Quantized topological terms in weak-coupling gauge theories with a global symmetry and their connection to symmetry-enriched topological phases

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We study the quantized topological terms in a weak-coupling gauge theory with gauge group G_g and a global symmetry G_s in d space-time dimensions. We show that the quantized topological terms are classified by a pair (G,ν_d), where G is an extension of G_s by G_g and ν_d an element in group cohomology $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. When $d=3$ and/or when G_g is finite, the weak-coupling gauge theories with quantized topological terms describe gapped symmetry enriched topological (SET) phases (i.e., gapped long-range-entangled phases with symmetry). Thus, those SET phases are classified by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, where $G/G_g=G_s$. We also apply our theory to a simple case $G_s=G_g=Z_2$, which leads to 12 different SET phases in $2+1$ dimensions [$(2+1)$D], where quasiparticles have different patterns of fractional $G_s=Z_2$ quantum numbers and fractional statistics. If the weak-coupling gauge theories are gapless, then the different quantized topological terms may describe different gapless phases of the gauge theories with a symmetry G_s, which may lead to different fractionalizations of G_s quantum numbers and different fractional statistics [if in $(2+1)$D].

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I. INTRODUCTION

For a long time, we thought that Landau symmetry breaking theory[1–3] described all phases and phase transitions. In 1989, through a theoretical study of high T_c superconducting model, we realized that there exists a new kind of order—topological order—which cannot be described by Landau symmetry breaking theory.[4–6] Recently, it was found that topological orders are related to long-range entanglements.[7,8] In fact, we can regard topological order as pattern of long-range entanglements[9] defined through local unitary (LU) transformations.[10–12]

The notion of topological orders and long-range entanglements leads to a more general and also more detailed picture of phases and phase transitions (see Fig. 1).[9] For gapped quantum systems without any symmetry, their quantum phases can be divided into two classes: short-range-entangled (SRE) states and long-range-entangled (LRE) states.

SRE states are states that can be transformed into direct product states via LU transformations. All SRE states can be transformed into each other via LU transformations. So all SRE states belong to the same phase [see Fig. 1(a)]; i.e., all SRE states can continuously deform into each other without closing the energy gap and without phase transition.

LRE states are states that cannot be transformed into direct product states via LU transformations. It turns out that, in general, different LRE states cannot be connected to each other through LU transformations. The LRE states that are not connected via LU transformations represent different quantum phases. Those different quantum phases are nothing but the topologically ordered phases.

Chiral spin liquids,[13,14] fractional quantum Hall states,[15,16] Z_2 spin liquids,[17–19] non-Abelian fractional quantum Hall states,[20–23] etc., are examples of topologically ordered phases. The mathematical foundation of topological orders is closely related to tensor category theory[9,10,24,25] and simple current algebra.[20,26] Using this point of view, we have developed a systematic and quantitative theory for nonchiral topological orders in two-dimensional (2D) interacting boson and fermion systems.[9,10,25] Also for chiral 2D topological orders with only Abelian statistics, we find that we can use integer K matrices to describe them.[27–32]

For gapped quantum systems with symmetry, the structure of phase diagram is even richer [see Fig. 1(b)]. Even SRE states now can belong to different phases. One class of nontrivial SRE phases for Hamiltonians with symmetry is the Landau symmetry breaking states. However, even SRE states that do not break the symmetry of the Hamiltonians can belong to different phases. The 1D Haldane phase for a spin-1 chain[33–36] and topological insulators[37–42] are nontrivial examples of phases with short-range entanglements that do not break any symmetry. We call these kinds of phases SPT phases. The term "SPT phase" may stand for symmetry protected topological phase,[35,36] since the known examples of those phases, the Haldane phase and the topological insulators, were already referred as topological phases. The term "SPT phase" may also stand for symmetry protected trivial phase, since those phases have no long-range entanglements and have trivial topological orders.

It turns out that there is no gapped bosonic LRE state in 1D.[11] So all 1D gapped bosonic states are either symmetry breaking states or SPT states. This realization led to a complete classification of all 1D gapped bosonic quantum phases.[43–45]

In Refs. 46 and 47, the classification of 1D SPT phase is generalized to any dimensions: For gapped bosonic systems in d space-time dimensions with an on-site symmetry G_s, we can construct distinct SPT phases that do not break the symmetry G_s from the distinct elements in $\mathcal{H}^d[G_s,U(1)]$, the d-cohomology class of the symmetry group G_s with $U(1)$ as coefficient. We see that we have a quite systematic understanding of SRE states with symmetry.[48,49]

For gapped LRE states with symmetry, the possible quantum phases should be much richer than SRE states. We may

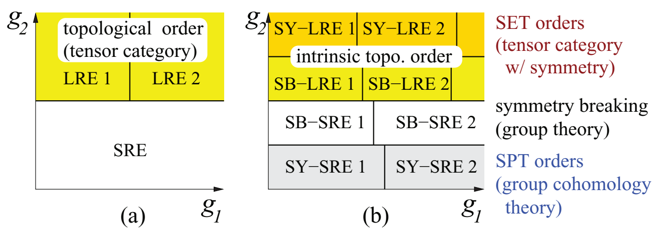

FIG. 1. (Color online) (a) The possible gapped phases for a class of Hamiltonians $H(g_1,g_2)$ without any symmetry. (b) The possible gapped phases for the class of Hamiltonians $H_{\text{symm}}(g_1,g_2)$ with symmetry. The yellow regions in (a) and (b) represent the phases with long-range entanglement. Each phase is labeled by its entanglement properties and symmetry breaking properties. SRE stands for short-range entanglement, LRE for long-range entanglement, SB for symmetry breaking, SY for no symmetry breaking. SB-SRE phases are the Landau symmetry breaking phases. The SY-SRE phases are the SPT phases. The SY-LRE phases are the SET phases.

call those phases symmetry enriched topological (SET) phases. The projective symmetry group (PSG) was introduced to study the SET phases.[50–52] The PSG describes how the quantum numbers of the symmetry group G_s get fractionalized on the gauge excitations.[51] When the gauge group G_g is Abelian, the PSG description of the SET phases can be be expressed in terms of group cohomology: The different SET states with symmetry G_s and gauge group G_g can be (partially) described by $\mathcal{H}^2(G_s,G_g)$.[53] Many examples of the SET states can be found in Refs. 48, 50, and 54–56.

Recently, Mesaros and Ran proposed a quite systematic understanding of a subclass of SET phases:[57] One can use the elements of $\mathcal{H}^d(G_s \times G_g,\mathbb{R}/\mathbb{Z})$ to describe the SET phases in d space-time dimensions with a finite gauge group G_g and a finite global symmetry group G_s. Here $\mathcal{H}^d(G_s \times G_g,\mathbb{R}/\mathbb{Z})$ is the group cohomology class of group $G_s \times G_g$. This result is based on the group cohomology theory of the SPT phases[47] and the Levin-Gu duality between the SPT phases and the "twisted" weak-coupling gauge theories.[58–60] Also, Essin and Hermele generalized the results of Refs. 50, 51, 54, and 55 and studied quite systematically the SET phases described by a $G_g = Z_2$ gauge theory.[53] They show that some of those SET phases can be classified by $\mathcal{H}^2(G_s,G_g)$.

In this paper, we develop a somewhat systematic understanding of SET phases, following a path-integral approach developed for the group cohomology theory of the SPT phases[47] and the topological gauge theory.[60,61] The idea is to classify quantized topological terms in weak-coupling gauge theory with symmetry. If the weak-coupling gauge theory happens to have a gap, then the different quantized topological terms will describe different SET phases. This allows us to obtain and generalize the results in Refs. 53 and 57. Since weak-coupling gauge theories only describe some topological ordered states, our theory only describes some of the SET states.

We show that quantized topological terms in symmetric weak-coupling gauge theory in d space-time dimensions with a gauge group G_g and a global symmetry group G_s can be described by a pair (G,ν_d), where G is an extension of G_s by G_g and ν_d is an element in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. (An extension of G_s by G_g is group G that contains G_g as a normal subgroup and satisfy $G/G_g = G_s$.) When G_g is finite or when $d = 3$, the weak-coupling gauge theory is gapped. In this case, (G,ν_d) describe different SET phases. Note that the extension G is nothing but the PSG introduced in Ref. 50. Also, when the symmetry group G_s contains antiunitary transformations, those antiunitary transformations will act nontrivially on $\mathbb{R}/\mathbb{Z}$: $x \to -x$, $x \in \mathbb{R}/\mathbb{Z}$.[47]

In Appendix B, we show that we can use $(y_0,y_1,\ldots,y_d)$ with

$$y_k \in \mathcal{H}^k[G_s,\mathcal{H}^{d-k}(G_g,\mathbb{R}/\mathbb{Z})] \tag{1}$$

to label the elements in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. However, such a labeling may not be one-to-one and it may happen that only some of $(y_0,y_1,\ldots,y_d)$ correspond to the elements in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. However, for every element in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, we can find a $(y_0,y_1,\ldots,y_d)$ that corresponds to it. If we choose a special extension $G = G_g \times G_s$, then we recover the result in Ref. 57 if G is finite: A set of SET states can be can be described by $(y_0,y_1,\ldots,y_d)$ with an one-to-one correspondence [see Eq. (A10)]:

$$\begin{aligned}\mathcal{H}^d(G_s \times G_g,\mathbb{R}/\mathbb{Z}) &= \oplus_{p=0}^{d}\mathcal{H}^{d-p}[G_s,\mathcal{H}^p(G_g,\mathbb{R}/\mathbb{Z})]\\ &= \oplus_{p=0}^{d}\mathcal{H}^{d-p}[G_g,\mathcal{H}^p(G_s,\mathbb{R}/\mathbb{Z})].\end{aligned} \tag{2}$$

The term $\mathcal{H}^d[G_s,\mathcal{H}^0(G_g,\mathbb{R}/\mathbb{Z})] = \mathcal{H}^d(G_s,\mathbb{R}/\mathbb{Z})$ describes the quantized topological terms associated with only the symmetry G_s which describes the SPT phases. The term $\mathcal{H}^0[G_s,\mathcal{H}^d(G_g,\mathbb{R}/\mathbb{Z})] = \mathcal{H}^d(G_g,\mathbb{R}/\mathbb{Z})$ describes the quantized topological terms associated with pure gauge theory. Other terms $\oplus_{p=1}^{d-1}\mathcal{H}^{d-p}[G_s,\mathcal{H}^p(G_g,\mathbb{R}/\mathbb{Z})]$ describe the quantized topological terms that involve both gauge theory G_g and symmetry G_s. Those terms describe how G_s quantum numbers get fractionalized on gauge-flux excitations.[57]

When G_g is Abelian, the different extensions, G, of G_s by G_g is classified by $\mathcal{H}^2(G_s,G_g)$. This reproduces a result in Ref. 53.

II. A SIMPLE FORMAL APPROACH

First let us describe a simple formal approach that allows us to quickly obtain the above results. We know that the SPT phases in d-dimensional discrete space-time are described by topological nonlinear σ models with symmetry G:

$$\mathcal{L} = \frac{1}{\lambda_s}[\partial g(x^\mu)]^2 + \mathrm{i}\, W_{\text{top}}(g), \quad g \in G, \tag{3}$$

where $\lambda_s \to \infty$, and the 2π-quantized topological term $\int W_{\text{top}}(g)$ is given by an element in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. Different elements in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ describe different SPT phases.[47] If we "gauge" the symmetry G, the topological nonlinear σ model will become a gauge theory,

$$\mathcal{L} = \frac{1}{\lambda_s}[(\partial - \mathrm{i}\,A)g(x^\mu)]^2 + \mathrm{i}\, W_{\text{top}}(g,A) + \frac{(F_{\mu\nu})^2}{\lambda}, \tag{4}$$

where $W_{\text{top}}(g,A)$ is the gauged topological term. For those topological terms that can be expressed in continuous field theory, $W_{\text{top}}(g,A)$ can be obtained from $W_{\text{top}}(g)$ by replacing ∂_μ with $\partial_\mu - \mathrm{i}\,A_\mu$. When G_s and G_g are finite, $W_{\text{top}}(g,A)$ can be constructed explicitly in discrete space-time.[62]

If we further integrate out g, we get a pure gauge theory with a topological term,

$$\mathcal{L} = \frac{(F_{\mu\nu})^2}{\lambda} + \mathrm{i}\, W_{\text{top}}(A). \tag{5}$$

This line of thinking suggests that the quantized topological term $\int \tilde{W}_{\text{top}}(A)$ in symmetric gauge theory is classified by the same $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ that classifies the 2π-quantized topological term $\int W_{\text{top}}(g)$.

Now let us consider topological nonlinear σ models with symmetry $G_s \times G_g$,

$$\mathcal{L} = \frac{1}{\lambda_s}[\partial g(x^\mu)]^2 + \mathrm{i}\, W_{\text{top}}(g), \quad g \in G = G_s \times G_g, \tag{6}$$

where the 2π-quantized topological term $\int W_{\text{top}}(g)$ is classified by $\mathcal{H}^d(G_s \times G_g,\mathbb{R}/\mathbb{Z})$. If we gauge only a subgroup G_g of the total symmetry group $G_s \times G_g$, we will get a gauge theory,

$$\mathcal{L} = \frac{1}{\lambda_s}[(\partial - \mathrm{i}\, A)g(x^\mu)]^2 + \mathrm{i}\, W_{\text{top}}(g,A) + \frac{(F_{\mu\nu})^2}{\lambda}, \tag{7}$$

with global symmetry G_s. This line of thinking suggests that the quantized topological term $\int W_{\text{top}}(g,A)$ is classified by the same $\mathcal{H}^d(G_s \times G_g,\mathbb{R}/\mathbb{Z})$.

We can generalize the above approach to obtain more general quantized topological terms in weak-coupling gauge theory with gauge group G_g and symmetry G_s. We start with a group G which is an extension of the symmetry group G_s by the gauge group G_g:

$$1 \to G_g \to G \to G_s \to 1. \tag{8}$$

In other words, G contains a normal subgroup G_g such that $G/G_g = G_s$. So we can start with a topological nonlinear σ models with symmetry G,

$$\mathcal{L} = \frac{1}{\lambda_s}[\partial g(x^\mu)]^2 + \mathrm{i}\, W_{\text{top}}(g), \quad g \in G, \tag{9}$$

where the 2π-quantized topological term $\int W_{\text{top}}(g)$ is classified by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. If we gauge only a subgroup G_g of the total symmetry group G, we will get a gauge theory,

$$\mathcal{L} = \frac{1}{\lambda_s}[(\partial - \mathrm{i}\, A)g(x^\mu)]^2 + \mathrm{i}\, W_{\text{top}}(g,A) + \frac{(F_{\mu\nu})^2}{\lambda}, \tag{10}$$

with global symmetry $G_s = G/G_g$. This line of thinking suggests that the quantized topological term $\int W_{\text{top}}(g,A)$ is classified by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$.

So more generally, the SET states in d-dimensional space-time with gauge group G_g and symmetry group G_s are labeled by the elements in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, where G is the extension of the symmetry group G_s by the gauge group G_g, provided that the symmetric gauge theory (9) is gapped in a small λ limit and $d \geqslant 3$. If the symmetric gauge Eq. (9) is gapless in the small λ limit, then $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ describes different gapless phases of the symmetric gauge theory.

The above approach is formal and hand-waving. When G is finite, we can rigorously obtain the above results, which is described in Ref. 62. In the following, we discuss such an approach assuming G_g is finite (but G_s can be finite or continuous). Then we discuss another approach that allows us to obtain the above result more rigorously for the case $G = G_s \times G_g$, where G_s, G_g can be finite or continuous.

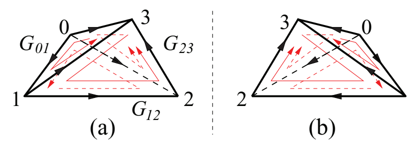

FIG. 2. (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.

III. AN EXACT APPROACH FOR FINITE G_g

This approach is based on the formal approach (10) discussed above, where G is an extension of the symmetry group G_s by the gauge group G_g: $G/G_g = G_s$. We make the above approach exact by putting the theory on space-time lattice of d dimensions.

A. Discretize space-time

We discretize the space-time M by considering its triangulation M_{tri} and define the d-dimensional gauge theory on such a triangulation. We call such a theory a lattice gauge theory. We call the triangulation M_{tri} a space-time complex, and a cell in the complex a simplex.

In order to define a generic lattice theory on the space-time complex M_{tri}, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure.[47,63] A branching structure is a choice of orientation of each edge in the d-dimensional complex so that there is no oriented loop on any triangle (see Fig. 2).

The branching structure induces a *local order* of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming edges, and the second vertex is the vertex with only one incoming edge, etc. So the simplex in Fig. 2(a) has the following vertex ordering: 0,1,2,3.

The branching structure also gives the simplex (and its subsimplexes) an orientation denoted by $s_{ij\cdots k} = 1,*$. Figure 2 illustrates two 3-simplices with opposite orientations $s_{0123} = 1$ and $s_{0123} = *$. The red arrows indicate the orientations of the 2-simplices which are the subsimplices of the 3-simplices. The black arrows on the edges indicate the orientations of the 1-simplices.

B. Gauged nonlinear σ model on space-time lattice

To put Eq. (10) on space-time lattice, we put the $g(x^\mu) \in G$ field on the vertices of the space-time complex, which becomes g_i, where i labels the vertices. We also put the gauge field on the edges ij which becomes $g_{ij} \in G_g$.

The action amplitude for a d cell $(ij\cdots k)$ is complex function of g_i and g_{ij}: $V_{ij\cdots k}(\{g_{ij}\},\{g_i\})$ The partition function is given by

$$Z = \sum_{\{g_{ij}\},\{g_i\}} \prod_{(ij\cdots k)} [V_{ij\cdots k}(\{g_{ij}\},\{g_i\})]^{s_{ij\cdots k}}, \tag{11}$$

where $\prod_{(ij\cdots k)}$ is the product over all the d cells $(ij\cdots k)$. If the above action amplitude $\prod_{(ij\cdots k)}[V_{ij\cdots k}(\{g_{ij}\},\{g_i\})]^{s_{ij\cdots k}}$ on closed space-time complex ($\partial M_{\text{tri}} = \emptyset$) is invariant under the gauge transformation

$$g_{ij} \to g'_{ij} = h_i g_{ij} h_j^{-1}, \quad g_i \to g'_i = h_i g_i h_i^{-1}, \quad h_i \in G_g, \tag{12}$$

then the action amplitude $V_{ij\cdots k}(\{g_{ij}\},\{g_i\})$ defines a gauge theory of gauge group G_g. If the action amplitude is invariant under the global transformation

$$g_{ij} \to g'_{ij} = h g_{ij} h^{-1}, \quad g_i \to g'_i = h g_i h^{-1}, \quad h \in G, \tag{13}$$

then the action amplitude $V_{ij\cdots k}(\{g_{ij}\},\{g_i\})$ defines a gauge theory with a global symmetry $G_s = G/G_g$. (We need to mod out G_g since when $h \in G_g$, it will generate a gauge transformation instead of a global symmetry transformation.)

Using a cocycle $\nu_d(g_0,g_1,\ldots,g_d) \in \mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, $g_i \in G$, we can construct an action amplitude $V_{ij\cdots k}(\{g_{ij}\},\{g_i\})$ that defines a gauge theory with gauge group G_s and global symmetry G_s. First, we note that the cocycle satisfies the cocycle condition

$$\nu_d(g_0,g_1,\ldots,g_d) = \nu_d(hg_0,hg_1,\ldots,hg_d), \quad h \in G,$$
$$\prod_i \nu_d(g_0,\ldots,\hat{g}_i,\ldots,g_{d+1}) = 1, \tag{14}$$

where $g_0,\ldots,\hat{g}_i,\ldots,g_{d+1}$ is the sequence $g_0,\ldots,g_i,\ldots,g_{d+1}$ with g_i removed. The gauge theory action amplitude is given by

$$V_{01\cdots d}(\{g_{ij}\},\{g_i\}) = 0, \quad \text{if } g_{ij}g_{jk} \neq g_{ik},$$
$$V_{01\cdots d}(\{g_{ij}\},\{g_i\}) = \nu_d(\tilde{g}_0 g_0,\tilde{g}_1 g_1,\ldots,\tilde{g}_d g_d), \quad \text{otherwise}, \tag{15}$$

where $\tilde{g}_i$ are given by

$$\tilde{g}_0 = 1, \quad \tilde{g}_1 = \tilde{g}_0 g_{01}, \quad \tilde{g}_2 = \tilde{g}_1 g_{12}, \quad \tilde{g}_3 = \tilde{g}_2 g_{23}, \ldots. \tag{16}$$

One can check that the above action amplitude $V_{01\cdots d}(\{g_{ij}\},\{g_i\})$ is invariant under the gauge transformation (12) and the global symmetry transformation (13). Thus, it defines a symmetric gauge theory.

We know that the action amplitude is nonzero only when $g_{ij}g_{jk} = g_{ik}$. The condition $g_{ij}g_{jk} = g_{ik}$ is the flat connection condition, and the corresponding gauge theory is in the weak-coupling limit (actually is at the zero-coupling). This condition can be implemented precisely only when G_g is finite. With the flat connection condition $g_{ij}g_{jk} = g_{ik}$, $\tilde{g}_i$'s and the gauge equivalent sets of g_{ij} have an one-to-one correspondence.

Since the total action amplitude $\prod_{(ij\cdots k)}[V_{ij\cdots k}(\{g_{ij}\},\{g_i\})]^{s_{ij\cdots k}}$ on a sphere is always equal to 1 if the gauge flux vanishes, $V_{ij\cdots k}(\{g_{ij}\},\{g_i\})$ describes a quantized topological term in weak-coupling gauge theory (or zero-coupling gauge theory). This way, we show that a quantized topological term in a weak-coupling gauge theory with gauge group G_g and symmetry group G_s can be constructed from each element of $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$.

When $G_g = \{1\}$ (or $G = G_s$),

$$V_{01\cdots d}(\{g_{ij}\},\{g_i\}) = \nu_d(g_0,g_1,\ldots,g_d) \tag{17}$$

become the action amplitude for the topological nonlinear σ model, describing the SPT phase labeled by the cocycle $\nu_d \in \mathcal{H}^d[G_s,\mathbb{R}/\mathbb{Z})$.[47]

When $G_s = \{1\}$ (or $G = G_g$),

$$V_{01\cdots d}(\{g_{ij}\},\{g_i\}) = \nu_d(\tilde{g}_0 g_0,\tilde{g}_1 g_1,\ldots,\tilde{g}_d g_d). \tag{18}$$

We can use the gauge transformation (12) to set $g_i = 1$ in the above and obtain

$$V_{01\cdots d}(\{g_{ij}\},\{g_i\}) = \nu_d(\tilde{g}_0,\tilde{g}_1,\ldots,\tilde{g}_d). \tag{19}$$

This is the topological gauge theory studied in Refs. 60 and 61.

IV. AN APPROACH BASED ON CLASSIFYING SPACE

In this section, we consider the cases where G_s,G_g can be finite or continuous. However, for the time being, we can only handle the situation where $G = G_s \times G_g$. Our approach is based on the classifying space.

A. Motivations and results

Let us first review some known results. To gain a systematic understand of SRE states with on-site symmetry G_s, we started with a nonlinear σ model,

$$\mathcal{L} = \frac{1}{\lambda_s}[\partial g(x^\mu)]^2, \quad g \in G_s, \tag{20}$$

with symmetry group G_s as the target space. The model can be in a disordered phase that does not break the symmetry G_s when λ is large. By adding different 2π quantized topological θ terms to the Lagrangian $\mathcal{L}$, we can get different Lagrangians that describe different disordered phases that do not break the symmetry G_s.[47] Those disordered phases are the symmetry protected topological (SPT) phases.[35,36] So we can use the quantized topological terms to classify the SPT phases. (In general, topological terms, by definition, are the terms that do not depend on space-time metrics.)

We know that gauge theory

$$\mathcal{L} = \frac{1}{\lambda}\text{Tr}(F_{\mu\nu}F^{\mu\nu}) \tag{21}$$

is one way to describe LRE states (i.e., topologically ordered states). In Refs. 60 and 61, different quantized topological terms in weak-coupling gauge theory with gauge group G_g and small λ in d space-time dimensions are constructed and classified, using the topological cohomology class $H^{d+1}(BG_g,\mathbb{Z})$ for the classifying space BG_g of the gauge group G_g. By adding those quantized topological terms to the above Lagrangian for the weak-coupling gauge theory, we may obtain different phases of the weak-coupling gauge theory.

In this section, we plan to combine the above two approaches by studying the quantized topological terms in the combined theory,

$$\mathcal{L} = \frac{1}{\lambda}\text{Tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{\lambda_s}[\partial g(x^\mu)]^2, \quad g \in G_s, \tag{22}$$

where F is the field strength with gauge group G_g, and $(\lambda,\lambda_s) \to$ (small,large). Such a theory is a gauge theory with symmetry G_s. We find that quantized topological terms in the combined theory can be constructed and classified by the topological cohomology class $H^{d+1}(BG_s \times BG_g,\mathbb{Z})$ for the classifying space of the product $G_g \times G_s$. Those quantized topological terms give us a somewhat systematic understanding of different phases of weak coupling gauge theories with symmetry. If those symmetric weak coupling gauge theories are gapped (for example, for finite gauge groups), then the theories will describe topologically ordered states with symmetry. Those SET phases in d space-time dimensions are described by elements in $H^{d+1}(BG_s \times BG_g,\mathbb{Z})$.

B. Gauge theory as a nonlinear σ model with classifying space as the target space

To obtain the above result, we follow closely the approaches used in Refs. 60 and 47. We obtain our result in two steps.

1. Symmetric weak-coupling gauge theory as the nonlinear σ model of $G_s \times BG_g$

As in Ref. 60, we may view a weak-coupling gauge theory with gauge group G_g as a nonlinear σ model with classifying space BG_g as the target space. So the symmetric weak-coupling gauge theory in Eq. (22) can be viewed as a nonlinear σ model with $G_s \times BG_g$ as the target space, where each path in the path integral is given by an *embedding* $\gamma : M_{\text{tri}} \to G_s \times BG_g$ from the space-time complex M_{tri} to $G_s \times BG_g$. We can study topological terms in our symmetric weak-coupling gauge theory by studying the topological terms in the corresponding nonlinear σ model.

Following Ref. 60, a total term S_{top} corresponds to evaluating a cocycle $\alpha_d \in Z(G_s \times BG_g,\mathbb{R}/\mathbb{Z})$ on the complex $\gamma(M_{\text{tri}}) \subset G_s \times BG_g$:

$$S_{\text{top}}[\gamma] = 2\pi \langle \alpha_d,\gamma(M_{\text{tri}})\rangle \mod 2\pi. \tag{23}$$

Such a topological term does not depend on any smooth deformation of γ and is thus "topological." (Note that the evaluation of the d-cocycle on any d-cycles [i.e., d-dimensional closed complexes] are equal to 0 mod 1 if the d-cycles are boundaries of some $(d+1)$D complex.)

Here we would like to stress that the cocycle α_d on the group manifold is *not* the ordinary topological cocycle. It has a symmetry condition,

$$\langle \alpha_d,c\rangle = \langle \alpha_d,c_g\rangle, \tag{24}$$

where c is a complex in G_s and c_g is the complex generated from c by the symmetry transformation $G_s \to gG_s$, $g \in G_s$. Also, since $\lambda_s \to \infty$ and $g(x^\mu)$ have large fluctuations in Eq. (22), $\langle \alpha_d,c\rangle$ only depend on the vertices $g_0,g_1,\ldots$ of c:

$$\langle \alpha_d,c\rangle = \nu(g_0,g_1,\ldots), \quad \nu(gg_0,gg_1,\ldots) = \nu(g_0,g_1,\ldots);\quad g,g_i \in G_s. \tag{25}$$

So, on G_s, α_d is actually a cocycle in the group cohomology $\mathcal{Z}(G_s,\mathbb{R}/\mathbb{Z})$,[47] while on BG_g, α_d is the usual cocycle in the topological cohomology $Z(BG_g,\mathbb{R}/\mathbb{Z})$.

Since, on G_s, α_d is a cocycle in the group cohomology $\mathcal{Z}(G_s,\mathbb{R}/\mathbb{Z})$, when G_s contains antiunitary symmetry, such antiunitary symmetry transformation will have a nontrivial action on $\mathbb{R}/\mathbb{Z}$: $x \to -x$, $x \in \mathbb{R}/\mathbb{Z}$.[47]

If two d-cocycles, $\alpha_d,\alpha'_d \in Z^d(BG_g,\mathbb{R}/\mathbb{Z})$, differ by a coboundary: $\alpha'_d - \alpha_d = \mathrm{d}\mu_d$, $\mu_d \in C^d(BG_g,\mathbb{R}/\mathbb{Z})$, then, the corresponding action amplitudes, $e^{\mathrm{i}S_{\text{top}}[\gamma]}$ and $e^{\mathrm{i}S'_{\text{top}}[\gamma]}$, can smoothly deform into each other without phase transition. So $e^{\mathrm{i}S_{\text{top}}[\gamma]}$ and $e^{\mathrm{i}S'_{\text{top}}[\gamma]}$, or α_d and α'_d, describe the same quantum phase. Therefore, we regard α_d and α'_d to be equivalent. The equivalent classes of the d-cocycles form the d cohomology class $H^d(G_s \times BG_g,\mathbb{R}/\mathbb{Z})$. We conclude that the topological terms in symmetric weak-coupling lattice gauge theories are described by $H^d(G_s \times BG_g,\mathbb{R}/\mathbb{Z})$ in d space-time dimensions.

To calculate $H^d(G_s \times BG_g,\mathbb{R}/\mathbb{Z})$, let us first calculate $H^d(G_s \times BG_g,\mathbb{Z})$. Using the the Künneth formula Eq. (A4) (with $M' = \mathbb{Z}$), we find that

$$\begin{aligned} &H^d(G_s \times BG_g,\mathbb{Z})\\ &\simeq \big[\oplus_{p=0}^{d} \mathcal{H}^p(G_s,\mathbb{Z}) \otimes_{\mathbb{Z}} H^{d-p}(BG_g,\mathbb{Z})\big]\\ &\quad\oplus\big[\oplus_{p=0}^{d+1} \mathrm{Tor}_1^{\mathbb{Z}}[\mathcal{H}^p(G_s,\mathbb{Z}),H^{d-p+1}(BG_g,\mathbb{Z})]\big]. \end{aligned} \tag{26}$$

In the above, we have used the fact that the cohomology on G_s is the group cohomology $\mathcal{H}$ and the cohomology on BG_g is the usual topological cohomology H.

In Appendix A, we show that [see Eq. (A6)]

$$H^d(X,\mathbb{R}/\mathbb{Z}) \simeq H^d(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \oplus \mathrm{Tor}_1^{\mathbb{Z}}[H^{d+1}(X,\mathbb{Z}),\mathbb{R}/\mathbb{Z}]. \tag{27}$$

Using

$$\begin{aligned} &\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} = \mathbb{R}/\mathbb{Z}, \quad \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} = 0,\\ &\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{R}/\mathbb{Z}) = 0, \quad \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_n,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n, \end{aligned} \tag{28}$$

we see that $H^d(X,\mathbb{R}/\mathbb{Z})$ has a form $H^d(X,\mathbb{R}/\mathbb{Z}) = \mathbb{R}/\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/\mathbb{Z} \oplus Z_{n_1} \oplus Z_{n_2} \oplus \cdots$. So the discrete part of $H^d(X,\mathbb{R}/\mathbb{Z})$ is given by

$$\mathrm{Dis}[H^d(X,\mathbb{R}/\mathbb{Z})] = Z_{n_1} \oplus Z_{n_2} \oplus \cdots = \mathrm{Tor}[H^{d+1}(X,\mathbb{Z})], \tag{29}$$

where we have used

$$H^{d+1}(X,\mathbb{Z}) = \mathrm{Free}[H^{d+1}(X,\mathbb{Z})] \oplus \mathrm{Tor}[H^{d+1}(X,\mathbb{Z})] \tag{30}$$

with $\mathrm{Tor}[H^{d+1}(X,\mathbb{Z})]$ the torsion part and $\mathrm{Free}[H^{d+1}(X,\mathbb{Z})]$ the free part of $H^{d+1}(X,\mathbb{Z})$. Therefore, we have

$$\begin{aligned} &\mathrm{Dis}[H^d(G_s \times BG_g,\mathbb{R}/\mathbb{Z})]\\ &\simeq \mathrm{Tor}\big[\big[\oplus_{p=0}^{d+1} \mathcal{H}^p(G_s,\mathbb{Z}) \otimes_{\mathbb{Z}} H^{d+1-p}(BG_g,\mathbb{Z})\big]\\ &\quad\oplus\big[\oplus_{p=0}^{d+2} \mathrm{Tor}_1^{\mathbb{Z}}(\mathcal{H}^p(G_s,\mathbb{Z}),H^{d-p+2}(BG_g,\mathbb{Z}))\big]\big]. \end{aligned} \tag{31}$$

Since $\mathcal{H}^d(G_s,\mathbb{Z}) = H^d(BG_s,\mathbb{Z})$, the above can be rewritten as

$$\begin{aligned} &\mathrm{Dis}[H^d(G_s \times BG_g,\mathbb{R}/\mathbb{Z})]\\ &\simeq \mathrm{Tor}\big[\big[\oplus_{p=0}^{d+1} H^p(BG_s,\mathbb{Z}) \otimes_{\mathbb{Z}} H^{d+1-p}(BG_g,\mathbb{Z})\big]\\ &\quad\oplus\big[\oplus_{p=0}^{d+2} \mathrm{Tor}_1^{\mathbb{Z}}(H^p(BG_s,\mathbb{Z}),H^{d-p+2}(BG_g,\mathbb{Z}))\big]\big]\\ &= \big[\oplus_{p=0}^{d+1} \mathrm{Tor}[H^p(BG_s,\mathbb{Z})] \otimes_{\mathbb{Z}} \mathrm{Tor}[H^{d+1-p}(BG_g,\mathbb{Z})]\big]\\ &\quad\oplus\big[\oplus_{p=0}^{d+1} \mathrm{Free}[H^p(BG_s,\mathbb{Z})] \otimes_{\mathbb{Z}} \mathrm{Tor}[H^{d+1-p}(BG_g,\mathbb{Z})]\big]\\ &\quad\oplus\big[\oplus_{p=0}^{d+1} \mathrm{Tor}[H^p(BG_s,\mathbb{Z})] \otimes_{\mathbb{Z}} \mathrm{Free}[H^{d+1-p}(BG_g,\mathbb{Z})]\big]\\ &\quad\oplus\big[\oplus_{p=0}^{d+2} \mathrm{Tor}_1^{\mathbb{Z}}(H^p(BG_s,\mathbb{Z}),H^{d-p+2}(BG_g,\mathbb{Z}))\big]. \end{aligned} \tag{32}$$

Each element in the above cohomology class describes a quantized topological term in the weakly coupled gauge theory with symmetry G_s.

2. Chern-Simons form

We note that

$$H^{d+1}(BG_s \times BG_g,\mathbb{Z}) = \left[\oplus_{p=0}^{d+1} H^p(BG_s,\mathbb{Z}) \otimes_{\mathbb{Z}} H^{d+1-p}(BG_g,\mathbb{Z})\right] \oplus \left[\oplus_{p=0}^{d+2} \text{Tor}_1^{\mathbb{Z}}[H^p(BG_s,\mathbb{Z}),H^{d-p+2}(BG_g,\mathbb{Z})]\right]. \quad (33)$$

So the result (32) is very close to our proposal that elements in $H^{d+1}(BG_s \times BG_g,\mathbb{Z})$ correspond to the quantized topological terms. The only thing missing is the free part of $H^d(BG_g,\mathbb{Z})$.

In fact, the free part of $H^{d+1}(BG_g,\mathbb{Z})$, denoted as Free$[H^{d+1}(BG_g,\mathbb{Z})]$, is nonzero only when $d =$ odd. So in the following we consider only $d =$ odd cases. The free part Free$[H^{d+1}(BG_g,\mathbb{Z})]$ corresponds to the Chern-Simons forms in d space-time dimensions.

To understand such a result, we first choose a $\omega \in$ Free$[H^{d+1}(BG_g,\mathbb{Z})]$. We can find integers K_i such that

$$-\omega + \frac{K_1}{\frac{d+1}{2}!(2\pi)^{\frac{d+1}{2}}} \text{Tr} F^{\frac{d+1}{2}} + \cdots \quad (34)$$

is an exact form $d\theta_d(A)$. Here $\theta_d(A)$ is called a Chern-Simons form in d dimensions.

We can use a Chern-Simons form $\theta_{d-p}(A)$ and a cocycle $\alpha_p \in \mathcal{H}^p(G_s,\mathbb{Z})$ to construct a quantized topological term,

$$S_{\text{top}}[\gamma] = 2\pi \langle \alpha_p \cup \theta_{d-p}(A), \gamma(M_{\text{tri}})\rangle \mod 2\pi. \quad (35)$$

Such kind of topological terms are labeled by the elements in

$$\oplus_{p=0}^{d+1} \mathcal{H}^p(G_s,\mathbb{Z}) \otimes_{\mathbb{Z}} \text{Free}[H^{d+1-p}(BG_g,\mathbb{Z})] = \oplus_{p=0}^{d+1} H^p(BG_s,\mathbb{Z}) \otimes_{\mathbb{Z}} \text{Free}[H^{d+1-p}(BG_g,\mathbb{Z})]. \quad (36)$$

Combining the above result with Eq. (32), we find that the elements in $H^{d+1}(BG_s \times BG_g,\mathbb{Z})$ correspond to the quantized topological terms.

V. AN EXAMPLE: $G_s = Z_2$ AND $G_g = Z_2$

In this section, we discuss a simple example with $G_s = Z_2$ and $G_g = Z_2$. There are two kinds of extensions G of $G_s = Z_2$ by $G_g = Z_2$: $G = Z_2 \times Z_2$ and $G = Z_4$. So the quantized topological terms and the SET phases are described by $\mathcal{H}^d(Z_2 \times Z_2,\mathbb{R}/\mathbb{Z})$ and $\mathcal{H}^d(Z_4,\mathbb{R}/\mathbb{Z})$ in d space-time dimensions.

In $d = 3$ space-time dimensions, we have

$$\mathcal{H}^3(Z_2 \times Z_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2^3, \quad \mathcal{H}^3(Z_4,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_4. \quad (37)$$

So there are 12 SET phases for weak-coupling Z_2 gauge theory with Z_2 symmetry. However, at this stage, it is not clear if those 12 SET phases are really distinct, since they could be smoothly connected via strong coupling gauge theory. Later, we will see that the 12 SET phases are indeed distinct, since they have distinct physical properties.

A. A K-matrix approach

To understand the physical properties of those 12 SET phases, we would like to use Levin-Gu duality to gauge the G_s and turn the theory into gauge theory with gauge group G.

Let us first consider the $G = Z_2 \times Z_2$ case. A $G = Z_2 \times Z_2$ gauge theory can be described by $U^4(1)$ mutual Chern-Simons theory:[54,64]

$$\mathcal{L} = \frac{1}{4\pi} K_{0,IJ} a_\mu^I \partial_\nu a_\lambda^J + \cdots, \quad (38)$$

with

$$K_0 = 2\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (39)$$

The G_s charge corresponds to the unit charge of a_μ^1 gauge field and the G_g gauge charge corresponds to the unit charge of a_μ^3 gauge field. The G_s flux excitation (in the $G = Z_2 \times Z_2$ gauge theory) corresponds to the end of the branch cut in the original theory along which we have a twist generated by a G_s symmetry transformation (see Ref. 58 for a detailed discussion about the symmetry twist). Such G_s flux corresponds to the flux of an a_μ^1 gauge field.

The eight types of quantized topological terms are given by

$$W_{\text{top}} = \frac{n_1}{2\pi} a_\mu^1 \partial_\nu a_\lambda^1 + \frac{n_{12}}{2\pi} a_\mu^1 \partial_\nu a_\lambda^3 + \frac{n_2}{2\pi} a_\mu^3 \partial_\nu a_\lambda^3, \quad (40)$$

$n_1 = 0,1,\ n_{12} = 0,1,\ n_2 = 0,1$. The total Lagrangian has a form

$$\mathcal{L} + W_{\text{top}} = \frac{1}{4\pi} K_{IJ} a_\mu^I \partial_\nu a_\lambda^J + \cdots, \quad (41)$$

with

$$K = \begin{pmatrix} 2n_1 & 2 & n_{12} & 0 \\ 2 & 0 & 0 & 0 \\ n_{12} & 0 & 2n_2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}. \quad (42)$$

Two K matrices are equivalent: $K_1 \sim K_2$ if $K_1 = U^T K_2 U$ for an integer matrix with $\det(U) = \pm 1$. We find $K(n_1,n_{12},n_2) \sim K(n_1+2,n_{12},n_2) \sim K(n_1,n_{12}+2,n_2) \sim K(n_1,n_{12},n_2+2)$. Thus, only $n_1,n_{12},n_2 = 0,1$ give rise to inequivalent K matrices.

A particle carrying l_I a_μ^I charge will have a statistics

$$\theta_l = \pi l_I (K^{-1})^{IJ} l_J. \quad (43)$$

A particle carrying l_I a_μ^I charge will have a mutual statistics with a particle carrying $\tilde{l}_I$ a_μ^I charge:

$$\theta_{l,\tilde{l}} = 2\pi l_I (K^{-1})^{IJ} \tilde{l}_J. \quad (44)$$

We note that *the G_s charge is identified with the unit a_μ^1 charge* and the G_g gauge charge is identified with the unit a_μ^3 charge. Using

$$K^{-1} = \frac{1}{4}\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & -2n_1 & 0 & -n_{12} \\ 0 & 0 & 0 & 2 \\ 0 & -n_{12} & 2 & -2n_2 \end{pmatrix}, \quad (45)$$

we find that the G_s charge (the unit a^1_μ charge) and the G_g gauge charge (the unit a^3_μ charge) remain bosonic after inclusion of the topological terms. This is actually a condition on the topological terms: The topological terms do not affect the statistics of the gauge charge.

The end of the branch cut in the original theory corresponds to π flux in a^1_μ. We note that a particle carrying l_I a^I_μ charge created a $l_2\pi$ flux in a^1_μ. So a unit a^2_μ charge always create a G_s twist. However, what is the G_s charge of the l_I particle?

To measure the G_s charge, we need to find the pure G_s twist. Let us assume that the pure G_s twist corresponds to $\mathbf{l}^v = (l^v_1, l^v_2, 0, 0)a^I_\mu$ charge. Then $l^v_2 = 1$ so that the $\mathbf{l}^v$ particle produces π a^1_μ flux. For a pure G_s twist, we also have

$$\pi(\mathbf{l}^v)^T K^{-1}\mathbf{l}^v = 0. \tag{46}$$

This allows us to obtain

$$(\mathbf{l}^v)^T = \left(\frac{n_1}{2}, 1, 0, 0\right). \tag{47}$$

Note that some times, $\mathbf{l}^v$ is not an allowed excitation. However, we can always use $\mathbf{l}^v$ to probe the G_s charge. Let

$$\mathbf{q} = 2K^{-1}\mathbf{l}^v = \begin{pmatrix} 1 \\ -n_1/2 \\ 0 \\ -n_{12}/2 \end{pmatrix}. \tag{48}$$

Moving a pure G_s twist around the l_I particle will induce a phase

$$2\pi \mathbf{l}^T K^{-1}\mathbf{l}^v = \pi \mathbf{q}^T \mathbf{l}. \tag{49}$$

We find that the G_s charge of the l_I particle is

$$G_s \text{ charge} = \mathbf{q}^T \mathbf{l} \bmod 2. \tag{50}$$

When $n_{12} = 0$, those gauge excitations have a trivial mutual statistics with the unit a^2_μ charge (i.e., the end of the branch cut). This means that those gauge excitations carry a trivial G_s quantum number. When $n_{12} = 1$, the unit a^4_μ charge (the gauge-flux excitation) has a $\pi/2$ mutual statistics with the unit a^2_μ charge (i.e., the end of the branch cut). This means that the unit a^4_μ charge carries a *fractional* G_s charge. Such a fractional-G_s-charge gauge excitation has a Bose/Fermi statistics if $n_2 = 0$ and a semion statistics if $n_2 = 1$. We see that both n_{12} and n_2 are measurable. n_1 is also measurable, which describes the G_s SPT phases.

To summarize, Tables I–VIII list the G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the Z_2 gauge theory which contains a topological term labeled by n_1, n_{12}, and n_2. The G_s charge is a Z_2 charge which is defined modular 2. The G_s twist = 0 means that there is no branch cut, and the G_s twist = 1 means that there is a branch cut with the G_s twist. The statistics in Tables I–VIII is defined as statistics $= \theta_l/\pi$. Thus, statistics $=$ 0 corresponds to Bose statistics, statistics $= 1$ corresponds to Fermi statistics, and statistics $= \pm 1/2$ corresponds to semion statistics, etc.

The G_g gauge excitations must have trivial mutual statistics with the G_s charge and are described by $(l_I) = (0,0,l_3,l_4)$. The G_g gauge sectors describe the four types of G_g gauge excitations:

the trivial excitation $(l_3,l_4) = (0,0) \to$ "0",

the G_g charge excitation $(l_3,l_4) = (1,0) \to$ "e",

the G_g vortex excitation $(l_3,l_4) = (0,1) \to$ "m",

the G_g charge vortex excitation $(l_3,l_4) = (1,1) \to$ "em".

We know that the above eight classes of SET states are classified by

$$\begin{aligned}\mathcal{H}^3(Z_2\times Z_2,\mathbb{R}/\mathbb{Z}) &= \mathcal{H}^3(G_s = Z_2,\mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^3(G_g = Z_2,\mathbb{R}/\mathbb{Z}) \\ &\quad \oplus \mathcal{H}^2(G_s = Z_2,\mathbb{Z}_2) \\ &= \mathbb{Z}_2^3,\end{aligned} \tag{51}$$

TABLE I. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1 n_{12} n_2) = (000)$.

$(l_1l_2l_3l_4)$	G_s charge	G_s twist	G_g gauge	Statistics
(0000)	0	0	0	0
(1000)	1	0	0	0
(0010)	0	0	e	0
(1010)	1	0	e	0
(0001)	0	0	m	0
(1001)	1	0	m	0
(0011)	0	0	em	1
(1011)	1	0	em	1
(0100)	0	1	0	0
(1100)	1	1	0	1
(0110)	0	1	e	0
(1110)	1	1	e	1
(0101)	0	1	m	0
(1101)	1	1	m	1
(0111)	0	1	em	1
(1111)	1	1	em	0

TABLE II. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1 n_{12} n_2) = (010)$.

$(l_1l_2l_3l_4)$	G_s charge	G_s twist	G_g gauge	statistics
(0000)	0	0	0	0
(1000)	1	0	0	0
(0010)	0	0	e	0
(1010)	1	0	e	0
(0001)	−1/2	0	m	0
(1001)	1/2	0	m	0
(0011)	−1/2	0	em	1
(1011)	1/2	0	em	1
(0100)	0	1	0	0
(1100)	1	1	0	1
(0110)	0	1	e	0
(1110)	1	1	e	1
(0101)	−1/2	1	m	−1/2
(1101)	1/2	1	m	1/2
(0111)	−1/2	1	em	1/2
(1111)	1/2	1	em	−1/2

TABLE III. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1 n_{12} n_2) = (100)$.

$(l_1l_2l_3l_4)$	G_s charge	G_s twist	G_g gauge	Statistics
(0000)	0	0	0	0
(1000)	1	0	0	0
(0010)	0	0	e	0
(1010)	1	0	e	0
(0001)	0	0	m	0
(1001)	1	0	m	0
(0011)	0	0	em	1
(1011)	1	0	em	1
(0100)	−1/2	1	0	−1/2
(1100)	1/2	1	0	1/2
(0110)	−1/2	1	e	−1/2
(1110)	1/2	1	e	1/2
(0101)	−1/2	1	m	−1/2
(1101)	1/2	1	m	1/2
(0111)	−1/2	1	em	1/2
(1111)	1/2	1	em	−1/2

From the Tables I–VIII, we see that $\mathcal{H}^3(G_g = Z_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$ (labeled by n_2) determine if the G_g gauge theory is a Z_2 gauge theory (for $n_2 = 0$) or a double-semion theory (for $n_2 = 1$). We also see that $\mathcal{H}^3(G_s = Z_2,\mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$ (labeled by n_1) describes the G_s SPT phases, and $\mathcal{H}^2(G_s = Z_2,\mathbb{Z}_2) = \mathbb{Z}_2$ (labeled by n_{12}) determines if the G_g gauge-flux excitations can carry a $1/2$ G_s charge.

From Tables I–VIII, we see that sometimes, a $1/2$ G_s charge can and can only appear on a gauge-flux excitation with $l_4 = 1$. This implies that the symmetry of the gauge-flux excitations is described by a nontrivial PSG $= Z_4$. In all eight phases, the G_g gauge-charge excitations (the a^3_μ charges) are always bosonic and always carry integer G_s charge. In other words, the symmetry of the gauge-charge excitations is described by a trivial PSG $= G_s \times G_g = Z_2 \times Z_2$.

Next, we consider the $G = Z_4$ case. We show that, in this case, the symmetry of the gauge-charge excitations is described by a nontrivial PSG $= Z_4$ (i.e., carries a fractional G_s charge). A $G = Z_4$ gauge theory can be described by $U^2(1)$ mutual Chern-Simons theory:

$$\mathcal{L} = \frac{1}{4\pi} K_{0,IJ} a^I_\mu \partial_\nu a^J_\lambda + \cdots, \tag{52}$$

with

$$K_0 = 4\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{53}$$

TABLE IV. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1 n_{12} n_2) = (110)$.

$(l_1l_2l_3l_4)$	G_s charge	G_s twist	G_g gauge	Statistics
(0000)	0	0	0	0
(1000)	1	0	0	0
(0010)	0	0	e	0
(1010)	1	0	e	0
(0001)	−1/2	0	m	0
(1001)	1/2	0	m	0
(0011)	−1/2	0	em	1
(1011)	1/2	0	em	1
(0100)	−1/2	1	0	−1/2
(1100)	1/2	1	0	1/2
(0110)	−1/2	1	e	−1/2
(1110)	1/2	1	e	1/2
(0101)	1	1	m	1
(1101)	0	1	m	0
(0111)	1	1	em	0
(1111)	0	1	em	1

TABLE V. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1 n_{12} n_2) = (001)$.

$(l_1l_2l_3l_4)$	G_s charge	G_s twist	G_g gauge	Statistics
(0000)	0	0	0	0
(1000)	1	0	0	0
(0010)	0	0	e	0
(1010)	1	0	e	0
(0001)	0	0	m	−1/2
(1001)	1	0	m	−1/2
(0011)	0	0	em	1/2
(1011)	1	0	em	1/2
(0100)	0	1	0	0
(1100)	1	1	0	1
(0110)	0	1	e	0
(1110)	1	1	e	1
(0101)	0	1	m	−1/2
(1101)	1	1	m	1/2
(0111)	0	1	em	1/2
(1111)	1	1	em	−1/2

TABLE VI. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1 n_{12} n_2) = (011)$.

$(l_1l_2l_3l_4)$	G_s charge	G_s twist	G_g gauge	Statistics
(0000)	0	0	0	0
(1000)	1	0	0	0
(0010)	0	0	e	0
(1010)	1	0	e	0
(0001)	−1/2	0	m	−1/2
(1001)	1/2	0	m	−1/2
(0011)	−1/2	0	em	1/2
(1011)	1/2	0	em	1/2
(0100)	0	1	0	0
(1100)	1	1	0	1
(0110)	0	1	e	0
(1110)	1	1	e	1
(0101)	−1/2	1	m	1
(1101)	1/2	1	m	0
(0111)	−1/2	1	em	0
(1111)	1/2	1	em	1

TABLE VII. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1 n_{12} n_2) = (101)$.

$(l_1l_2l_3l_4)$	G_s charge	G_s twist	G_g gauge	Statistics
(0000)	0	0	0	0
(1000)	1	0	0	0
(0010)	0	0	e	0
(1010)	1	0	e	0
(0001)	0	0	m	−1/2
(1001)	1	0	m	−1/2
(0011)	0	0	em	1/2
(1011)	1	0	em	1/2
(0100)	−1/2	1	0	−1/2
(1100)	1/2	1	0	1/2
(0110)	−1/2	1	e	−1/2
(1110)	1/2	1	e	1/2
(0101)	−1/2	1	m	1
(1101)	1/2	1	m	0
(0111)	−1/2	1	em	0
(1111)	1/2	1	em	1

A unit G_g gauge charge corresponds to the unit charge of a^1_μ gauge field and a G_g gauge-flux excitation corresponds to two-unit charge of a^2_μ gauge field. Note that a unit G_g gauge charge carries 1/2 G_s charge. In other words, the symmetry of the gauge-charge excitations is described by a nontrivial PSG = Z_4. Two-unit charge of a^1_μ gauge field carries no G_g gauge charge, but a unit of G_s charge.

The four types of quantized topological terms are given by

$$W_{\text{top}} = \frac{m_1}{2\pi} a^1_\mu \partial_\nu a^1_\lambda, \tag{54}$$

$m_1 = 0,1,2,3$. The total Lagrangian has a form

$$\mathcal{L} + W_{\text{top}} = \frac{1}{4\pi} K_{IJ} a^I_\mu \partial_\nu a^J_\lambda + \cdots, \tag{55}$$

with

$$K = \begin{pmatrix} 2m_1 & 4 \\ 4 & 0 \end{pmatrix}, \quad K^{-1} = \frac{1}{8}\begin{pmatrix} 0 & 2 \\ 2 & -m_1 \end{pmatrix}. \tag{56}$$

Since moving the G_s charge (two units of a^1_μ charge) around a unit a^2_μ charge induced a phase π, a unit a^2_μ charge corresponds to the end of the branch cut in the original theory along which we have a G_s symmetry twist. However, fusing two unit a^2_μ charges gives a nontrivial G_g gauge excitation, a unit of G_g gauge flux (described by two-unit charge of a^2_μ gauge field). Therefore, a unit a^2_μ charge does not correspond to a pure G_s twist. It is a bound state of G_s twist, G_g gauge excitation, and G_s charge.

To calculate the G_s charge for a generic quasiparticle with $l_I a^I_\mu$ charge, first we assume that that the G_s charge has the following form:

$$G_s\text{charge} = \mathbf{l}^T\mathbf{q}. \tag{57}$$

The vector $\mathbf{q}$ must satisfy $(2,0)\mathbf{q} = \pm 1$ so that two units of a^1_μ charge carry a G_s charge 1. To obtain another condition on $\mathbf{q}$, we note that the trivial quasiparticles are given by $\mathbf{l} = (K_{11},K_{12}) = (2m_1,4)$ and $\mathbf{l} = (K_{21},K_{22}) = (4,0)$. So we require that $(2m_1,4)\mathbf{q} = 0$ or 2. We find that $\mathbf{q}$ has four choices:

$$\begin{aligned} &\mathbf{q}^T = (1/2, -m_1/4), \quad \mathbf{q}^T = (-1/2, m_1/4), \\ &\mathbf{q}^T = (1/2,(2-m_1)/4), \quad \mathbf{q}^T = (-1/2,(2+m_1)/4). \end{aligned} \tag{58}$$

We may choose $\mathbf{q}^T = (1/2,-m_1/4)$ and obtain Tables IX–XII, which list the G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the Z_2 gauge theory with Z_2 symmetry which contain a topological term labeled by m_1 and a mixing of the gauge G_g and symmetry G_s described by $G = Z_4$. Other choices of $\mathbf{q}$ sometimes regenerate the above four states and sometimes generate new states.

From Tables I–XII, we see the patterns of G_s charges, G_s twists, and statistics are all different, except the

TABLE VIII. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state where $(n_1 n_{12} n_2) = (111)$.

$(l_1l_2l_3l_4)$	G_s charge	G_s twist	G_g gauge	Statistics
(0000)	0	0	0	0
(1000)	1	0	0	0
(0010)	0	0	e	0
(1010)	1	0	e	0
(0001)	−1/2	0	m	−1/2
(1001)	1/2	0	m	−1/2
(0011)	−1/2	0	em	1/2
(1011)	1/2	0	em	1/2
(0100)	−1/2	1	0	−1/2
(1100)	1/2	1	0	1/2
(0110)	−1/2	1	e	−1/2
(1110)	1/2	1	e	1/2
(0101)	1	1	m	1/2
(1101)	0	1	m	−1/2
(0111)	1	1	em	−1/2
(1111)	0	1	em	1/2

TABLE IX. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state $m_1 = 0$ with $\mathbf{q}^T = (1/2,-m_1/4)$.

(l_1l_2)	G_s charge	G_s twist	G_g gauge	Statistics
(00)	0	0	0	0
(20)	1	0	0	0
(10)	1/2	0	e	0
(30)	−1/2	0	e	0
(02)	0	0	m	0
(22)	1	0	m	0
(12)	1/2	0	em	1
(32)	−1/2	0	em	1
(01)	0	1	0	0
(21)	1	1	0	1
(11)	1/2	1	e	1/2
(31)	−1/2	1	e	−1/2
(03)	0	1	m	0
(23)	1	1	m	1
(13)	1/2	1	em	−1/2
(33)	−1/2	1	em	1/2

TABLE X. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state $m_1 = 1$ with $\mathbf{q}^T = (1/2, -m_1/4)$.

$(l_1 l_2)$	G_s charge	G_s twist	G_g gauge	statistics
(00)	0	0	0	0
(20)	1	0	0	0
(10)	1/2	0	e	0
(30)	−1/2	0	e	0
(02)	−1/2	0	m	−1/2
(22)	1/2	0	m	−1/2
(12)	0	0	em	1/2
(32)	1	0	em	1/2
(01)	−1/4	1	0	−1/8
(21)	3/4	1	0	7/8
(11)	1/4	1	e	3/8
(31)	−3/4	1	e	−5/8
(03)	−3/4	1	m	7/8
(23)	1/4	1	m	−1/8
(13)	−1/4	1	em	3/8
(33)	3/4	1	em	−5/8

$(n_1 n_{12} n_2) = (010)$ state and the $m_1 = 0$ state: The two states are related by an exchange $e \leftrightarrow m$. Thus, the construction produces 11 different Z_2 gauge theories with Z_2 symmetry.

Let us examine the quasiparticles without the G_s twist. We see that six states contain quasiparticles with bosonic and fermionic statistics. Those six states are described by standard $G_g = Z_2$ gauge theory. However, the $G_s = Z_2$ symmetry is realized differently. Some states contain quasiparticles with fractional $G_s = Z_2$ charge, while others contain quasiparticles without fractional $G_s = Z_2$ charge. In some states, the fermionic quasiparticles carry fractional $G_s = Z_2$ charge while in other states, the fermionic quasiparticles carry integer $G_s = Z_2$ charge.

TABLE XI. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state $m_1 = 2$ with $\mathbf{q}^T = (1/2, -m_1/4)$.

$(l_1 l_2)$	G_s charge	G_s twist	G_g gauge	Statistics
(00)	0	0	0	0
(20)	1	0	0	0
(10)	1/2	0	e	0
(30)	−1/2	0	e	0
(02)	1	0	em	1
(22)	0	0	em	1
(12)	−1/2	0	m	0
(32)	1/2	0	m	0
(01)	−1/2	1	0	−1/4
(21)	1/2	1	0	3/4
(11)	0	1	e	1/4
(31)	1	1	e	−3/4
(03)	1/2	1	em	−1/4
(23)	−1/2	1	em	3/4
(13)	1	1	m	−3/4
(33)	0	1	m	1/4

TABLE XII. The G_s charges, the G_s twists, the G_g gauge sectors, and the statistics of the 16 kinds of quasiparticles/defects in the SET state $m_1 = 3$ with $\mathbf{q}^T = (1/2, -m_1/4)$.

$(l_1 l_2)$	G_s charge	G_s twist	G_g gauge	Statistics
(00)	0	0	0	0
(20)	1	0	0	0
(10)	1/2	0	e	0
(30)	−1/2	0	e	0
(02)	1/2	0	m	1/2
(22)	−1/2	0	m	1/2
(12)	1	0	em	−1/2
(32)	0	0	em	−1/2
(01)	−3/4	1	0	−3/8
(21)	1/4	1	0	5/8
(11)	−1/4	1	e	1/8
(31)	3/4	1	e	−7/8
(03)	−1/4	1	m	5/8
(23)	3/4	1	m	−3/8
(13)	1/4	1	em	1/8
(33)	−3/4	1	em	−7/8

The other six states contain quasiparticles with semion statistics. Those states are twisted Z_2 gauge theory, which is also known as double-semion theory.[10,24] Again some of those states have fractional $G_s = Z_2$ charge while others are without fractional $G_s = Z_2$ charge. Sometimes, the semions only carry integer $G_s = Z_2$ charges, or only fractional $G_s = Z_2$ charges, or both integer and fractional $G_s = Z_2$ charges. Those results agree with those obtained in Refs. 65 and 66.

B. Comparison with group cohomology construction

In Ref. 57, SET phases are constructed using group cohomology, generalizing the toric code to include global symmetry. The physical excitations in phases with the group extension given by $G = G_s \times G_g = \mathbb{Z}_2 \times \mathbb{Z}_2$ were also explored there, and it is of interest to compare with the results above using a K matrix.

The group cohomology $\mathcal{H}^3(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The generators of each of the $\mathbb{Z}_2$ in the cohomology group are given by

$$\omega_{11}(x,y,z) = \exp\left(\frac{\pi i}{2} x_1 (y_1 + z_1 - \overline{y_1 + z_1})\right), \quad (59)$$

$$\omega_{22}(x,y,z) = \exp\left(\frac{\pi i}{2} x_2 (y_2 + z_2 - \overline{y_2 + z_2})\right), \quad (60)$$

$$\omega_{12}(x,y,z) = \exp\left(\frac{\pi i}{2} x_1 (y_2 + z_2 - \overline{y_2 + z_2})\right), \quad (61)$$

where $x,y,z \in \mathbb{Z}_2 \times \mathbb{Z}_2$, and $x = (x_1, x_2)$, where $x_{1,2} = \{0,1\}$, and similarly for y and z. Also, $\overline{a+b} = a + b \mod 2$. Note that

$$\begin{aligned} &\mathcal{H}^3(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) \\ &= \mathcal{H}^3[\mathbb{Z}_2, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^2[\mathbb{Z}_2, \mathcal{H}^1(\mathbb{Z}_2, \mathbb{R}/\mathbb{Z})] \\ &\quad \oplus \mathcal{H}^1[\mathbb{Z}_2, \mathcal{H}^2(\mathbb{Z}_2, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^3(\mathbb{Z}_2, \mathbb{R}/\mathbb{Z})] \\ &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_2 \\ &= \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_1 \times \mathbb{Z}_2. \end{aligned} \quad (62)$$

A phase is then characterized by three-cocycles of the form

$$\Omega(x,y,z) = \omega_{11}^{n_1}(x,y,z)\omega_{22}^{n_2}(x,y,z)\omega_{12}^{n_{12}}(x,y,z), \quad (63)$$

where $n_{1,12,2} = \{0,1\}$, and they can be precisely identified with the n_1, n_{12}, n_2 in Eq. (42). This can be easily checked by computing the modular S matrix from the group cycles and comparing with the matrix of mutual statistics obtained from the K matrix. More explicitly, using the methods detailed in Refs. 61, 67, and 68, the modular S matrix evaluated on the cocycle $\Omega(x,y,z)$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ lattice gauge theory is given by

$$S_{(g,\alpha)(h,\beta)}(n_1,n_{12},n_2) = \frac{1}{4}\exp\left(-\pi i\left(\left[\sum_i^2 \alpha_i h_i + \beta_i g_i\right] + n_1 g_1 h_1 + n_2 g_2 h_2 + \frac{n_{12}}{2}(g_1 h_2 + h_1 g_2)\right)\right), \quad (64)$$

where g,h,α,β are all two-component vectors whose components each taking values $\in\{0,1\}$. Here $g,h \in \mathbb{Z}_2 \times \mathbb{Z}_2$ are the flux excitations and α,β denote irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which correspond to charge excitations. The phase factor appearing in the modular matrix is related to the mutual statistics obtained in Eq. (44). It is clear that the phase factor indeed takes the form of Eq. (44) if we interpret $(\alpha_1,g_1,\alpha_2,g_2)$ and $(\beta_1,h_1,\beta_2,h_2)$ as our charge vectors l,l', respectively:[6,69]

$$S_{l,l'}(n_1,n_{12},n_2) = \tfrac{1}{4}\exp(-2\pi i l^T K^{-1} l'). \quad (65)$$

We can thus immediately read off the inverse of the K matrix from Eq. (64) to be

$$K^{-1} = \frac{1}{4}\begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 2n_1 & 0 & n_{12} \\ 0 & 0 & 0 & 2 \\ 0 & n_{12} & 2 & 2n_2 \end{pmatrix}, \quad (66)$$

which, up to a convention for the sign of n_1, n_{12}, n_3, is precisely Eq. (42).

In Ref. 57 the G_s charges of both flux and charge excitations of the gauge group G_g are computed, by explicitly constructing the G_s symmetry transformation operator and the (pair) creation operators (i.e., ribbon operators) of the excitations. In the language of the K-matrix construction, the gauge charge and flux excitations correspond to charges of a_3 and a_4, respectively; i.e., violation of vanishing flux in a plaquette corresponds to a_4 charges, and the a_3 charges correspond to the product of gauge variables along the ribbon connecting the pair of excitations at the end points of the ribbon. G_s charge fluctuations are also possible in the cocycle model, but it does not contain G_s-flux excitation by construction there. An a_2 charge would correspond to a field configuration in Ref. 57, which does not return to its original value after traversing a loop. Therefore we can compare the G_s charges of excitations with those in Ref. 57 when $l_2 = 0$.

Let us elaborate further on the conversion of gauge charges between the two descriptions. In Ref. 57 excited states with a pair of quasiparticle excitations are specified by $|h,h_g,\tilde{g},u_A\rangle$, where $h,h_g \in G_g$, $\tilde{g},u_A \in G_s$, and u_A corresponds to the field configuration at one of the two quasiparticle sites A,B connected by the ribbon operator. It satisfies the constraint $u_A u_B^{-1} = \tilde{g}$. Flux excitations are given by h, whereas charge fluctuations are given by h_g, and G_s charges are given by a mixture of $\tilde{g},u_A$. The charge fluctuations are, however, expressed in a different basis compared to the K-matrix description. To convert to the K matrix description, we again have to do the transformation (suppose we focus on the quasiparticle located at the end B, and fixing u_A at the other end)

$$|h,\alpha_g,\beta_s,u_A\rangle = \frac{1}{|G_g \times G_s|}\sum_{h_g,\tilde{g}} \rho_{\alpha_g}(h_g)\rho_{\beta_s}(\tilde{g})|h,h_g,\tilde{g},u_A\rangle, \quad (67)$$

where $\rho_{\alpha_g}(g)$ corresponds to characters of representations of $G_g = \mathbb{Z}_2$, and $\rho_{\beta_s}(\tilde{g})$ corresponds to that of $G_s = \mathbb{Z}_2$.[70] One can check that in terms of the diagonalized basis vectors of the G_s transformation as specified in Table II in Ref. 57, the G_s charge matches up with the result obtained in the K-matrix formulation given above.

The most important observation is that it is found in Ref. 57 (see Table II there) that only in the case where n_{12} and l_4 (i.e., flux charge $h = 1$ there) are *both* nonvanishing that charge fractionalization occurs. In fact, the G_s transformation U for the flux charge squares to -1, which is indeed the statement that the G_s charge is halved. This is in perfect agreement with the results in the previous section [see Eq. (50) or Tables I–VIII].

We note also that since the modular S matrix descending from the 3-cocycles agrees with that of the K matrix, the braiding statistics in Ref. 57 have to agree with that obtained using the K matrix when we turn off l_2 accordingly.

VI. SUMMARY

In this paper, we studied the quantized topological terms in a weak-coupling gauge theory with gauge group G_g and a global symmetry G_s in d-dimensional space-time. We showed that the quantized topological terms are classified by a pair (G,ν_d), where G is an extension of G_s by G_g and ν_d is an element in group cohomology $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. When $d = 3$ and/or when G_g is finite, the weak-coupling gauge theories with quantized topological terms describe gapped SET phases. Thus, those SET phases are classified by $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, where $G/G_g = G_s$. This result generalized the PSG description of the SET phases.[50,51,54,55] It also generalized the recent results in Refs. 53 and 57. We also apply our theory to a simple case $G_s = G_g = Z_2$ to understand the physical meanings of the $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ classification. Roughly, for the trivial extension $G = G_s \times G_g$, $\mathcal{H}^d(G_g \times G_s,\mathbb{R}/\mathbb{Z})$ describes different ways in which the quantum number of G_s becomes fractionalized on gauge-flux excitations, while the nontrivial extensions G describe different ways in which the quantum number of G_s become fractionalized on gauge-charge excitations.

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APPENDIX A: CALCULATING $H^*(X,\mathbb{R}/\mathbb{Z})$ FROM $H^*(X,\mathbb{Z})$

We can use the Künneth formula (see Ref. 71, page 247),

$$H^d(X\times X',M\otimes_R M') \simeq \left[\oplus_{p=0}^{d} H^p(X,M)\otimes_R H^{d-p}(X',M')\right] \oplus \left[\oplus_{p=0}^{d+1}\mathrm{Tor}_1^R(H^p(X,M),H^{d-p+1}(X',M'))\right], \quad (A1)$$

to calculate $H^*(X,M)$ from $H^*(X,Z)$. Here R is a principal ideal domain and M,M' are R modules such that $\mathrm{Tor}_1^R(M,M')=0$. Note that $\mathbb{Z}$ and $\mathbb{R}$ are principal ideal domains, while $\mathbb{R}/\mathbb{Z}$ is not. A R module is like a vector space over R (i.e., we can "multiply" a vector by an element of R.) For more details on principal ideal domain and R module, see the corresponding Wiki articles.

The tensor-product operation $\otimes_R$ and the torsion-product operation Tor_1^R have the following properties:

$$\begin{aligned} A\otimes_{\mathbb{Z}} B &\simeq B\otimes_{\mathbb{Z}} A,\\ \mathbb{Z}\otimes_{\mathbb{Z}} M &\simeq M\otimes_{\mathbb{Z}}\mathbb{Z}=M,\\ \mathbb{Z}_n\otimes_{\mathbb{Z}} M &\simeq M\otimes_{\mathbb{Z}}\mathbb{Z}_n=M/nM,\\ \mathbb{Z}_m\otimes_{\mathbb{Z}}\mathbb{Z}_n &= \mathbb{Z}_{(m,n)},\\ (A\oplus B)\otimes_R M &= (A\otimes_R M)\oplus(B\otimes_R M),\\ M\otimes_R(A\oplus B) &= (M\otimes_R A)\oplus(M\otimes_R B), \end{aligned} \quad (A2)$$

and

$$\begin{aligned} \mathrm{Tor}_1^R(A,B) &\simeq \mathrm{Tor}_1^R(B,A),\\ \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},M) &= \mathrm{Tor}_1^{\mathbb{Z}}(M,\mathbb{Z})=0,\\ \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_n,M) &= \{m\in M|nm=0\},\\ \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_m,\mathbb{Z}_n) &= \mathbb{Z}_{(m,n)},\\ \mathrm{Tor}_1^R(A\oplus B,M) &= \mathrm{Tor}_1^R(A,M)\oplus\mathrm{Tor}_1^R(B,M),\\ \mathrm{Tor}_1^R(M,A\oplus B) &= \mathrm{Tor}_1^R(M,A)\oplus\mathrm{Tor}_1^R(M,B), \end{aligned} \quad (A3)$$

where (m,n) is the greatest common divisor of m and n. These expressions allow us to compute the tensor-product $\otimes_R$ and the torsion-product Tor_1^R.

If we choose $R=M=\mathbb{Z}$, then the condition $\mathrm{Tor}_1^R(M,M')=\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},M')=0$ is always satisfied. So we have

$$H^d(X\times X',M') \simeq \left[\oplus_{p=0}^{d} H^p(X,\mathbb{Z})\otimes_{\mathbb{Z}} H^{d-p}(X',M')\right] \oplus \left[\oplus_{p=0}^{d+1}\mathrm{Tor}_1^{\mathbb{Z}}(H^p(X,\mathbb{Z}),H^{d-p+1}(X',M'))\right]. \quad (A4)$$

Now we can further choose X' to be the space of one point and use

$$H^d(X',M')) = \begin{cases} M', & \text{if} \quad d=0,\\ 0, & \text{if} \quad d>0, \end{cases} \quad (A5)$$

to reduce Eq. (A4) to

$$H^d(X,M)\simeq H^d(X,\mathbb{Z})\otimes_{\mathbb{Z}} M\oplus \mathrm{Tor}_1^{\mathbb{Z}}(H^{d+1}(X,\mathbb{Z}),M), \quad (A6)$$

where M' is renamed as M. The above is a form of the universal coefficient theorem which can be used to calculate $H^*(BG,M)$ from $H^*(BG,\mathbb{Z})$ and the module M.

Now, let us choose $M=\mathbb{R}/\mathbb{Z}$ and compute $H^d(BG,\mathbb{R}/\mathbb{Z})$ from $H^d(BG,\mathbb{Z})$. Note that $H^d(BG,\mathbb{Z})$ has a form $H^d(BG,\mathbb{Z})=\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}\oplus Z_{n_1}\oplus Z_{n_2}\oplus\cdots$. A $\mathbb{Z}$ in $H^d(BG,\mathbb{Z})$ will produce a $\mathbb{R}/\mathbb{Z}$ in $H^d(BG,\mathbb{R}/\mathbb{Z})$ since $\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}=\mathbb{R}/\mathbb{Z}$. A $\mathbb{Z}_n$ in $H^{d+1}(BG,\mathbb{Z})$ will produce a $\mathbb{Z}_n$ in $H^d(BG,\mathbb{R}/\mathbb{Z})$ since $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_n,\mathbb{R}/\mathbb{Z})=\mathbb{Z}_n$. So we see that $H^d(BG,\mathbb{R}/\mathbb{Z})$ has a form $H^d(BG,\mathbb{R}/\mathbb{Z})=\mathbb{R}/\mathbb{Z}\oplus\cdots\oplus\mathbb{R}/\mathbb{Z}\oplus Z_{n_1}\oplus Z_{n_2}\oplus\cdots$ and

$$\mathrm{Dis}[H^d(X,\mathbb{R}/\mathbb{Z})]\simeq \mathrm{Tor}[H^{d+1}(X,\mathbb{Z})], \quad (A7)$$

where $\mathrm{Dis}[H^d(X,\mathbb{R}/\mathbb{Z})]$ is the discrete part of $H^d(X,\mathbb{R}/\mathbb{Z})$.

If we choose $M=\mathbb{R}$, we find that

$$H^d(X,\mathbb{R})\simeq H^d(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}. \quad (A8)$$

So $H^d(X,\mathbb{R})$ has the form $\mathbb{R}\oplus\cdots\oplus\mathbb{R}$ and each $\mathbb{Z}$ in $H^d(X,\mathbb{Z})$ gives rise to a $\mathbb{R}$ in $H^d(X,\mathbb{R})$. Since $H^d(BG,\mathbb{R})=0$ for $d=$ odd, we have

$$H^d(BG,\mathbb{Z})=\mathrm{Tor}[H^d(BG,\mathbb{Z})], \quad \text{for} \quad d=\text{odd}. \quad (A9)$$

Using the Künneth formula Eq. (A4) we can also rewrite $H^d(G_s\times G_g,\mathbb{R}/\mathbb{Z})$ as

$$\begin{aligned} &\mathcal{H}^d(G_s\times G_g,\mathbb{R}/\mathbb{Z})\\ &= \mathcal{H}^{d+1}(G_s\times G_g,\mathbb{Z})\\ &= \left[\oplus_{p=0}^{d+1}\mathcal{H}^p(G_s,\mathbb{Z})\otimes_{\mathbb{Z}}\mathcal{H}^{d+1-p}(G,\mathbb{Z})\right]\\ &\quad\oplus\left[\oplus_{p=0}^{d+2}\mathrm{Tor}_1^{\mathbb{Z}}[\mathcal{H}^p(G_s,\mathbb{Z}),\mathcal{H}^{d-p+2}(G,\mathbb{Z})]\right]\\ &= \mathcal{H}^d(G_s,\mathbb{R}/\mathbb{Z})\oplus\mathcal{H}^d(G_g,\mathbb{R}/\mathbb{Z})\\ &\quad\oplus\left[\oplus_{p=1}^{d-1}\mathcal{H}^{d-p}(G_s,\mathbb{Z})\otimes_{\mathbb{Z}}\mathcal{H}^p(G_g,\mathbb{R}/\mathbb{Z})\right]\\ &\quad\oplus\left[\oplus_{p=1}^{d-1}\mathrm{Tor}_1^{\mathbb{Z}}[\mathcal{H}^{d-p+1}(G_s,\mathbb{Z}),\mathcal{H}^p(G_g,\mathbb{R}/\mathbb{Z})]\right]\\ &= \mathcal{H}^d(G_s,\mathbb{R}/\mathbb{Z})\oplus\mathcal{H}^d(G_g,\mathbb{R}/\mathbb{Z})\\ &\quad\oplus\left[\oplus_{p=1}^{d-1}\mathcal{H}^{d-p}[G_s,\mathcal{H}^p(G_g,\mathbb{R}/\mathbb{Z})]\right]\\ &= \oplus_{p=0}^{d}\mathcal{H}^{d-p}[G_s,\mathcal{H}^p(G_g,\mathbb{R}/\mathbb{Z})], \end{aligned} \quad (A10)$$

where we have used $\mathcal{H}^n(G,\mathbb{R}/\mathbb{Z})=\mathcal{H}^{n+1}(G,\mathbb{Z})$ for $n>0$, and $\mathcal{H}^1(G,\mathbb{Z})=0$ for compact or finite group G. We also used the universal coefficient theorem (A6)

$$\begin{aligned} &\mathcal{H}^{d-p}[G_s,\mathcal{H}^p(G_g,\mathbb{R}/\mathbb{Z})]\\ &= \mathcal{H}^{d-p}(G_s,\mathbb{Z})\otimes_{\mathbb{Z}}\mathcal{H}^p(G_g,\mathbb{R}/\mathbb{Z})\\ &\quad\oplus\mathrm{Tor}_1^{\mathbb{Z}}[\mathcal{H}^{d-p+1}(G_s,\mathbb{Z}),\mathcal{H}^p(G_g,\mathbb{R}/\mathbb{Z})]. \end{aligned} \quad (A11)$$

APPENDIX B: A LABELING SCHEME OF SET STATES DESCRIBED BY WEAK-COUPLING GAUGE THEORY

The Lyndon-Hochschild-Serre spectral sequence $\mathcal{H}^x[G_s,\mathcal{H}^y(G_g,\mathbb{R}/\mathbb{Z})]\Rightarrow\mathcal{H}^{x+y}(G,\mathbb{R}/\mathbb{Z})$ may help us to calculate the group cohomology $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ in terms of $\mathcal{H}^x[G_s,\mathcal{H}^y(G_g,\mathbb{R}/\mathbb{Z})]$. We find that $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ contains a chain of subgroups,

$$\{0\}=H^{d+1}\subset H^d\subset\cdots\subset H^1\subset H^0=\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z}), \quad (B1)$$

such that H^k/H^{k+1} is a subgroup of a factor group of $\mathcal{H}^k[G_s,\mathcal{H}^{d-k}(G_g,\mathbb{R}/\mathbb{Z})]$,

$$H^k/H^{k+1} \subset \mathcal{H}^k[G_s,\mathcal{H}^{d-k}(G_g,\mathbb{R}/\mathbb{Z})]/\mathcal{H}^k, \quad k=0,\dots,d, \tag{B2}$$

where $\mathcal{H}^k$ is a subgroup of $\mathcal{H}^k[G_s,\mathcal{H}^{d-k}(G_g,\mathbb{R}/\mathbb{Z})]$. Note that G_s has a nontrivial action on $\mathcal{H}^{d-k}(G_g,\mathbb{R}/\mathbb{Z})$ as determined by the structure $G_s = G/G_g$. We also have

$$\begin{aligned} H^0/H^1 &\subset \mathcal{H}^0[G_s,\mathcal{H}^d(G_g,\mathbb{R}/\mathbb{Z})], \\ H^d/H^{d+1} &= H^d = \mathcal{H}^d(G_s,\mathbb{R}/\mathbb{Z})/\mathcal{H}^d. \end{aligned} \tag{B3}$$

In other words, the elements in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ can be one-to-one labeled by $(x_0,x_1,\dots,x_d)$ with

$$x_k \in H^k/H^{k+1} \subset \mathcal{H}^k[G_s,\mathcal{H}^{d-k}(G_g,\mathbb{R}/\mathbb{Z})]/\mathcal{H}^k. \tag{B4}$$

If we want to use $(y_0,y_1,\dots,y_d)$ with

$$y_k \in \mathcal{H}^k[G_s,\mathcal{H}^{d-k}(G_g,\mathbb{R}/\mathbb{Z})] \tag{B5}$$

to label the elements in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, then such a labeling may not be one-to-one and it may happen that only some of $(y_0,y_1,\dots,y_d)$ correspond to the elements in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$. However, for every element in $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$, we can find a $(y_0,y_1,\dots,y_d)$ that corresponds to it.

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