$U(1) \times SU(2)$ gauge invariance leading to charge and spin conductivity of Dirac fermions in graphene

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Gauge symmetries have been identified in graphene and associated with specific physical properties. For instance, the $U(1)$ gauge group is related to electrodynamics in $(1+2)$-dimensional [$(1+2)$D] space-time and non-Abelian gauge groups can describe curvature and torsion. Here we demonstrate that the Dirac Lagrangian for massless electrons near the Dirac points is also invariant under the group $SU(2)$ related to local spin rotations, leading to the correct spin-orbit interactions and a rigorous definition for the spin-current density. Furthermore, we computed the charge and spin conductivity within the framework of Kubo linear response theory, using the algebra of relativistic Dirac spinors in $(1+2)$D space-time. The minimal value of electrical conductivity is predicted to be $\pi q^2/h$, in agreement with typical experimental findings.

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I. INTRODUCTION

Graphene is a two-dimensional allotrope of carbon which displays very unusual electronic properties and promises to revolutionize the industry of electronic devices, leading to a new era in the field of spintronics. Therefore, a formal expression for the principal actor in spintronics, namely, the spin-current density, should be precisely defined for electrons in graphene. One of the most interesting aspects of graphene is that, being a genuine two-dimensional material, it has a peculiar band structure in which electrons emulate the behavior of massless Dirac fermions in a $(1+2)$-dimensional [$(1+2)$D] relativistic space-time near the so-called Dirac points of the Brillouin zone of a honeycomb lattice,[1,2] providing an interesting bridge between condensed matter and relativistic high-energy physics.[3] The behavior of Dirac fermions under the influence of magnetic fields is dramatically distinct from ordinary nonrelativistic particles. For instance, an anomalous integer quantum Hall effect was experimentally measured[4,5] in agreement with theoretical predictions for massless Dirac fermions. The spin degree of freedom leads to the spin-orbit coupling and spin-current effects in graphene, which has been studied both experimentally and theoretically.[6–10] Besides that, we propose a solution to the problem of the missing π in previous attempts to predict theoretically the electrical conductivity of graphene. Usual theoretical predictions based on the Kubo formula produce a minimal conductivity of $4q^2/\pi h$, while current experiments point to a value of at least $\pi q^2/h$,[11] where q is the carrier charge. As a matter of fact, already in 1986, Fradkin[12] studied the critical behavior of disordered degenerate semiconductors within a mean-field theory valid when the number of degeneracy points is large and concluded that in 2D the conductivity appears to be a universal number proportional to q^2/h but independent of band structure details. Remarkably, experimental measurements of graphene charge conductivity corroborate such a conclusion, but the proportionality constant is dependent on graphene cleanliness from an experimental viewpoint and on the conductivity calculation technique or on the assumed scattering mechanisms on the theoretical side. The ac conductivity of massless Dirac fermions was theoretically predicted in Refs. 13, 14, and 15, yielding $4q^2/(\pi h)$ in the $\omega \to 0$ limit. On the other side, theories assuming the presence of screened Coulomb scatterers predict a minimal conductivity about $4q^2/h$, which is π times larger than the previous result and is in fairly good agreement with current experiments. Indeed, numerical calculations were also employed to obtain the conductivity of finite-size systems in which randomly distributed Coulomb scatterers are present[16] and a minimum value per massless Dirac fermion channel is found to be $q^2/(\pi h)$. The effect of impurities was also considered in Ref. 17, showing that when both short and long-range scatterers are present the minimal conductivity must be greater than $4q^2/(\pi h)$. In the context of two-dimensional d-wave superconductors, which present many theoretical similarities with graphene, Lee[18] demonstrated that the conductivity of low-lying quasiparticles is independent of the scattering rate but proportional to q^2/h. Going further, conductivity calculations for clean graphene have also been performed and related in the current literature, yielding conflicting results.[13,19–22] The theoretical values of conductivity for clean graphene usually depend on limiting processes and the order in which these limits are taken. For example, the dc conductivity is usually taken as the limit of an ac conductivity of a graphene sheet of infinite extent. The infinite extent limit can be interchanged with the zero frequency limit. Furthermore, clean and infinite extent graphene does not exist in practice. However, throughout the present contribution cleanliness and the infinite extent of graphene are our main assumptions, together with the Dirac Lagrangian density.

A deeper understanding of the underlying physics of graphene is provided by the study of its gauge symmetries. The concept of gauge invariance is a cornerstone of modern quantum field theories.[23,24] A considerable number of gauge symmetries have been identified and associated with specific physical properties in graphene.[25] For instance, the $U(1)$ gauge group is related to electrodynamics in $(1+2)$D space-time and non-Abelian gauge groups can be used to describe curvature and torsion.[26] It is our aim in this paper to demonstrate that the Dirac Lagrangian for massless electrons near the Dirac points is invariant under the $SU(2)$ gauge group related to

local spin rotations. As a consequence, the Rashba spin-orbit interaction will emerge quite naturally and a rigorous definition for the spin-current density will be obtained. Furthermore, a careful calculation based on relativistic Dirac spinors within the framework of Kubo linear response allowed us to correctly obtain the value of the electric conductivity, apparently solving the problem of the missing π.

In the context of a nonrelativistic theory of electrons, it was previously shown that a precise definition for the spin-current density can be achieved through the study of gauge symmetries.[27] Indeed, the Pauli-Schrödinger Lagrangian density describing nonrelativistic electrons is invariant under the $U(1) \times SU(2)$ gauge group, where the $U(1)$ sector is related to the usual electromagnetic potentials A_μ and the $SU(2)$ group describes local rotations of the spin quantization axis, leading to the introduction of non-Abelian gauge potentials identified with the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$.[27,28] As a physical consequence of the invariance under the $SU(2)$ gauge group, one obtains the interactions between the electron spin and the electromagnetic fields. Motivated by the earlier success obtained by gauging rotations in the spin space in nonrelativistic theories, we follow closely the lines of reasoning given in Ref. 27 and apply the same gauge principle to electrons in graphene. At this point, we emphasize that electrons living in monolayer graphene are essentially nonrelativistic, despite the fact they behave effectively as massless relativistic particles near the Dirac points. To corroborate that fact, notice the Fermi velocity v_F, which plays the role of the speed of light in graphene, is lower than the true speed of light c by two orders of magnitude; i.e., $v_F \approx c/300 = 10^6 \mathrm{m/s}$. Therefore, we believe the underlying theory should be invariant under local rotations in the spin space.

The content of our paper is organized as follows: In Sec. II the $U(1) \times SU(2)$ gauge invariance of the Dirac Lagrangian describing massless fermions in graphene is demonstrated, resulting in precise definitions for charge and spin current densities and correctly predicting the spin-orbit interactions. In Sec. III we use the Kubo linear response theory in order to obtain the charge and spin conductivities of clean graphene. Finally, in the last section, we summarize the main results of the present contribution.

II. $U(1) \times SU(2)$ GAUGE INVARIANCE OF THE DIRAC LAGRANGIAN IN (1 + 2)D SPACE-TIME

Our starting point is the Dirac Lagrangian density describing electrons in graphene near the corners of the Brillouin zone in a (1 + 2)D space-time, written as

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi, \tag{1}$$

where $\psi = (\xi_\uparrow, \xi_\downarrow, \zeta_\uparrow, \zeta_\downarrow)^T$ is an eight-component Dirac spinor, $\bar{\psi} = \psi^\dagger \gamma^0$ is the adjoint spinor, $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2)$ are the Dirac matrices, $\partial_\mu = \partial/\partial x^\mu$ is the derivative operator, $x^\mu = (x^0 = v_F t,\ x^1,\ x^2)$ are the space-time coordinates in 1 + 2D, the Fermi velocity $v_F (= 1$ in properly chosen units) plays the role of the speed of light c, and the index μ runs from 0 to 2. We are using the Einstein convention of summing over repeated indices throughout this paper. The two-component spinor $\xi_\sigma(\zeta_\sigma)$ describes an electron at the $\mathbf{K}(\mathbf{K}')$ Dirac point with genuine spin $\sigma = (\uparrow, \downarrow)$. Usually one defines an index known as the valley pseudospin α, corresponding to the Dirac points $\mathbf{K}(\alpha = +1)$ and $\mathbf{K}' = -\mathbf{K}(\alpha = -1)$; i.e, the spinors ξ and ζ correspond to valley pseudospin $+1$ and -1, respectively. Besides the genuine spin σ and the valley pseudospin α there is an additional sublattice pseudospin, associated with the Pauli matrices $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$, such that for an electron at $\mathbf{K}$ with spin σ we have $\tau_z \xi_\sigma = \pm 1 \xi_\sigma$, where the eigenvalue $+1$ (-1) represents an electron located at the sublattice A (B). This way, the Dirac matrices satisfying the anticommuting relation, $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$, where $g^{\mu\nu} = \mathrm{diag}(1, -1, -1)$ is the Minkowski metric tensor, can be explicitly represented by $\gamma^0 = \mathrm{diag}(\tau_z, \tau_z, \tau_z, \tau_z)$, $\gamma^1 = \mathrm{diag}(i\tau_y, i\tau_y, -i\tau_y, -i\tau_y)$, and $\gamma^2 = \mathrm{diag}(-i\tau_x, -i\tau_x, -i\tau_x, -i\tau_x)$.

Now we consider a general $U(1) \times SU(2)$ gauge transformation of the spinor ψ acting on phase and spin spaces, as follows:

$$\psi' = \exp\left(i\frac{\sigma^a \Lambda_a}{2}\right)\psi, \quad \bar{\psi}' = \bar{\psi}\exp\left(-i\frac{\sigma^a \Lambda_a}{2}\right), \tag{2}$$

$\Lambda_a(x^\mu)$ being a space-time-dependent four-vector and $\sigma^a = (\sigma^0, \vec{\sigma})$, $\sigma^0 = \mathbf{1}$ is the identity matrix, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices acting on the genuine spin space. The requirement that the Lagrangian density (1) must be invariant under the above transformation is easily achieved replacing the ordinary derivatives ∂_μ with covariant ones, given as[23,24]

$$D_\mu = \partial_\mu + iqA_\mu - ig\vec{\sigma} \cdot \mathbf{W}_\mu,$$

where A_μ and $\mathbf{W}_\mu$ are the gauge fields associated with the $U(1)$ and $SU(2)$ gauge groups, respectively, q is the electric charge, and g is the spin coupling constant. Clearly, $A_\mu = (A_0, -\mathbf{A})$ is the electromagnetic potential. In order to obtain the correct electron spin coupling to the electromagnetic fields in graphene we must specify $\mathbf{W}_\mu$ in the following way:

$$\mathbf{W}_0 = \mathbf{B}, \tag{3}$$

$$\mathbf{W}_i = \tfrac{1}{2}\hat{\mathbf{n}}_i \times \mathbf{E}, \tag{4}$$

where $\hat{\mathbf{n}}_i$ is a unit vector in the in-plane direction $i = 1,2$ and $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and electric fields, respectively.

The gauge invariant version of Eq. (1) is written explicitly as follows:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu + iqA_\mu - ig\vec{\sigma} \cdot \mathbf{W}_\mu)\psi. \tag{5}$$

The charge and spin-current densities are obtained by applying Noether's theorem[23,24] for an infinitesimal $U(1) \times SU(2)$ gauge transformation $\delta\psi = i(\sigma^a \Lambda_a/2)\psi$, leading to the expression

$$J^{\mu a} = (J^{\mu 0}, \vec{J}^\mu) = [q\bar{\psi}\gamma^\mu\psi, g\bar{\psi}\gamma^\mu\vec{\sigma}\psi], \tag{6}$$

where the index $a = 0$ denotes the electric current density $J^{\mu 0}$, while for $a = 1,2,3$ we get the components of spin-current density $\vec{J}^\mu$. The above result is just the expected one from physical intuition. Notice that the velocity operator for a Dirac fermion is given simply by $v^\mu = \gamma^0\gamma^\mu$ and the spin-current density $\vec{J}^\mu$ is just the product of the density operator $\psi^\dagger\psi$ with $v^\mu\vec{\sigma}$ arranged in the correct matrix ordering. It is worth mentioning that the spin of electrons in graphene can be thought of as a kind of weak hypercharge and the genuine spin space as a "weak isospace" outside the 1 + 2D space-time.

The charge and spin-current densities $J^{\mu 0}$ and $\vec{J}^{\mu}$ can be used to recast Eq. (5) into the following form:

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - J^{\mu 0}A_{\mu} + \vec{J}^{\mu}\cdot\mathbf{W}_{\mu}. \tag{7}$$

Clearly, the last term in the above equation describes the interaction between the spin-current density and the $SU(2)$ gauge potentials, whose components are related to the electric and magnetic fields, in accordance with Eqs. (3) and (4). From the above Lagrangian density, the Dirac Hamiltonian density is readily obtained:

$$\mathcal{H}_D = -i\bar{\psi}\vec{\gamma}\cdot\nabla\psi + J^{\mu 0}A_{\mu} - \vec{J}^{\mu}\cdot\mathbf{W}_{\mu}. \tag{8}$$

The last term on the right-hand side of Eq. (8) can be expanded writing the Dirac spinor in a convenient form,

$$\psi_{\alpha=+1} = \begin{pmatrix}\xi_{\uparrow}\\ \xi_{\downarrow}\end{pmatrix}, \quad \psi_{\alpha=-1} = \begin{pmatrix}\zeta_{\uparrow}\\ \zeta_{\downarrow}\end{pmatrix}, \tag{9}$$

where α is the valley index. This way, the spin-current density $\vec{J}^{\mu}$ becomes

$$\vec{J}^{0} = g\sum_{\alpha}\psi_{\alpha}^{\dagger}\vec{\sigma}\psi_{\alpha}, \tag{10}$$

$$\vec{J}^{1} = g\sum_{\alpha}\alpha\psi_{\alpha}^{\dagger}\tau_{x}\vec{\sigma}\psi_{\alpha}, \tag{11}$$

$$\vec{J}^{2} = g\sum_{\alpha}\psi_{\alpha}^{\dagger}\tau_{y}\vec{\sigma}\psi_{\alpha}, \tag{12}$$

making it possible to express the last term of Eq. (8) in the following way:

$$\mathcal{H}_{\text{int}} = -g\sum_{\alpha}\psi_{\alpha}^{\dagger}\Big[\vec{\sigma}\cdot\mathbf{B} + \frac{1}{2}\sigma_{z}(\alpha\tau_{x}E_{y} - E_{x}\tau_{y}) - \frac{1}{2}E_{z}(\alpha\tau_{x}\sigma_{y} - \sigma_{x}\tau_{y})\Big]\psi_{\alpha}. \tag{13}$$

The term $-g\vec{\sigma}\cdot\mathbf{B}$ is the Zeeman splitting, which is known to be very small in graphene even in magnetic fields as high as $|\mathbf{B}| = 10$ T[3]. The next terms represent the spin-orbit interactions. The coupling constant g must be identified with the Bohr magneton $\mu_B = q\hbar/2m_c$ of the electrons in graphene, provided the value of electron's cyclotron mass m_c. The Rashba coupling is given by the term $-\frac{g}{2}E_z(\alpha\tau_x\sigma_y - \sigma_x\tau_y)$, describing the effect of an electric field perpendicular to the graphene sheet and has been extensively studied in the current literature.[29–31] Our gauge invariant formulation also predicts the effect of in-plane electric fields, usually related to intrinsic spin-orbit effects. Notice that the integer quantum Hall effect is not taken into account by the $SU(2)$ gauge group, since it comes entirely from the $U(1)$ electromagnetic potential A_{μ} coupled to electrons via electric charge q. The presence of spin-orbit coupling is responsible for opening a gap near the Dirac points, which can be controlled by the magnitude of the external electric field E_z perpendicular to the graphene layer. The torque equations on the magnetization vector $\vec{J}^{0}$ can be straightforwardly obtained from a continuity equation in covariant form, $D_{\mu}\vec{J}^{\mu} = 0$. Writing it explicitly in terms of ordinary derivatives, we obtain

$$\partial_{\mu}\vec{J}^{\mu} = 2g\vec{J}^{\mu}\times\mathbf{W}_{\mu}. \tag{14}$$

Observing Eq. (14) it is clear that the spin-current density $\vec{J}^{\mu}$ is not a conserved quantity in the presence of electromagnetic fields, as usual.

III. CHARGE AND SPIN CONDUCTIVITIES FROM KUBO LINEAR RESPONSE THEORY

As the last goal in the present contribution, we want to compute the charge and spin conductivities for monolayer graphene using the Kubo linear response theory and the correct relativistic spinors. Assuming that the externally applied fields are homogeneous in space, the current $J^{\mu a}(x)$ generated by the potential $\mathcal{A}_{\mu a} = (A_{\mu}, \mathbf{W}_{\mu})$ is given by

$$J^{\mu a}(x) = \int_{-\infty}^{\infty} dt' K^{\mu a\nu b}(t-t')\mathcal{A}_{\nu b}(t'). \tag{15}$$

The kernel of the above equation is directly obtained from the averaged current-current correlation function, as

$$K^{\mu a\nu b}(t-t') = -\frac{i}{\hbar}\theta(t-t')\int d^{2}x'\langle[J^{\mu a}(x), J^{\nu b}(x')]\rangle, \tag{16}$$

where $[A,B] = AB - BA$ is the commutator, $\theta(t-t')$ is the Heaviside function, and $\langle\cdots\rangle$ denotes thermal average. The Fermi level of undoped monolayer graphene is exactly at the Dirac point, where the density of states vanishes, leading to the complete absence of free carriers, which would imply a vanishing Boltzmann conductivity. However, graphene shows a minimum conductivity observed experimentally, which comes from vacuum fluctuations allowing for creation/annihilation of electron-hole pairs. In order to calculate the charge and spin conductivities of graphene from Eq. (16) we express the Dirac spinor in second quantized form,

$$\psi(x) = \sum_{\alpha\mathbf{k}s}(u_{\alpha\mathbf{k}}c_{\alpha\mathbf{k}s}e^{-ikx} + v_{\alpha\mathbf{k}}d_{\alpha\mathbf{k}s}^{\dagger}e^{ikx}), \tag{17}$$

where $kx = k_0 t - \mathbf{k}\cdot\mathbf{x}$, the fermionic operator $c_{\alpha\mathbf{k}s}(d_{\alpha\mathbf{k}s}^{\dagger})$ destroys (creates) an electron (hole) with valley index α, momentum $\hbar\mathbf{k}$, and spin s, and $u_{\alpha\mathbf{k}}$ and $v_{\alpha\mathbf{k}}$ are Dirac spinors in momentum space, given by

$$u_{\alpha\mathbf{k}} = v_{\alpha\mathbf{k}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \alpha e^{i\alpha\varphi_{\mathbf{k}}}\end{pmatrix}, \tag{18}$$

with $\tan\varphi_{\mathbf{k}} = k_y/k_x$. It is worth pointing out that massless Dirac fermions obey the dispersion relation of the form $k_0^2 - \mathbf{k}^2 = 0$. Taking into account that for undoped graphene at low temperatures the ground state is the vacuum of electrons and holes, thermal averaging of any quantum operator $\hat{\mathcal{O}}$ is simply given by $\langle 0|\hat{\mathcal{O}}|0\rangle$, where $|0\rangle$ stands for the vacuum state. A straightforward calculation yields

$$K^{\mu a\nu b}(t-t') = -i\frac{g_a g_b}{\hbar}\text{tr}(\sigma^{a}\sigma^{b}) \times\sum_{\alpha\mathbf{k}}[\bar{v}_{\alpha\mathbf{k}}\gamma^{\mu}u_{\alpha,-\mathbf{k}}\bar{u}_{\alpha,-\mathbf{k}}\gamma^{\nu}v_{\alpha\mathbf{k}}e^{-2ik_0(t-t')} - \bar{u}_{\alpha\mathbf{k}}\gamma^{\mu}v_{\alpha,-\mathbf{k}}\bar{v}_{\alpha,-\mathbf{k}}\gamma^{\nu}u_{\alpha\mathbf{k}}e^{2ik_0(t-t')}], \tag{19}$$

where tr stands for trace, $g_a = q$ for $a = 0$, and $g_a = g$ for $a = 1,2,3$. A few relevant mathematical relations concerning Dirac spinors and γ matrices in $(1+2)$D space-time allowing

us to proceed the computation of the above expression are given as

$$\sum_{\alpha} u_{\alpha\mathbf{k}}\bar{u}_{\alpha\mathbf{k}} = \sum_{\alpha} v_{\alpha\mathbf{k}}\bar{v}_{\alpha\mathbf{k}} = \frac{\gamma^{\mu}k_{\mu}}{2k_0}, \tag{20}$$

$$\mathrm{tr}(\gamma^{\mu}\gamma^{\alpha}\gamma^{\nu}\gamma^{\beta}) = 4(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}), \tag{21}$$

$$\mathrm{tr}(\sigma^{a}\sigma^{b}) = 2\delta_{ab}, \tag{22}$$

δ_{ab} being the Kronecker δ function. After a little bit of algebra we get the expression

$$K^{\mu a\nu b}(t-t') = \frac{g_a^2}{\pi^2\hbar}\delta_{ab}\int_0^{\infty} kdk\,\sin[2k(t-t')] \times \int_0^{2\pi} d\varphi(2g^{\mu\nu} - \hat{k}^{\mu}\hat{k}^{\nu\prime} - \hat{k}^{\nu}\hat{k}^{\mu\prime}), \tag{23}$$

where $\hat{k}^{\mu} = (1, \cos\varphi, \sin\varphi)$ and $\hat{k}^{\mu\prime} = (1, -\cos\varphi, -\sin\varphi)$ are $(1+2)$D relativistic vectors such that $\hat{k}^{\mu}\hat{k}_{\mu} = 0$. Performing the integration of the variable φ leads to

$$K^{\mu a\nu b}(t-t') = \frac{g_a^2}{\pi\hbar}\delta_{ab}(g^{\mu\nu} - \delta^{\mu\nu})\int_0^{\infty} kdk\,\sin[2k(t-t')] = \frac{g^2}{4\hbar}\delta_{ab}(g^{\mu\nu} - \delta^{\mu\nu})\frac{\partial}{\partial t'}\delta(t-t'). \tag{24}$$

The response function $K^{\mu a\nu b}$ can be directly related to the conductivity, which is the physical quantity actually measured experimentally. To do that we substitute Eq. (24) into Eq. (15) and perform an integration by parts to obtain

$$J^{\mu a} = -\frac{g_a^2}{4\hbar}\delta_{ab}(g^{\mu\nu} - \delta^{\mu\nu})\frac{\partial\mathcal{A}_{\nu b}}{\partial t}. \tag{25}$$

For spatially invariant fields, generalized homogeneous "electric" fields are given by $F^b_{0\nu} = \partial\mathcal{A}_{\nu b}/\partial t$, making it possible to define the conductivity as $\sigma^{\mu a\nu b} = J^{\mu a}/F^b_{0\nu}$. This way, the fundamental result for the charge and spin conductivity is obtained:

$$\sigma^{\mu a\nu b} = -\frac{\pi g^2}{2h}\delta_{ab}(g^{\mu\nu} - \delta^{\mu\nu}). \tag{26}$$

For $\mu = \nu = 1$ or $\mu = \nu = 2$ we obtain $\sigma^{ab} = \pi g_a^2\delta_{ab}/h$. The minimal electrical conductivity $\sigma^0 = \pi q^2/h$ is obtained by setting $a = b = 0$ and corresponds to the response of the system to an applied electric field of the form $E_x = E_0\theta(t)$. Notice that, strictly speaking, our result (26) is not the same as the dc conductivity value usually calculated in the current literature,[13–15,21,25] since the frequency limit $\omega \to 0$ was never taken and the external stimulus $F^a_{0\nu}$ is of the form $E_0\theta(t)$, whose spectrum clearly differs from a genuine dc spectrum given by the Dirac function $\delta(\omega)$ in the frequency ω. We attribute mainly to this fact the differences between our theoretical prediction for conductivity and those based on the dc limit of the Kubo formula. It is important to notice that at times t much larger than the time scale of the slowest relaxation mechanism in graphene, it becomes unimportant if the stimulus was switched on at $t = 0$ or $t = -\infty$. Our theoretical prediction agrees well with current experimental findings for the most clean graphene samples which can be achieved in practice. Furthermore, the spin conductivity $\sigma^s = \pi g^2/h$ is obtained by making $\mu = \nu = 1$ and $a = b = 3$, which is to be compared with values recently obtained also using $SU(2)$ gauge symmetry principles.[32,33] It corresponds to the ratio between the spin current J^{xz} propagating in the x direction with spins aligned in the z direction, perpendicular to the graphene layer, and a "spin electric" field E^s_x, given by $F^z_{10} = \frac{1}{2}\partial E_y/\partial t$. Notice that the electric field E_y is a gauge potential with respect to the $SU(2)$ gauge symmetry and, therefore, it is its time derivative which produces the spin electric field. Assuming that the above theory is correct, we expect that a spin current will produce spin accumulation at the edges of a graphene sheet. Notice that the spin current J^{xz} is generated by the (time derivative of) electric field E_y, and therefore we call the spin accumulation described above as a form of the spin Hall effect. In order to measure such effect we envisage an experiment in which an in-plane electric field varying linearly in time is applied to graphene and the spin accumulation at the edges of a graphene sheet is measured by means of Kerr rotation, for instance. For a detailed discussion of the usual spin Hall effect the reader is referred to Ref. 34.

IV. CONCLUSION

In summary, we demonstrated that the Dirac Lagrangian for massless electrons near the Dirac points must be invariant under the $SU(2)$ gauge group related to local rotations of the spin quantization axis, allowing us to put forward a rigorous definition for the spin-current density. As a consequence, the Zeeman and Rashba spin-orbit interactions were obtained quite naturally. Finally, we solved the problem of the missing π in the theoretical prediction of electrical conductivity in graphene by correctly computing the conductivity within the framework of Kubo linear response theory, using the algebra of relativistic Dirac spinors in $(1+2)$D space-time. The minimal value of electrical conductivity is predicted to be $\pi q^2/h$, in agreement with current experimental results.

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