Transverse force on a vortex and vortex mass: Effects of free bulk and vortex-core bound quasiparticles

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The paper reassesses the old but still controversial problem of the transverse force on a vortex and the vortex mass. The transverse force from free bulk quasiparticles on the vortex, both in the Bose and the Fermi liquids, originates from the Aharonov-Bohm effect. However, in the Fermi liquid, one should take into account peculiarities of the Aharonov-Bohm effect for BCS quasiparticles described by two-component spinor wave functions. There is no connection between the transverse force (either from free bulk quasiparticles or from vortex-core bound quasiparticles) and the spectral flow in the vortex core in superfluid Fermi liquid, in contrast to widely known claims. In fact, there is no steady spectral flow in the core of the moving vortex, and the analogy with the Andreev bound states in the superconductor–normal-metal–superconductor junction, where the spectral flow is really possible, is not valid in this respect. The role of the backflow on the vortex mass is clarified. The backflow is an inevitable consequence of a mismatch between the currents inside and outside the vortex core and restores the conservation of the particle number (charge) violated by this mismatch. In the Fermi liquid, the backflow compensates the current through the core bound states, which is a source of the vortex mass (the Kopnin mass). This results in renormalization of the Kopnin vortex mass by a numerical factor.

I. INTRODUCTION

Discussions and debates on the transverse force on a vortex in superfluids (neutral and charged) continue during many decades and have been a topic of reviews and books.[1–6] They focused on quasiparticle contributions to this force, which are connected with geometrical (Aharonov-Bohm-Berry) phases in the superfluid around the vortex. Although in most practical cases the vortex can be considered as a massless object governed by the gyroscopic dynamics, the concept and the magnitude of the vortex mass was also vividly discussed in the Bose and the Fermi superfluids.[2,6–15]

Despite a huge literature on this subject, there still remain some issues, which require further clarification, especially for the Fermi superfluids. This paper addresses these issues. In particular, the paper rederives and discusses the origin of the transverse force from core states (the Kopnin-Kravtsov force) and the part of the transverse force from the free quasiparticles in the bulk of the Fermi superfluid, which seemed not to follow from the simple semiclassical approach based on the Aharonov-Bohm effect. Volovik[16] suggested that these transverse forces originate from the spectral flow in the vortex core (see also Chap. 25 in his book[6]). This interpretation was widely accepted[2,3] and was a basis for the claim that the spectral flow in the vortex core presumably revealed in mutual-friction measurements experimentally models the cosmological baryogenesis in the early Universe.[17]

The spectral flow concept is known both in mathematics[18] and physics. According to its mathematical definition, the spectral flow is a number of eigenstates of an operator with eigenvalues passing zero value at tuning of some parameter, on which the operator (and correspondingly its eigenstates) depends. A physical example of the spectral flow is the flow of the Andreev bound states in the ballistic superconductor–normal-metal–superconductor (SNS) junction.[19] The energy of the Andreev state linearly depends on the superfluid phase

difference between the superconductors forming the junction. When the phase difference monotonously varies in time (the ac Josephson effect), the discrete energy levels cross the whole superconducting gap passing the zero value of the energy. So, in this example the parameter governing the spectral flow is the phase difference and the operator corresponds to the Bogoliubov–de Gennes equations, which determine the Andreev bound states inside the gap. As was discovered long ago,[20,21] the Andreev bound states exist also in cores of vortices in Fermi superfluids, and Volovik argued that the process of vortex motion is accompanied by a steady shift of core bound-state levels from negative-energy continuum to the positive-energy continuum, i.e., by the spectral flow across the superconducting gap similar to that in the SNS junction. Any crossing of the gap by a bound state leads to transfer of the momentum, which leads to the transverse Kopnin-Kravtsov force. So momentum transfer from the vortex moving with the relative velocity $\boldsymbol{v}_L - \boldsymbol{v}_n$ with respect to the normal component (or to impurities in superconductors) is realized not simply via jumps of particles between energy levels caused by collisions, but via motion of energy levels themselves in the energy space.

This paper argues that the spectral flow can not be responsible for any part of the transverse force simply because it is absent in a core of a moving vortex, as already was noticed by Stone[19] in the past. On the other hand, all kinds of transverse forces can be understood within common approaches such as the scattering theory and the partial-wave expansion without any reference to the spectral flow. In particular, the part of the transverse force from scattering of free bulk quasiparticles in the Fermi superfluid, which was presumed to originate from the spectral flow, directly follows from peculiarities of the Aharonov-Bohm effect for BCS quasiparticles described by two-component spinor wave functions. Such conclusions led to a necessity to reassess Volovik's arguments in favor of

the spectral flow and to analyze why the analogy with the SNS junction, where the spectral flow definitely exists, is not applicable in this respect.

Addressing the vortex mass, this paper revises different contributions to it, compares them, and discusses possible effects of the vortex mass on vortex motion. In particular, the paper analyzes the so-called backflow vortex mass. The backflow mass is related with the kinetic energy of a superflow around the vortex core, which inevitably appears any time when the current density inside the core differs from that outside the core, and the intensity of the backflow is determined from the continuity of the total fluid current.

Let us present the nomenclature of various forces, which enter the equation of vortex motion:

$$mn_s\kappa[\hat{z} \times (\boldsymbol{v}_L - \boldsymbol{v}_s)] = \boldsymbol{F}_n + \boldsymbol{F}_c + \frac{d\boldsymbol{P}}{dt}. \quad (1)$$

The left-hand side is the Magnus force, which transfers momentum between the superfluid and the vortex. Here $\boldsymbol{v}_L$ and $\boldsymbol{v}_s$ are the vortex and the superfluid velocities, m is the particle mass, n_s is the superfluid density, and $\kappa = h/m$ is the circulation quantum. The force

$$\boldsymbol{F}_n = -D(\boldsymbol{v}_L - \boldsymbol{v}_n) - D'[\hat{z} \times (\boldsymbol{v}_L - \boldsymbol{v}_n)] \quad (2)$$

transfers momentum between the normal component (the gas of free quasiparticles) and the vortex. Here $\boldsymbol{v}_n$ is the normal velocity. The coefficients D and D',

$$D = \frac{1}{3h^3} \int \frac{\partial f_0(\varepsilon)}{\partial \epsilon} p^2 \sigma_\perp v_G \, d_3 \boldsymbol{p},$$
$$D' = \frac{1}{3h^3} \int \frac{\partial f_0(\varepsilon)}{\partial \epsilon} p^2 \sigma_\perp v_G \, d_3 \boldsymbol{p} \quad (3)$$

are determined by the longitudinal (transport) and the transverse cross sections $\sigma_\parallel$ and $\sigma_\perp$, which will be determined further in the paper. Here $f_0(\epsilon)$ is the equilibrium Fermi distribution function of energy ϵ of free quasiparticles, and v_G is the projection of the quasiparticle group velocity on the plane normal to the vortex line. The transverse force proportional to D' is the Iordanskii force.

The force $\boldsymbol{F}_c$ transfers momentum from the quasiparticles occupying bound states in the vortex core to impurities in superconductors or to free bulk quasiparticles constituting the bulk normal component of the ^{3}He superfluid. The force has also two components, longitudinal and transverse to the relative normal velocity $\boldsymbol{v}_n - \boldsymbol{v}_L$, the latter called the Kopnin-Kravtsov force.[22]

Finally, $d\boldsymbol{P}/dt$ is the inertial force, which is a product of the vortex mass and the vortex acceleration $d\boldsymbol{v}_L/dt$, the momentum $\boldsymbol{P}$ being the momentum of the vortex dependent on $\boldsymbol{v}_L$.

The theory presented in this paper assumes that the quasiparticle mean-free path is much longer than the core size and therefore it can not be used for high temperatures. The whole paper addresses neutral superfluids, although the results for the Fermi superfluids are relevant also for type-II s-wave superconductors since the effects of magnetic fields usually are not essential for vortex dynamics.[2] In superconductors the normal velocity $\boldsymbol{v}_n$ usually vanishes in the coordinate frame related to the crystal lattice.

The paper starts from Sec. II reminding the old results for semiclassical scattering of quasiparticles by a vortex. This shows the connection of the transverse force with the Aharonov-Bohm effect for quasiparticles. Section III considers the scattering of BCS quasiparticles on the basis of the Bogoliubov–de Gennes equations. The analysis is done using the geometric optics and the partial-wave method. It demonstrates that the whole transverse force from free bulk quasiparticles is fully explained by the Aharonov-Bohm effect without referring to the concept of spectral flow. But one must take into account the peculiarities of the Aharonov-Bohm effect for BCS quasiparticles described by two-component spinor wave functions. Section IV reminds properties of bound states in the vortex core in the Fermi superfluid focusing on the role of superfluid motion outside the core. Sections V and VI consider various contributions to the vortex mass in the Bose and the Fermi liquids, respectively. Section VII discusses the derivation of the transverse force and the vortex mass from the Boltzmann equation focusing on the effect of superfluid transport past the vortex and on the comparison with the analysis of the previous sections. Section VIII analyzes possible effects of the vortex mass on vortex dynamics. A concluding discussion of the results and the shortcomings of the spectral flow interpretation of the transverse force is presented in Sec. IX. Two appendixes address more special issues: the simplified derivation of the spectrum of bound states for a core with linear growth of the gap as a function of the distance from the axis (Appendix A) and the derivation of the vortex mass for a core with linear growth of density in the Bose superfluid (Appendix B).

II. TRANSVERSE FORCE FROM THE SEMICLASSICAL SCATTERING THEORY (GEOMETRIC OPTICS)

It is useful to start from the simplest approach to this problem based on the semiclassical scattering theory, which was used for rotons by Lifshitz and Pitaevskii[23] long ago. The theory is based on the geometric optics. A quasiparticle moves along a well-defined trajectory and its motion is described by variation of the position vector $\boldsymbol{R}$ and the momentum $\boldsymbol{p}$ of the quasiparticle in time. The classical Hamilton equations for these are

$$\frac{d\boldsymbol{R}}{dt} = \frac{\partial \epsilon}{\partial \boldsymbol{p}}, \quad \frac{d\boldsymbol{p}}{dt} = -\frac{\partial \epsilon}{\partial \boldsymbol{R}}. \quad (4)$$

Here

$$\epsilon(\boldsymbol{p}) = \epsilon_0(\boldsymbol{p}) + \boldsymbol{p} \cdot \boldsymbol{v}_v \quad (5)$$

is the energy of the quasiparticle in the moving fluid, ϵ_0 is the quasiparticle energy in the resting fluid, and $\boldsymbol{v}_v$ is the velocity induced by a rectilinear vortex:

$$\boldsymbol{v}_v = \frac{[\kappa \times \boldsymbol{r}]}{2\pi r^2}, \quad (6)$$

where $\boldsymbol{r}$ is a position vector in the plane normal to the vortex line (the projection of $\boldsymbol{R}$ on that plane). In order to simplify discussion, we assume that the quasiparticle moves in the normal plane, so its momentum $\boldsymbol{p}$ lies in this plane.

The vortex velocity field produces a force $\nabla(\boldsymbol{p} \cdot \boldsymbol{v}_v)$ on the quasiparticle. The force may be considered as weak
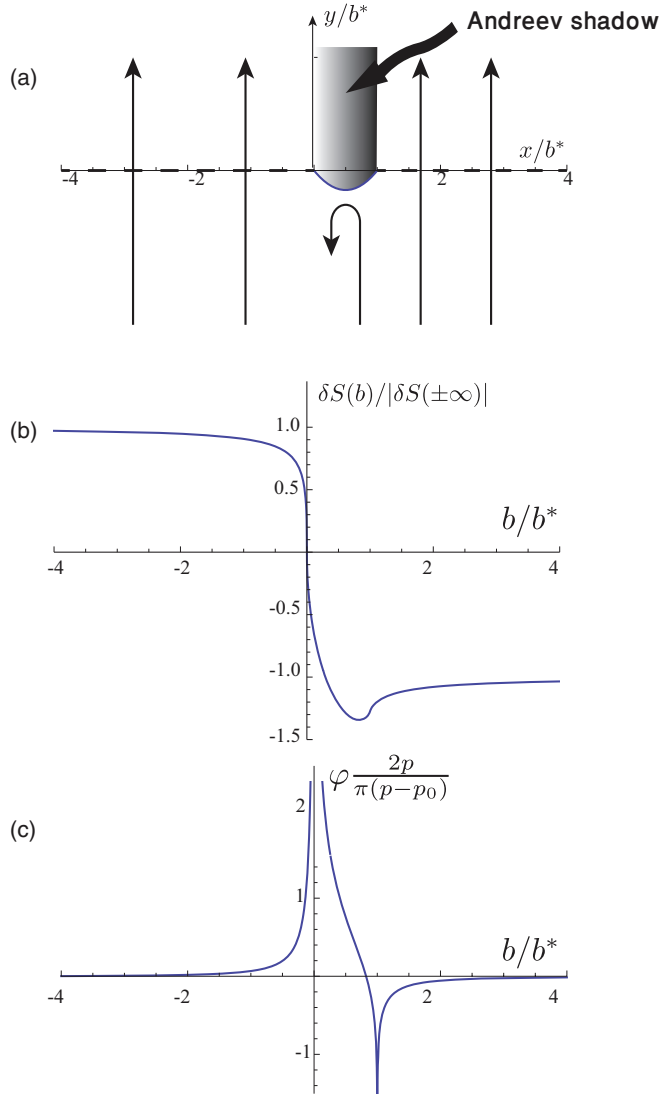
FIG. 1. (Color online) Semiclassical scattering of rotons. (a) Rotons at trajectories with impact parameters $b = x < 0$ or $b > b^*$ move past the vortex. Rotons at trajectories with $0 < b < b^*$ are fully reflected by the vortex. The shaded area (Andreev shadow) is classically forbidden for rotons. (b) Action variation $\delta S(b)$ along the trajectory as a function of the impact parameter b (dimensionless variables). (c) Scattering angle $\varphi(b)$ as a function of the impact parameter b (dimensionless variables).

and the quasiparticle trajectory as nearly rectilinear. Suppose that the trajectory is parallel to the y axis [Fig. 1(a)] and its impact parameter (the distance between the vortex line and the trajectory) is $b = x$. Then Eq. (4) gives

$$\frac{dy}{dt} = v_G, \quad \frac{d\boldsymbol{p}}{dt} = -\boldsymbol{\nabla}(\boldsymbol{p} \cdot \boldsymbol{v}_v). \quad (7)$$

Here $\boldsymbol{v}_G = \partial \epsilon_0(\boldsymbol{p})/\partial \boldsymbol{p}$ is the quasiparticle group velocity in the resting fluid, which is in our case approximately parallel to the axis y. Excluding time from these equations, one has a differential equation determining the quasiparticle momentum variation along the trajectory:

$$\frac{d\boldsymbol{p}}{dy} = -\frac{1}{v_G}\boldsymbol{\nabla}(\boldsymbol{p} \cdot \boldsymbol{v}_v). \quad (8)$$

Integration of this equation assuming that the group velocity v_G does not vary along the trajectory yields

$$\boldsymbol{p}(y) = \boldsymbol{p} - \frac{p}{v_G}\boldsymbol{v}_v(b, y), \quad (9)$$

where $\boldsymbol{p} = \boldsymbol{p}(-\infty)$ is the momentum at $y = -\infty$.

The scattering angle φ between the final and the initial momenta of the quasiparticle determines the momenta $p(1 - \cos\varphi)$ and $p\sin\varphi$, which are longitudinal and transverse with the respect to the incident momentum $\boldsymbol{p}$. The momenta are transferred by the scattered quasiparticle to the vortex. Correspondingly, the longitudinal and the transverse forces on the vortex from quasiparticles [see Eqs. (2) and (3)] are determined by the longitudinal (transport)

$$\sigma_\parallel = \int_{-\pi}^{\pi} \sigma(\varphi)(1 - \cos\varphi)d\varphi \approx \int_{-\infty}^{\infty} \frac{\varphi(b)^2}{2}db, \quad (10)$$

and the transverse

$$\sigma_\perp = \int_{-\pi}^{\pi} \sigma(\varphi)\sin\varphi \, d\varphi \approx \int_{-\infty}^{\infty} \varphi(b)db \quad (11)$$

effective cross sections. Here

$$\sigma(\varphi) = \frac{db}{d\varphi} \quad (12)$$

is the differential cross section. In our analysis we assume that the scattering angle $\varphi \approx -p_x/p$ is small.

In the Hamilton-Jacobi theory, the momentum is connected with the classical action: $\boldsymbol{p} = \partial S/\partial \boldsymbol{r}$. Then $p_x = \partial \delta S(b)/\partial b$, where

$$\delta S(b) = \int_{-\infty}^{\infty} [p(y) - p]dy = -\frac{p}{v_G}\int_{-\infty}^{\infty} \nabla_y v_{vy}dy \quad (13)$$

is the variation of the classical action along the trajectory, which is a function of the impact parameter b. This yields

$$\sigma_\perp = -\frac{1}{p}\int_{-\infty}^{\infty} \frac{\partial \delta S(b)}{\partial b}db = \frac{\delta S(-\infty) - \delta S(+\infty)}{p}. \quad (14)$$

Bearing in mind that the velocity induced by the vortex is $\boldsymbol{v}_v = (\kappa/2\pi)\boldsymbol{\nabla}\phi(\boldsymbol{r})$ where the phase $\phi = \arctan(y/x)$ is the azimuthal angle for the two-dimensional position vector $\boldsymbol{r}$ [see Eq. (6)], one obtains that

$$\delta S(b) = -\frac{p\kappa}{2\pi v_G}\int_{-\infty}^{\infty} \frac{b}{b^2 + y^2}dy = -\text{sign}b\frac{p\kappa}{2v_G}. \quad (15)$$

Eventually, Eq. (14) yields the transverse cross section[24]

$$\sigma_\perp = \frac{\kappa}{v_G}. \quad (16)$$

So, we have obtained for the transverse cross section a universal expression, which looks valid for any quasiparticle spectrum. The cross section is proportional to the total variation of the classical action around the vortex line. Because of correspondence of the classical action to the quantum mechanical phase, this points out connection of the transverse force with the geometric phase, or the Aharonov-Bohm effect.[24] Eqnarray (16) yields a correct transverse cross section for phonons even though the semiclassical theory is not valid for phonons: there are no well-defined classical trajectories for phonons except for large impact parameters b at which the

scattering angle φ is negligible. In the case of phonons, the group velocity v_G is the sound velocity c_s.

The simple expression (14) for the transverse cross section does not depend on how the scattering angle varies as a function of the impact parameter because an integrand in Eq. (14) is a derivative of the action. But it does assume that the action is a continuous function of the impact parameter. Now we shall check it for rotons in superfluid ^{4}He.

The energy spectrum for rotons is $\epsilon_0(\boldsymbol{p}) = \Delta + (p - p_0)^2/2\mu$, where Δ is the roton gap and μ is the roton mass. According to the energy conservation law following from the Hamilton equations [Eq. (4)], one has

$$\Delta + \frac{[p(y) - p_0]^2}{2\mu} + \boldsymbol{p}(y)\boldsymbol{v}_v(y) = \Delta + \frac{(p - p_0)^2}{2\mu}. \quad (17)$$

The right-hand side is the energy far from the vortex line, where $p = p(-\infty)$. The variation of the roton group velocity along the trajectory with the impact parameter b is given by

$$v_G(y) = \frac{p(y) - p_0}{\mu} = \frac{1}{\mu}\sqrt{(p - p_0)^2 - 2\mu v_{vy}}$$
$$= v_G\sqrt{1 - \frac{bb^*}{b^2 + y^2}}. \quad (18)$$

Here the characteristic scattering length

$$b^* = \frac{\kappa\mu p}{\pi(p - p_0)^2} \quad (19)$$

is introduced and the $v_G = v_G(-\infty) = (p - p_0)/\mu$ is the roton group velocity far from the vortex line. The asymptotic expression (9) obtained at constant v_G is valid for large impact parameters $|b| \gg b^*$.

In the classical scattering theory the point $y = 0$ on the trajectory is a turning point: At $y < 0$ the quasiparticle approaches to the scattering center (vortex line in our case), while at $y > 0$ the quasiparticles move away from the vortex line. Equation (18) shows that for impact parameters $0 < b < b^*$, the quasiparticle can not reach the usual turning point $y = 0$ since at $y = -y^*$, where $y^* = \sqrt{b^*b - b^2}$, the group velocity vanishes, and the quasiparticle starts to move back to $y = -\infty$ without an essential change of its momentum. This is Andreev reflection well known in the theory of superconductivity. At the point $y = -y^*$, $p = p_0$ and the transition between two branches of the roton spectrum with $p > p_0$ (positive branch, parallel momentum, and group velocity) and $p < p_0$ (negative branch, antiparallel momentum, and group velocity) occurs. Due to the Andreev reflection the shadow region is formed near the vortex line which is not available for the roton classical trajectories. That shadow (Andreev shadow) region is shown in Fig. 1(a).

Let us find the classic action variation first for trajectories with impact parameters $b > b^*$ or $b < 0$, when there is no Andreev reflection and the incident roton stays at the same branch after the collision. Taking into account variation of the roton group velocity along the trajectory [Eq. (18)], the variation of the action along the trajectory for the incident

momentum $p > p_0$ is

$$\delta S(b) = \int_{-\infty}^{\infty} [p(y) - p]\,dy$$
$$= (p - p_0)\int_{-\infty}^{\infty}\left(\sqrt{1 - \frac{b^*b}{b^2 + y^2}} - 1\right)dy$$
$$= 2\,\text{sign}(b)(p - p_0)\left[(b - b^*)F\left(\frac{b^*}{b}\right) - bE\left(\frac{b^*}{b}\right)\right], \quad (20)$$

where

$$F(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}},$$
$$E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\theta}\,d\theta \quad (21)$$

are complete elliptic integrals of the first and the second order, respectively. In the limits $b \to \pm\infty$, Eq. (20) reduces to Eq. (15).

In the interval $0 < b < b^*$ trajectory ends at the Andreev reflection point with the coordinate $y = -y^*$. The incident roton with momentum $p = p_0 + (p - p_0) > p_0$ returns after the Andreev reflection to $y = -\infty$ at the other branch with the same energy but a slightly different momentum $p_- = p_0 - (p - p_0) < p_0$. The variation of the action along the whole path is

$$\delta S(b) = \int_{-\infty}^{-y^*} p(y)\,dy + \int_{-y^*}^{-\infty} p_-(y)\,dy$$
$$- \int_{-\infty}^{a} p\,dy - \int_{a}^{-\infty} p_-\,dy$$
$$= 2(p - p_0)\left[\int_{-\infty}^{-y^*}\left(\sqrt{1 - \frac{b^*b}{b^2 + y^2}} - 1\right)dy - y^* - a\right]. \quad (22)$$

Here a is an undefined constant, which does not depend on b and therefore has no effect on the scattering angle φ. Choosing $a = 0$ one eliminates any discontinuity of $S(b)$ at $b = 0$ and $b = b^*$.[25] Introducing the angle variable again one obtains the expression

$$\delta S(b) = 2(p - p_0)\left[(b - b^*)F\left(\phi, \frac{b^*}{b}\right) - bE\left(\phi, \frac{b^*}{b}\right)\right] \quad (23)$$

in terms of incomplete elliptic integrals

$$F(\phi, m) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}},$$
$$E(\phi, m) = \int_0^{\phi} \sqrt{1 - m\sin^2\theta}\,d\theta, \quad (24)$$

where $\phi = \arcsin\sqrt{b/b^*}$.

In Figs. 1(b) and 1(c), the action $\delta S(b)$ and the scattering angle $\varphi(b) = -\partial\delta S(b)/\partial b$ are plotted as functions of the impact parameter b (in dimensionless variables). The angle φ has weak singularities at $b = 0$ and b^*, which are integrable in the integral for the transverse cross section $\sigma_\perp$ [Eq. (11)].

Substituting the transverse cross section (16) into the expression for the parameter D', which determines the Iordanskii force [see Eq. (3)], one obtains $D' = -\kappa m n_n$, where n_n is the normal particle density. This rather simple and universal expression tempts to claim its universal topological origin since κ in this expression is a topological charge. However, in Sec. III we shall see that the expression is not universal. For quasiparticles in a BCS superconductor with energy much exceeding the gap, an additional small factor appears in this expression.

III. SCATTERING OF BULK FREE BCS QUASIPARTICLES BY A VORTEX IN FERMI SUPERFLUIDS

A. The Bogoliubov–de Gennes equations

The wave function of quasiparticles in the BCS theory is a spinor with two components,

$$\psi(\boldsymbol{R}) = \begin{pmatrix} u(\boldsymbol{R}) \\ v(\boldsymbol{R}) \end{pmatrix}, \tag{25}$$

which are determined from the Bogoliubov–de Gennes equations[21]

$$-\frac{\hbar^2}{2m}\left(\nabla^2 + k_F^2\right)u(\boldsymbol{R}) + \Delta(\boldsymbol{R})e^{i\theta(\boldsymbol{R})}v(\boldsymbol{R}) = \epsilon u(\boldsymbol{R}), \tag{26}$$

$$\frac{\hbar^2}{2m}\left(\nabla^2 + k_F^2\right)v(\boldsymbol{R}) + \Delta(\boldsymbol{R})e^{-i\theta(\boldsymbol{R})}u(\boldsymbol{R}) = \epsilon v(\boldsymbol{R}). \tag{27}$$

Here k_F is the Fermi wave number, and the gap $\Delta(\boldsymbol{R})$ can vary in space. The equations correspond to the Hamiltonian with the density

$$\mathcal{H} = \frac{\hbar^2}{2m}\left(|\nabla u|^2 - k_F^2|u|^2\right) - \frac{\hbar^2}{2m}\left(|\nabla v|^2 - k_F^2|v|^2\right)$$
$$+ \Delta(\boldsymbol{R})e^{i\theta(\boldsymbol{R})}u^*v + \Delta(r)e^{-i\theta(\boldsymbol{R})}v^*u. \tag{28}$$

If a superfluid is at rest, the order parameter phase θ is a constant and the solution of the Bogoliubov–de Gennes equations is a plane wave

$$\begin{pmatrix} u_0 \\ v_0 e^{i\theta} \end{pmatrix} e^{i\boldsymbol{k}\cdot\boldsymbol{R}}, \tag{29}$$

where

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2}\left(1 + \frac{\xi}{\epsilon_0}\right)} \\ \sqrt{\frac{1}{2}\left(1 - \frac{\xi}{\epsilon_0}\right)} \end{pmatrix}. \tag{30}$$

The energy is given by the well-known BCS quasiparticle spectrum $\epsilon_0 = \pm\sqrt{\xi^2 + \Delta^2}$. Here $\xi = (\hbar^2/2m)(k^2 - k_F^2) \approx \hbar v_F(k - k_F)$ is the quasiparticle energy in the normal Fermi liquid and $v_F = k_F/m$ is the Fermi velocity. The two wave numbers

$$k_\pm = \sqrt{k_F^2 \pm 2\sqrt{\epsilon_0^2 - \Delta^2}} \tag{31}$$

correspond to the particlelike (+) and the holelike (−) branches of the spectrum.

The Bogoliubov–de Gennes equations are written for the wave function of quasiparticles, and, as in the case of the Schrödinger equation, there is the continuity equation for the probability $|u|^2 + |v|^2$ to find the quasiparticle in some point in space:

$$\frac{\partial(|u|^2 + |v|^2)}{dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{g}, \tag{32}$$

where

$$\boldsymbol{g} = -\frac{i\hbar}{2m}(u^*\boldsymbol{\nabla}u - u\boldsymbol{\nabla}u^*) + \frac{i\hbar}{2m}(v^*\boldsymbol{\nabla}v - v\boldsymbol{\nabla}v^*) \tag{33}$$

is the probability flux. Equation (32) is a manifestation of the conservation law for the number of quasiparticles. But, the number of quasiparticles and the probability flux $\boldsymbol{g}$ are not the same as the number of particles (charge) and the particle current $\boldsymbol{j}$. The Hamiltonian of Eq. (28) is not gauge invariant and there is no conservation law for the particle number. The Bogoliubov–de Gennes equations lead to the following equation for time variation of the particle density $|u|^2 - |v|^2$:

$$\frac{\partial(|u|^2 - |v|^2)}{dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{j} + 2i\Delta(e^{-i\theta}v^*u - e^{i\theta}vu^*), \tag{34}$$

where

$$\boldsymbol{j} = -\frac{i\hbar}{2m}(u^*\boldsymbol{\nabla}u - u\boldsymbol{\nabla}u^*) - \frac{i\hbar}{2m}(v^*\boldsymbol{\nabla}v - v\boldsymbol{\nabla}v^*) \tag{35}$$

is the particle current. Eqnarray (34) contains a source (the last term in the right-hand side) related with possible changing of the total particle number. Globally, the number of particles is of course a conserved quantity. The source in the continuity equation for the particle density corresponds to conversion of the superfluid part of the liquid to the normal one and vice versa in inhomogeneous states. In order to restore the global conservation law one should solve the Bogoliubov–de Gennes equations together with the self-consistency equation for the order parameter proportional to the gap. This property of the Bogoliubov–de Gennes equations is well known in the theory of superconductivity.[26]

B. Superfluid motion in the Bogoliubov–de Gennes equations

Superfluid velocity is determined by the order-parameter phase gradient

$$\boldsymbol{v}_s = \frac{\kappa_c}{2\pi}\boldsymbol{\nabla}\theta, \tag{36}$$

where $\kappa_c = h/2m$ is the circulation quantum for the Cooper-pair condensate and m is the particle mass. Assuming constant absolute value of the gap Δ and gradient of phase, the solution of the Bogoliubov–de Gennes equations is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_0 e^{i(\boldsymbol{k}+\boldsymbol{\nabla}\theta_1)\cdot\boldsymbol{R}} \\ v_0 e^{i(\boldsymbol{k}-\boldsymbol{\nabla}\theta_2)\cdot\boldsymbol{R}} \end{pmatrix}. \tag{37}$$

Here we introduced separate phases θ_1 and θ_2 for two spinor components. Their sum determines the order-parameter phase $\theta = \theta_1 + \theta_2$. The spinor (37) corresponds to the energy (neglecting terms of the second order in phase gradients)

$$\epsilon = \epsilon_0(k) + \frac{\hbar\kappa_c}{2\pi}\boldsymbol{k}\cdot\boldsymbol{\nabla}\theta + \frac{\partial\epsilon_0}{\partial\boldsymbol{k}}\cdot\frac{\boldsymbol{\nabla}\theta_1 - \boldsymbol{\nabla}\theta_2}{2}$$
$$= \epsilon_0(k) + \hbar\boldsymbol{k}\cdot\left[\boldsymbol{v}_s + \frac{\xi}{\epsilon_0}\frac{\kappa_c}{2\pi}(\boldsymbol{\nabla}\theta_1 - \boldsymbol{\nabla}\theta_2)\right]. \tag{38}$$

It looks as if the phase difference $\theta_1 - \theta_2$ were of no importance since it can be removed by redefinition of the wave vector $\boldsymbol{k}$. Choosing $\theta_1 = \theta_2$ one obtains the expression for the quasiparticle energy following from the Galilean invariance and well known from textbooks on superconductivity[21]: $\epsilon = \epsilon_0 + \hbar \boldsymbol{k} \cdot \boldsymbol{v}_s$. But, another choice is required in the theory of quasiparticle scattering by a vortex: either $\theta_1 = 0$ or $\theta_2 = 0$. This is dictated by cyclic boundary conditions for spinor components on the closed path around the vortex (see Secs. III C and III D).

For the choice $\theta_1 = \theta_2$, Eq. (35) yields the following expression for the current in the plane-wave state:

$$ \boldsymbol{j} = \frac{\hbar \boldsymbol{k}}{m} + N(\boldsymbol{k}) \boldsymbol{v}_s. \qquad (39) $$

So, the superfluid velocity contribution to the current is proportional to the charge $N(\boldsymbol{k}) = |u_0|^2 - |v_0|^2$ in the state.

C. Scattering of free BCS quasiparticles by a vortex: Simple approach

The mutual-friction force has been calculated for pure type-II superconductors long ago.[27,28] Since the BSC theory describes also the superfluid ^{3}He and the effect of the magnetic field is insignificant for mutual friction in type-II superconductors, these calculations are relevant also for singular vortices in the superfluid ^{3}He. In this section, we use simple approaches: geometric optics for low energies $\epsilon_0 - \Delta \ll \Delta$ and perturbation theory for high energies $\epsilon_0 \gg \Delta$. A more accurate theory based on the partial-wave expansion will be considered in the next section.

When the energy of the quasiparticles is close to the energy gap of the superconductor ($\xi \ll \Delta$), the BCS quasiparticle spectrum $\epsilon_0 \approx \Delta + v_F^2 \hbar^2 (k - k_F)^2 / 2\Delta$ is identical to the roton spectrum with the roton minimum momentum replaced by the Fermi momentum $\hbar k_F$ and the roton mass μ replaced by Δ / v_F^2, where $v_F = \hbar k_F / m$ is the Fermi velocity. So, the semiclassical theory for rotons can be directly applied to such BCS quasiparticles, and the transverse cross section for them is given by Eq. (16), in which the circulation quantum κ is replaced by the circulation quantum $\kappa_c = h/2m$ for the Cooper-pair condensate and the group velocity for the BCS quasiparticles is $v_G = v_F \xi / \epsilon_0$. Figure 1 illustrating scattering of rotons by the vortex is relevant also for low-energy quasiparticles scattered by the vortex. The phenomenon of the nearly 180% reflection of quasiparticles from the area of the Andreev shadow shown in the figure is important for description of zero-temperature superfluid turbulence.[29,30]

If the quasiparticle energy is much larger than the superconducting gap, the group velocity v_G approaches to the Fermi velocity v_F and the method of classical trajectories yields the transverse cross section κ_c / v_F. This result does not look reasonable because the cross section being small still does not vanish in the limit $\Delta \to 0$. Indeed, the partial-wave calculations[27,28] yielded that in the limit of small Δ / ξ the transverse cross section differed from the semiclassical result of Eq. (16) by the factor $\Delta^2 / 2\xi^2$. This also followed from the solution of the Bogoliubov–de Gennes equations in the Born approximation[31] as shown below.

Let us consider the perturbation theory with respect to the gap Δ and the superfluid velocity $\boldsymbol{v}_s = (\kappa_c / 2\pi) \nabla \theta$. For the sake of simplicity the wave vector $\boldsymbol{k}$ lies in the plane normal to the vortex axis. In our case the superfluid velocity is the velocity $\boldsymbol{v}_v$ induced by the vortex. In the zero-order approximation $u \sim \exp(i\boldsymbol{k} \cdot \boldsymbol{r})$ and $v = 0$. In the first-order approximation the second Bogoliubov–de Gennes equation (27) yields

$$ v = \left\{ \frac{\Delta \exp(-i\theta)}{\xi(k) + E(k)} + \frac{\Delta \exp(-i\theta)}{[\xi(k) + E(k)]^2} \frac{\hbar^2}{m} (\boldsymbol{k} \cdot \nabla \theta) \right\} e^{i\boldsymbol{k} \cdot \boldsymbol{r}}. \qquad (40) $$

The first term in curly brackets yields a correction to the quasiparticle energy $\propto \Delta^2$, but does not contribute to scattering which is determined by the order-parameter phase gradients. So we keep only the second term proportional to $\nabla \theta$. Inserting it to the first Bogoliubov–de Gennes equation (26), one obtains the following equation for the correction u' to the quasiparticle amplitude $u \sim 1$:

$$ (\nabla^2 + k^2) u' = (\boldsymbol{k} \cdot \nabla \theta) \frac{\Delta^2}{2\xi^2} e^{i\boldsymbol{k} \cdot \boldsymbol{r}}. \qquad (41) $$

This equation is similar to the wave equation for the sound wave propagating past the vortex[4,31] and using this analogy one easily obtains the expression for the transverse cross section:

$$ \sigma_\perp = \frac{\Delta^2}{2\xi^2} \frac{\pi}{k_F} = \frac{\Delta^2}{2\xi^2} \frac{\kappa_c}{v_F}. \qquad (42) $$

The cross section vanishes at $\Delta \to 0$ as expected. But, the question where the geometric optics went wrong still remains. The answer is that the cyclic boundary conditions were violated with the choice $\theta_1 = \theta_2 = \theta/2$. Let us move the spinor given by Eq. (37) along a closed path around the vortex line. After closing the path, the phase θ obtains the shift 2π but the shifts of the phases θ_1 and θ_2 are equal to π. So the periodic boundary conditions for the spinor components u and v are violated. They are satisfied only if either θ_1 or θ_2 vanishes. According to Eq. (38), this modifies the expression for the quasiparticle energy in the vortex velocity field

$$ \epsilon = \epsilon_0(k) + (\hbar \boldsymbol{k} \pm m \boldsymbol{v}_G) \cdot \boldsymbol{v}_v. \qquad (43) $$

Then the value of p in Eq. (15) must be replaced by $\hbar k \pm m v_G$. Choosing $-$ for quasiparticles and $+$ for quasiholes (this is dictated by a physically reasonable condition that the cross section vanishes far from the Fermi surface) one obtains the transverse cross section[27,28]

$$ \sigma_\perp = \frac{\kappa_c}{v_G} - \frac{\kappa_c}{v_F} = \frac{\kappa_c}{v_F} \left(\frac{\epsilon_0}{\sqrt{\epsilon_0^2 - \Delta^2}} - 1 \right). \qquad (44) $$

In the limit $\epsilon_0 \gg \Delta$, this agrees with the expression (42) obtained from the perturbation theory with respect to Δ. A more rigorous partial-wave analysis of the next section confirms this result for any ratio Δ / ϵ_0. This provides an explanation for shortcoming of the naive geometric-optics analysis: It ignored peculiarities of the Aharonov-Bohm effect for BCS quasiparticles and used an improper definition for the quasiparticle phase shift along the trajectory. We shall continue the discussion of this issue in the end of the next section.

D. Partial-wave analysis of scattering of free BCS quasiparticles by a vortex

The partial-wave analysis in the cylindric coordinates r, ϕ, z uses expansion of the spinor components in eigenfunctions $e^{il\phi}$ of the orbital moment. In the presence of a vortex, the phase of the order parameter $\Delta e^{i\theta}$ around the vortex is $\theta = \phi$ and the partial-wave expansion for the wave function is

$$u = \sum_l u_l e^{il\phi}, \quad v = \sum_l v_l e^{i(l-1)\phi}, \qquad (45)$$

where u_l and v_l must satisfy the Bogoliubov–de Gennes equations for partial waves:

$$-\frac{\hbar^2}{2m}\left(\frac{d^2 u_l}{dr^2} + \frac{1}{r}\frac{du_l}{dr} - \frac{l^2 u_l}{r^2}\right) + \Delta u_l = \left(\epsilon + \frac{\hbar^2 k_F^2}{2m}\right) v_l,$$

$$\frac{\hbar^2}{2m}\left(\frac{d^2 v_l}{dr^2} + \frac{1}{r}\frac{dv_l}{dr} - \frac{(l-1)^2 v_l}{r^2}\right) + \Delta u_l = \left(\epsilon - \frac{\hbar^2 k_F^2}{2m}\right) v_l. \qquad (46)$$

In our case, the orbital number l is not an ideal quantum number since the two components of the spinor correspond to two different orbital numbers l and $l-1$.

In order to find the scattering phases, we shall look for the semiclassical solution of the Bogoliubov–de Gennes equations for the scaled amplitudes $U_l = u_l\sqrt{r}$ and $V_l = v_l\sqrt{r}$:

$$-\frac{\hbar^2}{2m}\left(\frac{d^2 U_l}{dr^2} - \frac{l^2 - 1/4}{r^2}U_l\right) + \Delta V_l = \left(\epsilon + \frac{\hbar^2 k_F^2}{2m}\right) U_l,$$

$$\frac{\hbar^2}{2m}\left(\frac{d^2 V_l}{dr^2} - \frac{(l-1)^2 - 1/4}{r^2}V_l\right) + \Delta U_l = \left(\epsilon - \frac{\hbar^2 k_F^2}{2m}\right) V_l. \qquad (47)$$

The semiclassical solution of the Bogoliubov–de Gennes equations (47) for partial waves is

$$\psi \sim \frac{1}{\sqrt{k_\pm}}\begin{pmatrix} \sqrt{\frac{1}{2}\left(1 + \frac{\sqrt{\epsilon_0^2 - \Delta^2}}{\epsilon_0}\right)} \\ \sqrt{\frac{1}{2}\left(1 - \frac{\sqrt{\epsilon_0^2 - \Delta^2}}{\epsilon_0}\right)} \end{pmatrix} e^{i\int^r k_\pm(r)dr}, \qquad (48)$$

where $\epsilon_0 = \epsilon - (l - 1/2)/2r^2$ and

$$k_\pm(r)^2 = k_F^2 - \frac{(l-1/2)^2}{r^2} \pm 2\sqrt{\left(\epsilon - \frac{l-1/2}{2r^2}\right)^2 - \Delta^2}. \qquad (49)$$

If a quasiparticle with the wave number k_+ is incident on the vortex line, it will be reflected either as a quasiparticle with the same number k_+ (usual reflection) or as a quasiparticle belonging to the holelike branch with $k_- < k_F$ (Andreev reflection). The usual reflection occurs at the turning point determined by the condition $k_+(r_r) = 0$. In the Andreev reflection point $r = r_a$, the radical in Eq. (49) vanishes, i.e., $\epsilon - \frac{l-1/2}{2r_a^2} \pm \Delta = 0$. The type of the reflection depends on which turning point is reached earlier: usual or Andreev reflections take place if $r_r > r_a$ or $r_r < r_a$, respectively.

In the following we shall look for partial waves with large orbital numbers l, which correspond to large impact parameters. Then only usual reflection is possible, and one

can expand the radical in Eq. (49) with respect to $(l - 1/2)/r^2$. Then

$$k^2 \approx k_\pm^2 - \frac{(l-1/2)^2 \pm (l-1/2)\epsilon/\sqrt{\epsilon^2 - \Delta^2}}{r^2}$$

$$\approx k_\pm^2 - \frac{(l - 1/2 \pm \epsilon/2\sqrt{\epsilon^2 - \Delta^2})^2}{r^2}, \qquad (50)$$

where $k_\pm$ are values of $k_\pm(r)$ at $r \to \infty$. The total phase accumulated after quasiparticle motion from very large r to the turning point and back to large r is

$$\Phi_l = 2\int_{r_t}^r \sqrt{k_\pm^2 - \frac{(l - 1/2 \pm \epsilon/2\sqrt{\epsilon^2 - \Delta^2})^2}{r^2}}dr$$

$$-\frac{\pi}{2} = 2k_\pm r - \pi\left|l - \frac{1}{2} \pm \frac{\epsilon}{2\sqrt{\epsilon^2 - \Delta^2}}\right| - \frac{\pi}{2}. \qquad (51)$$

Here the phase shift $-\pi/2$ originates from the close vicinity of the reflection point where the semiclassical approach becomes invalid.[32] In order to find the phase shift from scattering of the quasiparticle (particle branch, the upper sign in the expressions above) by the vortex one should subtract the phase shift $\Phi_{l0} = 2k_+ r - \pi(|l| + 1/2)$ of the l partial-wave function in the uniform state. This follows from the expansion of a plane wave in partial waves. Then the scattering phase shift is

$$\delta_l = \frac{\Phi_l - \Phi_{l0}}{2} = -\frac{\pi}{2}\left|l - \frac{1}{2} + \frac{\epsilon}{2\sqrt{\epsilon^2 - \Delta^2}}\right| + |l|\frac{\pi}{2}$$

$$= \frac{\pi}{4}\left(1 - \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}}\right)\mathrm{sign}l. \qquad (52)$$

The variation of the classical action along the quasiparticle trajectory is connected with the quantum-mechanical scattering phase shift by the relation $\delta S(b) = 2\hbar\delta_l$, where $b \approx l/k_F$. Thus, one obtains that

$$\delta S(\pm\infty) = \mp\frac{\hbar k_F \kappa_c}{2}\left(\frac{1}{v_G} - \frac{1}{v_F}\right). \qquad (53)$$

Inserting it into Eq. (14) yields the transverse cross section Eq. (44) obtained after correction of the geometric-optics expression. In the case of the hole branch (the lower sign in the expressions above) one should subtract from the phase shift Φ_l in the vortex state the phase shift $\Phi_{(l-1)0} = 2k_- r - \pi(|l - 1| + 1/2)$ of the $(l - 1)$ partial-wave function in the uniform state.

The second term $\propto 1/v_F$ in the transverse cross section (44) was considered as anomalous and interpreted in the terms of spectral flow[6,16] although its original derivation from the partial-wave analysis[27] did not used this concept (see further discussion in Sec. IX). The analysis presented here demonstrates that it can be explained within the framework of the scattering theory taking into account peculiarities of the Aharonov-Bohm effect for BCS quasiparticles.

IV. BOUND VORTEX-CORE STATES AND THE CURRENT IN THE CORE

A. Bound Andreev states in a planar SNS junction

For the analysis of the role of the core states in vortex dynamics it is useful to consider a simplified approach to them based on geometric optics. Such an approach was suggested

by Stone.[19] He also used the model of a normal vortex core exploiting its analogy with the one-dimensional (1D) problem of Andreev bound states in the ballistic SNS junction. We also shall investigate this analogy. In the past, a number of authors addressed the question as to whether and what Josephson current is possible through such a junction in full absence of the order parameter in the normal layer.[33–35] They concluded that the Josephson current is possible due to phase coherence of the Andreev states, which are sensitive to the phase difference on the junction.

We consider a normal layer of the width L, which is perpendicular to the axis y. A superfluid in superconducting regions $y < 0$ and $y > L$ moves with the velocity $\boldsymbol{v}_s$. Let us look for a state with the energy

$$\epsilon = \epsilon_0 + \hbar \boldsymbol{k} \cdot \boldsymbol{v}_s \approx \epsilon_0 + \hbar \boldsymbol{k}_0 \cdot \boldsymbol{v}_s, \qquad (54)$$

where $|\epsilon_0| < \Delta$ and the wave vector $\boldsymbol{k}_0(k_x, k_f, k_z)$ has a modulus equal to the Fermi wave number k_F, so that the component k_y is equal to $k_f = \sqrt{k_F^2 - k_x^2 - k_z^2}$. The wave function, which satisfies the Bogoliubov–de Gennes equations, is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A e^{im(\boldsymbol{v}_s \cdot \boldsymbol{R})/\hbar + im\epsilon_0 y/\hbar^2 k_f} \\ B e^{-im(\boldsymbol{v}_s \cdot \boldsymbol{R})/\hbar - im\epsilon_0 y/\hbar^2 k_f} \end{pmatrix} e^{i\boldsymbol{k}_0 \cdot \boldsymbol{R}} \qquad (55)$$

inside the normal layer $0 < y < L$,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_- e^{i\theta_-/2 + im(\boldsymbol{v}_s \cdot \boldsymbol{R})/\hbar} \\ v_- e^{-i\theta_-/2 - im(\boldsymbol{v}_s \cdot \boldsymbol{R})/\hbar} \end{pmatrix} e^{i\boldsymbol{k}_0 \cdot \boldsymbol{R} + \frac{m\sqrt{\Delta^2 - \epsilon_0^2}}{\hbar^2 k_f} y} \qquad (56)$$

inside the superconductor at $y < 0$, and

$$\begin{pmatrix} u \\ v \end{pmatrix} = C \begin{pmatrix} u_+ e^{i\theta_+/2 + im(\boldsymbol{v}_s \cdot \boldsymbol{R})/\hbar} \\ v_+ e^{-i\theta_+/2 - im(\boldsymbol{v}_s \cdot \boldsymbol{R})/\hbar} \end{pmatrix} e^{i\boldsymbol{k}_0 \cdot \boldsymbol{R} - \frac{m\sqrt{\Delta^2 - \epsilon_0^2}}{\hbar^2 k_f} y} \qquad (57)$$

inside the superconductor at $y > L$. Here

$$u_\pm = v_\mp = \sqrt{\frac{1}{2} \left(1 \pm i \frac{\sqrt{\Delta^2 - \epsilon_0^2}}{\epsilon_0} \right)}, \qquad (58)$$

and the constants A and B are determined from the boundary conditions at the interface $y = 0$:

$$A = u_- e^{i\theta_-/2}, \quad A = v_- e^{-i\theta_-/2}. \qquad (59)$$

One can find the constant C from the boundary conditions at the interface $y = L$ only for discrete values of the energy ϵ_0 satisfying the following Bohr-Sommerfeld condition:[19]

$$\frac{2m\epsilon_0 L}{\hbar^2 k_f} = 2\pi \left(s + \frac{1}{2} \right) + (\theta_+ - \theta_-) - 2 \arcsin \frac{\epsilon_0}{\Delta}, \qquad (60)$$

with integer s. At small energy $\epsilon_0 \ll \Delta$ this yields the spectrum of the Andreev bound states:

$$\epsilon_0 = \left(\frac{2mL}{\hbar^2 k_f} + \frac{1}{\Delta} \right)^{-1} \left[2\pi \left(s + \frac{1}{2} \right) + (\theta_+ - \theta_-) \right]. \qquad (61)$$

The wave function given by Eqs. (55)–(57) assumes that only the Andreev reflection occurs at the core boundaries, so the wave vector is always close to the Fermi surface and its normal component $k_y \approx k_f$ does not change a sign at the reflection. This is a valid assumption in the weak-coupling

limit when the superconducting gap Δ is small compared to the Fermi energy $\epsilon_F = \hbar^2 k_F^2 / 2m$. In this approximation the boundary conditions at the interfaces $y = 0$ and L require continuity only of two spinor components, ignoring the continuity conditions for spinor gradients. It is worthwhile to note that this approximation does not provide exact continuity of the quasiparticle current at the interfaces as required by the conservation law for quasiparticles. This can be achieved only in the next approximation taking into account the possibility of usual reflection changing the direction of the momentum normal to the layers. However, for our goals we need not the quasiparticle current but the genuine particle current, and the used approximation is sufficient.

Now let us find the contribution of bound states to the momentum of the liquid. Since in the bound states particle and hole components are equal in amplitude, the charge $N(\boldsymbol{k}) = |u|^2 - |v|^2$ in these states vanishes, and according to Eq. (39) the current normal to the layers in any bound state is about $\hbar k_f / m$. Let us consider the case of a wide normal layer $L \gg \hbar k_F / m \Delta$ when there is a large number of bound states and the sum over them can be replaced by an integral. Then the total contribution of the bound states to the current is simply a product of $\hbar k_f / m$ and a difference of the number of states with the wave vectors in opposite directions. At $T = 0$ all states with $\epsilon = \epsilon_0 + \hbar \boldsymbol{k}_0 \cdot \boldsymbol{v}_s < 0$ are filled. This yields that the total momentum in the bound states (per unit area of the SNS junction) normal to the layers is

$$P_{bs} = - \int_{k < k_F} \frac{\hbar^2 k_f^2 v_s}{\delta \epsilon} \frac{d\boldsymbol{k}_\parallel}{4\pi^2} = -Lnmv_s, \qquad (62)$$

where $\boldsymbol{k}_\parallel(k_x, k_z)$ is the component of the wave vector in the layer plane, n is the particle density, and

$$\delta \epsilon = \pi \frac{\hbar^2 k_f}{mL} \qquad (63)$$

is the distance between discrete levels. The momentum corresponds to the current density $j_{bs} = P_{bs}/Lm = -nv_s$ inside the normal layer. This shows that quasiparticles in core bound states play a role of the effective normal component, which exists even at zero temperature.[6]

Note that the phase difference $\theta_+ - \theta_-$ between the super-conducting banks of the SNS junction has no effect on the current in the continuum limit. This is because the main effect of the phase difference is a shift of the levels in the forbidden gap without any essential change of their number. The shift of the levels leads to an entry of a new level on one side of the gap and an exit of an old level on another side (Fig. 2). Normally these two events are not fully synchronized and the number of level can fluctuate with the ± 1 level. This fluctuation is not essential in the limit of a large number of levels. On the other hand, exactly this small variation of the level number leads to the Josephson effect in the SNS junction,[33–35] which is beyond the scope of this work. The shift of levels constitutes the phenomenon of spectral flow, which arises if the phase difference $\theta_+ - \theta_-$ monotonously increases or decreases and the levels cross the forbidden gap.

One should also consider the contribution of the continuum delocalized states with negative energy, which are fully occupied in the ground state. For a delocalized state ($|\epsilon_0| > \Delta$)
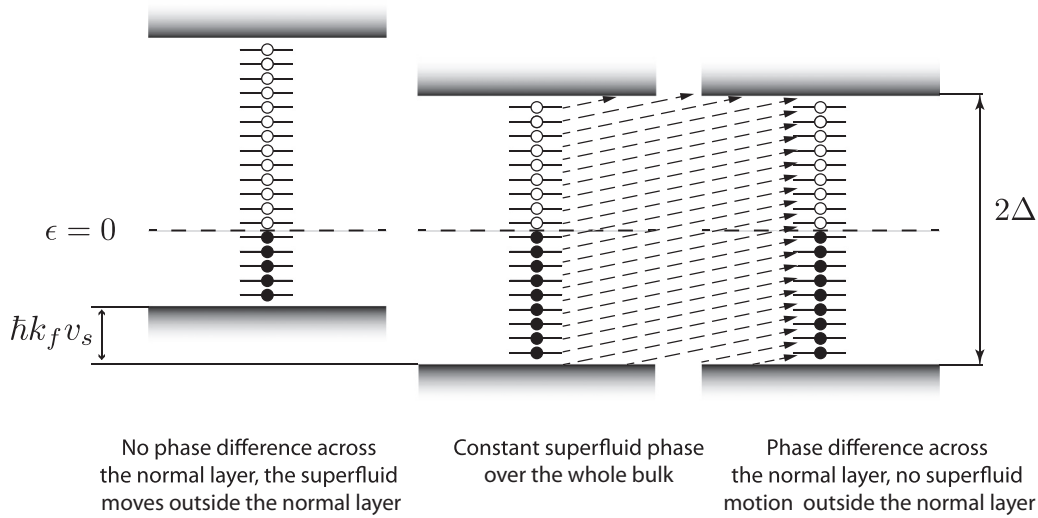
FIG. 2. Bound states inside the superconducting gap in the SNS junction. Occupied levels below the Fermi level $\epsilon = 0$ are shown by black circles, while empty levels above the Fermi level are shown by white ones. Center: No phase difference across the normal layer, no superfluid velocity inside superconducting areas. Right: There is phase difference $\theta_+ - \theta_-$ across the normal layer but still no superfluid velocity inside superconducting areas. Arrowed dashed lines show shift of level relatively to the gap and the Fermi level. Some levels exit from the gap at the top of the gap while some new levels enter the gap at the gap bottom. Left: No phase difference across the normal layer, but there is the superfluid velocity v_s at the bulk of superconductors. This shifts the gap with respect to the Fermi level and changes the number of occupied levels.

of a quasiparticle propagating from $y = -\infty$ to ∞, the spinor in the normal layer $0 < y < L$ is given by the same expression as Eq. (55) for the bound state, whereas in superconducting layers the states are described by spinors

$$t \begin{pmatrix} u_0 e^{i\theta_+/2 + im(\mathbf{v}_s \cdot \mathbf{R})/\hbar} \\ v_0 e^{-i\theta_+/2 - im(\mathbf{v}_s \cdot \mathbf{R})/\hbar} \end{pmatrix} e^{i\mathbf{k}_0 \cdot \mathbf{R} + i \frac{m\sqrt{\epsilon_0^2 - \Delta^2}}{\hbar^2 k_f} y} \qquad (64)$$

for $y > L$ and

$$e^{i\mathbf{k}_0 \cdot \mathbf{R}} \left[\begin{pmatrix} u_0 e^{i\theta_-/2 + im(\mathbf{v}_s \cdot \mathbf{R})/\hbar} \\ v_0 e^{-i\theta_-/2 - im(\mathbf{v}_s \cdot \mathbf{R})/\hbar} \end{pmatrix} e^{i \frac{m\sqrt{\epsilon_0^2 - \Delta^2}}{\hbar^2 k_f} y} \right.$$
$$\left. + r \begin{pmatrix} v_0 e^{i\theta_-/2 + im(\mathbf{v}_s \cdot \mathbf{R})/\hbar} \\ u_0 e^{-i\theta_-/2 - im(\mathbf{v}_s \cdot \mathbf{R})/\hbar} \end{pmatrix} e^{-i \frac{m\sqrt{\epsilon_0^2 - \Delta^2}}{\hbar^2 k_f} y} \right] \qquad (65)$$

for $y < 0$. Here t and r are amplitudes of transmission and reflection ($|t|^2 + |r|^2 = 1$) which are determined from the continuity of spinor components (but not their derivatives!)[35] at $y = 0$ and L. As well as for bound states, the analysis considers only the Andreev reflection neglecting probability of usual reflection, which changes the direction of the wave vector. The amplitudes of the spinor components in the normal layer [see Eq. (55)] are

$$A = t u_0 e^{i\theta_+/2 + im(\sqrt{\epsilon_0^2 - \Delta^2} - \epsilon_0)L/\hbar^2 k_f},$$
$$B = t v_0 e^{-i\theta_+/2 + im(\sqrt{\epsilon_0^2 - \Delta^2} + \epsilon_0)L/\hbar^2 k_f}. \qquad (66)$$

The transmission probability is

$$|t|^2 = \frac{\epsilon_0^2 - \Delta^2}{\epsilon_0^2 - \Delta^2 \cos^2[\epsilon_0 mL/\hbar^2 k_f - (\theta_+ - \theta_-)/2]}. \qquad (67)$$

The transmission probability differs from unity in the small energy interval of the order $\Delta \ll \epsilon_F$, and the effect of reflection is not essential for the contribution of delocalized states to the

supercurrent. The latter can be found by summation of Eq. (39) over the whole continuum of free bulk states. The whole particle density is accumulated in delocalized but not bound states. Neglecting reflection for the continuum states, the density and the current in the normal and the superconducting areas do not differ essentially. So, the whole ensemble of delocalized quasiparticles is a liquid of nearly constant density n moving with the spatially uniform velocity v_s. This points out nearly ideal transparency of the ballistic normal layers for the supercurrent of delocalized quasiparticles. Note that scattering of continuum states by impurities is impossible since all continuum states are fully occupied.

Summing the momenta in localized and delocalized states inside the normal layer, one obtains that the total momentum and the current eventually vanish there (with accuracy of the small parameter of weak coupling Δ/ϵ_F). Keeping in mind the presence of the current $n v_s$ in superconducting layers, this violates the conservation law for the particle number since backflow in our 1D geometry is impossible and the current must be constant along the direction normal to the layers. A proper conclusion from this is that the superfluid transport (but not diffusive transport with dissipation!) with high superfluid velocity in this one-dimensional geometry is impossible. But, this does not rule out the superfluid transport with very low superfluid velocities $v_s \leqslant \hbar/mL$ when discreteness of the Andreev bound states and the phase difference across the normal layer can not be ignored. This returns us again to the problem of the Josephson effect in the SNS junction.[33–35]

B. Bound vortex-core states: Ballistic normal core

Now let us consider bound states in a normal core of a vortex. A reliable assumption is that a quasiparticle inside the core, where the superconducting order parameter vanishes, moves along an approximately straight trajectory back and
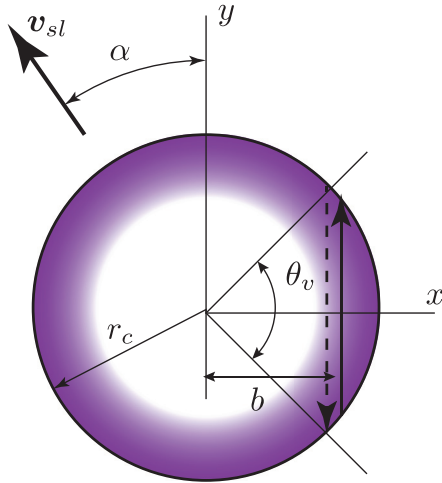
FIG. 3. (Color online) Bound state in the normal core. The vertical solid arrowed line shows the trajectory of the quasiparticle and the vertical dashed arrowed line shows the trajectory of the quasihole after Andreev reflection of the quasiparticle at the core boundary. Note that the picture is purely schematic, and in fact the analysis was done for the case when the impact parameter b is much less than the core radius r_c but still much larger than the interatomic distance $1/k_F$.

forth, reversing its direction of motion via Andreev reflection at the boundary of the core. The trajectory is chosen to be parallel to the y axis. For trajectories with impact parameters much less than the core radius, the bound states are similar to those in the SNS junction with the normal-layer width L equal to the core diameter $2r_c$. On the other hand, the phase difference $\theta_+ - \theta_- = \theta_v + \theta_s$ consists from the phase difference induced by the vortex, $\theta_v = \pi - 2\arcsin(b/r_c) \approx \pi - 2b/r_c$, and the phase difference θ_s produced by the superflow past the vortex. Here $b = l/k_f$ is the impact parameter and l is the quantum number of the discrete angular momentum. Geometry of the process is shown in Fig. 3.

Eventually the energy of the bound state in the normal core for the chiral zero-crossing branch $s = -1$ depends on the orbital quantum number l and is $\epsilon(l) = \epsilon_0(l) + \hbar k_f v_{sl} \cos\alpha$, where

$$\epsilon_0(l) = \left(1 + \frac{\hbar^2 k_f}{2mr_c\Delta}\right)^{-1} \frac{\hbar^2 k_f}{2mr_c}\left(-\frac{b}{r_c} + \frac{\theta_s}{2}\right), \quad (68)$$

and α is the angle between the trajectory (the axis y) and the local superfluid velocity v_{sl} (see Fig. 3), which is different from the superfluid velocity v_s far from the vortex in the presence of the backflow (see below). Introducing the angular momentum $L_z = \hbar l$ of the bound state, there is a frequency

$$\omega_0 = \frac{\partial \epsilon_0}{\partial L_z} = \frac{\hbar}{2mr_c^2}\left(1 + \frac{\hbar^2 k_f}{2m\Delta r_c}\right)^{-1}, \quad (69)$$

with which the trajectory slowly rotates around the vortex axis. The phase difference from the superflow outside the core can be presented in the form of a dipole field $\theta_s = (1 + \hbar^2 k_f/2mr_c\Delta)(4m/\hbar)\boldsymbol{r}_c \cdot \boldsymbol{v}_\theta$, where $\boldsymbol{r}_c$ is the vector of the modulus r_c directed normally to the cylindric border of the core and the superfluid velocity $\boldsymbol{v}_\theta$ is one more superfluid velocity different in general from the asymptotic velocity $\boldsymbol{v}_s$

and the local velocity $\boldsymbol{v}_{sl}$. Introducing the isotropic part of the spectrum

$$\epsilon_{00} = -\omega_0 L_z, \quad (70)$$

the whole spectrum becomes

$$\epsilon = \epsilon_{00} + \hbar\boldsymbol{k} \cdot (\tilde{\boldsymbol{v}}_s - \boldsymbol{v}_L), \quad (71)$$

where the presence of the vortex velocity $\boldsymbol{v}_L$ points out that the calculation is done for the coordinate frame connected with vortex and

$$\tilde{\boldsymbol{v}}_s = \boldsymbol{v}_\theta + \boldsymbol{v}_{sl} \quad (72)$$

is the effective superfluid velocity taking into account the superfluid velocity $\boldsymbol{v}_{sl}$ outside the core and the contribution of the phase difference θ_s. All three velocities in Eq. (72) are parallel to each other.

For the analysis of the vortex mass one needs to know the contribution of bound states to the total momentum of the liquid if the fluid flows past the vortex with the superfluid velocity $\boldsymbol{v}_s$. As well as in the case of the SNS junction, every bound state has a momentum of the magnitude about $\hbar k_f$ directed along the bound-state trajectory. Taking into account that the energy interval between levels is $\delta\epsilon = \hbar\omega_0$ and integrating over all directions and the wave number component k_z along the vortex axis, one obtains that the total momentum is

$$\boldsymbol{P}_{bs} = -\frac{1}{2}\int_{-k_F}^{k_F}\frac{dk_z}{2\pi}\int d\alpha \, \cos^2\alpha \, \frac{\hbar k_f^2}{\omega_0}(\boldsymbol{v}_{sl} - \boldsymbol{v}_L)$$

$$= -\frac{\pi\hbar n}{\omega_0}(\boldsymbol{v}_{sl} - \boldsymbol{v}_L). \quad (73)$$

The expression was derived for large r_c neglecting corrections of the order $\hbar^2 k_f/2m\Delta r_c$ in Eq. (69). The coefficient before the velocity $\boldsymbol{v}_L$ is the Kopnin vortex mass $\mu_K = \pi\hbar n/\omega_0$. Using Eq. (69) for ω_0 in the limit of large core $r_c \gg \hbar^2 k_F/m\Delta$ and assuming that the momentum is uniformly distributed over the area πr_c^2 of the core, this momentum corresponds to the current $\boldsymbol{j}_{bs} = \boldsymbol{P}_{bs}/\pi r_c^2 m = -2n(\boldsymbol{v}_{sl} - \boldsymbol{v}_L)$ inside the core.

The momentum in the core depends only on the local superfluid velocity $\boldsymbol{v}_{sl}$ outside the core and not on the phase difference θ_s. The reason is the same as in the case of the SNS junction: the phase difference shifts energy levels but does not change their number since any crossing of the zero energy by a level is compensated by an entry or an exit of an level at the bottom of the forbidden gap. Meanwhile, it is variation of the phase difference that governs the spectral flow. In contrast to the SNS case, where the phase difference across the normal layer can vary monotonously, the phase difference across the normal core of the moving vortex can only oscillate without monotonously crossing the forbidden gap. This rules out the steady spectral flow. The oscillation is related with rotation of the bound state with angular velocity ω_0 and dependence of the level position with respect to the gap on the α-dependent phase difference θ_s in Eq. (68).[19]

In the model of the normal core, the Andreev reflection for all bound states occurs at the core boundary, and the energies of bound states are easily determined analytically from the semiclassical Bohr-Sommerfeld condition. Meanwhile, the more realistic model of the core with linear growth of the

order parameter in the core considered in Appendix A shows that although the concept of well-defined trajectory (geometric optics) works well, one can not use the semiclassical approach for description of motion along the trajectory and the Bohr-Sommerfeld condition is invalid. Despite this, the model of the normal core gives a qualitatively correct energy spectrum, different from that from more accurate theories only by a numerical factor. On the other hand, this model allows a simple analytical analysis of the backflow effect on the vortex mass, which would require less transparent numerical calculations in more realistic models.

V. VORTEX MASS IN THE BOSE LIQUID: BACKFLOW AND COMPRESSIBILITY CONTRIBUTIONS

In an ideal liquid a singular vortex line has no own inertia and can not move with respect to the liquid, in which the vortex line is immersed (Helmholtz's theorem). But, this statement is exact only in the limit of an infinitely thin line. Taking into account the finite size of the vortex core, the vortex line can move with its own velocity $\boldsymbol{v}_L$ different from that of the surrounding liquid and there is an inertial force proportional to the vortex line acceleration $d\boldsymbol{v}_L/dt$.

A naive estimation for the vortex mass is to deduce it from the picture of a cylinder without its own mass moving through a perfect fluid assuming that the cylinder has a radius equal to the core radius.[10] Then classic hydrodynamics tells that the cylinder induces a dipole velocity field around it (backflow):

$$\boldsymbol{V}_{bf}(\boldsymbol{r}) = \frac{\kappa}{2\pi}\nabla\theta_{bf} = -r_c^2 \nabla\left[\frac{\boldsymbol{v}_{bf}\cdot\boldsymbol{r}}{r^2}\right]. \tag{74}$$

The condition of the absence of the radial current through the core (cylinder) boundary in the coordinate frame moving the vortex velocity $\boldsymbol{v}_L$ requires that the constant velocity $\boldsymbol{v}_{bf}$, which determines the backflow, is $\boldsymbol{v}_{bf} = \boldsymbol{v}_L - \boldsymbol{v}_s$. Here $\boldsymbol{v}_s$ is the superfluid velocity far from the vortex core. We consider the case of $T = 0$ when $n_s = n$. The kinetic energy of the backflow is given by

$$\mu_v \frac{(\boldsymbol{v}_L - \boldsymbol{v}_s)^2}{2} = \frac{mnr_c^4}{2}\int_{r>r_c} d\boldsymbol{r}^2 \left|\nabla\left[\frac{(\boldsymbol{v}_L - \boldsymbol{v}_s)\cdot\boldsymbol{r}}{r^2}\right]\right|^2$$

$$= \pi r_c^2 mn \frac{(\boldsymbol{v}_L - \boldsymbol{v}_s)^2}{2}. \tag{75}$$

So this yields the vortex mass μ_v equal to $\mu_{\text{core}} = \pi r_c^2 mn$, which is a mass per unit length of the liquid inside a cylinder of the radius equal to the core radius r_c.[10] Later we shall call μ_{core} a core mass (in contrast to a more general term vortex mass taking into account all possible contributions to the mass of the vortex).

The vortex mass can be determined from calculation of the vortex-velocity-dependent contribution to the energy or the momentum. Naturally, both calculations should yield the same mass. Sometimes it is simpler to calculate the momentum.[2] But, calculation of the momentum of the backflow has a subtlety well known in classical hydrodynamics. The direct way to estimate the momentum of the potential velocity field in an incompressible liquid is to integrate the expression for

the momentum by parts. For the backflow this yields

$$\boldsymbol{P} = mn\frac{\kappa}{2\pi}\int\nabla\theta_{bf}(\boldsymbol{r})\,d\boldsymbol{r}$$

$$= mn\frac{\kappa}{2\pi}\left(\int_{r=r_c}\theta_{bf}\boldsymbol{n}\,dS - \int_{r\to\infty}\theta_{bf}\boldsymbol{n}\,dS\right), \tag{76}$$

where integration is reduced to integrals over the cylindric surfaces of radius r_c and of infinite radius and $\boldsymbol{n}$ is a normal to these surfaces. Strictly mathematically speaking, this yields zero since surface integrals do not depend on surface radii for the backflow field. However, one should take into account that any finite momentum $\boldsymbol{P}$ in an incompressible liquid means that the whole liquid moves with the velocity $\boldsymbol{P}/mnV$ inversely proportional to the volume V. One should take into account this tiny velocity simply by deleting the contribution from the distant surface. This yields the momentum called in hydrodynamics Kelvin impulse (see Sec. 119 in the textbook by Lamb[36]):

$$\boldsymbol{P}_K = mn\frac{\kappa}{2\pi}\int_{r=r_c}\theta_{bf}\boldsymbol{n}\,dS = \mu_{\text{core}}\boldsymbol{v}_{bf}. \tag{77}$$

In classical hydrodynamics they justify using the Kelvin impulse for an object moving through an incompressible liquids by considering the momentum transferred to the object when making it to move from rest.[36] But, in quantum hydrodynamics the justification looks simpler. Local perturbations of the velocity field can not change the phase at infinity. So, the boundary condition at infinity is not vanishing velocity, but vanishing phase, i.e., the potential of the velocity field. On the basis of it, the integral over the distant surface in the expression for the momentum should be ignored.

The calculation of the vortex mass assumed that a moving core is impenetrable for the fluid as a real rigid cylinder though the cylinder itself has no mass. In reality the vortex core is not empty and the superfluid will flow through the core, thus producing a reduced backflow field.[14] So, our simple calculation provides only an upper bound on the vortex mass related to the core. For illustration of this effect let us consider the model of a partially filled core with constant particle density $n(1 - \lambda)$ inside characterized by the parameter $\lambda < 1$. Inside the core, the liquid moves with the constant velocity $\boldsymbol{v}_{\text{in}}$, which corresponds to the phase $\theta_{\text{in}} = 2\pi(\boldsymbol{v}_{\text{in}}\cdot\boldsymbol{r})/\kappa$. The continuity of the phases θ_{in} inside the core and the phase $\theta_{\text{out}} = 2\pi[(\boldsymbol{v}_s - \boldsymbol{v}_L)\cdot\boldsymbol{r}]/\kappa + \theta_{bf}$ outside the core together with continuity of the radial flow at the core boundary yield

$$\boldsymbol{v}_{\text{in}} = \boldsymbol{v}_s - \boldsymbol{v}_L - \boldsymbol{v}_{bf}, \quad \boldsymbol{v}_{bf} = -\frac{\lambda}{2 - \lambda}(\boldsymbol{v}_s - \boldsymbol{v}_L). \tag{78}$$

In the coordinate frame moving with the vortex velocity this gives the momentum

$$\boldsymbol{P}_L = \pi r_c^2 mn(1 - \lambda)\boldsymbol{v}_{\text{in}} + (S - \pi r_c^2)nm(\boldsymbol{v}_s - \boldsymbol{v}_L) + P_K$$

$$= nm(\boldsymbol{v}_s - \boldsymbol{v}_L)\left(S - \pi r_c^2\frac{2\lambda}{2 - \lambda}\right), \tag{79}$$

where S is the whole area occupied by the liquid. In order to see the value of the vortex mass, one needs to know the

momentum in the arbitrary coordinate frame:

$$
\begin{aligned}
\boldsymbol{P} &= \boldsymbol{P}_L + mn\big[S - \pi r_c^2 + (1-\lambda)\pi r_c^2\big]\boldsymbol{v}_L \\
&= mn\left(S - \pi r_c^2 \frac{2\lambda}{2-\lambda}\right)\boldsymbol{v}_s + mn\pi r_c^2 \frac{\lambda^2}{2-\lambda}\boldsymbol{v}_L.
\end{aligned} \quad (80)
$$

The vortex mass $\mu_v = \mu_{\text{core}}\lambda^2/(2-\lambda)$ is a factor before the vortex velocity $\boldsymbol{v}_L$. If the density suppression $\Delta n = n\lambda$ in the core is small, the vortex mass is quadratic in Δn.

Strictly speaking the model of constant density in the core is not relevant for singular vortices in Bose superfluids where the density must vanish on a vortex axis. Therefore, in Appendix B we derive the vortex mass for the Bose superfluid using a more realistic model with linear in r growth of the density in the core. On the other hand, the model of constant density in the core is relevant for continuous vortices in the Fermi liquids, namely, for estimation of the effect of superfluid density suppression on the vortex mass. However, this contribution is small compared to the effect of bound states in the core (see Sec. VI).

But in the Bose liquid the most important contribution to the vortex mass is connected with finite compressibility of the liquid. The cross term in the kinetic energy of the velocity field $\boldsymbol{v}_s(\boldsymbol{r}) - \boldsymbol{v}_L = \boldsymbol{v}_v(\boldsymbol{r}) + \boldsymbol{v}_s - \boldsymbol{v}_L$ in the coordinate frame moving with vortex produces the density variation in accordance with the Bernoulli law

$$
\delta n = -mn\frac{\partial n}{\partial P}\boldsymbol{v}_v(\boldsymbol{r})\cdot(\boldsymbol{v}_s - \boldsymbol{v}_L) = -\frac{n}{c_s^2}\boldsymbol{v}_v(\boldsymbol{r})\cdot(\boldsymbol{v}_s - \boldsymbol{v}_L),
$$
$$(81)$$

where $\partial n/\partial P = 1/mc_s^2$ is the fluid compressibility, P is the pressure, and c_s is the sound velocity. The density variation leads to the energy contribution[12,13]

$$
\begin{aligned}
\mu_{\text{com}}\frac{(\boldsymbol{v}_L - \boldsymbol{v}_s)^2}{2} &= \int_{r>r_c} d\boldsymbol{r}^2 \frac{\partial^2 E}{\partial n^2}\frac{\delta n^2}{2} \\
&= \int_{r>r_c} d\boldsymbol{r}^2 \frac{\partial \mu}{\partial n}\frac{\delta n^2}{2} = \frac{\varepsilon}{c_s^2}\frac{(\boldsymbol{v}_L - \boldsymbol{v}_s)^2}{2},
\end{aligned}
$$
$$(82)$$

where $\mu = \partial E/\partial n$ is the chemical potential, $\partial \mu/\partial n = mc_s^2/n$, and

$$
\varepsilon = \frac{mn\kappa^2}{4\pi}\ln\frac{R}{r_c}
$$

is the static vortex energy per unit vortex-line length. Like the vortex energy, the vortex mass is determined by a logarithmically divergent integral, which has to be cut off at some hydrodynamic scale R, e.g., the intervortex distance. In the Bose superfluid, according to the Gross-Pitaevskii theory, the core radius $r_c \sim \kappa/c_s$ is also determined by the sound velocity c_s and, as a consequence, the compressibility mass is by the logarithmic factor larger than the core mass $\mu_{\text{core}} = \pi r_c^2 mn$.

VI. VORTEX MASS IN THE FERMI SUPERFLUID

The two contributions to the vortex mass (from the backflow and the liquid compressibility) in the Bose superfluid in principle are relevant also in the Fermi superfluid. However, the compressibility becomes inessential in the weak-coupling

limit despite a large logarithmic factor. The difference with the Bose superfluid is that while in the Bose superfluid the sound velocity goes down (compressibility goes up) in the weak-interaction limit, in the Fermi superfluid the sound velocity remains high being always of the order of the Fermi velocity. But, the most important difference between the Bose and the Fermi superfluids comes from bound core states, which contribute not only to the mutual-friction force but also to the vortex mass.[2]

Earlier we derived the momentum $\boldsymbol{P}_{bs}$ in the ground state in the presence of the superflow past the vortex [Eq. (73)]. The factor before the vortex velocity $\boldsymbol{v}_L$ in this expression is the Kopnin mass $\mu_K = \pi\hbar n/\omega_0$. However, the full vortex mass is not reduced to the Kopnin mass. The current $\boldsymbol{j}_{bs} = \boldsymbol{P}_{bs}/\pi mr_c^2$ in the bound states exists only inside the core and must transform to the superfluid current outside the core. The latter current forms the backflow velocity field, which must be determined from the continuity equation for the total fluid. As a result, the Kopnin mass will be renormalized by the backflow effect.

In analogy with the analysis of the backflow for the Bose liquid, the local superfluid velocity $\boldsymbol{v}_{sl} = \boldsymbol{v}_s + \boldsymbol{v}_{bf}$ at the core boundary differs from the superfluid velocity far from the vortex and the continuity of the current at the core boundary is

$$
\boldsymbol{j}_{bs} = \frac{\hbar}{\omega_0 mr_c^2}n(\boldsymbol{v}_L - \boldsymbol{v}_s - \boldsymbol{v}_{bf}) = n\boldsymbol{v}_{bf}. \quad (83)
$$

Note that the current in the continuum of delocalized states does not affect this condition because it has no discontinuity at the core boundary and contributes the same term $mn\boldsymbol{v}_{sl}$ to the two sides of this equation. The latter yields

$$
\boldsymbol{v}_{bf} = \frac{\mu_K}{\mu_{\text{core}} + \mu_K}(\boldsymbol{v}_L - \boldsymbol{v}_s), \quad (84)
$$

and the total momentum including the backflow momentum (Kelvin impulse) $\boldsymbol{P}_K$ given by Eq. (77) is

$$
\boldsymbol{P}_{bs} + \mu_{\text{core}}\boldsymbol{v}_{bf} = \frac{2\mu_{\text{core}}\mu_K}{\mu_{\text{core}} + \mu_K}(\boldsymbol{v}_L - \boldsymbol{v}_s). \quad (85)
$$

According to this expression, the Kopnin mass μ_K is renormalized by the factor $2\mu_{\text{core}}/(\mu_K + \mu_{\text{core}})$ equal to $\frac{4}{3}$ for the value of ω_0 given by Eq. (69) in the limit of large-core radius $r_c \gg \hbar^2 k_f/2m\Delta$. The most important outcome of this analysis is not this numerical factor, which depends on the model of the core anyway, but a more adequate insight into the origin of the vortex mass. If the Kopnin mass μ_K is much smaller than the core mass μ_{core}, the Kopnin mass is renormalized by the factor 2, i.e., the backflow gives the same contribution as the bare Kopnin mass. The case of small normal density of bound states is realized for a vortex with a continuous core in superfluid ³He when the core radius r_c essentially exceeds the coherence length $\xi_c = \hbar v_F/\Delta$ and $\mu_K \sim \mu_{\text{core}}\xi_c/r_c$. Addressing this case, Volovik[6,37] arrived to an incorrect conclusion that the contribution of the backflow to the vortex mass is negligible compared to the bare Kopnin mass. The reason for it was that Volovik used the condition of continuity of the superfluid component [see Eq. (24.16) in his book[6]], whereas only the total particle number of the liquid but not its superfluid part is conserved in the presence of the Andreev reflection. In fact, Volovik estimated the backflow effect from weak suppression

of the superfluid density inside the continuous core considered for the Bose liquid in the previous section. He ignored the backflow induced by the current in bound states.

VII. BOLTZMANN EQUATION FOR THE CORE-STATES QUASIPARTICLES: THE KOPNIN-KRAVTSOV FORCE AND THE VORTEX MASS

If there are impurities in superconductors or collisions of bound quasiparticles with free bulk quasiparticles in superfluid ^{3}He, the bound states produce not only the vortex mass, but also the mutual-friction force (Kopnin-Kravtsov force). In this case one should use the Boltzmann equation.[2,19] Let us write the Boltzmann equation in the continuum of semiclassical states bound in the core and characterized by the two Hamiltonian-conjugate quantities "angle α-moment L_z":

$$\frac{\partial f}{\partial t} - \frac{\partial \epsilon}{\partial \alpha}\frac{\partial f}{\partial L_z} + \frac{\partial \epsilon}{\partial L_z}\frac{\partial f}{\partial \alpha} = \frac{\partial f}{\partial t}\bigg|_{\text{col}}. \tag{86}$$

The collision term in the right-hand side in the relaxation-time approximation is

$$\frac{\partial f}{\partial t}\bigg|_{\text{col}} = -\frac{f - f_n(\epsilon, \boldsymbol{v}_n)}{\tau}. \tag{87}$$

It takes into account elastic collisions with impurities in superconductors (then $\boldsymbol{v}_n$ is the velocity of the crystal lattice) or with bulk free quasiparticles in superfluids. Here

$$f_n(\epsilon, \boldsymbol{v}_n) = \frac{1}{e^{\frac{\epsilon - \hbar \boldsymbol{k}\cdot(\boldsymbol{v}_n - \boldsymbol{v}_L)}{T}} + 1} = \frac{1}{e^{\frac{\epsilon_{00} - \hbar \boldsymbol{k}\cdot(\boldsymbol{v}_n - \tilde{\boldsymbol{v}}_s)}{T}} + 1} \tag{88}$$

is the distribution function for bound states, which are in the equilibrium with the normal component.

The equilibrium distribution function in the collision term has a small anisotropic part if the superfluid part moves with respect to the normal part of the liquid. This is a well-known property of the Boltzmann equation in superconductors.[6,19,38] Note that Kopnin[2] used the different Boltzmann equation, which follows from that used in the paper if the superfluid velocity $\tilde{\boldsymbol{v}}_s$ is replaced by the normal velocity $\boldsymbol{v}_n = 0$. This difference does not lead to the difference in the Kopnin-Kravtsov force and the Kopnin mass since they do not depend on the relative velocity $\tilde{\boldsymbol{v}}_s - \boldsymbol{v}_n$. But, in general it could be important, e.g., for nonstationary phenomena when the distribution function varies in time.

We expand the distribution functions around the isotropic equilibrium distribution function $f_0(\epsilon_{00})$:

$$f(\boldsymbol{p}) = f_0(\epsilon_{00}) + f_1(\epsilon, \boldsymbol{v}_n). \tag{89}$$

The zero-approximation function $f_0(\epsilon_{00})$ is the equilibrium Fermi distribution function equal to f_n at $\boldsymbol{v}_n = \boldsymbol{v}_s = \boldsymbol{v}_L$. The equation for the first-order correction linear in the relative velocities is

$$\hbar\omega_0 \boldsymbol{k}\cdot[(\tilde{\boldsymbol{v}}_s - \boldsymbol{v}_L)\times\hat{z}]\frac{\partial f_0}{\partial \epsilon} - \omega_0\frac{\partial f_1}{\partial \alpha}$$
$$= -\frac{1}{\tau}\left[f_1 - \hbar\boldsymbol{k}\cdot(\boldsymbol{v}_n - \tilde{\boldsymbol{v}}_s)\frac{\partial f_0}{\partial \epsilon}\right]. \tag{90}$$

Its solution is

$$f_1 = \frac{\partial f_0}{\partial \epsilon}\hbar\bigg[\boldsymbol{k}\cdot(\tilde{\boldsymbol{v}}_s - \boldsymbol{v}_L)$$
$$- \frac{\omega_0\tau\boldsymbol{k}\cdot[(\boldsymbol{v}_n - \boldsymbol{v}_L)\times\hat{z}] + \boldsymbol{k}\cdot(\boldsymbol{v}_n - \boldsymbol{v}_L)}{1 + \omega_0^2\tau^2}\bigg]. \tag{91}$$

The total momentum in the vortex-core bound states[2,9] is given by

$$\boldsymbol{P}_{bs} = \frac{1}{2}\int_{-k_F}^{k_F}\frac{dk_z}{2\pi}\int d\alpha\int_{L_z^{\min}}^{L_z^{\max}}\frac{dL_z}{2\pi}\boldsymbol{k}f(\alpha, L_z). \tag{92}$$

Here $L_z^{\max} = L_0 + \hbar k_f r_c\theta_s/2$ and $L_z^{\min} = -L_0 + \hbar k_f r_c\theta_s/2$ are maximal and minimal values of the angular momentum in the bound state, which differ from $\pm L_0 = \pm\Delta/\omega_0$ because of the phase shift θ_s. This may look as if the momentum depends on the phase shift θ_s, contrary to what was calculated for the ground state in Sec. IV B. Indeed, the anisotropic part of the distribution function f_1 obtained from the Boltzmann equation depends on the θ_s dependent $\tilde{\boldsymbol{v}}_s$ given by Eq. (72). This is a natural result since the Boltzmann equation takes into account only events near the Fermi surface, while entries and exits of the bound states to and from the forbidden gap at the top and at the bottom of the gap are also important, as was demonstrated above. In fact, these events are accounted for with direction-dependent limits $L_z^{\max}$ and $L_z^{\min}$ of the integral in Eq. (92). One may change variable in this integral, introducing the modified angular momentum $L_z' = L_z - \hbar k_f r_c\theta_s/2$ so that

$$\boldsymbol{P}_{bs} = \int_{-k_F}^{k_F}\frac{dk_z}{4\pi}\int d\alpha\int_{-L_0}^{L_0}\frac{dL_z'}{2\pi}\boldsymbol{k}$$
$$\times\left[f_1 - \frac{\partial f_0}{\partial \epsilon}\hbar\boldsymbol{k}\cdot(\boldsymbol{v}_\theta - \boldsymbol{v}_L)\right]. \tag{93}$$

The second term in brackets cancels the θ_s-dependent term in f_1, and eventually after using the distribution function given by Eq. (91) only the local superfluid velocity $\boldsymbol{v}_{sl}$ appears in the expression for the total momentum of core bound states. One can use the modified angular momentum L_z' as a new variable instead of L_z from the very beginning in the Boltzmann equation (86) itself with the same result: the phase difference θ_s drops out from all expressions and the effective superfluid velocity $\tilde{\boldsymbol{v}}_s$ reduces to the local superfluid velocity $\boldsymbol{v}_{sl}$ outside the core.

In the limit of zero temperature $\partial f_0/\partial \epsilon = -\delta(\epsilon)$ and the momentum in the bound states is

$$\boldsymbol{P}_{bs} = \frac{\pi\hbar n}{\omega_0}\left\{\boldsymbol{v}_L - \boldsymbol{v}_{sl} + \frac{\omega_0\tau[(\boldsymbol{v}_n - \boldsymbol{v}_L)\times\hat{z}] + \boldsymbol{v}_n - \boldsymbol{v}_L}{1 + \omega_0^2\tau^2}\right\}. \tag{94}$$

The expression reduces to Eq. (73) in the limit $\tau \to \infty$. The part of the momentum linear in $\boldsymbol{v}_L$ determines the Kopnin mass taking into account the effect of collisions. The mass becomes a tensor:

$$\hat{\mu}_K = \frac{\pi\hbar n\tau}{1 + \omega_0^2\tau^2}\begin{pmatrix}\omega_0\tau & -1\\ 1 & \omega_0\tau\end{pmatrix}. \tag{95}$$

Kopnin and Vinokur[15] called the term in the momentum transverse to the relative velocity $\boldsymbol{v}_n - \boldsymbol{v}_L$ *transverse vortex*

mass. This term, however, does not lead to a conservative inertial force, which follows from some Hamiltonian. It determines a high-frequency correction to the dissipative (longitudinal) mutual-friction force, which has its counterpart in the dissipative function (see the next section).

Repeating the process of renormalization of the Kopnin mass by the backflow effect, one obtains the same renormalization factor $2\mu_{core}/(\mu_K + \mu_{core})$ as obtained in the previous section without collisions. In this factor, the Kopnin mass $\mu_K = \pi\hbar n/\omega_0$ is the scalar mass in the limit $\tau \to \infty$.

In the case of frequent collisions ($\tau \ll 1/\omega_0$) the velocity $\boldsymbol{v}_L$ drops out from the expression (94) for the momentum, and the Kopnin mass vanishes. This is because in this limit, the effect of reflections from the walls of the core is fully suppressed by frequent collisions with impurities or quasiparticles. This does not mean the total absence of the vortex mass but its absence *in our approximation*, which neglected effects of the order Δ/ϵ_F. It is worthwhile to note that the small $\omega_0\tau$ does not necessarily invalidate the assumption that the mean-free path l_f of quasiparticles is much longer than the core radius mentioned in the Introduction. Indeed, since $\tau = l_f/v_F$ and $\omega_0 \sim \hbar/mr_c^2$, the condition $\omega_0\tau \ll 1$ reduces to the condition $l_f/r_c \ll \epsilon_F/\Delta$. In the weak-coupling limit ϵ_F/Δ is very large so even large l_f/r_c can satisfy this condition.

The contribution of the bound states to the mutual-friction force [see Eq. (1)] is determined by the momentum transferred from bound states confined in the vortex core to normal quasiparticles or impurities via collisions:

$$\boldsymbol{F}_c = \frac{1}{2}\int_{-k_F}^{k_F}\frac{dk_z}{2\pi}\int d\alpha \int_{L_z^{\min}}^{L_z^{\max}}\frac{dL_z}{2\pi}\boldsymbol{k}\frac{\partial f}{\partial t}\bigg|_{col}. \tag{96}$$

Substituting f_1 from Eq. (91), the core contribution to the mutual-friction force is

$$\boldsymbol{F}_c = \pi\hbar n\frac{\omega_0\tau(\boldsymbol{v}_n - \boldsymbol{v}_L) - [(\boldsymbol{v}_n - \boldsymbol{v}_L) \times \hat{z}]}{1 + \omega_0^2\tau^2}. \tag{97}$$

The force component transverse to $\boldsymbol{v}_n - \boldsymbol{v}_L$ is the Kopnin-Kravtsov force. Uniting this force with the Magnus force in the left-hand side of Eq. (1) at $T = 0$ ($n_s = n$), one obtains the total transverse force

$$\boldsymbol{F}_\perp = mn\kappa_c[\hat{z} \times (\boldsymbol{v}_L - \boldsymbol{v}_s)] - mn\kappa_c\frac{[\hat{z} \times (\boldsymbol{v}_L - \boldsymbol{v}_n)]}{1 + \omega_0^2\tau^2}$$
$$= mn_M\kappa_c[\hat{z} \times \boldsymbol{v}_L] - mn\kappa_c\left\{[\hat{z} \times \boldsymbol{v}_s] - \frac{[\boldsymbol{v}_n \times \hat{z}]}{1 + \omega_0^2\tau^2}\right\}, \tag{98}$$

where

$$n_M = \frac{\omega_0^2\tau^2}{1 + \omega_0^2\tau^2} \tag{99}$$

is the density determining the effective Magnus force on the vortex. In the limit $\omega_0\tau \to 0$, the Kopnin-Kravtsov force compensates the Magnus force, and the total transverse force vanishes.

For a better understanding of the Kopnin-Kravtsov force, let us derive it by replacing in the integral of Eq. (96) the collision term by the left-hand side of the Boltzmann equation (86), which is the divergence of the flow of quasiparticles in the phase space $\{\alpha, L_z\}$ and corresponds to the Liouville equation.

After integration by parts, the Kopnin-Kravtsov force is

$$\boldsymbol{F}_c = \frac{1}{2}\int_{-k_F}^{k_F}\frac{dk_z}{2\pi}\int d\alpha \int_{L_z^{\min}}^{L_z^{\max}}\frac{dL_z}{2\pi}\boldsymbol{k}\left(-\frac{\partial\epsilon}{\partial\alpha}\frac{\partial f}{\partial L_z} + \frac{\partial\epsilon}{\partial L_z}\frac{\partial f}{\partial\alpha}\right)$$
$$= -\frac{1}{2}\int_{-k_F}^{k_F}\frac{dk_z}{2\pi}\int d\alpha \left\{\boldsymbol{k}\frac{\partial\epsilon}{\partial\alpha}f(L_z)\bigg|_{L_z^{\min}}^{L_z^{\max}}\right.$$
$$\left. + \int_{L_z^{\min}}^{L_z^{\max}}\frac{dL_z}{2\pi}[\hat{z} \times \boldsymbol{k}]\omega_0 f(L_z)\right\}. \tag{100}$$

Here we took into account that $\partial\epsilon/\partial L_z$ does not depend on α and $\partial\epsilon/\partial\alpha$ does not depend on L_z. The first term in the final expression is the momentum flux in the L_z space through the upper and the lower boundaries of the gap and the second term is the momentum transfer from the external force driving the vortex at the process of the bound-state rotation with the angular velocity $\omega_0 = \partial\epsilon/\partial L_z$. While the isotropic part of the distribution function contributes to the first term, only the anisotropic part provides the second term. Restricting ourselves with the case of $\omega_0\tau \to 0$ when the solution of the Boltzmann equation $f = f_n$ is given by Eq. (88), one obtains the Kopnin-Kravtsov force for this case. So the origin of the Kopnin-Kravtsov force looks clear and does not require a reference to the artificial concept of spectral flow.

VIII. EFFECT OF VORTEX MASS ON VORTEX DYNAMICS

Taking into account all forces discussed above, the general equation describing free motion of the vortex in the resting liquid ($\boldsymbol{v}_s = \boldsymbol{v}_n = 0$) is

$$\mu_v\frac{d\boldsymbol{v}_L}{dt} - mn_M\kappa[\hat{z} \times \boldsymbol{v}_L] = -\gamma\boldsymbol{v}_L - \mu_\perp\left[\hat{z} \times \frac{d\boldsymbol{v}_L}{dt}\right], \tag{101}$$

where n_M is given by Eq. (99) and κ must be replaced by κ_c in the Fermi superfluid. At zero temperature, n_M varies from $n_M = n$ for superclean superconductors down to $n_M = 0$ for moderately dirty superconductors. The right-hand side of the equation contains two dissipative forces. The second of them is connected with the transverse vortex mass originated from core bound states.[15] It determines a high-frequency correction to the dissipative (longitudinal) mutual-friction force, which does not appear in the Hamiltonian but has its counterpart in the dissipative function. In order to see it, let us derive the time variation of the kinetic energy of the vortex:

$$\frac{dE}{dt} = \frac{d}{dt}\left(\mu_v\frac{v_L^2}{2}\right) = -2F_D. \tag{102}$$

Here the dissipative function is

$$F_D = \frac{\gamma v_L^2}{2} + \frac{\mu_\perp}{2}\left[\boldsymbol{v}_L \times \frac{d\boldsymbol{v}_L}{dt}\right]\cdot\hat{z}. \tag{103}$$

The contribution of the transverse mass to the dissipative function is not positively defined. Therefore, the equation of motion as given by Eq. (101) makes sense only if the transverse-mass contribution is small compared to that of the usual friction force $\propto \gamma$.

Without dissipation, Eq. (101) is analogous to the equation of motion of a charged particle in a magnetic field. The vortex rotates around a circular orbit with the angular

velocity $\omega_c = mn_M\kappa/\mu_v$, which is an analog of the cyclotron frequency. The frequency ω_c also characterizes the frequency of an ac process at which the vortex-mass effect can compete with the transverse Magnus force. In the Bose liquid with $n_M = n$ and the vortex mass $\mu_v \sim mnr_c^2$, the frequency ω_c is on the order of $c_s^2/\kappa \sim \kappa/r_c^2$. A phonon with such a high frequency has a wavelength comparable with the core radius r_c. If the vortex moves around a circumference of the radius r_0, which exceeds the core radius, the linear velocity $\omega_c r_0$ exceeds the value of the critical velocity $c_s \sim \kappa/r_c$. Hardly this rotation is of practical importance. In the Fermi liquid, the frequency ω_c is of the same order as the frequency ω_0, which determines the distance $\hbar\omega_0$ between core energy levels. Although the latter is small in comparison with the gap Δ in the weak-coupling limit, the frequency itself is rather high. This is true both in the pure limit when $n_M = n$ and $\mu_v \sim mnr_c^2$ and in the dirty limit when $n_M = \omega_0^2\tau^2 n$ and $\mu_v \sim \omega_0^2\tau^2 mnr_c^2$. In all, it is not simple to reveal the vortex mass in superfluids and superconductors, although some experimental evidence of the vortex mass in superconducting thin films has been recently reported.[39]

IX. DISCUSSION AND CONCLUSIONS

Since our analysis does not reveal the spectral flow in the core of the moving vortex, let us discuss the arguments by Volovik[6,16] in favor of its existence. Deriving the spectral flow Volovik considered the bound state angular momentum $L_z' = \hat{z} \cdot [(\boldsymbol{r} - (\boldsymbol{v}_L - \boldsymbol{v}_n)t) \times \boldsymbol{p}]$ around the axis, which moves together with the thermal bath (normal component). Here $\boldsymbol{r}$ is the position vector with the origin on the symmetry axis of the moving vortex. Volovik's angular momentum varies in time:

$$\frac{dL_z'}{dt} = -\hat{z} \cdot [(\boldsymbol{v}_L - \boldsymbol{v}_n) \times \boldsymbol{p}]. \tag{104}$$

Since the energy of the bound state is proportional to the angular momentum, Volovik concluded that the energy levels move in the energy space and cross the zero energy level with the rate proportional to $\boldsymbol{v}_L - \boldsymbol{v}_n$. The problem with this argument is that the position of the bound-state energy with respect to the gap depends on the angular momentum about the symmetry axis of the vortex in the coordinate frame moving together with the vortex. Then the angular momentum is conserved and provides a good quantum number, which determines the energy of the bound state. In any other coordinate frame with the reference axis, which does not coincide with the vortex axis, the angular momentum is not conserved and is not a quantum number. Moreover, deriving Eq. (104), Volovik assumed that the momentum $\boldsymbol{p}$ of the bound state does not vary in time. Meanwhile, in a genuine bound state the momentum $\boldsymbol{p}$ rotates with the angular velocity ω_0 and vanishes in average. As a result, average dL_z'/dt vanishes also, and the angular momentum determined with reference to any axis does not differ from the angular momentum around the vortex symmetry axis. This is a direct consequence of the theorem of mechanics, which tells that for a system with vanishing velocity of the center of mass, the angular momentum does not depend on the choice of the reference axis. So the vortex motion with respect to the thermal bath does not lead to the spectral flow.

Volovik stressed that his derivation was only for continuum limit $\omega_0\tau \to 0$ when levels are strongly broadened and in fact
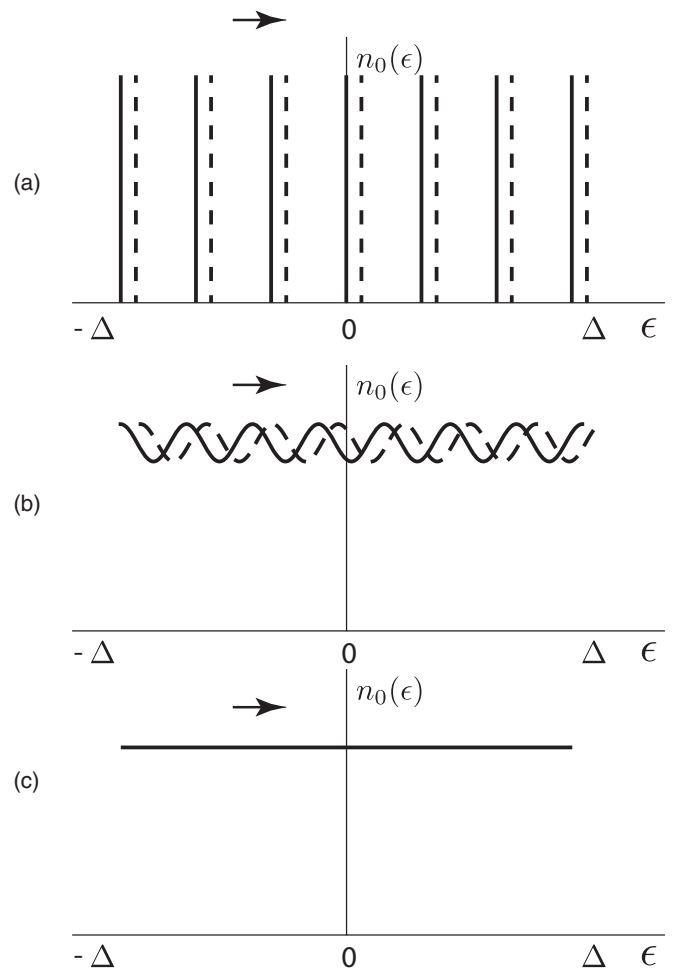

FIG. 4. Effect of shift of energy levels on the density of states $n_0(\epsilon)$ at various $\omega_0\tau$. The density of states is shown by solid lines before the shift and by dashed lines after the shift. (a) $\omega_0\tau \to \infty$: The density of states is a chain of sharped peaks. (b) $\omega_0\tau \ll 1$: Very broad peaks strongly overlap and cause only weak oscillations of the density of states, which are still noticeable in principle. (c) $\omega_0\tau = 0$: The plot of the density of states is totally flat and its shift does not lead to any physical consequence.

cease to be discrete levels. Originally, the spectral flow concept was considered only for discrete levels. In the continuum limit, the very concept of the spectral flow becomes ambiguous. This is illustrated in Fig. 4, which shows the effect of the level shift on the density of states $n_0(\epsilon)$ for various $\omega_0\tau$. Without collisions ($\omega_0\tau \to \infty$), the density of states is a chain of very narrow peaks and a shift of the levels with respect to the forbidden gap is a clear effect [Fig. 4(a)]. For very small but still finite $\omega_0\tau$, the effect of level shift on the density of states is much weaker but still noticeable [Fig. 4(b)]. In the extreme case $\omega_0\tau = 0$ when oscillations of the density of states are totally undetectable the level shift does not lead to any effect and can not influence any physical process. Without taking into account whatever tiny oscillations of the density of states, it is impossible even to define it.

Altogether this puts in question not the Kopnin-Kravtsov force itself but the connection of the force with the spectral flow. So, the claim that the experiment on mutual-friction force

confirms the spectral flow[6,17] is not justified. It is the Kopnin-Kravtsov force, which was revealed in the experiment, but not the spectral flow.

Arguing for the spectral flow in vortex dynamics they frequently draw an analogy with the Andreev reflection bound states in the superconductor-normal-metal–superconductor (SNS) junctions. Although this analogy is useful indeed for the bound states in the model of normal core,[19] it fails with respect to the role of spectral flow. In the SNS system the spectral flow really exists if the phase difference between two superconducting banks monotonously varies in time, as, e.g., in the ac Josephson effect. But, the superfluid phase difference across the core of the moving vortex does not vary in time in average. Therefore, the spectral flow exists in the former case, but is totally absent in the latter.

The absence of spectral flow in the core automatically rules out the spectral flow in the continuum of delocalized states also suggested by Volovik.[6] Indeed, in stationary processes these spectral flows should be equal. Otherwise, there were accumulation or depletion of states at the borders between localized and delocalized states. It is shown in this paper that the whole transverse force on the vortex from delocalized states in Fermi superfluids can be explained by peculiarities of the Aharonov-Bohm effect for BCS quasiparticles without referring to the spectral flow.

The paper clarifies also the role of the backflow on the vortex mass. The backflow is a ubiquitous phenomenon, which arises from mismatching of currents inside and outside the vortex core, either due to suppression of the fluid density in the Bose liquid, or due to to currents through core bound states in the Fermi liquid. Its existence follows from the conservation law for the particle number (charge). In the Fermi liquid the backflow leads to renormalization of the Kopnin vortex mass by a numerical factor both for singular and continuous vortices.

APPENDIX A: BOUND STATES IN A CORE WITH LINEAR GROWTH OF SUPERFLUID DENSITY

We consider a quasiparticle inside the core, which moves back and forth along an approximately straight trajectory parallel to the y changing its direction of motion via Andreev reflection. One can refer to the Bogoliubov–de Gennes equations in the 1D case eliminating fast oscillations of the wave function by the transformation $u = \tilde{u}e^{ik_F y}$, $v = \tilde{v}e^{ik_F y}$. For the sake of simplicity, we assume that there are no components of the wave vector parallel to the z axis. Neglecting second derivatives of $\tilde{u}$ and $\tilde{v}$, the Bogoliubov–de Gennes equations are

$$-i\hbar v_F \frac{d\tilde{u}(b,y)}{dy} + \frac{\Delta r}{r_c}e^{i\theta}\tilde{v}(b,y) = \epsilon\tilde{u}(b,y),$$
$$i\hbar v_F \frac{d\tilde{v}(b,y)}{dy} + \frac{\Delta r}{r_c}e^{-i\theta}\tilde{u}(b,y) = \epsilon\tilde{v}(b,y). \quad (A1)$$

Here $r = \sqrt{b^2 + y^2}$, and the linear dependence of the gap $\Delta(r) = \Delta r/r_c$ on the distance r is assumed. In the absence of superfluid motion through the core the phase θ coincides with the azimuthal angle $\phi = \arctan(y/b)$, and the Bogoliubov–de Gennes equations become

$$-i\hbar v_F \frac{d\tilde{u}(b,y)}{dy} + \frac{\Delta(b+iy)}{r_c}\tilde{v}(b,y) = \epsilon\tilde{u}(b,y),$$
$$i\hbar v_F \frac{d\tilde{v}(b,y)}{dy} + \frac{\Delta(b-iy)}{r_c}\tilde{u}(b,y) = \epsilon\tilde{v}(b,y). \quad (A2)$$

The normalized solution of the Bogoliubov–de Gennes equations is

$$\tilde{u} = -\tilde{v} = \frac{e^{-y^2/2\tilde{r}^2}}{2\sqrt{\pi}\tilde{r}} \quad (A3)$$

with the energy equal to

$$\epsilon_0 = -\frac{b\Delta}{r_c} = -\omega_0 L_z. \quad (A4)$$

Here the length $\tilde{r} = \sqrt{r_c\xi_c}$ is the geometric average of the core radius r_c and the coherence length $\xi_c = \hbar v_F/\Delta$ with all three lengths being of the same order of magnitude, $L_z = \hbar k_F b$ is the angular momentum of the bound state, and the frequency

$$\omega_0 = \frac{\Delta}{\hbar k_F r_c} \quad (A5)$$

gives the angular velocity of slow trajectory rotation around the vortex axis, in accordance with the canonical relation equating the rotation velocity to $\partial\mathcal{H}/\partial L_z$.

The energy spectrum given by Eq. (A4) insignificantly differs from the spectrum obtained in the original paper[20] and in the book by de Gennes[21] more accurately using the partial-wave analysis and a more realistic variation of the gap Δ in the space. This agreement confirms a simple picture of the bound states assuming well-defined trajectories of quasiparticle motion. However, it is necessary to stress that although the trajectory is well defined in the sense that the impact parameter is well defined, the motion along the trajectory can not be described semiclassically. In particular, our solution shows that there are no well-defined Andreev reflection points. Using the semiclassical approach and the Bohr-Sommerfeld condition for calculation of energy levels, one obtains a totally wrong spectrum, which is not linear in the angular momentum. So, the semiclassical theory of motion along the trajectory of the bound state is valid only for the model of the totally normal core but not for more realistic models with nonzero order parameter in the core.

APPENDIX B: VORTEX MASS OF A CORE WITH LINEAR GROWTH OF SUPERFLUID DENSITY IN THE BOSE SUPERFLUID

According to the Gross-Pitaevskii theory in the vortex core, the density grows linearly with the distance r from the vortex axis. Extrapolating this dependence up to the core radius r_c and approximating the density outside the core by the constant value n, the continuous density distribution is

$$n(r) = \begin{cases} n\frac{r}{r_c} & \text{at } r < r_c, \\ n & \text{at } r > r_c. \end{cases} \quad (B1)$$

The liquid mass inside the core,

$$\tilde{m}_{\text{core}} = 2\pi \frac{mn}{r_c} \int_0^{r_c} r^2 \, dr = \frac{2}{3}\mu_{\text{core}}, \qquad \text{(B2)}$$

is by the factor $\frac{2}{3}$ less than the core mass μ_{core} estimated under the assumption that the liquid density is not suppressed inside the core. If the superfluid moves past the vortex, the continuity equation in the coordinate frame related to the vortex is

$$\nabla[n(r)\nabla\theta] = n(r)\nabla^2\theta + \nabla n(r) \cdot \nabla\theta = 0, \qquad \text{(B3)}$$

where the phase θ determines the velocity field: $\boldsymbol{v}_s(\boldsymbol{r}) - \boldsymbol{v}_L = (\kappa/2\pi)\nabla\theta(\boldsymbol{r})$. From symmetry, all fields are dipole fields, and the phase in the cylindric coordinate system is $\theta(\boldsymbol{r}) = \theta(r)\cos\phi$, where ϕ is the azimuthal angle with respect to the velocity $\boldsymbol{v}_s - \boldsymbol{v}_L$. The one-dimensional function $\theta(r)$ is determined from the equation

$$\frac{d\theta^2}{dr^2} + \left[\frac{1}{r} + \frac{1}{n(r)}\frac{dn(r)}{dr}\right]\frac{d\theta}{dr} - \frac{\theta}{r^2} = 0. \qquad \text{(B4)}$$

Inside the core, Eq. (B4) yields that $\theta \propto r^\alpha$ with the exponent $\alpha = (\sqrt{5} - 1)/2 < 1$. This means that the velocity (but not the current!) has a weak integrable singularity at $r = 0$. The continuity of the azimuthal component of the superfluid velocity at the core boundary $r = r_c$ is satisfied by the following phase distribution outside and inside the core (the azimuthal angle dependence is omitted):

$$\theta_{\text{out}} = \frac{2\pi}{\kappa}\left(v_s r - \frac{v_{bf} r_c^2}{r}\right), \quad \theta_{\text{in}} = \frac{2\pi}{\kappa}(v_s - v_{bf})\frac{r^\alpha}{r_c^{\alpha-1}}. \qquad \text{(B5)}$$

Here v_s is the superfluid velocity far from the vortex in the coordinate frame moving with the vortex and v_{bf} is the amplitude of the backflow velocity field given by Eq. (74).

Continuity of the radial velocity gives the condition

$$\frac{d\theta_{\text{in}}}{dr} = \frac{2\pi}{\kappa}\alpha(v_s - v_{bf}) = \frac{d\theta_{\text{out}}}{dr} = \frac{2\pi}{\kappa}(v_{bf} + v_s). \qquad \text{(B6)}$$

This yields the relation

$$v_{bf} = -v_s\frac{1 - \alpha}{1 + \alpha}. \qquad \text{(B7)}$$

The total momentum includes the momentum $\boldsymbol{P}_{\text{in}}$ inside the core, the momentum of transport superfluid velocity $\boldsymbol{v}_s$ outside the core, and the Kelvin impulse of the backflow velocity field outside the core [Eq. (77)]:

$$\begin{aligned}
\boldsymbol{P}_L &= \frac{m\kappa}{2\pi}\int_{r<r_c} n(r)\nabla\theta_{\text{in}}\,d\boldsymbol{r} \\
&\quad + mn\big(S - \pi r_c^2\big)(\boldsymbol{v}_s - \boldsymbol{v}_L) + \boldsymbol{P}_K \\
&= mn\left[S - \pi r_c^2\left(\frac{\alpha}{2 + \alpha} + \frac{1 - \alpha}{1 + \alpha}\right)\right](\boldsymbol{v}_s - \boldsymbol{v}_L), \quad \text{(B8)}
\end{aligned}$$

where the superfluid velocity $\boldsymbol{v}_s$ was replaced by the relative velocity $\boldsymbol{v}_s - \boldsymbol{v}_L$. The last step is to transform the momentum $\boldsymbol{P}_L$ in the coordinate frame moving with the vortex to the momentum in the arbitrary coordinate frame:

$$\begin{aligned}
\boldsymbol{P} &= \boldsymbol{P}_L + \big[mn\big(S - \pi r_c^2\big) + \tilde{m}_{\text{core}}\big]\boldsymbol{v}_L \\
&= mn\left[S - \pi r_c^2\left(\frac{\alpha}{2 + \alpha} + \frac{1 - \alpha}{1 + \alpha}\right)\right]\boldsymbol{v}_s + \mu_v\boldsymbol{v}_L. \quad \text{(B9)}
\end{aligned}$$

Here μ_v is the vortex mass. Taking into account the value $\alpha = (\sqrt{5} - 1)/2$ the vortex mass is

$$\begin{aligned}
\mu_v &= \pi r_c^2 mn\left[\frac{\alpha}{2 + \alpha} + \frac{1 - \alpha}{1 + \alpha} - \frac{1}{3}\right] \\
&= \pi r_c^2 mn\left[2\sqrt{5} - 4\frac{1}{3}\right] = 0.139\mu_{\text{core}}. \qquad \text{(B10)}
\end{aligned}$$

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