

**Electron-hole pairing in topological insulator heterostructures in the quantum Hall state**K. V. Germash<sup>1</sup> and D. V. Fil<sup>2</sup><sup>1</sup>*Karazin Kharkov National University, Svobody Square, 4, Kharkov 61022, Ukraine*<sup>2</sup>*Institute for Single Crystals, National Academy of Sciences of Ukraine, Lenin Avenue 60, Kharkov 61001, Ukraine*

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A thin film of a topological insulator (TI) on a dielectric substrate and a bulk TI–dielectric film–bulk TI structure are considered as natural double-well heterostructures suitable for realizing the counterflow superconductivity. The effect is connected with pairing of electrons and holes belonging to different surfaces of TI and the transition of a gas of electron-hole pairs into a superfluid state. The case of TI heterostructures subjected to a strong perpendicular magnetic field is considered. It is shown that such systems are characterized by two critical temperatures—a mean-field temperature of pairing and a much smaller temperature of the superfluid transition. The dependence of the critical temperatures on the magnetic field is computed. The advantages of TI based structures in comparison with GaAs heterostructures as well as graphene based heterostructures are discussed.

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**I. INTRODUCTION**

During the last two decades spontaneous interlayer phase coherence in quantum Hall bilayers was the subject of comprehensive investigations. According to the theoretical predictions,<sup>1–6</sup> a double-layer electron system subjected to a strong magnetic field directed perpendicular to the layers should demonstrate unusual transport behavior at the total filling factor of Landau levels close to  $\nu_T = 1$ . Such behavior is connected with the interlayer pairing of electrons and holes belonging to the zeroth Landau level (the formation of stable magnetoexcitons) and the transition of the magnetoexciton gas into a superfluid state. The superfluid state is expected to reveal itself in a flow of antiparallel electrical supercurrents in adjacent layers and in the vanishing of the Hall voltage. The effect was realized in AlGaAs heterostructures by a number of groups.<sup>7–9</sup> A huge increase in the counterflow conductivity and a strong lowering of the Hall voltage was observed at temperatures below 1 K. Nevertheless, in these experiments a state with infinite counterflow conductivity was not registered. It can be accounted for the presence of unbound vortices,<sup>10–13</sup> but the question is still open.

The typical magnetic field used for the observation of the effect in AlGaAs heterostructures is  $B \sim 2$  T. At such a field one can fulfill the condition for the filling factor  $\nu_T = 1$  as well as the requirement for the magnetic length  $\ell = \sqrt{\hbar c / eB}$  to be larger or of order of the interlayer distance  $d$ . But this field does not provide smallness of the Coulomb energy  $E_c = e^2 / \epsilon \ell$  ( $\epsilon$  is the dielectric constant of the matrix) compared to the distance between Landau levels  $\hbar\omega_c = \hbar^2 / m_* \ell^2$  ( $m_*$  is the effective mass of carriers). In that case the validity of the lowest Landau-level approximation (commonly used in theoretical studies) is questionable. The opposite inequality  $E_c < \hbar\omega_c$  can be achieved at higher magnetic fields that corresponds to smaller  $\ell$ , but smaller  $\ell$  require smaller  $d$ . Then, the interlayer tunneling amplitude increases, that is a negative factor for the counterflow superconductivity.<sup>14–16</sup>

Graphene systems open new prospects for realizing the magnetoexciton superfluidity in bilayers.<sup>17–22</sup> In graphene the relation between the Coulomb energy and the distance between Landau levels does not depend on the magnetic field. The zeroth Landau level in graphene is separated from the nearest

positive and negative levels by the gap  $\Delta E_{01} = \sqrt{2} \hbar v_F / \ell$ , where  $v_F \approx 10^6$  m/s is the Fermi velocity in graphene. The inequality  $E_c < \Delta E_{01}$  is equivalent to  $\epsilon > \alpha_{\text{eff}} / \sqrt{2}$ , where  $\alpha_{\text{eff}} = e^2 / \hbar v_F \approx 2.2$  is the effective fine-structure constant for suspended graphene. The latter is fulfilled in graphene-based heterostructures, where SiO<sub>2</sub>, Al<sub>2</sub>O<sub>3</sub>, or BN compounds are used as the dielectric parts.<sup>23,24</sup>

As was shown in Refs. 19 and 20, an imbalance of filling factors of graphene layers is required for realizing the magnetoexciton superfluidity in bilayer graphene structures. It is connected with an additional fourfold degeneracy of Landau levels due to the spin and valley degrees of freedom. This behavior is in similarity with one for the  $\nu_T = 2$  quantum Hall bilayers.<sup>25</sup> The imbalance can be created by an electrostatic field applied perpendicular to the graphene layers. The change in the electrical field should follow the change in the magnetic field. It is required to keep the ratio  $E/B$  close to the value  $E/B = \alpha c / \epsilon$ , where  $\alpha \approx 1/137$  is the fine-structure constant. For instance, for  $\epsilon = 4$  and  $B \approx 1$  T the electrical field  $E \approx 5 \times 10^3$  V/cm is required.

The discovery of topological insulators (see Refs. 26 and 27 and references therein) stimulates new proposals toward realizing the superfluidity of spatially indirect excitons.<sup>28–32</sup> The idea<sup>28–32</sup> is that the surface of the topological insulator (TI) may work as a natural two-dimensional conductor, while the interior of TI works as a dielectric. The electron spectrum of the TI surface states is similar to the graphene spectrum: it contains Dirac cones. The TI surface should therefore demonstrate the same quantum Hall behavior as graphene. On the TI surface the number of Dirac cones is odd, in particular, the most studied three-dimensional TI Bi<sub>2</sub>Se<sub>3</sub> belongs to a so-called one-cone family. Thus, TI systems have the same advantage as graphene systems—the smallness of the Coulomb energy comparing to the distance between Landau levels, but, at the same time, they are free from the disadvantage caused by the additional degeneracy of Landau levels in graphene.

In this paper we analyze two types of structures: a TI film on a dielectric substrate, and a bulk TI–dielectric film–bulk TI heterostructure. In Sec. II we obtain the zero-temperature phase diagram in the coordinates “the ratio  $d/\ell$ –the dielectric constants.” It is shown that for the TI film on a substrate

the range of allowed  $d/\ell$  ( $d$  is the TI film thickness) is restricted from below and from above and it shrinks under increase in the dielectric constant of the substrate  $\varepsilon_s$ . For known TIs the dielectric constant  $\varepsilon_{TI}$  is large ( $\sim 10^2$ ) and the typical situation corresponds to  $\varepsilon_s/\varepsilon_{TI} \lesssim 0.1$ . In this case the state with electron-hole pairing is expected in the range  $d/\ell = 0.5-2$ . For the TI–dielectric film–TI structure the range of allowed  $d/\ell$  ( $d$  is the dielectric film thickness) is restricted only from above and it shrinks under decrease in the dielectric constant of the dielectric film  $\varepsilon_d$ . For  $\varepsilon_d/\varepsilon_{TI} \lesssim 0.1$  the maximum allowed  $d/\ell$  is small ( $\lesssim 0.2$ ).

In Sec. III the mean-field critical temperature  $T_{mf}$  and the temperature of the superfluid transition  $T_s$  as the functions of the magnetic field are computed. The temperature  $T_{mf}$  is given by the self-consistence equation for the order parameter. The temperature  $T_s$  is the Berezinskii-Kosterlitz-Thouless transition temperature. We find that the strong inequality  $T_s \ll T_{mf}$  is fulfilled. For the TI film structures the superfluid state is reached at temperatures higher than for the TI–dielectric film–TI structures. The temperature  $T_s$  is a nonmonotonic function of the magnetic field  $B$ . At fixed  $B$  it increases under decrease in  $\varepsilon_s$  and under increase in  $\varepsilon_d$ .

## II. ZERO-TEMPERATURE PHASE DIAGRAM

Let us consider the electron surface states of a one-cone TI in a quantizing magnetic field directed perpendicular to the surface. The low-energy Hamiltonian has the form  $H_0 = \pm v_F(p_x \sigma_y - p_y \sigma_x) + m \sigma_z$ , where  $\sigma_i$  are the Pauli matrices that act in the spin space,  $\mathbf{p} = -i\hbar \nabla + e\mathbf{A}/c$  is the momentum operator,  $\mathbf{A}$  is the vector potential,  $m$  is the Zeeman splitting, and  $v_F$  is the Fermi velocity that is the material parameter (typically,  $v_F \approx 5 \times 10^5$  m/s). The eigenproblem is given by the Dirac equation

$$\begin{pmatrix} m & \mp i v_F (P_- + \frac{e}{c} A_-) \\ \pm i v_F (P_+ + \frac{e}{c} A_+) & -m \end{pmatrix} \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix} = E \begin{pmatrix} \Psi_\uparrow \\ \Psi_\downarrow \end{pmatrix}, \quad (1)$$

where  $P_\pm = -i\hbar(\partial_x \pm i\partial_y)$  and  $A_\pm = A_x \pm iA_y$ . The upper (lower) sign in Eq. (1) corresponds to the top (bottom) surface.

The eigenproblem (1) yields the following energies for the Landau levels:

$$E_0 = -m, \quad E_{\pm N} = \pm \sqrt{2 \left( \frac{\hbar v_F}{\ell} \right)^2 N + m^2},$$

where  $N = 1, 2, \dots$ . The eigenfunctions are presented in the Appendix.

It is implied that at zero magnetic field the Fermi level is tuned to the Dirac point  $E = 0$ . It can be done by the appropriate doping of TI.<sup>33</sup> In the magnetic field the zeroth Landau level is shifted from zero. But if the number of surface carriers is conserved, the Fermi level is shifted as well and it coincides with the zeroth Landau level. We note that in the case considered the Zeeman splitting  $m = g\mu_B B/2$  (where  $\mu_B$  is the Bohr magneton, and  $g$  is the gyromagnetic ratio) is small in comparison with the distance between the Landau levels (e.g.,  $m/E_1 \approx 3 \times 10^{-3}$  at  $B = 1$  T).

The Coulomb interaction Hamiltonian for the electrons in the zeroth Landau level reads

$$H = \frac{1}{2} \sum_{i,i'} \int d^2 r d^2 r' V_{i,i'}(|\mathbf{r} - \mathbf{r}'|) \hat{\rho}_i(\mathbf{r}) \hat{\rho}_{i'}(\mathbf{r}'), \quad (2)$$

where  $V_{i,i'}(r)$  is the potential of the Coulomb interaction between electrons located on the  $i$  and  $i'$  working surfaces,

$$\hat{\rho}_i(\mathbf{r}) = \sum_{X_1, X_2} \Phi_{0, X_1}^*(\mathbf{r}) \Phi_{0, X_2}(\mathbf{r}) c_{i, X_1}^\dagger c_{i, X_2} \quad (3)$$

is the electron density operator in the second quantization representation,  $c_{i, X}^\dagger$  ( $c_{i, X}$ ) is the creation (annihilation) operator for the electron in the zeroth Landau level on the surface  $i$ ,  $\Phi_{0, X}(\mathbf{r})$  is the eigenfunction [see the Appendix, Eq. (A1)], and  $:\hat{O}:$  means the normal ordering of an operator  $\hat{O}$ .

Substituting (A1) and (3) into (2) we obtain the following expression for the Coulomb interaction Hamiltonian:

$$H = \frac{1}{2S} \sum_{ii'} \sum_{X, X', \mathbf{q}} V_{ii'}(\mathbf{q}) e^{-q^2 \ell^2 / 2 + i q_x (X' - X)} \times c_{i, X+q_y \ell^2 / 2}^\dagger c_{i', X'-q_y \ell^2 / 2}^\dagger c_{i', X'+q_y \ell^2 / 2} c_{i, X-q_y \ell^2 / 2}, \quad (4)$$

where  $V_{ii'}(\mathbf{q})$  are the Fourier components of the potential and  $S$  is the area of the system.

In what follows we neglect the influence of outer boundaries on the interaction between electrons on the working surfaces and consider the model heterostructure “an infinitely thick dielectric 1—the working surface 1—a dielectric 2 of thickness  $d$ —the working surface 2—an infinitely thick dielectric 3.” In the general case the dielectrics 1, 2, and 3 are characterized by different dielectric constants  $\varepsilon_1$ ,  $\varepsilon_2$ , and  $\varepsilon_3$ , correspondingly. For such a structure the quantities  $V_{ii'}(\mathbf{q})$  read as

$$V_{11}(\mathbf{q}) = \frac{4\pi e^2}{q} \frac{\varepsilon_2 + \varepsilon_3 + (\varepsilon_2 - \varepsilon_3)e^{-2qd}}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_2 + \varepsilon_1) - (\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_1)e^{-2qd}}, \quad (5)$$

$$V_{22}(\mathbf{q}) = \frac{4\pi e^2}{q} \frac{\varepsilon_2 + \varepsilon_1 + (\varepsilon_2 - \varepsilon_1)e^{-2qd}}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_2 + \varepsilon_1) - (\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_1)e^{-2qd}}, \quad (6)$$

$$V_{12}(\mathbf{q}) = \frac{8\pi e^2}{q} \frac{\varepsilon_2 e^{-qd}}{(\varepsilon_2 + \varepsilon_3)(\varepsilon_2 + \varepsilon_1) - (\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_1)e^{-2qd}}. \quad (7)$$

The pairing of electrons of surface 1 with holes of surface 2 is characterized by the order parameter

$$\Delta_X = \langle \Psi | c_{1X}^\dagger c_{2X} | \Psi \rangle, \quad (8)$$

where  $|\Psi\rangle$  is the many-particle wave function. In (8) the relation between the electron annihilation and hole creation operator  $c_{iX} = h_{iX}^\dagger$  is taken into account. We consider the many-particle wave function

$$|\Psi\rangle = \prod_X (u c_{1X}^\dagger h_{2X}^\dagger + v) |\text{vac}\rangle = \prod_X (u c_{1X}^\dagger + v c_{2X}^\dagger) |0\rangle \quad (9)$$

that is an analog of the wave function introduced in the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity. The  $u$ - $v$  coefficients satisfy the relation  $|u|^2 + |v|^2 = 1$ .

We parametrize them as  $u = \cos(\theta_0/2)$  and  $v = e^{i\varphi_0} \sin(\theta_0/2)$ . In Eq. (9)  $|0\rangle$  is the state with the empty zeroth Landau level, and  $|\text{vac}\rangle$  is a “vacuum” state defined as  $|\text{vac}\rangle = \prod_X c_{2X}^+ |0\rangle$ .

The energy of the state (9)  $E = \langle \Psi | H | \Psi \rangle$  reads

$$E = \frac{S}{8\pi\ell^2} \left( W \cos^2 \theta_0 - \left[ \frac{J_{11} + J_{22}}{2} (1 + \cos^2 \theta_0) + (J_{11} - J_{22}) \cos \theta_0 \right] - J_{12} \sin^2 \theta_0 \right), \quad (10)$$

where

$$W = \frac{1}{2\pi\ell^2} \lim_{q \rightarrow 0} \left[ \frac{V_{11}(q) + V_{22}(q)}{2} - V_{12}(q) \right] = \frac{e^2 d}{\varepsilon_2 \ell^2} \quad (11)$$

is the energy (per particle) of the direct Coulomb interaction, and

$$J_{ik} = \frac{1}{2\pi} \int_0^\infty q V_{ik}(q) e^{-\frac{q^2 \ell^2}{2}} dq \quad (12)$$

are the energies of the intralayer and interlayer exchange interaction.

In the state (9) the filling factors of the zeroth Landau level on surfaces 1 and 2 are

$$\nu_{1(2)} = \frac{1 \pm \cos \theta_0}{2}. \quad (13)$$

The difference  $\tilde{\nu} = \nu_1 - \nu_2 = \cos \theta_0$  (the filling factor imbalance) is determined by the condition of minimum of the energy (10) and it may vary under variation in the magnetic field.

The minimum is reached at

$$\cos \theta_0 = \begin{cases} 1, & \text{at } \frac{J_{11} - J_{22}}{2(W + J_{12}) - J_{11} - J_{22}} > 1; \\ -1, & \text{at } \frac{J_{11} - J_{22}}{2(W + J_{12}) - J_{11} - J_{22}} < -1; \\ \frac{J_{11} - J_{22}}{2(W + J_{12}) - J_{11} - J_{22}}, & \text{otherwise} \end{cases} \quad (14)$$

[it follows from the direct computations that  $2(W + J_{12}) - J_{11} - J_{22} > 0$  for any  $d$  and  $\varepsilon_i$ ]. According to Eq. (14) in the general case the filling factor imbalance depends on the ratio between  $\varepsilon_i$  and on the parameter  $d/\ell$ .

In the state (9) the modulus of the order parameter is equal to  $|\Delta| = \sin \theta_0/2$ . Zero imbalance ( $\cos \theta_0 = 0$ ) corresponds to the maximum order parameter. This case is realized in the symmetric heterostructures ( $\varepsilon_1 = \varepsilon_3$ ). In the asymmetric heterostructures the imbalance is nonzero which results in the lowering of the order parameter. If the imbalance becomes maximum ( $\tilde{\nu} = \pm 1$ ) the order parameter goes to zero. The direct evaluation of (14) shows that the imbalance increases under decrease in  $d/\ell$  and it reaches the maximum at some nonzero value of that ratio. Thus, there is the critical  $d/\ell$  below which electron-hole pairing does not occur in the asymmetric system.

A restriction on the parameter  $d/\ell$  also comes from the dynamical stability condition (the condition for the collective mode spectrum to be real valued). To obtain the spectrum of excitations we follow the approach of Ref. 34 and consider the many-particle wave function that accounts for the fluctuations of the phase and modulus of the order parameter

$$|\Psi\rangle = \prod_X \left( \cos \frac{\theta_X}{2} c_{1X}^+ + e^{i\varphi_X} \sin \frac{\theta_X}{2} c_{2X}^+ \right) |0\rangle. \quad (15)$$

In the quadratic approximation the energy of the fluctuations can be presented in the diagonal form

$$E_{fl} = \frac{1}{2} \sum_q (m_z^*(q) \varphi^*(q)) \mathbf{K}(q) \begin{pmatrix} m_z(q) \\ \varphi(q) \end{pmatrix}, \quad (16)$$

where

$$m_z(q) = \frac{1}{2} \sqrt{\frac{2\pi\ell^2}{S}} \sum_X (\cos \theta_X - \cos \theta_0) e^{-iqX} \quad (17)$$

and

$$\varphi(q) = \sqrt{\frac{2\pi\ell^2}{S}} \sum_X \varphi_X e^{-iqX} \quad (18)$$

are the Fourier components of the fluctuations of the filling factor imbalance and the phase of the order parameter, correspondingly. The matrix  $\mathbf{K}$  in (16) is

$$\mathbf{K}(q) = \begin{pmatrix} K_{zz}(q) & 0 \\ 0 & K_{\varphi\varphi}(q) \end{pmatrix} \quad (19)$$

with the components

$$K_{zz}(q) = 2(H(q) - F_S(q) + F_D(0) + \cot^2 \theta_0 \Xi(q)), \quad (20)$$

$$K_{\varphi\varphi}(q) = \frac{1}{2} \sin^2 \theta_0 \Xi(q), \quad (21)$$

where

$$H(q) = \frac{1}{2\pi\ell^2} \left( \frac{V_{11}(q) + V_{22}(q)}{2} - V_{12}(q) \right) e^{-q^2 \ell^2 / 2}, \quad (22)$$

$$\Xi(q) = F_D(0) - F_D(q), \quad (23)$$

$$F_S(q) = \frac{1}{4\pi} \int_0^\infty p J_0(pq\ell^2) (V_{11}(p) + V_{22}(p)) e^{-p^2 \ell^2 / 2} dp, \quad (24)$$

and

$$F_D(q) = \frac{1}{2\pi} \int_0^\infty p J_0(pq\ell^2) V_{12}(p) e^{-p^2 \ell^2 / 2} dp. \quad (25)$$

In (24) and (25)  $J_0(q)$  is the Bessel function.

The quantities  $m_z(q)$  and  $\varphi(q)$  are the canonically conjugated variables. They satisfy the equations of motion  $\hbar\dot{\varphi}(q) = K_{zz}(q)m_z(q)$ ,  $\hbar\dot{m}_z(q) = -K_{\varphi\varphi}(q)\varphi(q)$  from which the following expression for the collective mode spectrum comes from

$$\Omega(q) = \sqrt{K_{zz}(q)K_{\varphi\varphi}(q)}. \quad (26)$$

Analysis of (26) shows that at  $d/\ell$  larger than some critical one the quantity  $\Omega(q)$  becomes imaginary valued at finite  $q$ . It signals for the dynamical instability of the system.

Thus, for the TI–dielectric film–TI structure (the symmetric structure) the range of existence of the superfluid state is given by the inequality  $d/\ell < \tilde{d}_{c1}$ , where  $\tilde{d}_{c1}$  is the function of the ratio  $\varepsilon_d/\varepsilon_{TI}$ . This function is shown in Fig. 1(a). One can see that the range of  $d/\ell$  suitable for realizing the magnetoexciton superfluidity broaden out under an increase in  $\varepsilon_d$ . For the TI film on a dielectric substrate (the asymmetric structure) the range of existence of the superfluid state is  $\tilde{d}_{c2} < d/\ell < \tilde{d}_{c1}$ , where  $\tilde{d}_{c1}$  and  $\tilde{d}_{c2}$  are the functions of two ratios:  $\varepsilon_1/\varepsilon_2$  and

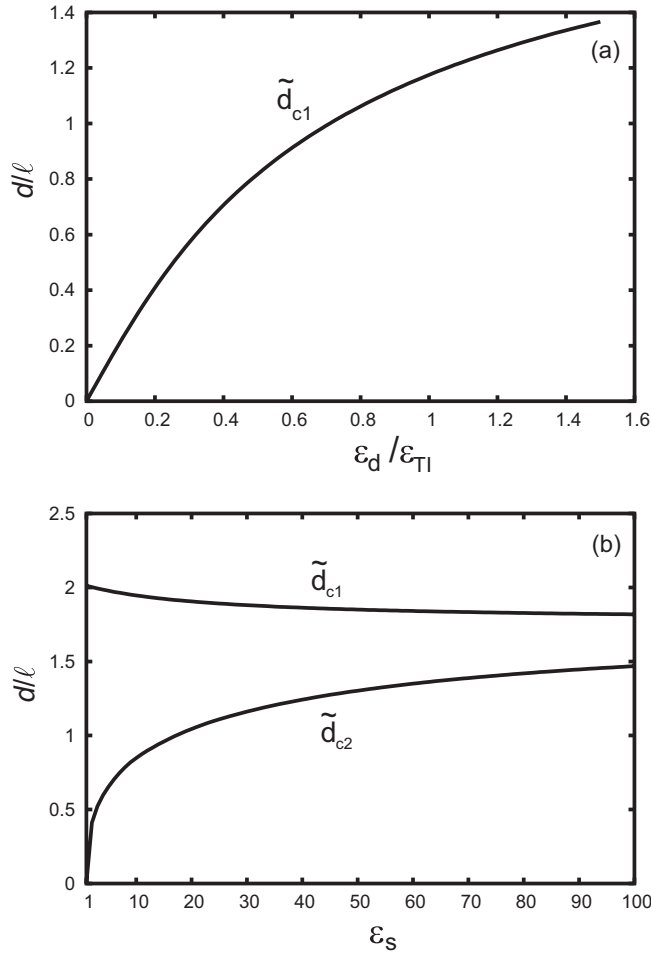


FIG. 1. The zero-temperature phase diagrams for the TI-dielectric film-TI heterostructure (a) and for the heterostructure “TI film on a dielectric substrate” (b).

$\epsilon_2/\epsilon_3$ . Here we specify  $\epsilon_1 = 1$ ,  $\epsilon_2 = \epsilon_{TI} = 80$  and present the dependencies of  $\tilde{d}_{c1}$  and  $\tilde{d}_{c2}$  on the dielectric constant of the substrate  $\epsilon_s$  [Fig. 1(b)]. One can see that the latter case is in some ways opposite to the previous one: the use of dielectric substrates with larger  $\epsilon$  results in shrinking of the range of allowed  $d/\ell$ . We emphasize that the filling factor imbalance can be controlled by the gate voltage applied to the system. Therefore, in systems subjected to an electrical field directed perpendicular to the working surfaces the lower and upper critical  $d/\ell$  will differ from ones presented in Fig. 1.

As was already mentioned in the Introduction, for graphene-based heterostructures, in difference with TI heterostructures, the use of the electrical gate is the necessary condition for realizing the magnetoexciton superfluidity. The difference is connected with the presence of only one Dirac cone on the surface of a TI as compared to four Dirac cones in graphene. The advantage of the one cone specifics of TI was discussed previously in connection with electron-hole pairing in the absence of a magnetic field.<sup>28–32</sup> In such systems the electric gate is in any case required and the advantage consists of a reduction of screening of the interlayer Coulomb attraction between electrons and holes. For the electron-hole pairing in the zeroth Landau level the screening is not so important.

Actually, the effect of screening is small if the Coulomb energy does not exceed the gap between the zeroth and  $N = 1$  Landau level. But just due to the one cone specifics of the TI the state with spontaneous interlayer phase coherence in the zeroth Landau level is stable with respect to ones without such coherence. Also, since in difference with graphene and GaAs heterostructures, the zeroth Landau level in the TI is completely spin polarized (see the Appendix), low-energy excitations connected with spin (and valley) degrees of freedom are forbidden. It allows us to consider the TI as a refined system for realizing the magnetoexciton superfluidity.

### III. FINITE-TEMPERATURE PROPERTIES

Let us consider finite-temperature behavior of the system in the framework of the mean-field approach.<sup>35,36</sup>

One obtains from (4) the following mean-field Hamiltonian:

$$H_{MF} = \sum_X \left[ \sum_i \epsilon_i c_{iX}^+ c_{iX} - (J_{12} \Delta c_{1X}^+ c_{2X} + \text{H.c.}) \right], \quad (27)$$

where  $\Delta = \langle c_{2X}^+ c_{1X} \rangle$  is the mean-field order parameter,

$$\epsilon_i = D_i - J_{ii} v_i - \mu \quad (28)$$

with  $v_i = \langle c_{iX}^+ c_{iX} \rangle$ , the mean-field filling factors,

$$D_1 = \frac{1}{2\pi\ell^2} \lim_{q \rightarrow 0} \left[ V_{11}(q) \left( v_1 - \frac{1}{2} \right) + V_{12}(q) \left( v_2 - \frac{1}{2} \right) \right], \quad (29)$$

$$D_2 = \frac{1}{2\pi\ell^2} \lim_{q \rightarrow 0} \left[ V_{22}(q) \left( v_2 - \frac{1}{2} \right) + V_{12}(q) \left( v_1 - \frac{1}{2} \right) \right], \quad (30)$$

and  $\mu$ , the chemical potential. In (29) and (30) the interaction with the positive background is taken into account.

The Hamiltonian (27) is diagonalized using the  $u$ - $v$  transformation

$$\begin{aligned} c_{1X} &= u\alpha_X + v^*\beta_X^+, \\ c_{2X} &= u^*\beta_X^+ - v\alpha_X, \end{aligned} \quad (31)$$

where  $u = \cos(\Theta/2)$  and  $v = \sin(\Theta/2)e^{i\varphi}$ . The condition of vanishing nondiagonal terms in the transformed Hamiltonian yields the following relations:

$$\sin \Theta = \frac{J_{12}|\Delta|}{\sqrt{\tilde{\epsilon}^2 + J_{12}^2|\Delta|^2}}, \quad \cos \Theta = \frac{\tilde{\epsilon}}{\sqrt{\tilde{\epsilon}^2 + J_{12}^2|\Delta|^2}}, \quad (32)$$

where

$$\tilde{\epsilon} = \frac{\epsilon_1 - \epsilon_2}{2} = \frac{1}{2} \left[ \left( W - \frac{J_{11} + J_{22}}{2} \right) \tilde{v} - \frac{J_{11} - J_{22}}{2} \right]. \quad (33)$$

The transformed Hamiltonian has the form

$$H_{MF} = \sum_X (E_\alpha \alpha_X^+ \alpha_X + E_\beta \beta_X^+ \beta_X) \quad (34)$$

with the spectrum

$$E_{\alpha(\beta)} = \sqrt{\tilde{\epsilon}^2 + J_{12}^2|\Delta|^2} \pm \frac{\epsilon_1 + \epsilon_2}{2}. \quad (35)$$

The condition  $\nu_1 + \nu_2 = 1$  yields the relation

$$1 = 1 + N_F(E_\alpha) - N_F(E_\beta), \quad (36)$$

where  $N_F(E)$  is the Fermi distribution function. The relations (35) and (36) lead to the condition

$$\epsilon_1 + \epsilon_2 = 0. \quad (37)$$

Note that Eq. (37) can be satisfied under the appropriate choice for the chemical potential  $\mu$ . It corresponds to that the chemical potential is determined by the relation  $\nu_1 + \nu_2 = 1$ . Under accounting Eq. (37) the spectrum (35) is reduced to  $E_\alpha = E_\beta = \mathcal{E} = \sqrt{\tilde{\epsilon}^2 + J_{12}^2 |\Delta|^2}$ .

Equation (36) is the first one in a set of three self-consistence equations. The other two equations read as

$$\begin{aligned} \tilde{\nu} &= -\frac{\tilde{\epsilon}}{\mathcal{E}} \tanh \frac{\mathcal{E}}{2T}, \\ \Delta &= \frac{J_{12}\Delta}{2\mathcal{E}} \tanh \frac{\mathcal{E}}{2T}. \end{aligned} \quad (38)$$

It follows from (38) that at  $\Delta \neq 0$  the following relation is fulfilled:

$$J_{12}\tilde{\nu} + 2\tilde{\epsilon} = 0. \quad (39)$$

Equation (39) yields

$$\tilde{\nu} = \frac{J_{11} - J_{22}}{2(W + J_{12}) - J_{11} - J_{22}} \quad (40)$$

that coincides with Eq. (14) under assumption that  $|J_{11} - J_{22}| < 2(W + J_{12}) - J_{11} - J_{22}$ . If the latter inequality is not fulfilled, Eq. (39) cannot be fulfilled as well, and the order parameter  $\Delta = 0$  (electron-hole pairing does not occur). Thus, in the paired state the filling factor imbalance does not depend on temperature and is given by Eq. (40).

Equations (38) yield the mean-field critical temperature of pairing<sup>35</sup>

$$T_{mf} = \frac{J_{12}}{2} \frac{|\tilde{\nu}|}{\ln \frac{1+|\tilde{\nu}|}{1-|\tilde{\nu}|}}, \quad (41)$$

where  $\tilde{\nu}$  is given by Eq. (40).

The temperature  $T_{mf}$  is the function of  $d/\ell$ . For a given sample the distance  $d$  is fixed and the parameter  $d/\ell$  depends only on the magnetic field. Therefore, it is instructive to present the critical temperature as the function of the magnetic field. We choose the  $B_d = \phi_0/\pi d^2$  units for  $B$ , where  $\phi_0 = hc/2e$  is the magnetic flux quantum [ $B/B_d = (d/\ell)^2$ ]. The quantity  $e^2/d$  is used as the energy unit. In computation  $T_{mf}$  we account for the dynamical stability condition, implying that  $T_{mf} = 0$  at  $d/\ell > \tilde{d}_{c1}$ . The result of computations for two types of heterostructures is presented in Fig. 2.

The critical temperature  $T_{mf}$  is not a temperature of the superfluid transition. The superfluid transition temperature is given by the Kosterlitz-Thouless equation

$$T_s = \frac{\pi}{2} \rho_s(T_s), \quad (42)$$

where  $\rho_s(T)$  is the superfluid stiffness. Equation (42) can be applied under assumption that the gas of bound electron-hole pairs exists. The latter requires  $T_s < T_{mf}$ . Evaluation

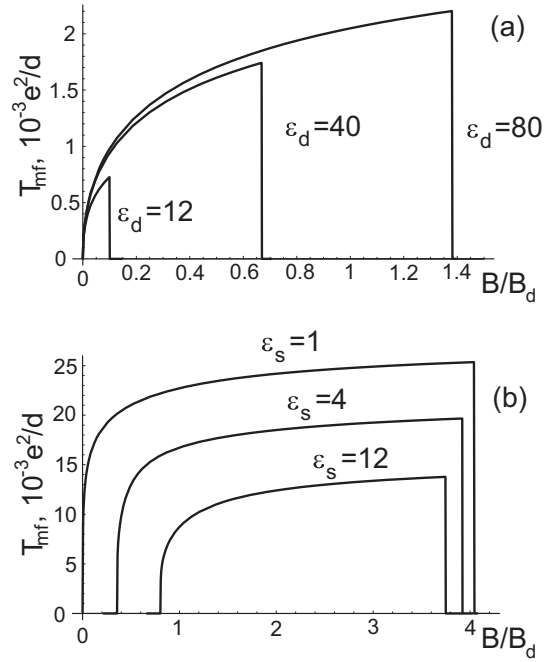


FIG. 2. The dependence of the mean-field critical temperature on the magnetic field for the TI-dielectric film-TI heterostructure (a) and for the heterostructure “TI film on a dielectric substrate” (b).

of  $T_s$  shows (see below) that, actually, the strong inequality  $T_s \ll T_{mf}$  is fulfilled. The temperature  $T_{mf}$  can therefore be interpreted as an analog of the ionization temperature. One can see from Fig. 2(b) that the dependence of  $T_{mf}$  on  $B/B_d$  is saturated at large  $B$ . It is connected with that  $T_{mf}$  is determined by the binding energy. The limit  $B \gg B_d$  corresponds to  $\ell \ll d$  and in the latter case the binding energy is determined in the main part by the interlayer distance  $d$ .

The superfluid stiffness is the coefficient of the expansion of the free energy in the gradient of the phase of the superfluid order parameter:

$$F = F_0 + \frac{S}{2} \rho_s (\nabla \varphi)^2.$$

We compute  $\rho_s(T)$  as follows. We consider the many-particle wave function

$$|\Psi\rangle = \prod_X \left( \cos \frac{\theta_X}{2} c_{1,X+Q_y \ell^2/2}^+ + e^{i(Q_x X + \varphi_X)} c_{2,X-Q_y \ell^2/2}^+ \right) |0\rangle. \quad (43)$$

Equation (43) describes the state with a uniform gradient of the phase of the order parameter:  $\nabla \varphi = \mathbf{Q} = (Q_x, Q_y)$ . To see that we neglect for a moment the fluctuations ( $\theta_X = \theta_0$  and  $\varphi_X = 0$ ) and define the space-dependent order parameter

$$\Delta(\mathbf{r}) = \sum_{X,X'} \psi_X^*(\mathbf{r}) \psi_{X'}(\mathbf{r}) \langle \Psi | c_{1X}^+ c_{2X'} | \Psi \rangle. \quad (44)$$

In (44)  $\psi_X(\mathbf{r}) = (\sqrt{\pi^{1/2} L_y \ell})^{-1} \exp[-iXy/\ell^2 - (x-X)^2/2\ell^2]$  is the one-particle wave function for the zeroth Landau level, and  $L_y$  is the size of the system in the  $y$  direction. The direct calculation yields

$$\Delta(\mathbf{r}) = \frac{\sin \theta_0}{2} e^{-Q^2 \ell^2/2} e^{i\mathbf{Q}\cdot\mathbf{r}}. \quad (45)$$

Equation (45) shows that  $\mathbf{Q}$  is indeed the gradient of the phase of the order parameter.

The free energy is given by the formula

$$F = E_0(Q) - TS = E_0(Q) + T \sum_{\mathbf{q}} \ln(1 - e^{-\Omega(\mathbf{q}, \mathbf{Q})/T}), \quad (46)$$

where

$$E_0(Q) = E_0(0) + \frac{S}{8\pi\ell^2} \sin^2 \theta_0 [F_D(0) - F_D(Q)] \quad (47)$$

is the energy of state (43),  $S$  is the entropy of the gas of elementary collective excitations,  $\Omega(\mathbf{q}, \mathbf{Q})$  is the spectrum of excitations, and  $E_0(0)$  is the energy given by Eq. (10).

At  $\mathbf{Q} = 0$  the spectrum  $\Omega(\mathbf{q}, 0) = \Omega(q)$  [Eq. (26)] is isotropic. Anisotropy of  $\Omega(\mathbf{q}, \mathbf{Q})$  is connected with the appearance of a specific direction in the system (the direction of the phase gradient).

Considering state (43) and repeating the same steps as in obtaining the spectrum (26) we find  $\Omega(\mathbf{q}, \mathbf{Q})$  for  $\mathbf{q}$  directed parallel to the  $x$  axis.

The energy of fluctuations has the form (16), where the matrix  $\mathbf{K}$  depends on  $\mathbf{Q}$ :

$$\mathbf{K}(\mathbf{q}, \mathbf{Q}) = \begin{pmatrix} K_{zz}(\mathbf{q}, \mathbf{Q}) & K_{z\varphi}(\mathbf{q}, \mathbf{Q}) \\ K_{z\varphi}^*(\mathbf{q}, \mathbf{Q}) & K_{\varphi\varphi}(\mathbf{q}, \mathbf{Q}) \end{pmatrix} \quad (48)$$

with  $\mathbf{q} = q\mathbf{i}_x$ ,

$$K_{zz}(\mathbf{q}, \mathbf{Q}) = 2(H(\mathbf{q}, \mathbf{Q}) - F_S(q) + F_D(Q) + \cot^2 \theta_0 \Xi(\mathbf{q}, \mathbf{Q})), \quad (49)$$

$$K_{\varphi\varphi}(\mathbf{q}, \mathbf{Q}) = \frac{1}{2} \sin^2 \theta_0 \Xi(\mathbf{q}, \mathbf{Q}), \quad (50)$$

$$K_{z\varphi}(\mathbf{q}, \mathbf{Q}) = i\tilde{K}_{z\varphi}(\mathbf{q}, \mathbf{Q}) = i \cos \theta_0 [F_D(|\mathbf{q} - \mathbf{Q}|) - F_D(|\mathbf{q} + \mathbf{Q}|)] / 2, \quad (51)$$

$$H(\mathbf{q}, \mathbf{Q}) = \frac{1}{2\pi\ell^2} \left( \frac{V_{11}(q) + V_{22}(q)}{2} - V_{12}(q) \cos(|\mathbf{q} \times \mathbf{Q}|/\ell^2) \right) e^{-q^2\ell^2/2}, \quad (52)$$

and

$$\Xi(\mathbf{q}, \mathbf{Q}) = F_D(Q) - \frac{F_D(|\mathbf{q} + \mathbf{Q}|) + F_D(|\mathbf{q} - \mathbf{Q}|)}{2}. \quad (53)$$

The spectrum has the form

$$\Omega(\mathbf{q}, \mathbf{Q}) = \sqrt{K_{zz}(\mathbf{q}, \mathbf{Q})K_{\varphi\varphi}(\mathbf{q}, \mathbf{Q})} + \tilde{K}_{z\varphi}(\mathbf{q}, \mathbf{Q}). \quad (54)$$

One can see that expression (54) is invariant with respect to rotation of the coordinate axes. The restriction  $\mathbf{q} = q\mathbf{i}_x$  can therefore be omitted and Eq. (54) yields the spectrum at the general  $\mathbf{q}$ . The spectrum Eq. (54) is anisotropic because of its dependence on the angle between  $\mathbf{q}$  and  $\mathbf{Q}$ .

Expanding (46) in  $\mathbf{Q}$  we arrive at the following expression for the superfluid stiffness:

$$\rho_s = \rho_{s0} + \delta\rho_s(T), \quad (55)$$

where

$$\begin{aligned} \rho_{s0} &= \frac{\sin^2 \theta_0}{8\pi\ell^2} F_D''(Q)|_{Q=0} \\ &= \frac{\ell^2}{32\pi^2} \sin^2 \theta_0 \int_0^\infty p^3 V_{12}(p) e^{-p^2\ell^2/2} dp \end{aligned} \quad (56)$$

is the zero-temperature superfluid stiffness, and

$$\begin{aligned} \delta\rho_s(T) &= \frac{1}{S} \sum_{\mathbf{q}} \left[ N_B(q) \frac{\partial^2 \Omega(\mathbf{q}, \mathbf{Q})}{\partial Q^2} \right. \\ &\quad \left. - \frac{1}{T} N_B(q)(1 + N_B(q)) \left( \frac{\partial \Omega(\mathbf{q}, \mathbf{Q})}{\partial Q} \right)^2 \right] \Big|_{Q=0} \end{aligned} \quad (57)$$

is its temperature correction. In (57)  $N_B(q) = (e^{\Omega(q)/T} - 1)^{-1}$  is the Bose distribution function.

One can show that  $\delta\rho_s(T) < 0$ . We note that Eq. (57) generalizes the expression for the superfluid stiffness.<sup>37</sup> The case of Ref. 37 corresponds to a Bose gas in a free space. In that case the Galilean transformation yields the following spectrum of excitations:  $\Omega(\mathbf{q}) = \Omega_0(q) + \hbar^2 \mathbf{q} \cdot \mathbf{Q} / M$  with  $\Omega_0(q)$ , the excitation spectrum in the reference frame, where the Bose gas is at rest, and  $M$  is the mass of the Bose particle. Then, Eq. (57) is reduced to the common one.<sup>37</sup> The dependence of the spectrum  $\Omega(\mathbf{q}, \mathbf{Q})$  [Eq. (54)] on  $\mathbf{Q}$  is more complicated and the first term in (57) should be taken into account.

The dependence of the superfluid transition temperature on the magnetic field is shown in Fig. 3. One can see that at all  $B$  the inequality  $T_s \ll T_{mf}$  is satisfied. The temperature  $T_s$  is a nonmonotonic function of the magnetic field. The superfluid

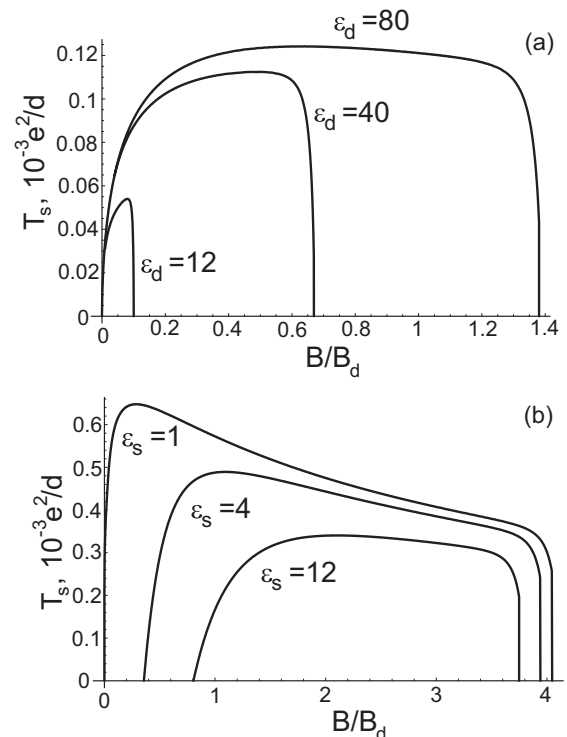


FIG. 3. The dependence of the superfluid transition temperature on the magnetic field for the TI-dielectric film-TI heterostructure (a) and for the heterostructure “TI film on a dielectric substrate” (b).

state can be realized in the whole range of allowed  $d/\ell$  ( $\tilde{d}_{c1} < d/\ell < \tilde{d}_{c2}$ ), but at  $d/\ell$  near the upper and lower range limit the temperature  $T_s$  approaches zero. The maximum temperature is reached in the middle of that range.

For the TI film structure the temperature of the superfluid transition is higher if a substrate with a lower dielectric constant  $\varepsilon_s$  is used. For the TI–dielectric film–TI structure the decrease in the dielectric constant of the film  $\varepsilon_d$  results in lowering the superfluid transition temperature.

#### IV. CONCLUSION

In conclusion, we have shown that TI heterostructures are suitable for realizing the superfluidity of spatially indirect magnetoexcitons. The structure “TI film on a substrate” is preferable to the TI–dielectric film–TI structure. The main disadvantage of the latter one is that the dielectric layer separating two TIs should be rather thin. For instance, for  $B = 1$  T,  $\varepsilon_d = 12$ , and  $\varepsilon_{TI} = 80$  the dielectric layer should not be thicker than 8 nm. For the same parameters the thickness of the TI film for the substrate with  $\varepsilon_s = 12$  can be up to 50 nm. The TI film structure has the problem of shorting two working surfaces through the side surface. But this problem can be resolved by depositing a magnetic insulator on the side surface. It opens a gap in the energy spectrum of the side surface states:  $E = \pm\sqrt{m^2 + v_F^2 p^2}$  (with  $m = J_H S_z \gg T_s$ , where  $J_H$  is the energy of the exchange coupling of the electron and ion spin, and  $S_z$  is the value of the spin of the magnetic ions) and prevents the interlayer leakage of the counterflow current through the side surface.

Taking  $d = 10$  nm we evaluate that the maximum temperature of the superfluid transition  $T_s$  is about 1 K for the TI film structures and is about 0.2 K for the TI–dielectric film–TI structures. Graphene heterostructures are characterized by slightly higher<sup>20</sup> temperatures of the transition into the superfluid state, but the disadvantage of graphene heterostructures is that

the magnetoexciton superfluidity can be realized only under application of the interlayer gate voltage.

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#### APPENDIX: LANDAU-LEVEL EIGENFUNCTIONS

In the Landau gauge  $\mathbf{A} = (0, xB, 0)$  Eq. (1) yields the following eigenfunctions: For the zeroth Landau level

$$\Phi_{0,x}(\mathbf{r}) = \frac{e^{-iky} e^{-(x-X)^2/2\ell^2}}{\pi^{1/4} \sqrt{L_y \ell}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (\text{A1})$$

where  $k$  is the wave number connected with the guiding center of the orbit  $X$  by the relation  $k = X/\ell^2$ .

For nonzero levels

$$\Phi_{\pm N,x}(\mathbf{r}) = \frac{e^{-iky} e^{-(x-X)^2/2\ell^2}}{\pi^{1/4} \sqrt{2^N N! L_y \ell}} \frac{1}{\sqrt{\frac{(E_{\pm N} - m)^2}{2N} + \left(\frac{\hbar v_F}{\ell}\right)^2}} \times \begin{pmatrix} (E_{\pm N} - m) H_{N-1}\left(\frac{x-X}{\ell}\right) \\ \pm \frac{\hbar v_F}{\ell} H_N\left(\frac{x-X}{\ell}\right) \end{pmatrix}, \quad (\text{A2})$$

where  $H_N(x)$  are the Hermite polynomials.

In a moderate magnetic field and in the absence of magnetic exchange interactions the strong inequality  $m \ll \hbar v_F/\ell$  is fulfilled and the relation between the components of the eigenfunction (A2) is practically the same as for the Landau-level eigenfunctions in graphene.

The important difference between graphene and a topological insulator is that the zeroth Landau level on the TI surface is completely spin polarized. Due to the square root dependence of the Landau-level energies on the magnetic field even in a rather small field the states with an admixture of the opposite spin polarization (e.g., the states in the  $N = 1$  Landau level) are separated from the zero level spin-polarized states by a large energy gap.

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