Electron-electron correlations in a dynamical impurity system with a Fermi edge singularity

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We study spatial correlations in the ground state of a one-dimensional electron gas coupled to a dynamic quantum impurity. The system displays a nontrivial many-body effect known as the Fermi edge singularity: transitions between discrete internal states of the impurity have a power-law dependence on the internal energies of the impurity states. We present compact formulas for the static current-current correlator and the pair-correlation function. These reveal that spatial correlations induced by the impurity decay slowly (as the third inverse power of distance) and have a power-law energy dependence, characteristic of the Fermi edge singularity.

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I. INTRODUCTION

The situation where a localized scatterer, with discreet internal quantum-mechanical degrees of freedom, is coupled to an electron gas is a common occurrence both in bulk condensed matter systems[1] and in nanometer-scale electronic devices.[2–4] Many nontrivial features of these systems can be explained with the aid of simple dynamic quantum impurity models.[5] For several such models, nonperturbative results have been obtained, and these results have played a major role in our understanding of strongly correlated electron systems.

In this context, a phenomenon known as the Fermi edge singularity[6] has been very influential. Its essential ingredients are a Fermi gas that is initially in a stationary state and a local scattering potential that is abruptly switched on at a time t_0. For times larger than t_0, the Fermi gas is no longer in a stationary state. To see just how far from stationary a typical initial state is, it is useful to consider the overlap between the ground states of the Hamiltonian before and after t_0. It is found that the overlap is zero, and this is known as the orthogonality catastrophe.[7] Since the initial state is far from stationary, the Fermi sea is severely shaken up;[8] the local scattering potential creates a multitude of particle-hole excitations.

The original theory[9,10] was formulated to account for singularities in the photoemission and photoabsorption spectra in some metals. For this problem the natural quantities to investigate are impurity transition rates. These where found to have a power-law dependence on the internal energies of the impurity levels. It was subsequently shown that the same physics applies to transport through a barrier containing an impurity[11] level, or through a quantum dot.[12,13] In recent work, nonequilibrium setups in which the Fermi sphere is replaced by a nonequilibrium distribution has been investigated.[14–20] Complications to the basic theory, for instance, what happens if the impurity induces resonant scattering, were also considered.[21] Apart from impurity transition rates, current and noise,[22] as well as the quench dynamics of the electron gas,[23,24] have been investigated.

These studies clearly indicate that the dynamical impurity induces significant correlations in the electron gas. A system that displays the Fermi edge singularity can therefore be expected to possess a nontrivially correlated ground state. This is a topic that is receiving significant attention in other impurity systems, such as the Kondo model.[25–27] Ground-state spatial correlations, however, have not yet been addressed in the context of the Fermi edge singularity. Correlations in time, such as current noise,[22] have been studied. Perhaps the reason is that some common realizations of the Fermi edge singularity rely on explicitly time-dependent Hamiltonians. In this work we consider a realization with a time-independent Hamiltonian. It allows us to investigate ground-state correlations. A model of the type that we consider has previously been considered in the literature on atom tunneling in metallic glasses.[28,29] As we discuss in Sec. II D, the low-energy physics of our model is qualitatively different from that of the Kondo model.

We consider a setup in which a localized impurity interacts with one-dimensional electrons with a linear dispersion relation.[30] An example of the type of impurity we have in mind is the charge qubit formed when a single electron is trapped in a double quantum dot.[31,32] Our model can also be realized with a Josephson charge qubit.[33] In the main text, we derive results for electrons in a single chiral channel, such as a quantum Hall edge state. For a two-state impurity, generalization to situations with both left- and right-moving electrons, or indeed, to more general multichannel setups, is possible and no new physics emerge, as we show in Appendix A. Our main result is a compact formula for the current-current correlator $\langle \delta j(x)\delta j(y)\rangle$ [Eq. (4.26)], that is valid at large separations. The impurity induces no additional correlations between electrons on the same side of the impurity. For electrons on opposite sides of the impurity, impurity-induced correlations decay as $1/|x-y|^3$. Owing to the Fermi-edge singularity, the correlator has a power-law dependence on the internal energies of the impurity. The correlations persist beyond the range of the interaction between the electron gas and the impurity. This is in contrast to the expectation values of single-particle observables such as the density $\langle \rho(x)\rangle$, which is unaffected by the impurity for x outside the range of the interaction. Because of the linear dispersion relation, the current-current correlator is essentially equal to the pair-correlation function $\langle \rho(x)\rho(y)\rangle$. This insight allows for a simple interpretation of our results. If an electron is detected at x to the left of the impurity, the likelihood of finding another at y to the right of the impurity is larger than in the absence of the impurity by an amount $\sim 1/|x-y|^3$ and has a power-law dependence on the internal energies of the impurity, characteristic of the Fermi edge singularity.

We present the work as follows: We define the model by means of its Hamiltonian (Sec. II A). Since we are dealing with one-dimensional electrons with a linear dispersion relation, it is convenient to bosonize them. In Sec. II B we collect the relevant bosonization results. We are interested in the model because it displays a Fermi edge singularity. We start Sec. II C by reviewing how the conventional theory of the Fermi edge singularity applies to the transition rates between internal states of the impurity. We also take the first step towards studying ground-state properties by considering the probability to find the impurity in a given internal state, if the system as a whole is in the ground state. These occupation probabilities turn out to play a central role in determining spatial correlations between electrons in the presence of the impurity. We show that the Fermi edge singularity is also manifested in the occupation probabilities. In Sec. II D we contrast the system we study with the Kondo model. As already stated, our main aim is to investigate spatial correlations among the electrons. But before doing so, we need to consider the expectation values of single-particle observables. In Sec. III we therefore investigate the average density $\langle\rho(x)\rangle$. We show that the total number of particles displaced by the impurity obeys a generalization of Friedel's sum rule. (A generalized Friedel sum rule has previously been shown to hold for another dynamic impurity system, namely, the Anderson impurity model.) Our results for the current-current correlator are presented in Sec. IV and for the pair-correlation function in Sec. V. Section VI contains a self-contained summary of our main results and conclusions.

II. MODEL

Our model describes a localized impurity with an internal state space spanned by the orthonormal basis $B = \{|\alpha\rangle|\alpha = 1, \ldots, M\}$. It interacts with a Fermi gas. In the main text we take this Fermi gas to consist of a single chiral channel. The model is inspired by experimental setups in which electrons in a coherent conductor are coupled to a charge qubit ($M = 2$).[17,30] Since it does not substantially complicate the analysis for a single channel, we consider arbitrary M. In Appendix A, we consider the multichannel case. It is shown that generalization of the results in the main text is straightforward for $M = 2$. We have not yet solved the most general situation, involving multiple channels and arbitrary M.

A. Hamiltonian

The Hamiltonian for our model is $H = H_0 + H_T$, where

$$H_0 = \sum_\alpha H_\alpha \otimes |\alpha\rangle\langle\alpha|, \quad H_T = \sum_{\alpha\beta} \gamma_{\alpha\beta}|\alpha\rangle\langle\beta|, \qquad (2.1)$$

$$H_\alpha = \int_{-L/2}^{L/2} dx\, \psi^\dagger(x)[-i\partial_x + v_\alpha(x)]\psi(x) + \varepsilon_\alpha. \qquad (2.2)$$

Here H_α describes the fermions when the impurity is in state $|\alpha\rangle$ and ε_α is the internal energy of the impurity state $|\alpha\rangle$.[34] The fermion creation and annihilation operators $\psi^\dagger(x)$ and $\psi(x)$ obey periodic boundary conditions on the interval $(-L/2,L/2]$. We will eventually send the system size L to infinity. The electrostatic potential $v_\alpha(x)$ depends on the state of the impurity. Since our aim in this work is to investigate many-body correlations that persist beyond the range of the interaction between the impurity and the Fermi gas, it is convenient to assume that all the potentials v_α have bounded support. We define the scattering region as the shortest interval $l = [x_-,x_+]$ such that $v_\alpha(x) = 0$ for all α and all $x \notin l$. The term H_T induces tunneling between impurity states. Without loss of generality, we set $\gamma_{\alpha\alpha} = 0$.

The ground state $|F_\alpha\rangle$ of the fermion Hamiltonian H_α is a Fermi sea in which all (the infinitely many) negative energy orbitals associated with the single-particle Hamiltonian $-i\partial_x + v_\alpha(x)$ are occupied, and all positive energy orbitals are empty.

The scattering states of $-i\partial_x + v_\alpha(x)$ are of the form $\exp[ikx - i\int_{-\infty}^{x} dx' v_\alpha(x')]$. This leads to an identification of

$$\delta_\alpha = -\frac{1}{2}\int_{-\infty}^{\infty} dx\, v_\alpha(x) \qquad (2.3)$$

as the scattering phase shift associated with the potential $v_\alpha(x)$. Phase shifts play an important role in impurity physics. The expression in Eq. (2.3) is exact for a linear dispersion relation, but only approximate beyond the linear approximation. It is important to note, however, that small v_α and δ_α, while sufficient, is not a necessary requirement for the validity of Eq. (2.3). What is required is that the potential changes slowly on the scale of the Fermi wavelength.

B. Bosonization

The infinite Fermi sea represented by a state such as $|F_\alpha\rangle$ must be handled with care. Finite observable quantities are often represented as the difference between infinite quantities. In such calculations, an uncontrolled rearrangement of terms can lead to incorrect answers.[35,36] For one-dimensional Fermi systems, such as the one we are studying, a standard and elegant method to treat the infinitely deep Fermi sea correctly is provided by bosonization.[37,38] In fact, one of the pioneering applications of bosonization was to the Fermi edge singularity.[39] Through bosonization, our model is mapped onto a spin-boson model[40] with a bath that is ohmic at low energies.[30] For our purposes the following bosonization results are required. The Fourier components of the density operator

$$\rho_q = \int_{-L/2}^{L/2} dx\, e^{-iqx}\psi^\dagger(x)\psi(x), \qquad (2.4)$$

with $q = 2\pi n/L, n = 0, \pm1, \pm2, \ldots$, obey bosonic commutation relations

$$[\rho_q,\rho_{q'}] = \frac{Lq}{2\pi}\delta_{q,-q'}. \qquad (2.5)$$

The fermion Hamiltonians H_α can be expressed in terms of ρ_q as $H_\alpha = h_\alpha + E_{0\alpha}$, where

$$h_\alpha = \sum_{q>0} \frac{2\pi}{L}\left(\rho_{-q} + \frac{v_{\alpha,-q}}{2\pi}\right)\left(\rho_q + \frac{v_{\alpha,q}}{2\pi}\right). \qquad (2.6)$$

Here $v_{\alpha,q} = \int_{-L/2}^{L/2} dx\, e^{-iqx} v_\alpha(x)$ are the Fourier components of the potentials $v_\alpha(x)$, and

$$E_{0\alpha} = \varepsilon_\alpha + \frac{\rho_0 v_{\alpha,0}}{L} - \sum_{q>0}\frac{2\pi}{L}\left|\frac{v_{\alpha,q}}{2\pi}\right|^2 \qquad (2.7)$$

is the ground-state energy of H_α as a function of ρ_0, the number of electrons in the system, up to an α-independent constant. We will work with a fixed number of particles. The number operator ρ_0, and therefore also $E_{0\alpha}$, can then be treated as real numbers instead of operators. Note that $E_{0\alpha}$ can be varied independently of h_α by adjusting the internal energy ε_α of the impurity state α. We define the energy differences

$$\omega_{\alpha\beta} = E_{0\alpha} - E_{0\beta}. \tag{2.8}$$

For $q > 0$,

$$\rho_q|F_\alpha\rangle = -\frac{v_{\alpha,q}}{2\pi}|F_\alpha\rangle, \tag{2.9}$$

i.e., the Fermi sea $|F_\alpha\rangle$ is an eigenstate (coherent state) of the bosonic annihilation operator ρ_q.

C. Fermi edge singularity

The Fermi edge singularity manifests as power-law singularities in inelastic transition rates between internal states of the impurity, as a function the energy differences $\omega_{\alpha\beta}$. Denote the many-body eigenstates of H_β by $|m_\beta\rangle$, and let $E_{m\beta}$ be the associated energies. Assuming an initial state $|F_\alpha\rangle \otimes |\alpha\rangle$, the total transition rate $W_{\beta\alpha}(\omega_{\beta\alpha})$ to configurations with the impurity in state $|\beta\rangle$, calculated to second order in H_T using Fermi's golden rule, is given by

$$\begin{aligned} W_{\beta\alpha}(\omega_{\beta\alpha}) &= 2\pi|\gamma_{\beta\alpha}|^2 \sum_m \delta(E_{m\beta} - E_{0\alpha})|\langle F_\alpha|m_\beta\rangle|^2 \\ &= |\gamma_{\beta\alpha}|^2 \sum_m \int_{-\infty}^{\infty} dt\, e^{i(E_{m\beta}-E_{0\alpha})t}|\langle F_\alpha|m_\beta\rangle|^2 \\ &= |\gamma_{\beta\alpha}|^2 \int_{-\infty}^{\infty} dt\, e^{i\omega_{\beta\alpha}t}\langle F_\alpha|e^{ih_\beta t}e^{-ih_\alpha t}|F_\alpha\rangle. \end{aligned} \tag{2.10}$$

The canonical theory of the Fermi edge singularity provides an expression (the so-called closed-loop factor)[10,12] for the expectation value $\langle F_\alpha|e^{ih_\beta t}e^{-ih_\alpha t}|F_\alpha\rangle$. The closed-loop factor is analytic in the upper half of the complex plain. For large $|t|$, its asymptotic behavior is of the form $(i\Lambda t)^{-\Delta_{\alpha\beta}^2}$, where

$$\Delta_{\alpha\beta} = -(\delta_\alpha - \delta_\beta)/\pi. \tag{2.11}$$

A branch cut in the lower half of the complex t plane is understood, and the branch with $\arg(t) = 0$ for real positive t is implied. Λ is a large energy scale. In our model, with an infinitely deep Fermi sea, it is of the order of the inverse of the range of the interaction of the impurity and the Fermi gas. The interpretation is that an interaction with range Λ^{-1} can excite particle-hole pairs with energies $\sim\Lambda$. In models where the inverse of the interaction range is larger than the Fermi energy, measured from the bottom of the Fermi sea, Λ is of the order of the Fermi energy. We further note that the energy difference $\omega_{\beta\alpha}$ plays the role of the x-ray frequency in the canonical theory. From the asymptotics and analyticity structure of the closed-loop factor, the asymptotic behavior of the rates $W_{\beta\alpha}$ can be extracted at small energy differences $\omega_{\beta\alpha}$:

$$W_{\alpha\beta}(\omega_{\beta\alpha}) = |\gamma_{\alpha\beta}|^2 C_{\alpha\beta}\Delta_{\alpha\beta}^2\theta(\omega)\left(\frac{\omega}{\Lambda}\right)^{\Delta_{\alpha\beta}^2}\frac{1}{\omega}\bigg|_{\omega=-\omega_{\beta\alpha}}. \tag{2.12}$$

$C_{\alpha\beta}$ is a constant that tends to a value of order 1 when $\Delta_{\alpha\beta} \to 0$. The result for the rate breaks down at energies $\omega_{\alpha\beta}$ corresponding to excitations with wavelengths $\omega_{\alpha\beta}^{-1}$ short enough to resolve the spatial structure of the interaction potentials $v_\alpha(x)$ and $v_\beta(x)$. For energies larger than Λ, the transition rate becomes exponentially suppressed. There is also another constraint. For perturbation theory in $\gamma_{\alpha\beta}$ to be valid, the ratio $W_{\beta\alpha}(\omega_{\beta\alpha})/\omega_{\alpha\beta}$ must be much smaller than one.[41] For $\Delta_{\alpha\beta}^2 < 2$ this imposes the constraint[42] that

$$\frac{\omega_{\alpha\beta}}{\Lambda} \gg \left(\frac{C_{\alpha\beta}\Delta_{\alpha\beta}^2|\gamma_{\alpha\beta}|^2}{\Lambda^2}\right)^{\frac{1}{2-\Delta_{\alpha\beta}^2}}. \tag{2.13}$$

The point $\omega_{\beta\alpha} = 0$ is known as the Fermi edge threshold. More information regarding the above statements can be found in Ref. 30, where we analyzed transition rates for essentially the same model in detail. The fact that for $\Delta_{\beta\alpha} < 1$ the transition rate $W_{\omega_{\beta\alpha}}$ becomes large close to the threshold, while for $\Delta_{\beta\alpha} > 1$ it becomes small close to the threshold, is an instance of what is known as a Schmidt transition.[41]

The transition rate discussed above is a dynamical property of the model. However, the Fermi edge singularity also manifests in ground-state properties. Denote the ground state of the full Hamiltonian by $|\mathrm{GS}\rangle$. We define n_α as the probability to find the impurity in state $|\alpha\rangle$ when the system is in $|\mathrm{GS}\rangle$

$$n_\alpha = \langle \mathrm{GS}|(I_F \otimes |\alpha\rangle\langle\alpha|)|\mathrm{GS}\rangle. \tag{2.14}$$

Here I_F is the identity operator in the many-fermion Hilbert space. Let λ be the index such that $|F_\lambda\rangle \otimes |\lambda\rangle$ is the ground state of H_0, the Hamiltonian in the absence of impurity tunneling. If the ground-state energy difference $\omega_{\alpha\lambda}$ between sectors H_α and H_λ of H_0 is much larger than the large energy scale Λ, we can ignore the dynamics of the fermions and n_α is obtained from the ground state of the effective impurity Hamiltonian

$$H_{\mathrm{imp}} = \sum_{\alpha\beta}(\omega_{\alpha\lambda}\delta_{\alpha\beta} + \gamma_{\alpha\beta})|\alpha\rangle\langle\beta|, \tag{2.15}$$

which yields $n_\alpha = |\gamma_{\alpha\lambda}|^2/\omega_{\alpha\lambda}^2$ for small $\gamma_{\alpha\lambda}$. For $\omega_{\alpha\lambda}$ smaller than Λ, the fermion dynamics cannot be ignored. The full ground state $|\mathrm{GS}\rangle$ contains contributions $|\psi_\alpha\rangle \otimes |\alpha\rangle$ with $\alpha \neq \lambda$, where $|\psi_\alpha\rangle$ is a complicated, many-fermion state. These contributions are still small if $\gamma_{\alpha\lambda}/\omega_{\alpha\lambda}$ is small, but the nontrivial many-body correlations contained in $|\psi_\alpha\rangle$ modify the simple inverse square $\omega_{\alpha\lambda}$ behavior of n_α. In the spirit of the original work on the Fermi edge singularity, we will perform a calculation of $|\psi_\alpha\rangle$ that is nonperturbative in the potentials v_α, thus including contributions with arbitrarily many particle-hole excitations.

It is straightforward to show that, to second order in H_T, n_α with $\alpha \neq \lambda$ is given by

$$n_\alpha = \int_{-\infty}^{0} \frac{d\omega}{2\pi}\frac{W_{\alpha\lambda}(\omega)}{(\omega_{\alpha\lambda} - \omega)^2}. \tag{2.16}$$

(See Appendix B for a derivation.) Note that since λ refers to the ground state of H_0, $\omega_{\alpha\lambda}$ is positive and the denominator of the integrand is therefore always nonzero. Using the expression of Eq. (2.11) for $W_{\alpha\lambda}$, one finds

$$n_\alpha(\omega_{\alpha\lambda}) = |\gamma_{\alpha\lambda}|^2 D_{\alpha\lambda}^2\Gamma\big(2 - \Delta_{\alpha\lambda}^2\big)\left(\frac{\omega_{\alpha\lambda}}{\Lambda}\right)^{\Delta_{\alpha\lambda}^2}\frac{1}{\omega_{\alpha\lambda}^2}. \tag{2.17}$$

Here $D_{\alpha\lambda}$ is a constant of order 1. The regime of validity of the above equation is the same as that of Eq. (2.11) for $W_{\alpha\lambda}$.

In particular, for $\Delta_{\alpha\lambda}^2 < 2$, it becomes invalid at sufficiently small $\omega_{\alpha\lambda}$ due to a breakdown in perturbation theory in H_T. For $\Delta_{\alpha\lambda}^2 > 2$ there are additional contributions that are less singular at small $\omega_{\alpha\lambda}$ that ensure the positivity of n_α. For large $\omega_{\alpha\lambda}$ there is a crossover from the power law of Eq. (2.17) to the inverse square law $n_\alpha \sim (\omega_{\alpha\lambda})^{-2}$, valid for $\omega_{\alpha\lambda} > \Lambda$.

D. Absence of a Kondo effect

Our model is designed to display the physics of the Fermi edge singularity. It is also, however, reminiscent of the Kondo model in that it describes electrons scattering off a "spin"—our impurity. There is a close connection between the Fermi edge singularity and the Kondo effect,[43] and the question should therefore be asked whether our model displays Kondo physics.

In the Kondo model,[1] the interaction between the impurity and the electron gas contains terms of the form $c^\dagger c'|\alpha\rangle\langle\beta|$, where $c^\dagger$ and c' create and annihilate electrons and $|\alpha\rangle\langle\beta|$ changes the state of the impurity. We can get the impurity interaction in our model (2.2) into this form by working in a basis that diagonalizes the unperturbed impurity Hamiltonian $H_{\rm imp} = \sum_{\alpha\beta}(\omega_{\alpha\lambda}\delta_{\alpha\beta} + \gamma_{\alpha\beta})|\alpha\rangle\langle\beta|$. This leads to Kondo-like terms with a dimensionless coupling strength of order $J = \delta\gamma/D$, where δ is of the order of the relative phase shifts induced by the potentials $v_\alpha - v_\beta$, γ is of the order of the tunneling amplitudes $\gamma_{\alpha\beta}$, and D is of the order of the level spacing of the impurity Hamiltonian $H_{\rm imp}$. If the Kondo-like terms do give rise to a Kondo effect, the associated Kondo temperature would then be of order $T_K = \Lambda e^{-D/\gamma\delta}$. In the Kondo model, the smaller the level spacing of the unperturbed impurity Hamiltonian, the better. In fact, if the unperturbed level spacing of the Kondo impurity is larger than the Kondo temperature, the Kondo effect disappears. This is precisely the fate of any potential Kondo effect in our model. In the small-γ regime, the level spacing D is of the order $\omega_{\alpha\lambda}$. For sufficiently small γ, this is larger than the Kondo temperature.

The perturbative results in γ that we obtain below diverge when the energy differences $\omega_{\alpha\lambda}$ become small. Could this divergence signal the onset of a Kondo effect? First, we note that, even in this case, there is an energy difference between impurity levels of order γ, and neglecting this energy difference would make the model trivial. This suggests that, in contrast to the Kondo model, where the dynamics of the impurity is entirely due to the interaction with the Fermi sea, in our model it is the interplay of the impurity's unperturbed internal dynamics and the interaction with the Fermi gas that is important.

However, this does not rule out the possibility that in the limit where all the $\omega_{\alpha\lambda} \to 0$, the low-energy physics of our model is governed by the same strong coupling fixed-point Hamiltonian as the Kondo model. In order to answer the question definitively, a renormalization group analysis must be performed. This has been done[29] for a two-state impurity, i.e., $M = 2$, in the context of atom tunneling in metallic glasses. Our model corresponds to the so-called "commutative model."[28] It was found that the commutative model's low-energy physics is qualitatively different from that of the Kondo model. In particular, the renormalized couplings in the commutative model do not scale into a strong coupling regime as is the case in the Kondo model.

To summarize, in the small γ and sufficiently large $\omega_{\alpha\lambda}$ regime that we analyze, no Kondo effect is expected. Our results diverge when $\omega_{\alpha\lambda} \to 0$, and one may wonder whether this is due to the onset of the Kondo effect. However, for a two-state impurity ($M = 2$), a renormalization group analysis indicates that this is not the case.

III. CURRENT AND FRIEDEL SUM RULE

Using the commutation relations between Fourier components of the density operator, the continuity equation $i[\rho(x),H] = \partial_x j(x)$ is straightforwardly derived where the current operator is

$$j(x) = \rho(x) + \sum_\alpha \frac{v_\alpha(x)}{2\pi}|\alpha\rangle\langle\alpha|. \tag{3.1}$$

As a preliminary to our goal of studying correlation functions, we consider the average current when the system is in an arbitrary stationary state. Let D be any density matrix that describes a stationary state of the system, i.e., $[H,D] = 0$. The expectation value of the commutator of any operator A with the Hamiltonian is zero, i.e.,

$$\langle[A,H]\rangle \equiv {\rm tr}\{D[A,H]\} = 0. \tag{3.2}$$

From the continuity equation it then follows that $\partial_x\langle j(x)\rangle = 0$. Integrating from $-\infty$ to x and using the explicit expression of Eq. (3.1) for $j(x)$, we obtain

$$\langle\rho(x)\rangle = \bar\rho - \sum_\alpha \frac{v_\alpha(x)}{2\pi} n_\alpha, \tag{3.3}$$

where $n_\alpha = \langle|\alpha\rangle\langle\alpha|\rangle$ is the probability to find the impurity in the state $|\alpha\rangle$ and $\bar\rho$ is the density at infinity, which is equal to the homogeneous density in the absence of the impurity. The total charge ΔN displaced by the impurity is obtained by subtracting $\bar\rho$ and integrating over x. In terms of the scattering phase shifts defined in Eq. (2.3), we obtain

$$\Delta N = \frac{1}{\pi}\sum_{\alpha=1}^M n_\alpha\,\delta_\alpha. \tag{3.4}$$

For the case of a static impurity, i.e., $M = 1$ and $n_1 = 1$, the result reduces to Friedel's well-known sum rule $\Delta N = \delta/\pi$.[1] Equation (3.4) represents a generalization of Friedel's sum rule to our dynamic impurity model. That such a generalization exists is not unexpected. A classic result in many-body theory is the proof by Langreth[44] that $\Delta N = \delta/\pi$ holds for another dynamic impurity system, namely, the Anderson impurity model.

Before considering correlation functions, we define the current-fluctuation operator $\delta j(x)$ through its Fourier transform $\delta j_q = \int_{-L/2}^{L/2} dx\, e^{-iqx}\delta j(x)$, where

$$\delta j_q = \sum_\alpha j_{\alpha q}|\alpha\rangle\langle\alpha| - \delta_{q0}j_{\lambda 0}, \quad j_{\alpha q} = \rho_q + \frac{v_{\alpha q}}{2\pi}. \tag{3.5}$$

This definition is such that the expectation value of δj_q with respect to the noninteracting ground state $|F_\lambda\rangle \otimes |\lambda\rangle$ is zero. From the continuity equation follows that the expectation value of δj_q with respect to the interacting ground state $|{\rm GS}\rangle$ is also zero, unless $q = 0$. Using the definition of $\Delta_{\alpha\lambda}$ in Eq. (2.11) and the fact that $\sum_\alpha n_\alpha = 1$, one furthermore finds

$$\langle \mathrm{GS}|\delta j_q|\mathrm{GS}\rangle = \delta_{0q}\sum_\alpha \Delta_{\alpha\lambda} n_\alpha. \tag{3.6}$$

For a finite interaction strength between the impurity and the Fermi gas, the relative phase shifts $\Delta_{\alpha\lambda}$ are finite, and hence in the thermodynamic limit $L\to\infty$, the Fourier transformed fluctuations operator $\delta j(x)$ has a zero expectation with respect to the full ground state:

$$\langle \mathrm{GS}|\delta j(x)|\mathrm{GS}\rangle = \frac{1}{L}\sum_\alpha \Delta_{\alpha\lambda} n_\alpha \to 0|_{L\to\infty}. \tag{3.7}$$

IV. CURRENT-CURRENT CORRELATORS

The similarity of the generalized Friedel sum rule of Eq. (3.4) to the static impurity result, $\Delta N = \delta/\pi$, begs the following question: Given the probabilities n_α, can the effect of the impurity on the electron gas perhaps be accounted for by a static effective potential $v_{\mathrm{eff}} = \sum_\alpha n_\alpha v_\alpha(x)$ instead of a dynamic impurity? In this section, we show that the answer is "No." We show this by examining the ground-state current-current correlator $\langle \mathrm{GS}|\delta j(x)\delta j(y)|\mathrm{GS}\rangle$, which can, in principle, be extracted from the statistics of repeated experiments in which the current is measured (see Appendix C).

A. Zeroth order

We write

$$\langle \mathrm{GS}|\delta j(x)\delta j(y)|\mathrm{GS}\rangle = c_0(x,y) + c_2(x,y) + \mathcal{O}(\gamma^4), \tag{4.1}$$

where c_n is of order of n in γ. The zero-order term

$$c_0(x,y) = \langle F_\lambda|\delta j(x)\delta j(y)|F_\lambda\rangle \tag{4.2}$$

becomes singular when $x\to y$. To deal with this singularity, it is convenient to define a regularized current operator

$$\delta j_r(x) = \frac{1}{L}\sum_q e^{iqx}e^{-r|q|}\delta j_q \simeq \int_{-L/2}^{L/2}\frac{dx'}{\pi}\frac{r}{(x-x')^2+r^2}\delta j(x), \tag{4.3}$$

the last line being valid for $|x|/L \ll 1$. At the end of the calculation the limit $r\to 0^+$ is taken and results independent of the regularization are obtained as long as $x\neq y$. From Eqs. (2.5) and (2.9) it follows that

$$\langle F_\lambda|\delta j_p \delta j_q|F_\lambda\rangle = \theta(p)\delta_{p,-q}\frac{Lp}{2\pi}. \tag{4.4}$$

Performing the inverse Fourier transform we then find

$$\langle F_\lambda|\delta j(x)\delta j(y)|F_\lambda\rangle = \frac{-i}{2\pi L}\partial_x\sum_{p>0} e^{ip(x-y+2ir)} = \frac{-i}{2\pi L}\partial_x[1-e^{i\frac{2\pi}{L}(x-y+2ir)}]^{-1}. \tag{4.5}$$

Taking the limits $L\to\infty$ and $r\to 0^+$, we find

$$c_0(x,y) = -\frac{1}{(2\pi)^2}\frac{1}{(y-x)^2}. \tag{4.6}$$

This result is independent of $v_\lambda(x)$ and represents the exact result for a static impurity.

B. Second order

We obtain the second-order correction $c_2(x,y)$ to the current-current correlator by expanding $|\mathrm{GS}\rangle$ in Eq. (4.1) to second order in γ. The perturbation expansion of eigenstates can be obtained from a perturbation expansion of the interaction picture time evolution operator describing the situation in which the perturbation is switched on adiabatically [Gell-mann Low theorem]. Thus to second order in the γ's,

$$|\mathrm{GS}\rangle = |F_\lambda\rangle\otimes|\lambda\rangle + \sum_\alpha |\psi_\alpha\rangle\otimes|\alpha\rangle + \sum_{\alpha_1\alpha_2}|\psi_{\alpha_1\alpha_2}\rangle\otimes|\alpha_2\rangle, \tag{4.7}$$

$$|\psi_\alpha\rangle = -i\sum_\alpha \gamma_{\alpha\lambda}\int_{-\infty}^{0}dt\, e^{\eta t}Q_{\alpha\lambda}(t)|F_\lambda\rangle, \tag{4.8}$$

$$|\psi_{\alpha_1\alpha_2}\rangle = -\sum_{\alpha_1\alpha_2}\gamma_{\alpha_2\alpha_1}\gamma_{\alpha_1\lambda}\int_{-\infty}^{0}dt_2\int_{-\infty}^{t_2}dt_1\, e^{\eta(t_1+t_2)}Q_{\alpha_2\alpha_1}(t_2) \times Q_{\alpha_1\lambda}(t_1)|F_\lambda\rangle\otimes|\alpha_2\rangle. \tag{4.9}$$

Here η is a small positive constant and the limit $\eta\to 0^+$ must be taken after the expectation value in Eq. (4.1) is evaluated. The operators $Q_{\alpha\beta}(t)$ are defined as

$$Q_{\alpha\beta}(t) = e^{iH_\alpha t}e^{-iH_\beta t}. \tag{4.10}$$

We also define the expectation value

$$P_{\alpha\beta}(t) \equiv \langle F_\lambda|Q_{\alpha\beta}(t)|F_\lambda\rangle = e^{i(\omega_{\alpha\lambda}-\omega_{\beta\lambda})t}\langle F_\lambda|e^{ih_\alpha t}e^{-ih_\beta t}|F_\lambda\rangle. \tag{4.11}$$

It is convenient to perform a Fourier transform

$$c_2(p,q) = \int_{-L/2}^{L/2}dx\int_{-L/2}^{L/2}dy\, e^{-ipx}e^{-iqy}c_2(x,y). \tag{4.12}$$

Expanding the ground state $|\mathrm{GS}\rangle$ in γ as in Eq. (4.7), we obtain

$$c_2(p,q) = \sum_\alpha c_2^\alpha(p,q),$$
$$c_2^\alpha(p,q) = \langle\psi_\alpha|\delta j_{\alpha p}\delta j_{\alpha q}|\psi_\alpha\rangle + \langle F_\lambda|\delta j_{\lambda p}\delta j_{\lambda q}|\psi_{\alpha\lambda}\rangle + \langle\psi_{\alpha\lambda}|\delta j_{\lambda p}\delta j_{\lambda q}|F_\lambda\rangle. \tag{4.13}$$

We now substitute $|\psi_\alpha\rangle$ and $|\psi_{\alpha\beta}\rangle$ from Eqs. (4.8) and (4.9) into Eq. (4.13). The individual terms in Eq. (4.13) are evaluated by exploiting the bosonic commutators $[\delta j_{\alpha q},\delta j_{\beta q'}] = Lq\delta_{q,-q'}/2\pi$ obeyed by the $j_{\alpha q}$ operators [cf. Eq. (2.5)]. For $q>0$, the $\delta j_{\alpha q}$ correspond to bosonic annihilation operators, with corresponding creation operators $\delta j_{\alpha q}^\dagger = \delta j_{\alpha -q}$. The state $|F_\lambda\rangle$ is a coherent state, i.e., an eigenstate of the annihilation operators [cf. Eq. (2.9)]:

$$\delta j_{\alpha q}|F_\lambda\rangle = \Delta_{\alpha\lambda}^q|F_\lambda\rangle,\quad \Delta_{\alpha\lambda}^q = \frac{v_{\alpha q}-v_{\lambda q}}{2\pi}. \tag{4.14}$$

The commutator of $\delta j_{\alpha q}$ with $Q_{\beta\gamma}(t)$ is easily calculated. The result is

$$[\delta j_{\alpha q}, Q_{\beta\gamma}] = (e^{iqt}-1)\Delta_{\beta\gamma}^q Q_{\beta\gamma}(t). \tag{4.15}$$

After some algebra, we obtain the expression $c_2^\alpha(p,q) = C_2^\alpha(p,q) + C_2^\alpha(-p,-q)^*$, where

$$C_2^\alpha(p,q) = \lim_{\eta\to 0^+} |\gamma_{\alpha\lambda}|^2 \Delta_{\alpha\lambda}^p \Delta_{\alpha\lambda}^q \int_{-\infty}^0 dt_1 \int_{-\infty}^{t_1} dt_2 \,\{e^{i(p+q)t_1} + \theta(p) e^{iqt_1}[e^{ipt_2} - e^{ipt_1}] + \theta(q) e^{ipt_1}[e^{iqt_2} - e^{iqt_1}]\} e^{\eta(t_1+t_2)} P_{\alpha\lambda}(t_2 - t_1). \tag{4.16}$$

We change integration variables to $t = t_1$ and $\tau = t_2 - t_1$ and perform the t integral to obtain

$$C_2^\alpha(p,q) = \lim_{\eta\to 0^+} \frac{|\gamma_{\alpha\lambda}|^2}{2\eta + i(p+q)} \Delta_{\alpha\lambda}^p \Delta_{\alpha\lambda}^q \int_{-\infty}^0 d\tau [1 + \theta(p)(e^{ip\tau} - 1) + \theta(q)(e^{iq\tau} - 1)] e^{\eta\tau} P_{\alpha\lambda}(\tau). \tag{4.17}$$

From Eq. (4.11) it follows that $P_{\alpha\lambda}(t)$ is related to the transition rate $W_{\alpha\lambda}$ by a Fourier transform,

$$|\gamma_\alpha \lambda|^2 P_{\alpha\lambda}(t) = \int_{-\infty}^0 \frac{d\omega}{2\pi} e^{i(\omega_{\alpha\lambda} - \omega)t} W_{\alpha\lambda}(\omega). \tag{4.18}$$

Substituting for $|\gamma_{\alpha\lambda}|^2 P_{\alpha\lambda}(\tau)$ from Eq. (4.18), performing the τ integral, and taking the $\eta \to 0^+$ limit, we find

$$C_2^\alpha(p,q) = \frac{\Delta_{\alpha\lambda}^p \Delta_{\alpha\lambda}^q}{p+q} \int_{-\infty}^0 \frac{d\omega}{2\pi} W_{\alpha\lambda}(\omega) \left\{ \frac{1}{\omega - \omega_{\alpha\lambda}} + \theta(p) \left[\frac{1}{\omega - \omega_{\alpha\lambda} - p} - \frac{1}{\omega - \omega_{\alpha\lambda}} \right] + \theta(q) \left[\frac{1}{\omega - \omega_{\alpha\lambda} - q} - \frac{1}{\omega - \omega_{\alpha\lambda}} \right] \right\} \tag{4.19}$$

and hence

$$c_2^\alpha(p,q) = \frac{\Delta_{\alpha\lambda}^p \Delta_{\alpha\lambda}^q}{p+q} \int_{-\infty}^0 \frac{d\omega}{2\pi} W_{\alpha\lambda}(\omega) \left\{ \mathrm{sgn}(p) \left[\frac{1}{\omega - \omega_{\alpha\lambda} - |p|} - \frac{1}{\omega - \omega_{\alpha\lambda}} \right] + \mathrm{sgn}(q) \left[\frac{1}{\omega - \omega_{\alpha\lambda} - |q|} - \frac{1}{\omega - \omega_{\alpha\lambda}} \right] \right\}. \tag{4.20}$$

By Fourier transforming back to $c_2^\alpha(x,y)$ we obtain

$$c_2^\alpha(x,y) = \int_{-\infty}^0 \frac{d\omega}{2\pi} W_{\alpha\lambda}(\omega) \int_{-\infty}^0 dz \int_0^\infty dz' [\Delta_{\alpha\lambda}(x+z)\Delta_{\alpha\lambda}(y+z') + \Delta_{\alpha\lambda}(x+z')\Delta_{\alpha\lambda}(y+z)] h_\alpha(z'-z), \tag{4.21}$$

where

$$\Delta_{\alpha\lambda}(x) = \int_{-\infty}^\infty \frac{dq}{2\pi} e^{iqx} \Delta_{\alpha\lambda}^q = \frac{v_\alpha(x) - v_\lambda(x)}{2\pi} \tag{4.22}$$

and

$$h_\alpha(z) = \int_0^\infty \frac{dq}{\pi} \sin(qz) \left(\frac{1}{\omega_{\alpha\lambda} - \omega} - \frac{1}{\omega_{\alpha\lambda} - \omega + |q|} \right). \tag{4.23}$$

For large z (Ref. 45),

$$h_\alpha(z) \simeq \frac{2}{\pi} \frac{1}{(\omega_{\alpha\lambda} - \omega)^3 z^3} = \frac{-1}{\pi} \partial_{\omega_{\alpha\lambda}} \frac{1}{(\omega_{\alpha\lambda} - \omega)^2 z^3}. \tag{4.24}$$

Since $\Delta_{\alpha\lambda}(x)$ is zero for $x \notin l$, i.e., outside the scattering region, the first term in the square brackets in Eq. (4.21) gives a nonzero contribution only when $x > x_-$ and $y < x_+$. The second term in the square brackets, on the other hand, only gives a nonzero contribution when $x < x_+$ and $y > x_-$. Thus for x and y both to the left ($<x_-$) or both to the right ($>x_+$) of the scattering region, $c_2^\alpha(x,y)$ is zero. It can be shown that this statement is true to all orders in H_T.

For x and y on opposite sides of the scattering region, and such that $|x - y| \gg x_+ - x_-$, we can use the large z expansion of $h_\alpha(z)$ [cf. Eq. (4.24)]. For $|z| \gg x_+ - x_-$, $h_\alpha(z)$ is a slowly varying function on the scale of $x_+ - x_-$. The leading-order behavior in $|x - y|$ of $c_2^\alpha(x,y)$ can be obtained by evaluating $h_\alpha(z' - z)$ at $|x - y|$:

$$c_2^\alpha(x,y) \simeq -\frac{1}{\pi} (\Delta_{\alpha\lambda})^2 \frac{1}{|x-y|^3} \partial_{\omega_{\alpha\lambda}} \int_{-\infty}^0 \frac{d\omega}{2\pi} \frac{W_{\alpha\lambda}(\omega)}{(\omega_{\alpha\lambda} - \omega)^2}, \tag{4.25}$$

where $\Delta_{\alpha\lambda} \equiv \int_{-l}^l dx\, \Delta_{\alpha\lambda}(x)$. Substituting into the above the result of Eq. (2.16), where the rate $W_{\alpha\lambda}$ was related to the occupation probability n_α of impurity level α, we obtain the simple final result

$$c_2^\alpha(x,y) \simeq -\frac{1}{\pi} (\Delta_{\alpha\lambda})^2 \frac{1}{|x-y|^3} \partial_{\omega_{\alpha\lambda}} n_\alpha(\omega_{\alpha\lambda}) \tag{4.26}$$

for $\min\{x,y\} \ll x_-$ and $\max\{x,y\} \gg x_+$.

For $\omega_{\alpha\lambda}$ not too large, i.e., not too far from the Fermi edge threshold, the energy dependence of the correlator can be obtained from the expression [Eq. (2.17)] for n_α that we discussed in the section on the Fermi edge singularity. This leads to

$$c_2^\alpha(x,y) \sim (\omega_{\alpha\lambda})^{\Delta_{\alpha\lambda}^2 - 3}. \tag{4.27}$$

For $\Delta_{\alpha\lambda}^2 < 3$ the result diverges close to the Fermi edge threshold. As with the rate $W_{\alpha\lambda}$ and the occupation probability n_α, the divergence signals a breakdown of our expansion in γ. Understanding this regime is an open problem that we are currently working on.

V. PAIR-CORRELATION FUNCTION

Although it can be measured directly (see Appendix C), the current-current correlator that we studied in the previous section is a rather abstract quantity. In this section we therefore relate it to another correlator, the pair-correlation function, that has an appealingly straightforward interpretation. The pair-correlation function is defined as

$$g(x,y) = \langle \mathrm{GS}|\psi^\dagger(y)\psi^\dagger(x)\psi(x)\psi(y)|\mathrm{GS}\rangle. \tag{5.1}$$

It measures the likelihood of finding an electron at x together with another electron at y. For $x \neq y$ it can also be written as $g(x,y) = \langle \mathrm{GS}|\rho(x)\rho(y)|\mathrm{GS}\rangle$. Since we have $j(x) = \rho(x)$ outside the scattering region [see Eq. (3.1)], our result for the current-current correlator can be used to obtain a formula for $g(x,y)$ away from the scatterer, namely,

$$g(x,y) = \bar{\rho}^2 - \frac{1}{(2\pi)^2}\frac{1}{(x-y)^2} + \sum_\alpha c_2^\alpha(x,y) + \mathcal{O}(\gamma^4). \tag{5.2}$$

The negative sign in front of the second term is due to Fermi statistics: Given that there is an electron at x, the likelihood of finding another electron at nearby position y is less than for uncorrelated distinguishable particles. The term $c_2(x,y)$, as well as the higher-order corrections, is zero if x and y refer to points on the same side of the scatterer, i.e., if $x,\ y < x_-$ or $x,\ y > x_+$.

For $\max\{x;y\} \gg x_+$ and $\min\{x;y\} \ll x_-$, the approximate result of Eq. (4.26) may be used for c_2^α. The sign of $c_2^\alpha(x,y)$ is determined by $-\partial_\omega n_\alpha(\omega)|_{\omega=\omega_{\alpha\lambda}}$. It is intuitively plausible that the probability n_α is a decreasing function of $\omega_{\alpha\lambda}$: the higher the excitation energy, the less likely it is to find the state $|\alpha\rangle$ occupied. Referring back to Eq. (2.17) in our discussion of the Fermi edge singularity, we see that this is indeed the case not too far from the Fermi edge threshold, provided $\Delta_{\alpha\lambda}^2 < 2$. Exact results for specific model interactions indicate that n_α remains a decreasing function of $\omega_{\alpha\lambda}$ also when $\omega_{\alpha\lambda}$ is further away from the threshold or $\Delta_{\alpha\lambda} > 2$. This leads to the conclusion that $c_2^\alpha(x,y)$ is always positive. Thus in the presence of the impurity, the pair-correlation function is larger than in the absence of the impurity. This a truly two-particle correlation effect. It cannot be accounted for by an increase in density around the impurity for the following reasons. The increase in pair correlations occurs outside the scattering region, where $\langle \mathrm{GS}|\rho(x)|\mathrm{GS}\rangle$ is unaffected by the presence of the impurity. Also, the pair-correlation function increases regardless of whether the impurity attracts or repel electrons. Finally, for x and y on opposite sides of the scatterer and $|x-y|$ sufficiently larger than the size of the scattering region, the increase in the pair-correlation function depends on distance $|x-y|$ rather than on the distance from x or y to the impurity.

VI. CONCLUSIONS

We studied a model in which a dynamical quantum impurity is coupled to a degenerate electron gas. The probability n_α to find the impurity in excited state $|\alpha\rangle$ if the system as a whole is in the ground state has an energy dependence [cf. Eq. (2.17)]

$$n_\alpha \sim (\omega_{\alpha\lambda})^{\Delta_{\alpha\lambda}^2 - 2}, \tag{6.1}$$

where $\omega_{\alpha\lambda}$ is the internal energy of impurity state $|\alpha\rangle$ minus the threshold energy of the Fermi edge singularity, and $\Delta_{\alpha\lambda}$ is the relative scattering phase shift that measures the strength of the interaction between impurity state $|\alpha\rangle$ and the Fermi gas. The power law in Eq. (6.1) is due to the Fermi edge singularity, a phenomenon that is usually discussed in the context of impurity transition rates. We note that the divergence of n_α at the Fermi edge threshold for $\Delta_{\alpha\lambda} < 2$ signals a breakdown in perturbation theory in the tunneling matrix elements $\gamma_{\alpha\beta}$ of the impurity.

Equation (6.1) establishes that the Fermi edge singularity manifests itself in ground-state properties of the impurity. How does it manifest itself in properties of the electron gas? We found that, for any stationary state, the average density of the electron gas is [cf. Eq. (3.3)]

$$\langle \rho(x)\rangle = \bar{\rho} - \sum_\alpha \frac{v_\alpha(x)}{2\pi} n_\alpha, \tag{6.2}$$

where $\bar{\rho}$ is the homogeneous density in the absence of the impurity, and $v_\alpha(x)$ is the electrostatic potential that the electron gas is subjected to when the impurity is in state $|\alpha\rangle$. This leads to a generalized Friedel sum rule [cf. Eq. (3.4)] $\Delta N = \sum_\alpha n_\alpha \delta_\alpha/\pi$ for the number of particles displaced by the impurity, where δ_α is the scattering phase shift associated with $v_\alpha(x)$. Two important points about these results are that (1) the effect of the Fermi edge singularity is contained in the occupation probabilities n_α; and (2) the average density is unaffected by the impurity in regions where the potentials $v_\alpha(x)$ are zero, i.e., outside the scattering region.

Having investigated the expectation values of single-particle observables, the next logical step is to look at correlation functions. We therefore calculated the static current-current correlator $\langle \mathrm{GS}|\delta j(x)\delta j(y)|\mathrm{GS}\rangle$ to second order in the impurity tunneling amplitudes $\gamma_{\alpha\beta}$. The zero-order result, which is also the full answer in the case of a static impurity, is [cf. Eq. (4.6)]

$$c_0(x,y) = -\frac{1}{(2\pi)^2(x-y)^2}. \tag{6.3}$$

We obtained an expression for the second-order correction $c_2(x,y)$ to this result [Eq. (4.21)] that is valid for all x and y. It is zero for x and y both to the left or both to the right of the scattering region. Taking x and y on different sides of the scattering region and the distance $|x-y|$ large compared to the size of the scattering region, we obtained the compact formula

$$c_2(x,y) \simeq -\frac{1}{\pi|x-y|^3}\sum_\alpha (\Delta_{\alpha\lambda})^2 \partial_{\omega_{\alpha\lambda}} n_\alpha(\omega_{\alpha\lambda}). \tag{6.4}$$

Thus, current-current correlations induced by the impurity show a slow (power-law) decay as a function of distance. These correlations are sub-leading: at large distances, correlations that are present also in the absence of the impurity decay more slowly (second vs third inverse power of distance). However, the impurity-induced correlations can be detected by varying the impurity parameters, as this leaves the leading-order correlations unaffected. Due to the appearance of n_α in Eq. (6.4), the impurity-induced correlations have a power-law energy dependence $\sim(\omega_{\alpha\lambda})^{\Delta_{\alpha\lambda}^2-3}$. Thus correlations are also sensitive to the Fermi edge singularity. The divergence at the Fermi edge threshold for $\Delta_{\alpha\lambda}^2 < 3$ again signals the breakdown of perturbation theory in $\gamma_{\alpha\lambda}$. Understanding correlations in the regime $(\Delta_{\alpha\lambda})^2 < 3$ and $\omega_{\alpha\lambda} \to 0$ is an open problem we are currently working on. In the regime where the Fermi edge singularity result for n_α [Eq. (6.1)] is valid, the correction $c_2(x,y)$ is positive. We have argued that n_α is always a decreasing function of $\omega_{\alpha\lambda}$. Thus we always expect a positive correction $c_2(x,y)$.

Outside the scattering region, the current-current correlator is simply related to the pair-correlation function $g(x,y) = \langle \mathrm{GS}|\rho(x)\rho(y)|\mathrm{GS}\rangle$ as

$$g(x,y) = \bar{\rho}^2 - \langle \mathrm{GS}|\delta j(x)\delta j(y)|\mathrm{GS}\rangle. \tag{6.5}$$

The pair-correlation function measures the likelihood to find an electron at x together with another electron at y. The positivity of $c_2(x,y)$ implies that given an electron at x, the likelihood of finding another one at y in the presence of the impurity increases by an amount proportional to $1/|x-y|^3$, for x and y on opposite sides of the scattering region, compared to when the impurity is absent. It is important to note that the increase occurs outside the scattering region, where the average density of electrons is unaffected by the impurity.

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APPENDIX A: GENERALIZATION TO MANY CHANNELS FOR A TWO-LEVEL IMPURITY

For a two-level impurity, i.e., $(M = 2)$, the results derived in the main text can be generalized to conductors with an arbitrary number of chiral channels. Since a nonchiral n-channel conductor can be thought of as a $2n$-channel chiral conductor with interchannel scattering at the impurity, this is simultaneously a generalization to nonchiral conductors.

We generalize H_α of Eq. (2.2) to

$$H_\alpha = \int_{-L/2}^{L/2} dx\, \Psi^\dagger(x)[-i\partial_x + V_\alpha(x)]\Psi(x) + \varepsilon_\alpha, \tag{A1}$$

where $\Psi(x)$ is now a vector of annihilation operators $\psi_a(x)$, with a enumerating the channels of the multichannel conductor. We emphasize that in what is to follow, we will assume a two-state impurity, and hence α takes on values 1 and 2. $V_\alpha(x)$ is a matrix in channel space.

Note that the results we derived in the main text only depend on the impurity potentials $v_\alpha(x)$ through $n_\alpha(\omega_{\alpha\lambda})$. As long as the energy $\omega_{\alpha\lambda}$ corresponds to particle-hole excitations with wavelengths much larger than the region in which $v_\alpha(x)$ is nonzero, the precise position dependence of $v_\alpha(x)$ does not matter. Outside the scattering region the same correlators are obtained for all potentials that produce the same phase shift δ_α and large energy scale Λ [cf. Eq. (2.12)]. In the multichannel problem, we therefore assume that $V_\alpha(x)$ has the form

$$V_\alpha(x) = A_\alpha v(x), \tag{A2}$$

where $v(x)$ is normalized such that

$$\int dx\, v(x) = 1 \tag{A3}$$

and represents a δ function, regularized to give the correct large energy scale Λ in power laws associated with the Fermi edge singularity. We note that the replacement of the actual potential by a δ-like potential is implicit in recent work on the Fermi edge singularity,[12–17] where time-evolution operators are replaced by scattering matrices.

The next step is to perform a unitary transformation

$$\Phi(x) = U^\dagger e^{iA_1\int_{-L/2}^{x} dx' v(x')}\Psi(x). \tag{A4}$$

Here U is the unitary matrix in channel space that diagonalizes $A_2 - A_1$, i.e.,

$$\sum_b [A_2 - A_1]_{ab} U_{bc} = 2\pi\Delta_c U_{ac}, \tag{A5}$$

where $2\pi\Delta_c$ are the eigenvalues of $A_2 - A_1$. The components of $\Phi(x)$ are denoted $\phi_a(x)$. In terms of the ϕ operators, the Hamiltonians H_1 and H_2 are given by

$$H_\alpha = \sum_a \int_{-L/2}^{L/2} dx\, \phi_a^\dagger(x)[-i\partial_x + v_{a\alpha}(x)]\phi_a(x) + \varepsilon_\alpha, \tag{A6}$$

with $v_{a1}(x) = 0$ and $v_{a2}(x) = 2\pi\Delta_a v(x)$. Thus we have nearly succeeded in transforming the problem of n coupled channels into a problem of n decoupled channels. We say "nearly" because, strictly speaking, if $\Psi(x)$ obeys periodic boundary conditions, then $\Phi(x)$ obeys a complicated boundary condition that couples channels. However, this change in boundary conditions is irrelevant in the limit $L \to \infty$: The dynamics of an excitation generated by the impurity is only affected after it has had time $\sim L$ to travel to the boundary. Thus the expectation values of time evolution operators such as $P_{\alpha\beta}(t)$ in Eq. (4.11) are modified only for times larger than L. This affects transition rates and ground-state correlations only for energies $\omega_{\alpha\lambda} < 1/L$. The only subtlety concerns the many-particle ground-state energies of H_1 and H_2. These change by a finite amount, since an infinite number of single-particle energies contribute to them. As a result, the energy difference ω_{12} between the ground state of H_1 and H_2 is offset from the value it assumes for periodic boundary conditions. However, since we view ω_{12} as an experimentally tunable parameter (it can be varied by varying the energy difference $\varepsilon_1 - \varepsilon_2$ between impurity levels), in terms of which our results are stated, we do not calculate the offset explicitly. Thus, for $L \to \infty$, the coupled many-channel problem

$$H = \sum_{\alpha=1,2} H_\alpha|\alpha\rangle\langle\alpha| + \gamma|1\rangle\langle 2| + \gamma^*|2\rangle\langle 1|, \tag{A7}$$

with H_α given by Eq. (A1), is equivalent to the decoupled problem defined by H_α given by Eq. (A6). For the latter problem, the calculation of correlators proceeds as in the main text, with a species of boson for each channel. The probability n to find the impurity in the excited state (since there is only one excited state, the index α is dropped) is

$$n(\omega) \sim \omega^{\sum_a \Delta_a^2 - 2} + \text{ less singular terms}, \tag{A8}$$

where $\omega = \omega_{21}$ or $\omega = \omega_{12}$ depending on whether the ground state of H_0 has the impurity in the state $|1\rangle$ or $|2\rangle$. Current and density operators acquire a channel index a:

$$\rho_a(x) = \phi_a^\dagger(x)\phi_a(x), \tag{A9}$$

$$j_a(x) = \rho_a(x) + \Delta_a v(x)|2\rangle\langle 2|. \tag{A10}$$

The current-current correlator now also has channel indices

$$\langle \mathrm{GS}|\delta j_a(x)\delta j_b(x)|\mathrm{GS}\rangle = \delta_{ab}c_0(x,y) + c_{2ab}(x,y). \tag{A11}$$

Here $c_0(x,y)$ is still given by Eq. (4.6). The second order in γ correction $c_{2ab}(x,y)$ is now asymptotically given by

$$c_{2ab}(x,y) = -\frac{1}{\pi}\Delta_a\Delta_b\frac{1}{|x-y|^3}\partial_\omega n(\omega). \tag{A12}$$

We note that $\Delta_a\Delta_b$ can in principle be positive or negative for $a \neq b$ so that interchannel correlations can occur with either sign, in contrast to correlations in the same channel.

APPENDIX B: THE RELATION BETWEEN IMPURITY TRANSITION RATES AND GROUND-STATE OCCUPATION PROBABILITIES

In this Appendix we derive a relation [Eq. (2.12)] between the ground-state occupation probability n_α and the transition rate $W_{\alpha\lambda}$ of the impurity. The starting point is the perturbation expansion of Eqs. (4.7) and (4.8) for the ground state. From these equations follow that, to second order in the γ's, n_α, with $\alpha \neq \lambda$, is given by

$$n_\alpha = \lim_{\eta\to 0^+}|\gamma_{\alpha\lambda}|^2\int_{-\infty}^0 dt\int_{-\infty}^0 dt'\, e^{\eta(t+t')}P_{\alpha\lambda}(t-t'). \tag{B1}$$

By changing integration variables to $T = (t+t')/2$ and $\tau = t - t'$, performing the T integral we then obtain

$$n_\alpha = \lim_{\eta\to 0^+}\frac{|\gamma_{\alpha\lambda}|^2}{2\eta}\int_{-\infty}^{\infty} d\tau\, e^{-\eta|\tau|}P_{\alpha\lambda}(\tau). \tag{B2}$$

Substituting for $|\gamma_{\alpha\lambda}|^2 P_{\alpha\lambda}(t)$ from Eq. (4.18) allows us to perform the τ integral. Taking the $\eta \to 0^+$ limit, we obtain

$$n_\alpha = \int_{-\infty}^0 \frac{d\omega}{2\pi}\frac{W_{\alpha\lambda}(\omega)}{(\omega_{\alpha\lambda}-\omega)^2}. \tag{B3}$$

APPENDIX C: MEASURABILITY OF THE CURRENT-CURRENT CORRELATOR

We present here what we find to be conceptually the simplest scheme to measure the current-current correlator. Other schemes may be more general or more practical. The Hamiltonian of Eq. (2.2) describes right-moving electrons propagating along the real line. A straightforward realization of such a conductor is a quantum Hall edge state. However, another realization is a semi-infinite wire running from $x = -\infty$ to $x = 0$, containing both left- and right-moving electrons. In this realization, the fermion creation operators $\psi^\dagger(x)$ and $\psi^\dagger(-x)$ both create an electron at a position $-|x|$, while the sign of x determines whether a left mover (positive sign) or a right mover (negative sign) is created. The physically measurable current at $-|x|$ corresponds to the operator

$$J(-|x|) = j(-|x|) - j(|x|), \tag{C1}$$

where $j(x)$ is the current operator defined in Eq. (3.1). In other words, the physically measured current equals the current of right movers minus the current of left movers.

In spectral representation, we write $J(-|x|)$ as

$$J(-|x|) = \sum_m J_m|m\rangle\langle m|, \tag{C2}$$

where J_m and $|m\rangle$ are the many-body eigenvalues and eigenstates of $J(-|x|)$.

By performing current measurements at $-|x|$ on a sufficient number of identical systems prepared in the ground state, the probability density

$$P(J) \equiv \sum_m \delta(J_m - J)|\langle \mathrm{GS}|m\rangle|^2 \tag{C3}$$

for an outcome J can approximately be determined. Given $P(J)$, its second moment

$$M_2(-|x|) \equiv \int_{-\infty}^{\infty} dJ\, J^2 P(J) \tag{C4}$$

can be extracted. From the definitions of $J(-|x|)$ and $P(J)$ it follows that

$$\begin{aligned} M_2(-|x|) = {}& \langle j(|x|)^2\rangle + \langle j(-|x|)^2\rangle \\ & - \langle j(-|x|)j(|x|)\rangle - \langle j(|x|)j(-|x|)\rangle. \end{aligned} \tag{C5}$$

In the above expression, expectation values are with respect to the ground state.

The expectation value $\langle j(x)^2\rangle$ turns out to be position independent for x outside the range of the impurity interaction, i.e., $x \notin l$. The proof is as follows:

$$\begin{aligned} \partial_x\langle j(x)^2\rangle &= \langle \partial_x j(x) j(x)\rangle + \langle j(x)\partial_x j(x)\rangle \\ &= i\langle[\rho(x),H]j(x)\rangle + i\langle j(x)[\rho(x),H]\rangle \\ &= i\langle[j(x),H]j(x)\rangle + i\langle j(x)[j(x),H]\rangle \\ &= 0. \end{aligned} \tag{C6}$$

In the second line we exploited the continuity equation, while in the third line we used the fact that $\rho(x) = j(x)$ [cf. Eq. (3.1)] for $x \notin l$. The last line is obtained by noting that the expectation value is with respect to $|\mathrm{GS}\rangle$, which is an eigenstate of H.

Thus $M_2(-|x|)$ can be measured and is given by

$$M_2(-|x|) = M + \langle j(-|x|)j(|x|)\rangle + \langle j(|x|)j(-|x|)\rangle, \tag{C7}$$

where M is independent of x. Our main result [Eq. (4.26)] then translates into a prediction for the x-dependent part of M_2:

$$M_2(-|x|) - M = -\frac{1}{4\pi|x|^3}\sum_\alpha(\Delta_{\alpha\lambda})^2\partial_{\omega_{\alpha\lambda}}n_\alpha(\omega_{\alpha\lambda}). \tag{C8}$$

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