$.

particular, chapter 3), and one obtains a set of Feynman rules which differ from the zero-temperature rules of the effective Lagrangian technique only in one respect: the periodicity condition imposed on the magnon fields alters the propagator, which, at finite temperature, is now given by

$$G(x) = \sum_{n=-\infty}^{\infty} \Delta(\vec{x}, x_4 + n\beta), \quad \vec{x} = (x_1, \dots, x_{d_s}). \quad (11)$$

Here, $\Delta(x)$ is the Euclidean propagator at zero temperature,

$$\begin{aligned} \Delta(x) &= \int \frac{dk_4 d^{d_s} k}{(2\pi)^{d_s+1}} \frac{e^{i\vec{k}\vec{x} - ik_4 x_4}}{\gamma \vec{k}^2 - ik_4 + \mu H} \\ &= \Theta(x_4) \int \frac{d^{d_s} k}{(2\pi)^{d_s}} e^{i\vec{k}\vec{x} - \gamma \vec{k}^2 x_4 - \mu H x_4}. \end{aligned} \quad (12)$$

An explicit expression for the thermal propagator, dimensionally regularized in the spatial dimension d_s , is

$$G(x) = \frac{1}{(4\pi\gamma)^{\frac{d_s}{2}}} \sum_{n=-\infty}^{\infty} \frac{1}{x_n^{\frac{d_s}{2}}} e^{-\frac{\vec{x}^2}{4\gamma x_n} - \mu H x_n} \Theta(x_n), \quad (13)$$

with

$$x_n \equiv x_4 + n\beta. \quad (14)$$

We restrict ourselves to the infinite volume limit and evaluate the free energy density z , defined by

$$z = -T \lim_{L \rightarrow \infty} L^{-d_s} \ln[\text{Tr exp}(-\mathcal{H}/T)]. \quad (15)$$

From a conceptual point of view it is quite remarkable that the effective field theory method can be applied to one-dimensional systems, such as ferromagnetic spin chains. In fact, ferromagnets represent a peculiar case. The crucial point is that, in the effective field theory framework, the perturbative evaluation of the partition function is based on the suppression of loop diagrams by some power of momentum. This suppression of loops depends on the spatial dimension d_s of the system as well as on the dispersion relation of its Goldstone bosons. Now for systems with a quadratic dispersion relation, such as the ferromagnet, each loop involves an integral of the type

$$\int d\omega d^{d_s} k \frac{1}{\omega - \gamma \vec{k}^2} \propto p^{d_s}, \quad (16)$$

related to ferromagnetic magnons circling in the loop. On dimensional grounds, the integral is proportional to d_s powers of momentum. While loops in three- (two-) dimensional ferromagnets are suppressed by three (two) powers of momentum, each loop in a Feynman diagram referring to ferromagnetic spin chains is still suppressed by one power of momentum p . The one-loop diagram 3 is of order p^3 , as it involves $\mathcal{L}_{\text{eff}}^2$ (p^2) and one loop (p). The two-loop diagram 4 is of order p^4 , as it involves one more loop compared to diagram 3.

This suppression rule lies at the heart of the organization of the Feynman graphs of the partition function referring to ferromagnetic spin chains depicted in Fig. 1. Now we also understand why the piece $\mathcal{L}_{\text{eff}}^6$ is not needed for the present study: the corresponding one-loop graph with a vertex from $\mathcal{L}_{\text{eff}}^6$ is of order p^7 , i.e., beyond the order we are concerned with here.

In the next section, we evaluate the partition function of the one-dimensional ideal ferromagnet in full generality up to order p^4 . The evaluation at order p^5 is much more involved. Specifically, the renormalization and numerical evaluation of the three-loop graph 5c turns out to be rather elaborate: the detailed account of this calculation will be presented elsewhere.⁹⁶ In the present study, we rather focus on the general structure of the low-temperature expansion and answer the question of which contributions in the thermodynamic quantities are due to noninteracting spin waves and which ones originate from the spin-wave interaction. Still, we also evaluate graph 5d, which is related to noninteracting magnons, in order to compare our results with the literature.

We emphasize that the suppression of loops in the case of ferromagnets is different from the loop suppression for systems with a linear dispersion relation. There, a loop corresponds to an integral of the type

$$\int d\omega d^{d_s} k \frac{1}{\omega^2 - c^2 \vec{k}^2} \propto p^{d_s-1}. \quad (17)$$

On dimensional grounds, the integral is proportional to d_s-1 powers of momentum. This means that for antiferromagnetic magnons, loops in three (two) spatial dimensions are suppressed by two (one) power of momentum and that the effective loop expansion perfectly works in these cases. However, in one spatial dimension, loops are not suppressed at all and that's why the effective Lagrangian method cannot be used to systematically analyze antiferromagnetic spin chains or any other one-dimensional (pseudo-)Lorentz-invariant system in terms of a loop expansion.

III. LOW-TEMPERATURE PROPERTIES OF FERROMAGNETIC SPIN CHAINS

We now consider those Feynman diagrams of Fig. 1 that contribute to the partition function up to order p^4 or, equivalently, up to order T^2 . We also include the diagram 5d, because we want to compare our results with the condensed matter literature which is restricted to noninteracting spin waves. Additional information on finite-temperature effective Lagrangians and the evaluation of the corresponding Feynman diagrams—going beyond the outline given in the previous section—can be found in Ref. 5 (see Sec. III and Appendix A). Again, we like to point out that we are considering one-dimensional *ideal* ferromagnets, i.e., ferromagnetic spin chains that are governed by the isotropic exchange interaction between nearest neighbors and by the interaction with a weak external magnetic field.

At leading order p^2 , the tree graph 2, which involves $\mathcal{L}_{\text{eff}}^2$, merely leads to a temperature-independent contribution to the free energy density,

$$z_2 = -\Sigma \mu H. \quad (18)$$

The leading temperature-dependent contribution comes from the one-loop graph 3 and is of order p^3 . It is related to a $(d_s + 1)$ -dimensional nonrelativistic free Bose gas and amounts to

$$z_3^T = -\frac{1}{2\pi^{\frac{1}{2}} \gamma^{\frac{1}{2}}} T^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{e^{-\mu H n \beta}}{n^{\frac{3}{2}}}. \quad (19)$$

At order p^4 , the first two-loop graph shows up. This contribution, associated with graph 4, is proportional to single space derivatives of the propagator at the origin,

$$z_4 \propto [\partial_{x_1} G(x)]_{x=0} [\partial_{x_1} G(x)]_{x=0} = 0, \quad (20)$$

and thus vanishes because the thermal propagator is invariant under parity, much like the Heisenberg Hamiltonian. As we have pointed out, the effective action—and thus the thermal propagator—inherits all the symmetries of the underlying Heisenberg model.

Finally, we include the one-loop graph 5d of order p^5 , which corresponds to noninteracting magnons. Here, the next-to-leading-order Lagrangian $\mathcal{L}_{\text{eff}}^4$ comes into play through a two-magnon vertex,

$$z_{5d} = -\frac{2l_3}{\Sigma} [\partial_{x_1}^4 G(x)]_{x=0}, \quad (21)$$

yielding the temperature-dependent contribution

$$z_{5d}^T = -\frac{3l_3}{4\pi^{\frac{1}{2}} \Sigma \gamma^{\frac{5}{2}}} T^{\frac{5}{2}} \sum_{n=1}^{\infty} \frac{e^{-\mu H n \beta}}{n^{\frac{5}{2}}}. \quad (22)$$

Gathering terms, the free energy density of the ferromagnetic spin chain becomes

$$z = -\Sigma \mu H - \frac{1}{2\pi^{\frac{1}{2}} \gamma^{\frac{1}{2}}} T^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{e^{-\mu H n \beta}}{n^{\frac{3}{2}}} - \frac{3l_3}{4\pi^{\frac{1}{2}} \Sigma \gamma^{\frac{5}{2}}} T^{\frac{5}{2}} \sum_{n=1}^{\infty} \frac{e^{-\mu H n \beta}}{n^{\frac{5}{2}}} + \mathcal{O}(p^5). \quad (23)$$

The contributions of order $T^{3/2}$ and $T^{5/2}$ originate from one-loop graphs and are both related to noninteracting spin waves. While the former is fully determined by the leading-order effective constants Σ and F ($\gamma = F^2/\Sigma$), the latter involves the next-to-leading-order constant l_3 .

It is quite remarkable that the spin-wave interaction does not yet manifest itself at next-to-leading order p^4 in the low-temperature expansion of the free energy density. The only potential candidate, the two-loop diagram 4 of order T^2 , turns out to be zero due to parity. This is the analog of Dyson's statement that, in the case of the three-dimensional ideal ferromagnet, there is no term of order T^4 in the low-temperature series of the free energy density. Likewise, there is no interaction term of order T^3 in the low-temperature series of the free energy density referring to the two-dimensional ideal ferromagnet. Regardless of the spatial dimension, the relevant two-loop diagram turns out to be zero due to parity.^{4,5,20,21} In the case of ferromagnetic spin chains, the spin-wave interaction enters through the three-loop graphs 5a–5c, yielding additional terms of order $p^5 \propto T^{5/2}$ in the series (23).

It is important to stress that our rigorous approach is completely systematic and does not resort to any kind of approximations or *ad hoc* assumptions. The structure of the above low-temperature series is an immediate consequence of the symmetries inherent in the one-dimensional ideal ferromagnet.

We now discuss the effect of a weak magnetic field. We thus expand the result (23) in the dimensionless parameter

$$\sigma = \mu H \beta = \frac{\mu H}{T}, \quad (24)$$

i.e., the ratio between the magnetic field $H = |\vec{H}| > 0$ and temperature. Retaining all terms up to quadratic in σ , we obtain

$$z = -\Sigma \mu H - \frac{1}{2\pi^{\frac{1}{2}} \gamma^{\frac{1}{2}}} T^{\frac{3}{2}} \left[\zeta\left(\frac{3}{2}\right) - 2\pi^{\frac{1}{2}} \sigma^{\frac{1}{2}} - \zeta\left(\frac{1}{2}\right) \sigma + \frac{1}{2} \zeta\left(-\frac{1}{2}\right) \sigma^2 + \mathcal{O}(\sigma^3) \right] - \frac{3l_3}{4\pi^{\frac{1}{2}} \Sigma \gamma^{\frac{5}{2}}} T^{\frac{5}{2}} \times \left[\zeta\left(\frac{5}{2}\right) - \zeta\left(\frac{3}{2}\right) \sigma + \frac{4}{3} \pi^{\frac{1}{2}} \sigma^{\frac{3}{2}} + \frac{1}{2} \zeta\left(\frac{1}{2}\right) \sigma^2 + \mathcal{O}(\sigma^3) \right] + \mathcal{O}(p^5). \quad (25)$$

A thorough discussion of the range of validity of this series will be given in Sec. IV. As it turns out, it would be inconsistent to take the limit $H \rightarrow 0$. Note that, due to our convention $H = |\vec{H}| > 0$, the parameter σ defined in Eq. (24) and the spontaneous magnetization at zero temperature (Σ) always take positive values. In particular, the above series for the free energy density does not depend on the direction of the magnetic field.

We also consider the low-temperature series for the energy density u , for the entropy density s , and for the heat capacity c_V of the ferromagnetic spin chain. They can be obtained from the thermodynamic relations

$$s = \frac{\partial P}{\partial T}, \quad u = Ts - P, \quad c_V = \frac{\partial u}{\partial T} = T \frac{\partial s}{\partial T}. \quad (26)$$

Because the system is homogeneous, the pressure is given by the temperature-dependent part of the free energy density,

$$P = z_0 - z, \quad (27)$$

and the thermodynamic quantities take the form

$$u = \frac{1}{2\pi^{\frac{1}{2}} \gamma^{\frac{1}{2}}} T^{\frac{3}{2}} \left[\sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} \right] + \frac{3l_3}{4\pi^{\frac{1}{2}} \Sigma \gamma^{\frac{5}{2}}} T^{\frac{5}{2}} \left[\sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} + \frac{3}{2} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{5}{2}}} \right] + \mathcal{O}(p^5),$$

$$s = \frac{1}{2\pi^{\frac{1}{2}} \gamma^{\frac{1}{2}}} T^{\frac{1}{2}} \left[\sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}} + \frac{3}{2} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} \right] + \frac{3l_3}{4\pi^{\frac{1}{2}} \Sigma \gamma^{\frac{5}{2}}} T^{\frac{3}{2}} \left[\sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} + \frac{5}{2} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{5}{2}}} \right] + \mathcal{O}(p^3),$$

$$c_V = \frac{1}{2\pi^{\frac{1}{2}} \gamma^{\frac{1}{2}}} T^{\frac{1}{2}} \left[\sigma^2 \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{-\frac{1}{2}}} + \sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}} + \frac{3}{4} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} \right] + \frac{3l_3}{4\pi^{\frac{1}{2}} \Sigma \gamma^{\frac{5}{2}}} T^{\frac{3}{2}} \left[\sigma^2 \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}} + 3\sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} + \frac{15}{4} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{5}{2}}} \right] + \mathcal{O}(p^3). \quad (28)$$

Again, for a weak magnetic field H , the series may be expanded in the parameter $\sigma = \mu H/T$,

$$\begin{aligned}
u &= \frac{1}{2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}} T^{\frac{3}{2}} \left[\frac{1}{2}\zeta\left(\frac{3}{2}\right) + \frac{1}{2}\zeta\left(\frac{1}{2}\right)\sigma - \frac{3}{4}\zeta\left(-\frac{1}{2}\right)\sigma^2 + \mathcal{O}(\sigma^3) \right] \\
&\quad + \frac{3l_3}{4\pi^{\frac{1}{2}}\Sigma\gamma^{\frac{5}{2}}} T^{\frac{5}{2}} \left[\frac{3}{2}\zeta\left(\frac{5}{2}\right) - \frac{1}{2}\zeta\left(\frac{3}{2}\right)\sigma - \frac{1}{4}\zeta\left(\frac{1}{2}\right)\sigma^2 + \mathcal{O}(\sigma^3) \right] + \mathcal{O}(p^5), \\
s &= \frac{1}{2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}} T^{\frac{1}{2}} \left[\frac{3}{2}\zeta\left(\frac{3}{2}\right) - 2\sqrt{\pi}\sigma^{\frac{1}{2}} - \frac{1}{2}\zeta\left(\frac{1}{2}\right)\sigma - \frac{1}{4}\zeta\left(-\frac{1}{2}\right)\sigma^2 + \mathcal{O}(\sigma^3) \right] \\
&\quad + \frac{3l_3}{4\pi^{\frac{1}{2}}\Sigma\gamma^{\frac{5}{2}}} T^{\frac{3}{2}} \left[\frac{5}{2}\zeta\left(\frac{5}{2}\right) - \frac{3}{2}\zeta\left(\frac{3}{2}\right)\sigma + \frac{4\sqrt{\pi}}{3}\sigma^{\frac{3}{2}} + \frac{1}{4}\zeta\left(\frac{1}{2}\right)\sigma^2 + \mathcal{O}(\sigma^3) \right] + \mathcal{O}(p^3), \\
c_V &= \frac{1}{2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}} T^{\frac{1}{2}} \left[\frac{3}{4}\zeta\left(\frac{3}{2}\right) + \frac{1}{4}\zeta\left(\frac{1}{2}\right)\sigma + \frac{3}{8}\zeta\left(-\frac{1}{2}\right)\sigma^2 + \mathcal{O}(\sigma^3) \right] \\
&\quad + \frac{3l_3}{4\pi^{\frac{1}{2}}\Sigma\gamma^{\frac{5}{2}}} T^{\frac{3}{2}} \left[\frac{15}{4}\zeta\left(\frac{5}{2}\right) - \frac{3}{4}\zeta\left(\frac{3}{2}\right)\sigma - \frac{1}{8}\zeta\left(\frac{1}{2}\right)\sigma^2 + \mathcal{O}(\sigma^3) \right] + \mathcal{O}(p^3),
\end{aligned} \tag{29}$$

where we have retained terms up to quadratic in the magnetic field.

Note that all terms in the above series for u , s and c_V originate from the two one-loop graphs displayed in Fig. 1. The explicit contribution due to the spin-wave interaction, entering at order $p^5 \propto T^{5/2}$ ($p^3 \propto T^{3/2}$) for u (s, c_V), will be considered in detail in Ref. 96. Here, we want to emphasize that there is no interaction term of order $p^4 \propto T^2$ in the energy density and no interaction term of order $p^2 \propto T$ in the entropy density and heat capacity.

Let us now turn to the magnetization. With the expression for the free energy density (23), the low-temperature expansion for the magnetization

$$\Sigma(T, H) = -\frac{\partial z}{\partial(\mu H)} \tag{30}$$

of ferromagnetic spin chains takes the form

$$\frac{\Sigma(T, H)}{\Sigma} = 1 - \tilde{\alpha}_0 T^{\frac{1}{2}} - \tilde{\alpha}_1 T^{\frac{3}{2}} + \mathcal{O}(p^3). \tag{31}$$

The coefficients $\tilde{\alpha}_i$ depend on the dimensionless ratio $\sigma = \mu H/T$ and are given by

$$\tilde{\alpha}_0 = \frac{1}{2\pi^{\frac{1}{2}}\Sigma\gamma^{\frac{1}{2}}} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}}, \quad \tilde{\alpha}_1 = \frac{3l_3}{4\pi^{\frac{1}{2}}\Sigma^2\gamma^{\frac{5}{2}}} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}}. \tag{32}$$

Finally, the susceptibility,

$$\chi(T, H) = \frac{\partial \Sigma(T, H)}{\partial(\mu H)}, \tag{33}$$

of ferromagnetic spin chains amounts to

$$\chi(T, H) = \tilde{\kappa}_0 T^{-\frac{1}{2}} + \tilde{\kappa}_1 T^{\frac{1}{2}} + \mathcal{O}(p), \tag{34}$$

with coefficients

$$\tilde{\kappa}_0 = \frac{1}{2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{-\frac{1}{2}}}, \quad \tilde{\kappa}_1 = \frac{3l_3}{4\pi^{\frac{1}{2}}\Sigma\gamma^{\frac{5}{2}}} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}}. \tag{35}$$

In what follows, we critically examine the range of validity of the series presented in this section and thereby put the above low-temperature expansions for a one-dimensional system on a firm basis, on the same footing as the low-temperature series for ferro- and antiferromagnets in three or two spatial dimensions.

IV. LOW-TEMPERATURE SERIES: RANGE OF VALIDITY

The range of validity of the low-temperature series presented in this work is restricted due to the Mermin-Wagner theorem.⁹⁷ The theorem states that, in one or two spatial dimensions, no spontaneous symmetry breaking at any finite temperature can occur in the O(3)-invariant Heisenberg model. Also, in the context of the ferromagnet, an energy gap is generated nonperturbatively and the correlation length of the magnons is no longer infinite. Still, even in one spatial dimension, the correlation length is large, being proportional to the inverse temperature,³⁹

$$\xi_{np} = a C_{\xi}^{(0)} \frac{JS^2}{T} \left[1 + C_{\xi}^{(1)} \frac{1}{\pi} \sqrt{\frac{T}{JS^3}} + \mathcal{O}(T) \right]. \tag{36}$$

Here, a is the spacing between two neighboring sites of the spin chain and the quantities $C_{\xi}^{(0)}$ and $C_{\xi}^{(1)}$ are dimensionless constants. According to Ref. 39, they take the values

$$C_{\xi}^{(0)} = 1.14 \pm 0.11, \quad C_{\xi}^{(1)} = 0.6514 \pm 0.0012. \tag{37}$$

On the other hand, the value for the first constant, quoted in Refs. 10 and 98, is $C_{\xi}^{(0)} = 1$, which is the number we will use for the estimate below. It is important to note that, unlike in two spatial dimensions,⁹⁹ the correlation length of magnons in one spatial dimension is no longer exponentially large.

Apart from the nonperturbatively generated correlation length ξ_{np} , ferromagnetic magnons are also characterized by the correlation length ξ which is related to the magnetic field. As long as the correlation length ξ of the Goldstone bosons is much smaller than the nonperturbatively generated correlation length ξ_{np} , our low-temperature series are valid: in this regime, the spin-waves are well-defined and represent the relevant

low-energy degrees of freedom. A natural way to define the correlation length ξ for ferromagnetic magnons is based on their dispersion relation,

$$\omega(k) = \gamma k^2 + \mu H + \mathcal{O}(k^4), \quad \gamma \equiv \frac{F^2}{\Sigma}, \quad (38)$$

leading to

$$\xi = \sqrt{\frac{\gamma}{\mu H}} = \frac{F}{\sqrt{\Sigma \mu H}}. \quad (39)$$

This quantity has dimension of length and tends to infinity if the magnetic field is switched off. Indeed, the correlation length of the magnons in a ferromagnetic spin chain is infinite at zero temperature.

For the low-temperature series to be valid, the ratio

$$x = \xi/\xi_{np} \quad (40)$$

of the two correlation lengths must be a small number. Using Eqs. (36) and (39), and expressing the effective constant γ in terms of the exchange integral J of the underlying theory as¹⁰⁰

$$\gamma = JSa^2, \quad (41)$$

we arrive at

$$\frac{\mu H}{T} = \frac{1}{S^3 x^2} \frac{T}{J}. \quad (42)$$

For $S = \frac{1}{2}$ and with the ratio $x = \frac{1}{10}$, we obtain the relation

$$\frac{\mu H}{T} = 800 \frac{T}{J}, \quad (43)$$

which can be interpreted as follows. The exchange integral J defines a scale in the underlying theory and for the effective expansion to be valid, the temperature has to be small with respect to this scale. For practical purposes, we may choose

$$\frac{T}{J} = \frac{1}{50} \implies \frac{\mu H}{T} = \sigma = 16. \quad (44)$$

The parameter σ can thus be large, i.e., the magnetic field need not be small compared to the temperature. What is essential, however, is that the magnetic field itself—much like the temperature—is small compared to the intrinsic scale J of the underlying theory.

It is important to note that we cannot completely switch off the magnetic field in our low-temperature expansions. Rather, we start running into trouble as soon as we choose a ratio $\mu H/T$, which is smaller than $800T/J$; we then leave the domain of validity of the low-temperature series derived in this work, because the effective calculation does not take into account the nonperturbative effect.

To illustrate the range of validity, we consider the two-dimensional domain defined by the parameters T/J and $\mu H/J$, which both have to be small for the effective expansion to be valid. In terms of these parameters the condition (43) takes the form

$$\frac{\mu H}{J} = 800 \frac{T^2}{J^2}. \quad (45)$$

This is the line plotted in Fig. 2. In the parameter space above that curve, the low-temperature series derived in this work are valid. In one spatial dimension, the parameter regime is thus quite restricted. In particular, note that the horizontal axis

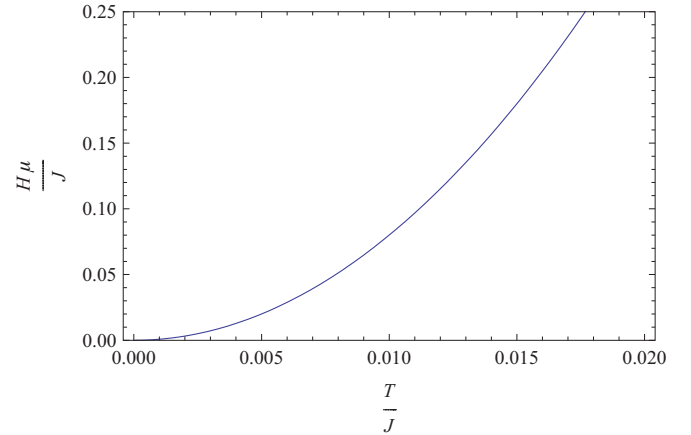


FIG. 2. (Color online) Range of validity of the effective low-temperature expansion of the ferromagnetic spin chain. The allowed parameter regime corresponds to the area above the curve and is restricted to very low temperatures.

corresponding to zero magnetic field is outside the allowed domain.

It is very instructive to compare this result with the range of validity of the analogous low-temperature series, referring to two-dimensional ideal ferromagnets. There, the nonperturbatively generated correlation length is exponentially large, the argument of the exponential being proportional to the inverse temperature,⁹⁹

$$\xi_{np} = C_\xi a S^{-\frac{1}{2}} \sqrt{\frac{T}{JS^2}} \exp\left(\frac{2\pi JS^2}{T}\right) \quad (d_s = 2), \quad (46)$$

where a is the spacing between two neighboring sites on the square lattice, and the quantity $C_\xi \approx 0.05$ is a dimensionless constant.

Following the same steps as before, for two-dimensional ideal ferromagnets, one derives the relation²⁰

$$\frac{\mu H}{T} = \frac{400S^4}{x^2} \frac{J^2}{T^2} \exp\left(-\frac{4\pi JS^2}{T}\right) \quad (d_s = 2), \quad (47)$$

or

$$\frac{\mu H}{T} = 2500 \frac{J^2}{T^2} \exp\left(-\pi \frac{J}{T}\right) \quad (d_s = 2), \quad (48)$$

for $S = \frac{1}{2}$ and $x = \frac{1}{10}$. In two spatial dimensions, it is also conceptually inconsistent to switch off the magnetic field; here, we start running into trouble as soon as the ratio $\mu H/T$ is smaller than the value given by the right-hand side (RHS) of Eq. (48).

Rewritten in terms of the parameters T/J and $\mu H/J$, the condition (48) implies

$$\frac{\mu H}{J} = 2500 \frac{J}{T} \exp\left(-\pi \frac{J}{T}\right) \quad (d_s = 2). \quad (49)$$

This is the line plotted in Fig. 3, indicating that the effective low-temperature series derived in Refs. 20 and 21 are valid in the parameter space above this line. As one can see, the allowed parameter regime is much larger than in one spatial dimension. Again, this is due to the fact that in two spatial dimensions, the nonperturbative correlation length is exponentially large.

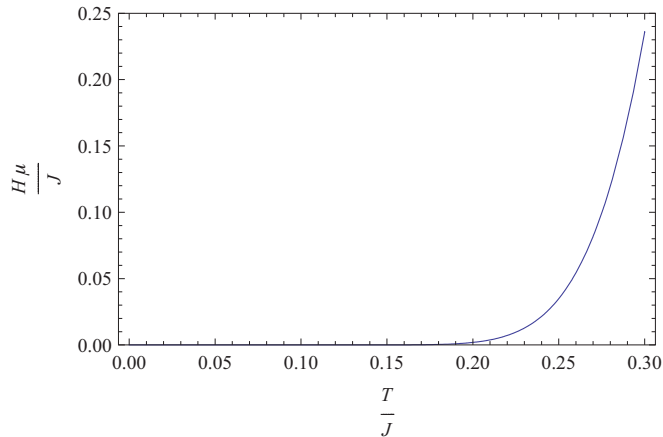


FIG. 3. (Color online) Range of validity of the effective low-temperature expansion of the two-dimensional ideal ferromagnet. The allowed parameter regime corresponds to the area above the curve.

In the previous section, we have considered the effect of a weak magnetic field by Taylor expanding the low-temperature series for z, u, s , and c_V in the small parameter $\sigma = \mu H/T$. In order to illustrate in which parameter regime the Taylor expansion does make sense for the free energy density, in Fig. 4, we have plotted the quantity Δ ,

$$\Delta = 1 - \frac{\sum_{n=1}^{\infty} \frac{e^{-\mu H n \beta}}{n^{3/2}}}{\zeta\left(\frac{3}{2}\right)}, \quad \sigma = \mu H \beta = 800 \frac{T}{J}, \quad (50)$$

as a function of the parameter T/J . The quantity Δ must be small, let us say $\Delta \leq 0.05$, in order for the Taylor series (25) to make sense: the leading term proportional to $\zeta\left(\frac{3}{2}\right)$ in the expansion (25) then makes up 95% or more of the full leading-order contribution displayed in the numerator of Eq. (50). However, according to Fig. 4, for that to be the case, the temperature has to be extremely small compared to the scale J . We thus conclude that the Taylor expansions for z , as well for u, s , and c_V , in the parameter σ only make sense in a very small domain. Still, we have provided these expansions in

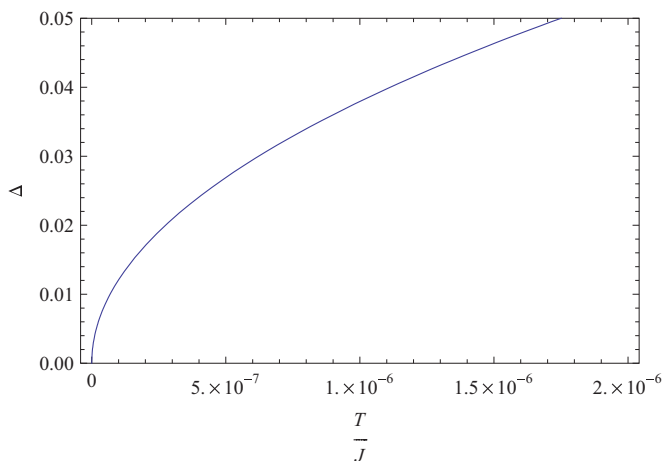


FIG. 4. (Color online) Ferromagnetic spin chain: the Taylor expansion for the free energy density in the parameter σ , Eq. (25), only makes sense at very low temperatures compared to the scale J .

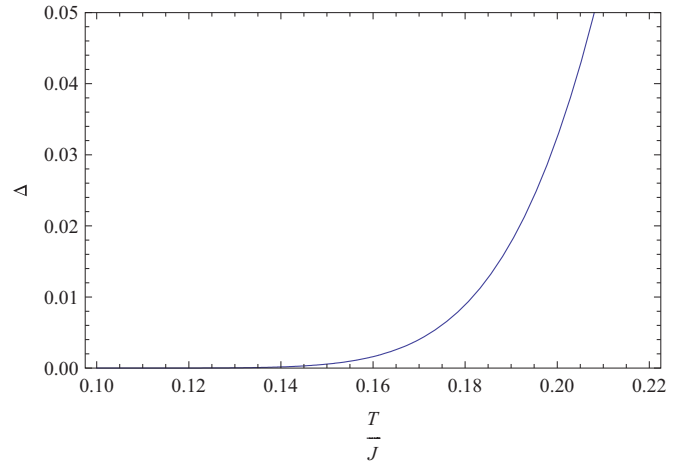


FIG. 5. (Color online) Two-dimensional ideal ferromagnet: the Taylor expansion for the free energy density in the parameter σ , Eq. (III.8) of Ref. 20, makes sense up to temperatures which are about one fifth of the scale J .

the previous section for completeness, and also because in the next section, we want to compare them with the literature. We have to stress, however, that their range of validity has never been thoroughly discussed in the literature, i.e., the leading terms in these Taylor series were given in Refs. 9, 10, and 19 without actually pointing out that these series are only valid in an extremely small parameter regime.

Again, it is instructive to also consider the situation in two-dimensional ideal ferromagnets. The analogous quantity Δ for the free energy density, according to Eq. (III.8) of Ref. 20, is

$$\Delta = 1 - \frac{\sum_{n=1}^{\infty} \frac{e^{-\mu H n \beta}}{n^2}}{\zeta(2)}, \quad (51)$$

$$\sigma = \mu H \beta = 2500 \frac{J^2}{T^2} \exp\left(-\pi \frac{J}{T}\right) \quad (d_s = 2).$$

This is the curve plotted in Fig. 5, indicating that the situation here is entirely different: in two spatial dimensions, where the nonperturbative correlation length is exponentially large, the parameter regime in which the Taylor expansion in σ for z , as well as for u, s , and c_V makes sense, is much larger: the temperature need not be tiny with respect to the scale J .

We have argued that the magnetic field cannot be switched off in the present study. While the case $\vec{H} = 0$ is beyond the reach of the effective expansion presented here, it is not beyond the reach of the effective field theory method. Rather, one has to establish a different type of systematic perturbative expansion, which can cope with a zero magnetic field. Work in this direction is in progress.

V. COMPARISON WITH THE LITERATURE

The thermodynamic properties of ferromagnetic spin chains have been analyzed with a variety of microscopic methods: spin-wave theory, Bethe ansatz, Schwinger-boson mean-field theory, Green functions, and renormalization group methods. Almost all of these references obtain a term of order T^2 in the low-temperature expansion of the free energy density, which appears to be in contradiction with the systematic effective

field theory calculation. In fact, we have argued that there is no T^2 term in this series, because the only potential candidate, the two-loop graph 4 of Fig. 1, vanishes due to parity. This is an exact statement and represents the analog of Dyson's statement that, in three spatial dimensions, there is no term of order T^4 in the free energy density of an ideal ferromagnet.

The essential point is to realize that most of the above references refer to ferromagnetic spin chains in zero external magnetic field. This is not the domain where the effective Lagrangian method presented in this work operates: for our expansions to be valid, the magnetic field is always different from zero. It should be pointed out that some references indeed consider a finite magnetic field and provide low-temperature expansions for this case (see, e.g., Ref. 40), but the magnetic field there is tiny compared to the temperature, such that these series lie outside the parameter regime of our effective expansions. Also, given the fact that the ferromagnetic spin chain is integrable, and exact results and analytical expressions can be obtained, it is rather surprising that low-temperature expansions in the presence of a weak (but not tiny or zero) magnetic field—which thus could be compared with our results—appear to be unavailable.

While the focus of the pioneering articles by Takahashi, Refs. 9 and 10 is on $H = 0$, the nonzero field case is also considered there. The method developed and advocated is modified spin-wave theory. As Takahashi states in Ref. 9, modified spin-wave theory is restricted to zero external magnetic field. Conventional spin-wave theory, on the other hand, should be used if one wants to study the low-temperature properties of ferromagnetic spin chains in the presence of a magnetic field. Indeed, conventional spin-wave theory predicts that there is no term of order T^2 in the free energy density of a one-dimensional ferromagnet,^{9,10}

$$z = -T \left[1.0421869 \left(\frac{T}{J} \right)^{\frac{1}{2}} + 0.0668971 \left(\frac{T}{J} \right)^{\frac{3}{2}} + \dots \right] \left(S = \frac{1}{2} \right), \quad (52)$$

in agreement with the systematic effective field theory result. However, two comments are in order here.

First, Takahashi's analysis is restricted to noninteracting spin waves; in accordance with the effective analysis, free magnon particles do not produce a term of order T^2 in the above series. The crucial point is that, as the effective analysis demonstrates, the spin-wave interaction does not lead to a T^2 term, either. This result is entirely new.

Second, Eq. (52)—in view of the way it was derived—cannot be trusted. Apparently, in order to obtain the above expression, Takahashi has taken the limit $H \rightarrow 0$, which appears to be conceptually inconsistent: conventional spin-wave theory does not operate in this sector. Likewise, as we have discussed at length in the previous section, the magnetic field in our effective low-temperature series cannot be switched off, either. Furthermore, the statements that—in the absence of a magnetic field—conventional spin-wave theory *applies to some degree* (see Ref. 10, p. 168), or that conventional spin-wave theory *is valid in some sense* (see Ref. 101, p. 156), are misleading in our opinion.

Takahashi's expression (52) can still be used to extract the effective low-energy coupling l_3 by taking the same (conceptually inconsistent) limit $H \rightarrow 0$ in our effective expansion (25). Matching the two expressions, we end up with

$$l_3 = \frac{\sqrt{\pi} c_{5/2}}{6\sqrt{2}\zeta(\frac{5}{2})} J a^3 \approx 0.0104 J a^3 \left(S = \frac{1}{2} \right), \quad (53)$$

where the quantity $c_{5/2} = 0.0668971$ is the second coefficient in Takahashi's expansion (52). For general spin S , the effective constant l_3 reads

$$l_3 = \frac{J S^2 a^3}{24}. \quad (54)$$

The ferromagnetic spin chain in nonzero magnetic field was also considered in Ref. 19, where another variant of conventional spin-wave theory, capable to deal with low-dimensional systems—spin-wave theory at constant order parameter—was invented. The authors also discuss the case $H \neq 0$ and obtain the following expansion for the magnetization $m(H)$:

$$\frac{m(H)}{S} = 1 - \frac{\zeta(\frac{1}{2})}{2S\sqrt{\pi}} \sqrt{t} - \frac{1}{2S} \sqrt{\frac{t}{v}} + \mathcal{O}(t, t^{3/2} v^{-1/2}), \quad (55)$$

$$t = \frac{T}{JS}, \quad v = \frac{H}{T}.$$

Indeed, Eq. (55) agrees with the leading terms of our effective result (31).

In fact, the authors of Ref. 19 provide a further term in the above expansion that is related to the spin-wave interaction:

$$\left(\frac{m(H)}{S} \right)_{\text{int}} = \frac{1}{16} \left(\frac{1}{2S} \sqrt{\frac{t}{v}} \right)^3. \quad (56)$$

Although our calculation does not yet include the impact of the interaction, we note that the above term appears to be not quite correct. More precisely, the authors of Ref. 19 conclude that the coefficient $(2S)^{-1} \sqrt{t/v}$ is close to unity. For $S = 1/2$ this implies $v = t$, which is equivalent to

$$\frac{\mu H}{J} = 2 \frac{T^2}{J^2}. \quad (57)$$

However, according to Eq. (45), we are clearly outside the domain where the spin-wave picture applies. Because the parameter $(2S)^{-1} \sqrt{t/v}$ is close to unity, the authors of Ref. 19 state that the leading fluctuation correction to the Hartree-Fock theory is not controlled by a small parameter. In fact, for $S = 1/2$ the second temperature-dependent term in Eq. (55) is close to -1 , and does not represent a small correction to the magnetization $m(H)/S$. This also seems to hint at an inconsistency related to the interaction contribution (56).

For the reader's convenience, expressing the effective constants γ and l_3 in terms of microscopic parameters according to Eqs. (41) and (54), we rewrite our series in a form where the $1/S$ expansion becomes manifest:

$$z = -\frac{S\mu H}{a} - \frac{1}{2\pi^{\frac{1}{2}} \sqrt{J} S a} T^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{e^{-\mu H n \beta}}{n^{\frac{3}{2}}} - \frac{1}{32\pi^{\frac{1}{2}} \sqrt{J^3 S^3} a} T^{\frac{5}{2}} \sum_{n=1}^{\infty} \frac{e^{-\mu H n \beta}}{n^{\frac{5}{2}}} + \mathcal{O}(p^5),$$

$$\begin{aligned}
u &= \frac{1}{2\pi^{\frac{1}{2}}\sqrt{J}Sa} T^{\frac{3}{2}} \left[\sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} \right] \\
&+ \frac{1}{32\pi^{\frac{1}{2}}\sqrt{J^3}S^3a} T^{\frac{5}{2}} \left[\sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} + \frac{3}{2} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{5}{2}}} \right] \\
&+ \mathcal{O}(p^5), \\
s &= \frac{1}{2\pi^{\frac{1}{2}}\sqrt{J}Sa} T^{\frac{1}{2}} \left[\sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}} + \frac{3}{2} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} \right] \\
&+ \frac{1}{32\pi^{\frac{1}{2}}\sqrt{J^3}S^3a} T^{\frac{3}{2}} \left[\sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} + \frac{5}{2} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{5}{2}}} \right] \\
&+ \mathcal{O}(p^3), \\
c_V &= \frac{1}{2\pi^{\frac{1}{2}}\sqrt{J}Sa} T^{\frac{1}{2}} \left[\sigma^2 \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{-\frac{1}{2}}} + \sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}} \right. \\
&+ \left. \frac{3}{4} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} \right] + \frac{1}{32\pi^{\frac{1}{2}}\sqrt{J^3}S^3a} T^{\frac{3}{2}} \left[\sigma^2 \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}} \right. \\
&+ \left. 3\sigma \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} + \frac{15}{4} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{5}{2}}} \right] + \mathcal{O}(p^3), \\
\frac{\Sigma(T, H)}{\Sigma} &= 1 - \frac{1}{2\pi^{\frac{1}{2}}\sqrt{J}S^3} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}} T^{\frac{1}{2}} \\
&- \frac{1}{32\pi^{\frac{1}{2}}\sqrt{J^3}S^5} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{3}{2}}} T^{\frac{3}{2}} + \mathcal{O}(p^3), \\
\chi(T, H) &= \frac{1}{2\pi^{\frac{1}{2}}\sqrt{J}Sa} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{-\frac{1}{2}}} T^{-\frac{1}{2}} \\
&+ \frac{1}{32\pi^{\frac{1}{2}}\sqrt{J^3}S^3a} \sum_{n=1}^{\infty} \frac{e^{-\sigma n}}{n^{\frac{1}{2}}} T^{\frac{1}{2}} + \mathcal{O}(p). \quad (58)
\end{aligned}$$

In contrast to Refs. 9, 10, and 19 we have carefully discussed the range of validity of the above series in the previous section, and thus have put these low-temperature expansions on safe grounds, on the same footing as the low-temperature series for ferro- and antiferromagnets in three and two spatial dimensions.

It is important to emphasize that all the theoretical references providing low-temperature series for ferromagnetic spin chains in the presence of a magnetic field, were restricted to *free* magnons so far (except for Ref. 19, which we have discussed above). The important question of whether the spin-wave interaction—in the case $H \neq 0$ —already shows up at order $T^{5/2}$ in the free energy density, or rather beyond, has been neglected. In other words, it remained unclear whether the series in Refs. 9 and 10 referring to the ideal magnon gas and derived within spin-wave theory, are indeed complete up to order $T^{5/2}$.

We have demonstrated that the spin-wave interaction starts manifesting itself at the three-loop level in the systematic effective expansion. In the low-temperature series of the free energy density, the corresponding three-loop graphs 5a–5c of Fig. 1, indeed lead to a contribution of order $p^5 \propto T^{5/2}$. The explicit evaluation is quite involved and will be presented

elsewhere.⁹⁶ Here, we rather wanted to draw the attention to the general structure of the low-temperature series in the presence of a weak external magnetic field and critically examine their range of validity, as well as compare our systematic results with the condensed matter literature, which is restricted (except for Ref. 19) to noninteracting spin waves.

VI. CONCLUSIONS

We have studied the low-temperature behavior of ferromagnetic spin chains in the presence of a weak external magnetic field. While these systems have been investigated by many authors using different techniques, such as Bethe ansatz, spin-wave theory, and Schwinger-boson mean-field theory, in the present study, we have made use of the systematic effective Lagrangian method. We have evaluated the low-temperature expansion of the partition function of ferromagnetic spin chains in a weak magnetic field up to two-loop order and derived the low-temperature series for the energy density, entropy density, heat capacity, magnetization, and susceptibility. Interestingly, the spin-wave interaction does not yet manifest itself at this order in the low-temperature expansions: the only two-loop graph turns out to be zero due to parity. The spin-wave interaction only enters at the three-loop level.

From a conceptual point of view, it is quite remarkable that the low-temperature properties of systems defined in one spatial dimension can be analyzed within the systematic effective loop expansion. One-dimensional systems, which display a linear, i.e., relativistic, dispersion law cannot be systematically analyzed with effective Lagrangians, because the loop counting breaks down. However, for systems with a quadratic dispersion relation like the ferromagnet, the method perfectly works, because loop graphs are still suppressed by one power of momentum.

We have carefully examined the range of validity of the low-temperature series that is quite restricted. Unlike in two spatial dimensions, where the nonperturbatively generated correlation length of the spin waves is exponentially large at low temperatures, in one spatial dimension, the nonperturbative correlation length is only proportional to the inverse temperature. As a consequence, both in one and two spatial dimensions, it is conceptually inconsistent to switch off the magnetic field in our series as we would then leave their domain of validity. We have confronted our results with those obtained by spin-wave theory and have pointed out that there are some misleading statements in the literature.

In the above microscopic studies, the magnons were considered as ideal Bose particles: the problem of the spin-wave interaction was neglected (except for Ref. 19 which we have discussed in detail) and it thus remained unclear whether the low-temperature series given in these articles are complete or will receive additional corrections due to the interaction. While we have argued that, in the presence of a magnetic field, the spin-wave interaction enters at order $p^5 \propto T^{5/2}$ in the free energy density of ferromagnetic spin chains, the explicit evaluation of the corresponding three-loop graphs has not been considered here: this will be the subject of Ref. 96.

The present study demonstrates that the effective Lagrangian technique is a very powerful tool to analyze the

general structure of the low-temperature expansion of the partition function for systems with a spontaneously broken global symmetry. Not only have we rigorously discussed the impact of the spin-wave interaction and a weak external magnetic field in a systematic manner, but also have we put our low-temperature series on a firm basis.

It would be very interesting to establish the effective Lagrangian method in the parameter regime where the magnetic field is zero. This quite nontrivial problem has been solved

in Ref. 69 for the two-dimensional antiferromagnet in zero magnetic and staggered field. Work on transferring these techniques to the ferromagnet is in progress.

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