Multipoint correlators of conformal field theories: Implications for quantum critical transport

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We compute three-point correlators between the stress-energy tensor and the conserved currents of conformal field theories (CFTs) in $2+1$ dimensions. We first compute the correlators in the large-flavor-number expansion of conformal gauge theories and then perform the computation using holography. In the holographic approach, the correlators are computed from an effective action on $(3+1)$-dimensional anti-de Sitter space (AdS$_4$) and depend upon the coefficient γ of a four-derivative term in the action. We find a precise match between the CFT and the holographic results, thus, fixing the values of γ. The CFTs of free fermions and bosons take the values $\gamma = 1/12, -1/12$, respectively, and so saturate the bound $|\gamma| \leqslant 1/12$ obtained earlier from the holographic theory; the correlator of the conserved gauge flux of $U(1)$ gauge theories takes intermediate values of γ. The value of γ also controls the frequency dependence of the conductivity and other properties of quantum critical transport at nonzero temperatures. Our results for the values of γ lead to an appealing physical interpretation of particlelike or vortexlike transport near quantum phase transitions of interest in condensed-matter physics. This paper includes Appendices reviewing key features of the AdS-CFT correspondence for condensed-matter physicists.

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I. INTRODUCTION

This paper is a contribution to the program of connecting strongly interacting condensed-matter systems to theories based upon the methods of gauge-gravity duality.[1,2] Such methods offer powerful tools to describe dynamics at nonzero temperatures and far from equilibrium in regimes far removed from any quasiparticle theory. But they have been rigorously established only for strongly interacting non-Abelian gauge theories, which are very different from those relevant for condensed-matter applications. For the latter, the simplest context in which the connections may be made are conformal field theories (CFTs) in $2+1$ dimensions,[3] which are dual to gravity theories on anti-de Sitter space (AdS$_4$). Myers *et al.*[4] proposed extending the gauge-gravity methods to a wider class of CFTs by viewing the gravity theory as a phenomenological effective field theory on $(3+1)$-dimensional anti-de Sitter space (AdS$_4$) with physical observables to be computed in the gravity theory at *tree level*. The effective field theory was expanded in powers of space-time gradients, and all terms with up to four gradients were retained; such a field theory was also considered earlier by Ritz and Ward.[5] In this paper, we pin down the values of some of the coupling constants in this holographic theory by a matching procedure based upon the computation of three-point correlators of the stress-energy tensor and the conserved currents at zero temperature (T).[6] For the case of linear-response functions of charge transport in a CFT at zero density, all needed four gradient couplings will be determined; we will review the arguments for this in Appendix A. This allows us to relate CFTs of interest in condensed matter to a specific holographic action. And it paves the way for predictions on the nonzero T and nonequilibrium dynamics for condensed-matter systems from holographic methods as illustrated in Fig. 1.

We have written this paper for readers with backgrounds in condensed-matter theory and knowledge of general relativity. Readers with no prior knowledge of gauge-gravity duality are referred to a recent review article[8] for an overall perspective and to Appendix C for a description of the correspondence between correlators of the CFT and the theory on AdS$_4$.

Although our results are quite general, it is useful to express them in the context of a particular CFT, which has numerous condensed-matter applications.[9–11] The matter sector has Dirac fermions ψ_α, $\alpha = 1 \cdots N_f$ and complex scalars z_a, $a = 1 \cdots N_s$. We will always take the large-N_f limit with N_s/N_f fixed and will use the symbol N_F to refer generically to either N_s or N_f. These matter fields are coupled to each other and a $U(1)$ gauge field a_i by a Lagrangian of the form

$$\mathcal{L} = \sum_{\alpha=1}^{N_f} i\overline{\psi}_\alpha \gamma^i D_i \psi_\alpha + \sum_{a=1}^{N_s}\left(|D_i z_a|^2 + s|z_a|^2 + \frac{u}{2}(|z_a|^2)^2\right) + \cdots, \quad (1.1)$$

where $D_i = \partial_i - i a_i$ is the gauge covariant derivative, the Dirac matrices obey $\mathrm{Tr}(\gamma^i\gamma^j) = 2\eta^{ij}$ where η^{ij} is the Minkowski metric, and the ellipses represent additional possible contact couplings between the fermions and the bosons. The scalar "mass" term s has to be tuned to reach the quantum critical point, which is described by a CFT at the renormalization-group (RG) fixed point; fermion mass terms can be removed by imposing discrete symmetries. So the scalar mass is the only relevant perturbation at the CFT fixed point, and only a single parameter has to be tuned to access the fixed point. All other couplings, such as u and the Yukawa coupling,

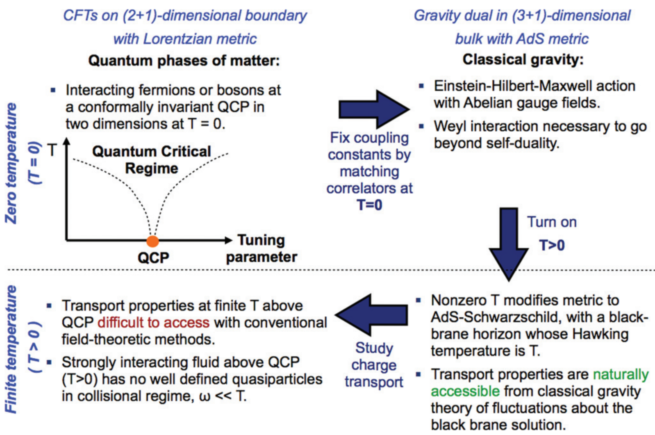

FIG. 1. (Color online) Illustration of the AdS-CFT correspondence in the context of quantum critical transport at finite temperatures. The present paper is concerned with the upper blue arrow: We fix couplings by matching correlators of the CFT to those of the gravity theory. The bottom blue arrow is addressed in Refs. 4 and 7, which computed the relevant conductivities and quasinormal modes of the gravity dual for general values of the couplings in Eq. (1.7).

reach values associated with the RG fixed point, and so their values are immaterial for the universal properties of interest in the present paper.

This CFT has three globally conserved currents. There is the $SU(N_s)$ scalar flavor current,

$$J^{\ell}_{s,i} = -i z^*_a T^{\ell}_{ab}(D_i z_b) + i(D_i z_a)^* T^{\ell}_{ab} z_b, \tag{1.2}$$

where T^{ℓ} are the generators of $SU(N_s)$ normalized by $\mathrm{Tr}(T^{\ell}T^m) = \delta^{\ell m}$. Similarly, there is the fermion $SU(N_f)$ flavor current,

$$J^{\ell}_{f,i} = \overline{\psi}_{\alpha} T^{\ell}_{\alpha\beta}\gamma_i \psi_{\beta}. \tag{1.3}$$

Finally, there is the topological $U(1)$ current,

$$J_{t,i} = \frac{1}{2\pi}\epsilon_{ijk}\partial^j a^k. \tag{1.4}$$

We will use the symbol J_i to generically refer to any one of these three currents. A basic property of the CFT (Ref. 12) is that the two-point correlator of a conserved current obeys

$$\langle J_i(\boldsymbol{k}) J_j(-\boldsymbol{k})\rangle = -C_J|\boldsymbol{k}|\left(\eta_{ij} - \frac{k_i k_j}{|\boldsymbol{k}|^2}\right), \tag{1.5}$$

where $\boldsymbol{k}$ is a space-time momentum, $\eta_{ij} = \mathrm{diag}(-1,1,1)$ is the Minkowski metric, and C_J is a dimensionless universal constant associated with the CFT and the current. Similarly, the stress-energy tensor T_{ij} of the CFT has the two-point correlator,[13]

$$\begin{aligned}\langle T_{ij}(\boldsymbol{k}) T_{uv}(-\boldsymbol{k})\rangle = C_T|\boldsymbol{k}|^3\Big(&\eta_{iu}\eta_{jv} + \eta_{ju}\eta_{iv} - \eta_{ij}\eta_{uv}\\ &+ \eta_{ij}\frac{k_u k_v}{|\boldsymbol{k}|^2} + \eta_{uv}\frac{k_i k_j}{|\boldsymbol{k}|^2}\\ &- \eta_{iu}\frac{k_j k_v}{|\boldsymbol{k}|^2} - \eta_{ju}\frac{k_i k_v}{|\boldsymbol{k}|^2} - \eta_{iv}\frac{k_j k_u}{|\boldsymbol{k}|^2}\\ &- \eta_{jv}\frac{k_i k_u}{|\boldsymbol{k}|^2} + \frac{k_i k_j k_u k_v}{|\boldsymbol{k}|^4}\Big),\end{aligned} \tag{1.6}$$

where C_T is another universal constant characterizing the CFT.

The primary focus of the present paper will be on the structure of the three-point correlator $\langle T_{i_1 j_1}(\boldsymbol{k}_1) J_{i_2}(\boldsymbol{k}_2) J_{i_3}(\boldsymbol{k}_3)\rangle$. The general form of this correlator for a CFT was specified by Osborn and Petkou[12] in position space: They showed that it was fully determined by the values of C_J, C_T, and a single additional constant. Such a position space correlator was matched to holographic results by Hofman and Maldacena,[6] and we follow their methods in Sec. VI. However, we will first perform this computation in momentum space. It is not a simple matter to take the Fourier transform of the earlier position space result,[12] and we, therefore, compute this correlator directly from the CFT and from its holographic partner.

Our purpose is to relate the conserved current correlators of the CFT (1.1) to the effective holographic theory of Refs. 4 and 5. The theory is defined on AdS_4 and has a (non-Abelian or Abelian) gauge field A_μ and corresponding gauge flux $F_{\mu\nu}$ associated with each of the conserved currents J_i. (Our convention is that greek indices run over all directions in the bulk, whereas, latin indices are used to denote boundary directions.) We note that there is no direct relationship between the bulk gauge field A_μ and the boundary gauge field a_i. As we review in Appendix A, the most general four-derivative action for linear transport of the CFT in each bulk gauge field is

$$S = \frac{1}{g_4^2}\int d^4x\sqrt{-g}\,\mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \gamma L^2 C_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\right], \tag{1.7}$$

where Tr is the trace over $SU(N_s)$ or $SU(N_f)$ indices (if present), $g_{\mu\nu}$ is the metric of AdS_4 with radius L, and $C_{\mu\nu\rho\sigma}$ is the Weyl curvature tensor; we will usually set $L = 1$, although it will be reinstated in some final results. We reiterate the conditions under which (1.7) constitutes the most-general tree-level effective holographic theory: for linear charge transport in CFTs in the absence of a chemical potential. As we review below, matching the two-point correlator of the current between (1.1) and (1.7) fixes the value of the coupling g_4. The coupling crucial for our purposes is γ; it was shown that the stability of the theory S requires $|\gamma| \leqslant 1/12$. The structure of the three-point correlator $\langle T_{i_1 j_1}(\boldsymbol{k}_1) J_{i_2}(\boldsymbol{k}_2) J_{i_3}(\boldsymbol{k}_3)\rangle$ is determined by γ, and so, γ plays the role of the additional constant noted by Osborn and Petkou[12] (the explicit relation to their constants is specified in Sec. VI). Comparison with the CFT computation yields the value of γ. An overview of the correlation functions needed to fix the values of the coupling constants in Eq. (1.7) is given in Fig. 2.

Our results for the values of γ for the currents in (1.2), (1.3), and (1.4) are

$$\begin{aligned}\gamma_f &= \frac{1}{12} + O(1/N_F), \quad \gamma_s = -\frac{1}{12} + O(1/N_F),\\ \gamma_t &= \frac{N_s - N_f}{12(N_s + N_f)} + O(1/N_F).\end{aligned} \tag{1.8}$$

It is interesting that the free CFT results (γ_f and γ_s at $N_F = \infty$) saturate the bound on γ in the large-N_F limit. We recall that a similar feature was observed in earlier computations of three-point correlators of the stress-energy tensor where the free-field results also saturate the bounds obtained from the holographic higher-derivative theory.[14,15]

Coupling	Correlator
G_N	$\frac{\langle TJJ\rangle}{\sqrt{\langle TT\rangle}\langle JJ\rangle}$
g_4^2	$\langle JJ\rangle$
γ	$\frac{\langle TJJ\rangle^{---}}{\langle TJJ\rangle^{--+}}$

FIG. 2. Correlators (with helicity projections) that fix the numerical values of the couplings in the holographic action specified by Eqs. (1.7) and (C22). These correlators are evaluated in the present paper in the boundary conformal field theory.

For $N_f = 0$, we have $\gamma_s = -\gamma_t$. This change in sign of γ is consistent with the expectations[4] of its transformation under particle-vortex duality and the interpretation of J_{ti} as the matter current in the dual theory. Further discussions on the physical consequences of these values of γ appear in Sec. VII.

We note that three-point correlators of CFTs have also played an important role in recent investigations of theories with higher-spin-conserved currents.[16] Our three-point correlator is similar, but our holographic considerations follow a different route.

The outline of the rest of the paper is as follows. In Sec. II, we describe the setting in which we perform our correlation function calculations. Section III presents the computation of the three-point correlator in the large-N_F limit of the CFT. In Sec. IV, we present the holographic computation of the three-point correlator implied by the AdS$_4$ action of Myers *et al.* at tree level. The two sets of results are matched in Sec. V. Section VI presents another derivation of our values of γ for the free-field theories, using the methods of Ref. 6. In Sec. VII, we explore some of the consequences of these results.

II. SETTING

In this section, we introduce our momentum space notation for the three-point correlators and briefly recapitulate the spinor-helicity projections that we perform in the CFT as well as in the holography computation. The momentum space expressions of the three-point correlator are obtained by Fourier transforming along the boundary directions,

$$\mathcal{K}^{i_1j_1i_2i_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \langle T^{i_1j_1}(\boldsymbol{k}_1)J^{i_2}(\boldsymbol{k}_2)J^{i_3}(\boldsymbol{k}_3)\rangle \equiv \int \langle \mathcal{T}\{T^{i_1j_1}(\boldsymbol{x}_1)J^{i_2}(\boldsymbol{x}_2)J^{i_3}(\boldsymbol{x}_3)\}\rangle \times \exp\left(i\sum_{m=1}^{3}\boldsymbol{k}_m\cdot\boldsymbol{x}_m\right)d^3x_m, \quad (2.1)$$

where $\mathcal{T}$ is the time-ordering symbol and the integral runs over the three flat directions on the boundary. (The time-ordered correlator is what we would get by computing the Euclidean correlation function and then Wick rotating to Lorentzian space.)

There are several advantages of working in momentum space. Some of these become apparent below, but let us comment on one immediate benefit. In (2.1), we have many free indices. In particular, for the stress tensor, the Ward identities tell us that, if we contract i_1 and j_1 in (2.1), this yields a known answer in terms of lower-point correlators. Similarly, (2.1) is symmetric in i_1,j_1 up to contact terms that, again, involve two-point functions. However, this still naively leaves us with 5 degrees of freedom in the stress tensor and 3 in each of the currents.

However, both the stress tensor and the currents are conserved. In position space, this leads to differential Ward identities. In momentum space, these identities become algebraic: They translate to the simple statement that, for $m = 1,2,3$, the contraction $k_{m,i_m}\mathcal{K}^{i_1j_1i_2i_3}$ is determined in terms of lower-point correlators.

This means that we can extract all the physical information in (2.1) by contracting the stress tensor with any two symmetric and traceless polarization tensors that are transverse to the momentum $\boldsymbol{k}_1$ and the two currents with polarization vectors that are transverse to $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$, respectively. So, instead, we can consider

$$\mathcal{K}(\boldsymbol{e}_1,\boldsymbol{k}_1,\boldsymbol{\epsilon}_2,\boldsymbol{k}_2,\boldsymbol{\epsilon}_3,\boldsymbol{k}_3) = e_{1,i_1j_1}\epsilon_{2,i_2}\epsilon_{3,i_3}\langle T^{i_1j_1}(\boldsymbol{k}_1)J^{i_2}(\boldsymbol{k}_2)J^{i_3}(\boldsymbol{k}_3)\rangle. \quad (2.2)$$

Here, $\boldsymbol{e}_1$ is a polarization tensor for the stress tensor, and $\boldsymbol{\epsilon}_2$ and $\boldsymbol{\epsilon}_3$ are polarization vectors for the currents. We choose these to be transverse to the momentum carried by the corresponding operator, and it is also convenient for us to choose them to be null,

$$\boldsymbol{\epsilon}_m\cdot\boldsymbol{k}_m = \boldsymbol{\epsilon}_m\cdot\boldsymbol{\epsilon}_m = 0. \quad (2.3)$$

We can choose the polarization tensor $\boldsymbol{e}_1$ to be just an outer product of two polarization vectors for $\boldsymbol{k}_1$,

$$e_{1,ij} = \epsilon_{1,i}\epsilon_{1,j}. \quad (2.4)$$

So, the use of momentum space drastically cuts down the number of independent indices that we need to deal with and allows us to directly engage with the physical quantities in (2.2).

To simplify the algebra even further, we will use the *spinor-helicity* formalism to write down explicit expressions for the polarization tensors and, later, to simplify the correlators. The spinor-helicity formalism was initially introduced to study four-dimensional scattering amplitudes as a means of efficiently encoding the kinematics of the external particles. (See Ref. 17 and references there.) It was adapted to the study of correlators in three-dimensional conformal field theories by Maldacena and Pimentel.[18]

Our conventions are different from those of Ref. 18, and we provide a detailed introduction to this formalism in Appendix D. Here, we excerpt a few of the essential details to help the reader parse the formulas in this paper.

Given a three vector $\boldsymbol{k} = (k_0,k_1,k_2)$, we consider the 2×2 matrix,

$$k_{\alpha\dot\alpha} = k_0\sigma^0_{\alpha\dot\alpha} + k_1\sigma^1_{\alpha\dot\alpha} + k_2\sigma^2_{\alpha\dot\alpha} + i|\boldsymbol{k}|\sigma^3_{\alpha\dot\alpha}, \quad (2.5)$$

where $|\boldsymbol{k}| \equiv \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}} = \sqrt{k_1^2+k_2^2-k_0^2}$. By construction, this 2×2 matrix has rank 1, and so, it can be decomposed into the outer product of a 2×1 and a 1×2 vector,

$$k_{\alpha\dot\alpha} = \lambda_\alpha\bar\lambda_{\dot\alpha}. \quad (2.6)$$

The λ and $\bar{\lambda}$ are called spinors, and instead of giving the momentum three vectors for each operator insertion, we can, instead, give these spinors.

We can define dot products of these spinors via

$$\begin{aligned}\langle\lambda_1,\lambda_2\rangle &= \epsilon^{\alpha\beta}\lambda_{1\alpha}\lambda_{2\beta} = \lambda_{1\alpha}\lambda_2^{\alpha},\\ \langle\bar{\lambda}_1,\bar{\lambda}_2\rangle &= \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{1\dot{\alpha}}\bar{\lambda}_{2\dot{\beta}} = \bar{\lambda}_{1\dot{\alpha}}\bar{\lambda}_2^{\dot{\alpha}}.\end{aligned} \tag{2.7}$$

Finally, one other advantage of this formalism is that the polarization vectors we referred to above can be written quite easily in terms of these spinors,

$$\epsilon^{+}_{\alpha\dot{\alpha}} = 2\frac{\bar{\lambda}^{\dagger}_{\alpha}\bar{\lambda}_{\dot{\alpha}}}{[\lambda,\bar{\lambda}]} = \frac{\bar{\lambda}^{\dagger}_{\alpha}\bar{\lambda}_{\dot{\alpha}}}{i|\boldsymbol{k}|},\quad \epsilon^{-}_{\alpha\dot{\alpha}} = 2\frac{\lambda_{\alpha}\lambda^{\dagger}_{\dot{\alpha}}}{[\lambda,\bar{\lambda}]} = \frac{\lambda_{\alpha}\lambda^{\dagger}_{\dot{\alpha}}}{i|\boldsymbol{k}|}, \tag{2.8}$$

where we have labeled the polarization vectors by a helicity that can be either positive or negative. We refer the interested reader to Appendix D for further details.

III. CFT COMPUTATION OF THREE-POINT CORRELATORS

In this section, we compute the three-point correlators of each of the conserved currents (1.2), (1.3), and (1.4) for the Lagrangian (1.1) with its couplings at the CFT fixed point. The stress-energy tensor is

$$T_{ij} = T_{s,ij} + T_{f,ij}, \tag{3.1}$$

which consists of a scalar bosonic contribution,

$$\begin{aligned}T_{s,ij} = \sum_{a=1}^{N_s}\Big(&(D_i z_a)^*(D_j z_a) + (D_j z_a)^*(D_i z_a)\\ &-\frac{1}{4}(\partial_i\partial_j + \eta_{ij}\partial^2)|z_a|^2\Big),\end{aligned} \tag{3.2}$$

and the fermionic contribution,

$$\begin{aligned}T_{f,ij} = \frac{i}{4}\sum_{\alpha=1}^{N_f}[&\overline{\psi}_{\alpha}\gamma_i(D_j\psi_{\alpha}) + \overline{\psi}_{\alpha}\gamma_j(D_i\psi_{\alpha})\\ &-(D_i^*\overline{\psi}_{\alpha})\gamma_j\psi_{\alpha} - (D_j^*\overline{\psi}_{\alpha})\gamma_i\psi_{\alpha}].\end{aligned} \tag{3.3}$$

We evaluate the correlators by summing over all possible Wick contractions of the constituent operators of $\langle TJJ\rangle$ defined in (2.1) in the limit of large-flavor number N_F. As expected, we see that the leading contractions with the flavor currents are those of the free CFT. For the topological currents, the first nonvanishing contractions appear at $O(1/N_F)$. All contractions involve tensor-valued one-loop integrations in momentum space, which we evaluate using Davydychev recursion relations.[19] Finally, the full tensor-valued expressions are contracted with the polarization or helicity operators defined in Sec. II to bring them to a form that facilitates comparison with the corresponding helicity projections from the holographic calculation (performed in Sec. IV).

We refer the readers to Appendix B for a review of the computations of the two-point functions $\langle JJ\rangle$ and $\langle TT\rangle$, leading to (1.5) and (1.6) and the final results after contracting with the corresponding polarization tensors.

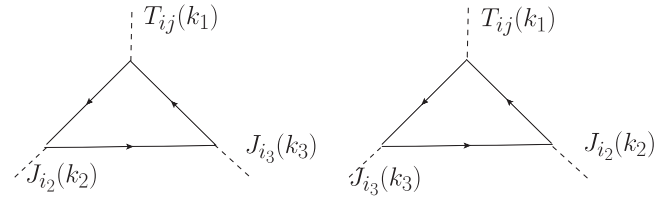

FIG. 3. One-loop triangle diagrams for the scalar contribution to $\langle TJJ\rangle$. The top corner of the respective triangles are (momentum-dependent) stress-tensor vertices, whereas, the bottom two corners represent current vertices.

A. $\langle TJJ\rangle$ for the $SU(N_s)$ scalar flavor current

Evaluating Wick's theorem for the scalar correlator yields two nonvanishing contractions depicted diagrammatically in Fig. 3.

The full expression for the two diagrams is as follows:

$$\begin{aligned}&\mathcal{K}_s^{i_1j_1i_2i_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\\ &= \int\frac{d^3P}{8\pi^3}\frac{4}{P^2(\boldsymbol{P}+\boldsymbol{k}_1)^2(\boldsymbol{P}-\boldsymbol{k}_2)^2}(2\boldsymbol{P}-\boldsymbol{k}_2)^{i_2}(2\boldsymbol{P}+\boldsymbol{k}_1)^{i_3}\\ &\quad\times\Big[\frac{1}{2}(\boldsymbol{P}-\boldsymbol{k}_2)^{i_1}(\boldsymbol{P}+\boldsymbol{k}_1)^{j_1} + \frac{1}{2}(\boldsymbol{P}-\boldsymbol{k}_2)^{j_1}(\boldsymbol{P}+\boldsymbol{k}_1)^{i_1}\\ &\quad+\frac{1}{8}[|\boldsymbol{k}_3|^2\eta^{i_1j_1} + (\boldsymbol{k}_1+\boldsymbol{k}_2)^{i_1}(\boldsymbol{k}_1+\boldsymbol{k}_2)^{j_1}]\Big],\end{aligned} \tag{3.4}$$

with $\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3 = 0$. The momentum dependence in the numerator of (3.4) comes from derivative operators of the fields at each vertex. We are only interested in certain polarization projections of this expression, and we now explain how this simplifies the momentum structure considerably.

Quite generally, a current insertion with momentum $\boldsymbol{k}$ at a vertex where one line brings in $\boldsymbol{P}$ (Fig. 4) and the other line carries away $\boldsymbol{P}+\boldsymbol{k}$ leads to an effective vertex: $(2P_i + k_i)$. However, since this correlator will finally be dotted with a transverse polarization vector, one can drop the k_i term on the right-hand side in the computations below. Also, here and below, we have dropped the $SU(N_s)$ generator T^{ℓ} because it only yields factors of unity after tracing over $SU(N_s)$ indices. Similarly, a stress-tensor insertion carrying momentum $\boldsymbol{k}$ at a vertex where one line brings in the loop momentum $\boldsymbol{P}$ (Fig. 4) and the other line carries away $\boldsymbol{P}+\boldsymbol{k}$ results in a vertex that we are finally going to contract with a polarization tensor

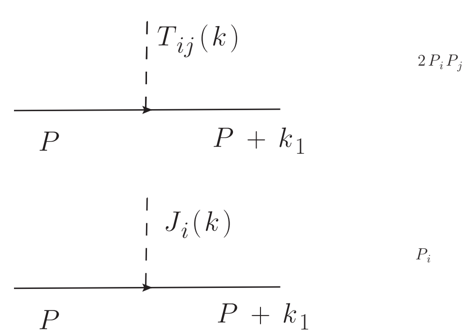

FIG. 4. Momentum structure of the stress tensor (top) and current vertex (bottom) after contracting with transverse and traceless polarization tensors.

that is *transverse and traceless*. Since this tensor will satisfy $e^{ij}\eta_{ij} = 0 = e^{ij}k_i$, we can drop the terms proportional to η_{ij} and the terms proportional to k_i and k_j above. Using this logic, the expressions for the effective stress tensor and current vertex, respectively, become quite simple (see Fig. 4), and from Eq. (3.4), we only need to consider

$$8N_s e_{i_1 j_1}\epsilon_{i_2}\epsilon_{i_3}\int\frac{d^3P}{8\pi^3}\left[\frac{P_{i_1}P_{j_1}(\boldsymbol{P}+\boldsymbol{k}_1)_{i_2}P_{i_3}}{P^2(\boldsymbol{P}+\boldsymbol{k}_1)^2(\boldsymbol{P}-\boldsymbol{k}_3)^2} + \frac{P_{i_1}P_{j_1}(\boldsymbol{P}+\boldsymbol{k}_1)_{i_3}P_{i_2}}{P^2(\boldsymbol{P}+\boldsymbol{k}_1)^2(\boldsymbol{P}-\boldsymbol{k}_2)^2}\right]. \tag{3.5}$$

These integrals can be performed by automating the Davydychev recursion relations.[19] The resulting expressions are quite lengthy as shown in the attached MATHEMATICA file in the Supplemental Material.[20] However, after we dot this answer with polarization tensors and rewrite it using the spinor-helicity formalism, our final answers are quite simple. The interested reader should, again, consult the MATHEMATICA file for details. We find the following results for N_s complex scalars, which we later compare to the results obtained from holography,

$$\frac{1}{N_s}\mathcal{K}_s^{+--} = \frac{-\langle\lambda_3,\lambda_2\rangle^4}{32\langle\lambda_2,\lambda_1\rangle^2\langle\lambda_3,\lambda_1\rangle^2|\boldsymbol{k}_1|^2|\boldsymbol{k}_2||\boldsymbol{k}_3|} \times\{[|\boldsymbol{k}_1|^2-(|\boldsymbol{k}_2|-|\boldsymbol{k}_3|)^2]^2(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)\}. \tag{3.6}$$

Contracting the stress tensor with a negative helicity polarization tensor and both the currents with negative helicity polarization vectors leads to

$$\frac{1}{N_s}\mathcal{K}_s^{---} = \frac{\langle\lambda_2,\lambda_1\rangle^2\langle\lambda_3,\lambda_1\rangle^2}{32|\boldsymbol{k}_1|^2}\times\left(\frac{8|\boldsymbol{k}_1|^3}{(|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^4}-\frac{1}{|\boldsymbol{k}_2|}-\frac{1}{|\boldsymbol{k}_3|}\right). \tag{3.7}$$

Contracting with a negative helicity for the stress tensor and one of the currents and a positive helicity for the second current, we find

$$\frac{1}{N_s}\mathcal{K}_s^{--+} = \frac{\langle\lambda_2,\lambda_1\rangle^4(-|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^2}{32\langle\lambda_3,\lambda_2\rangle^2|\boldsymbol{k}_1|^2|\boldsymbol{k}_2||\boldsymbol{k}_3|(|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^2} \times[(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)(|\boldsymbol{k}_1|^2+|\boldsymbol{k}_2|^2+|\boldsymbol{k}_3|^2) +2(|\boldsymbol{k}_2|^2+|\boldsymbol{k}_3|^2)|\boldsymbol{k}_1|]. \tag{3.8}$$

It is worthwhile to point out that all the answers above have the correct Lorentz transformation properties on the boundary and have the correct dimensions. They are also symmetric in particles 2 and 3 when those particles have the same helicity.

B. $\langle TJJ\rangle$ for the $SU(N_f)$ fermion flavor current

Now, we turn to the computation of the three-point correlator $\mathcal{K}_f$ for the fermion current J_f. The nonvanishing contractions from Wick's theorem are again given by Fig. 3 with fermion-loop propagators and the current and stress-tensor vertices carrying additional Dirac matrix structures instead of derivative operators as is the case for scalars. The full expression for $\mathcal{K}_f$ is given by

$$\mathcal{K}_f^{i_1j_1i_2i_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = -\frac{1}{4}[\Upsilon^{i_1u_3i_2v_2i_3u_2}\eta^{j_1v_3}+i_1\leftrightarrow j_1] \times\int\frac{d^3P}{8\pi^3}\frac{(2\boldsymbol{P}+\boldsymbol{k}_1-\boldsymbol{k}_2)_{v_3}(\boldsymbol{P}-\boldsymbol{k}_2)_{u_3}P_{v_2}(\boldsymbol{P}+\boldsymbol{k}_1)_{u_2}}{P^2(\boldsymbol{P}+\boldsymbol{k}_1)^2(\boldsymbol{P}-\boldsymbol{k}_2)^2}, \tag{3.9}$$

with a trace over six Dirac matrices given by

$$\Upsilon^{i_1u_3i_2v_2i_3u_2} = 2\,\mathrm{Tr}[\gamma^{i_1}\gamma^{u_3}\gamma^{i_2}\gamma^{v_2}\gamma^{i_3}\gamma^{u_2}]. \tag{3.10}$$

Again, the momentum integral can be performed using the Davydychev recursion relations, and the trace over Dirac matrices can be carried out using standard identities of the Clifford algebra. After contracting with polarization vectors—the reader should consult the attached MATHEMATICA file in the Supplemental Material[21] for details—and simplifying further, we get

$$\frac{1}{N_f}\mathcal{K}_f^{+--} = \frac{-\langle\lambda_3,\lambda_2\rangle^4}{64\langle\lambda_2,\lambda_1\rangle^2\langle\lambda_3,\lambda_1\rangle^2|\boldsymbol{k}_1|^2|\boldsymbol{k}_2||\boldsymbol{k}_3|} \times[(|\boldsymbol{k}_1|^2-(|\boldsymbol{k}_2|-|\boldsymbol{k}_3|)^2)^2(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)], \tag{3.11}$$

$$\frac{1}{N_f}\mathcal{K}_f^{---} = \frac{\langle\lambda_2,\lambda_1\rangle^2\langle\lambda_3,\lambda_1\rangle^2}{64|\boldsymbol{k}_1|^2}\times\left(\frac{-16|\boldsymbol{k}_1|^3}{(|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^4}-\frac{1}{|\boldsymbol{k}_2|}-\frac{1}{|\boldsymbol{k}_3|}\right), \tag{3.12}$$

$$\frac{1}{N_f}\mathcal{K}_f^{--+} = \frac{\langle\lambda_2,\lambda_1\rangle^4(-|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^2}{64\langle\lambda_3,\lambda_2\rangle^2|\boldsymbol{k}_1|^2|\boldsymbol{k}_2||\boldsymbol{k}_3|(|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^2} \times[(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)|\boldsymbol{k}_1|^2+(|\boldsymbol{k}_2|-|\boldsymbol{k}_3|)^2 \times(2|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)], \tag{3.13}$$

where we used the same conventions for the helicity superscripts as in the scalar case (3.6).

C. $\langle TJJ\rangle$ for the $U(1)$ topological current

The contractions involving two topological currents (1.4) necessarily involve two gauge-field insertions, and the leading diagrams of the $1/N_F$ expansion are shown in Fig. 5. Although there is no bare dynamics in the gauge sector of Eq. (1.1), the gauge field picks an order $1/N_F$ dynamical renormalization from fluctuations of the scalars and fermions[10] and takes the well-known "overdamped" form

$$D_{u_2v_2}(\mathbf{q}) = \langle a_{u_2}a_{v_2}\rangle = \frac{16}{(N_s+N_f)}\frac{1}{|\mathbf{q}|}\left(\eta_{u_2v_2}-\zeta\frac{q_{u_2}q_{v_2}}{\mathbf{q}^2}\right), \tag{3.14}$$

where ζ is a gauge-fixing parameter that should not appear in the expression for any physical observable. With this gauge

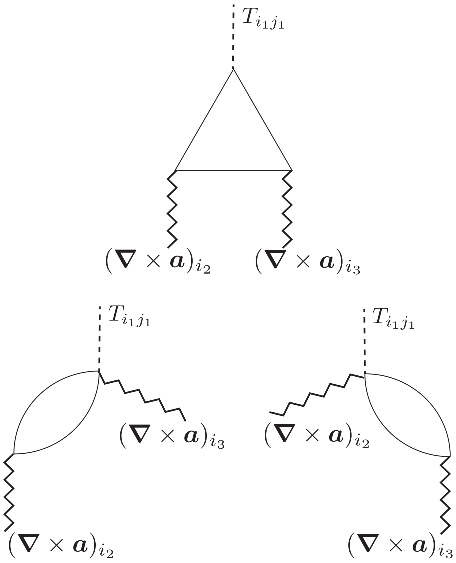

FIG. 5. Feynman diagrams contributing to the three-point correlator of J_t. The full lines are the bosonic or fermionic matter fields, and the zigzag line is the a_i propagator.

propagator, the diagrams in Fig. 5 lead to the expressions,

$$\begin{aligned}
&\mathcal{K}_t^{i_1j_1i_2i_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\\
&\quad= \left(\frac{8}{\pi(N_f+N_s)}\right)^2 \epsilon^{i_2}_{u_2v_2}\epsilon^{i_3}_{u_3v_3}\frac{k_2^{u_2}k_3^{u_3}}{|\boldsymbol{k}_2||\boldsymbol{k}_3|}\\
&\quad\times\Bigg[N_f\Bigg\{\mathcal{K}_f^{i_1j_1v_2v_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\\
&\quad+\frac{|\boldsymbol{k}_3|}{32}[\eta^{v_2j_1}\eta^{i_1v_3}+\eta^{v_2i_1}\eta^{j_1v_3}]\\
&\quad+\frac{|\boldsymbol{k}_2|}{32}[\eta^{v_3j_1}\eta^{i_1v_2}+\eta^{v_3i_1}\eta^{j_1v_2}]\Bigg\}\\
&\quad+N_s\Bigg\{\mathcal{K}_s^{i_1j_1v_2v_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\\
&\quad+\frac{|\boldsymbol{k}_2|}{16}[\eta^{v_2j_1}\eta^{i_1v_3}+\eta^{v_2i_1}\eta^{j_1v_3}]\\
&\quad+\frac{|\boldsymbol{k}_3|}{16}[\eta^{v_3j_1}\eta^{i_1v_2}+\eta^{v_3i_1}\eta^{j_1v_2}]\Bigg\}\Bigg],
\end{aligned}\tag{3.15}$$

where the terms proportional to $\mathcal{K}_s$ and $\mathcal{K}_f$, respectively, originate from the top diagram in Fig. 5. The other terms proportional to products of the metric originate from the loops involving only two internal propagators; these terms are analytic in two of the momenta and give rise to contact terms when Fourier transformed back to position space. A discussion of the nature of these terms appears in Sec. V. These contact terms drop out of the final polarization contractions that are compared to the results from holography.

IV. HOLOGRAPHIC COMPUTATION OF THREE-POINT CORRELATORS

In this section, we compute the three-point correlators discussed above from the bulk theory using AdS-CFT.

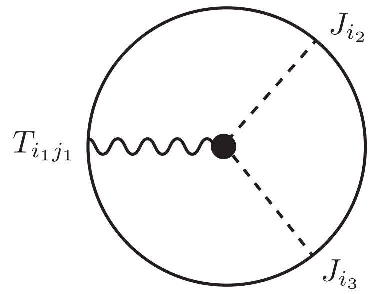

FIG. 6. Witten diagram illustrating the holographic computation. The disk represents AdS_4, and the CFT is on its boundary. The holographic coordinate z is the radial direction. The wavy line is a bulk graviton $h_{\mu\nu}$, and the dashed line is the gauge field A_μ.

We work with the Poincaré patch of AdS,

$$ds^2 = \frac{dz^2 + \eta^{ij}dx_i dx_j}{z^2}, \tag{4.1}$$

where i,j run over the three boundary directions and we have set the AdS radius to 1. So, all dimensionful quantities that follow are measured in these units.

The computation of the correlator requires us to evaluate the bulk action at nonlinear order in the presence of certain solutions to the linearized equations of motion. This corresponds to evaluating the "Witten diagram" in Fig. 6, which requires a three-point bulk interaction between the gauge fields and the fluctuations of the metric.

A. Evaluation of the bulk action

The first step in our computation is to write down the nonlinear three-point interaction terms in the action. We can simplify our calculation by realizing that we are only interested in evaluating this action "on-shell" (when the gauge field and metric perturbation satisfy linearized equations of motion), and so there are various terms that we can drop as we do below.

The relevant part of the action is as follows:

$$\begin{aligned}
S = \frac{1}{g_4^2}\int d^4x\sqrt{-g}\Big[&-\frac{1}{4}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\\
&+\gamma C_{\mu\nu\rho\sigma}F_{\alpha\beta}F_{\gamma\delta}g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta}\Big].
\end{aligned}\tag{4.2}$$

First, we need to expand the Weyl tensor term in terms of the metric perturbation. We will use the conformal transformation properties of the Weyl tensor to write

$$C_{\alpha\beta\gamma\delta}\left(\frac{\eta_{\mu\nu}}{z^2}+h_{\mu\nu}\right) = \frac{1}{z^2}C_{\alpha\beta\gamma\delta}(\eta_{\mu\nu}+z^2h_{\mu\nu}), \tag{4.3}$$

where the Weyl tensor is written as a function of the metric. For convenience, we define

$$\widetilde{h}_{\mu\nu} = z^2 h_{\mu\nu}. \tag{4.4}$$

In what follows below, we use the notation that

$$\begin{aligned}
C_{\alpha\beta\gamma\delta} &\equiv C_{\alpha\beta\gamma\delta}\left(\frac{\eta_{\mu\nu}}{z^2}+h_{\mu\nu}\right),\\
\widetilde{C}_{\alpha\beta\gamma\delta} &\equiv C_{\alpha\beta\gamma\delta}(\eta_{\mu\nu}+\widetilde{h}_{\mu\nu}),
\end{aligned}\tag{4.5}$$

with similar conventions for other quantities, such as the Riemann and Ricci tensors. (A tilde comes on top of quantities evaluated in the flat-space background metric with the perturbation $\widetilde{h}$.)

We can choose a gauge—both in flat space and in AdS—where the metric fluctuation obeys

$$\widetilde{h}^{z\mu} = 0. \tag{4.6}$$

It is easy to check that solutions to the equations of motion must be transverse and traceless,

$$\widetilde{h}_{\mu\nu}\eta^{\mu\nu} = 0 = \partial_\rho h_{\mu\nu}\eta^{\mu\rho}. \tag{4.7}$$

If we know that we only have to evaluate the interaction vertex on wave functions that obey (4.6) and (4.7), we can simplify the expressions for the Riemann tensor, the Ricci tensor, and the Ricci scalar in the linearized theory,

$$\begin{aligned}\widetilde{R}_{\alpha\mu\beta\nu} &= \tfrac{1}{2}(\widetilde{h}_{\alpha\nu,\mu\beta} + \widetilde{h}_{\mu\beta,\nu\alpha} - \widetilde{h}_{\mu\nu,\alpha\beta} - \widetilde{h}_{\alpha\beta,\mu\nu}),\\ \widetilde{R}_{\alpha\beta} &= -\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\partial_\nu\widetilde{h}_{\alpha\beta},\\ \widetilde{R} &= 0.\end{aligned} \tag{4.8}$$

From this, we can obtain the Weyl tensor, which is as follows: (here d is the *boundary dimension*, and so $d+1$ is the bulk dimension):

$$\begin{aligned}\widetilde{C}_{\alpha\mu\beta\nu} &= \widetilde{R}_{\alpha\mu\beta\nu} - \frac{2}{d-1}(\eta_{\alpha[\beta}\widetilde{R}_{\nu]\mu} - \eta_{\mu[\beta}\widetilde{R}_{\nu]\alpha})\\ &\quad + \frac{2}{d(d-1)}\widetilde{R}\eta_{\alpha[\beta}\eta_{\nu]\mu}\\ &= \frac{1}{2}(\widetilde{h}_{\alpha\nu,\mu\beta} + \widetilde{h}_{\mu\beta,\nu\alpha} - \widetilde{h}_{\mu\nu,\alpha\beta} - \widetilde{h}_{\alpha\beta,\mu\nu}\\ &\quad + \Box\{\eta_{\alpha[\beta}\widetilde{h}_{\nu]\mu} - \eta_{\mu[\beta}\widetilde{h}_{\nu]\alpha}\}),\end{aligned} \tag{4.9}$$

where we have defined $\Box \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$ and have used $d = 3$.

However, this expression can be simplified considerably. With the understanding that i,j,k,l run over the *boundary directions* and with z representing the radial direction, we need the following components of the Weyl tensor:

$$\begin{aligned}2\widetilde{C}_{zizj} &= \frac{1}{2}\left[\sum_l \partial_l^2 - \partial_z^2\right]\widetilde{h}_{ij},\\ 2\widetilde{C}_{zijk} &= \partial_k\partial_z\widetilde{h}_{ij} - \partial_j\partial_z\widetilde{h}_{ik},\\ 2\widetilde{C}_{ijkl} &= \frac{1}{2}\left[\partial_z^2 - \sum_l \partial_l^2\right](\eta_{ik}\widetilde{h}_{jl} - \eta_{il}\widetilde{h}_{kj} - \eta_{jk}\widetilde{h}_{il} + \eta_{jl}\widetilde{h}_{ik}).\end{aligned} \tag{4.10}$$

In evaluating the first two lines, we have used the conditions (4.6) and (4.7). In evaluating the last line, we have used the fact that the Weyl tensor vanishes identically in three dimensions. This might suggest that only the additional term involving the z derivatives survives; however, one needs to be careful about the factor in front of the Laplacian, which is dimension dependent. When we take all of this into account, we get the expression above. [This almost—but not quite—agrees with the results of Ref. 18. In particular, the first line of (4.10) does not agree with the first line of (2.12) of Ref. 18, in general, and neither does the last line. However, the expressions do agree if we are evaluating this tensor on a solution of the form $h_{ij} = \epsilon_{ij}e^{-|\boldsymbol{k}|z + i\boldsymbol{k}\cdot\boldsymbol{x}}$, which was the case under consideration in that paper.]

With these results for the flat-space Weyl tensor, the expression for the Weyl tensor in AdS is also fixed by the relation (4.3). We should point out that, although we have not written all the nonzero components above, the components that we have written and the symmetries of the Weyl tensor fix everything.

To evaluate the interaction vertex, we also note the fact that, for the evaluation of the three-point function under consideration, the non-Abelian terms in the field strength are unimportant. So, in what follows below, we simply take

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{4.11}$$

and choose a gauge where

$$A_z = 0, \qquad \partial_i A^i = 0. \tag{4.12}$$

To finally evaluate the interaction vertex in AdS, we will use the explicit forms of the wave functions for the gauge field and for the graviton. These are given by

$$\begin{aligned}A_i &= \epsilon_i e^{-|\boldsymbol{k}|z}e^{i\boldsymbol{k}\cdot\boldsymbol{x}},\\ h_{ij} &= \frac{1}{z^2}e_{ij}e^{-|\boldsymbol{k}|z}(1+|\boldsymbol{k}|z)e^{i\boldsymbol{k}\cdot\boldsymbol{x}},\\ \widetilde{h}_{ij} &= e_{ij}e^{-|\boldsymbol{k}|z}(1+|\boldsymbol{k}|z)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}.\end{aligned} \tag{4.13}$$

See Ref. 22 for further details on the notation. Below, we use R_m to denote the radial part of the wave function of the mth particle,

$$R_1 = (1+|\boldsymbol{k}_1|z)e^{-|\boldsymbol{k}_1|z}, \quad R_2 = e^{-|\boldsymbol{k}_2|z}, \quad R_3 = e^{-|\boldsymbol{k}_3|z}, \tag{4.14}$$

and use the notation $\dot{f} \equiv \frac{\partial f}{\partial z}$.

We now need to evaluate the variation in the action for first order in the metric perturbation h and for second order in the gauge field. This is appropriate since we wish to compute a three-point function involving one stress tensor and two currents. Since the Weyl tensor vanishes in pure AdS and we have no gauge-field background either, the variation in the Weyl-gauge term is simply its value in the presence of the perturbation,

$$\frac{g_4^2}{\gamma}\delta S_1 = \int dz\sqrt{-g}C_{\mu\nu\rho\sigma}F_{\alpha\beta}F_{\gamma\delta}g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta}; \tag{4.15}$$

here and below, we drop the integral over the flat three-dimensional space time whose role is to ensure conservation of momentum with $\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = 0$.

Let us now evaluate the different contractions that appear in the expression above keeping track of the numerical factors.

(1) First of all, we note that, given the expressions for the wave functions in (4.13) above, we can always replace a derivative $\partial_j \to ik_j$. Each term in the contraction has two such spatial derivatives leading to an overall minus sign.

(2) Second, the Weyl tensor is antisymmetric under the interchange of the first two or the third and fourth indices.

Since the field strength is also antisymmetric, we get a factor of 4 by summing over these permutations.

(3) Finally, there is a factor of $\frac{1}{2}$ in (4.10), but we have to keep in mind that we need to sum over the two possible permutations of the gauge fields in the Witten diagram.

Therefore, we have

$$\begin{aligned}\delta S_{11} &= C_{zizj}F^{zi}F^{zj} + C_{izjz}F^{iz}F^{jz}\\ &\quad + C_{izzj}F^{iz}F^{zj} + C_{zijz}F^{zi}F^{jz}\\ &= -2z^6(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(|\boldsymbol{k}_1|^2R_1 + \ddot{R}_1)\dot{R}_2\dot{R}_3,\\ \delta S_{12} &= C_{zijk}F^{zi}F^{jk} + C_{izjk}F^{iz}F^{jk}\\ &\quad + C_{ijzk}F^{ij}F^{zk} + C_{ijkz}F^{ij}F^{kz}\\ &= -2z^6[\{(\boldsymbol{k}_1\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{k}_3\cdot\boldsymbol{\epsilon}_1)\\ &\quad - (\boldsymbol{k}_1\cdot\boldsymbol{k}_3)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)\dot{R}_1\dot{R}_2R_3\\ &\quad + \{(\boldsymbol{k}_1\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{k}_2\cdot\boldsymbol{\epsilon}_1)\\ &\quad - (\boldsymbol{k}_1\cdot\boldsymbol{k}_2)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)\}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\dot{R}_1R_2\dot{R}_3],\\ \delta S_{13} &= C_{ijkl}F^{ij}F^{kl} = -4z^6[(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_1)\\ &\quad - (\boldsymbol{k}_2\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{k}_3\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)\\ &\quad - (\boldsymbol{k}_3\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{k}_2\cdot\boldsymbol{\epsilon}_3)(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_1) + (\boldsymbol{k}_2\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{k}_3\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_3)]\\ &\quad \times(\ddot{R}_1 + |\boldsymbol{k}_1|^2R_1)R_2R_3. \end{aligned}\tag{4.16}$$

Let us make a comment about the overall power of z. We get four factors of z^2 from the four inverse metric components that are required to raise the indices of F. However, we get one factor of $\frac{1}{z^2}$ from C. This is what leads to the overall z^6 outside. Also, we caution the reader that, when we write $\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2$ above and other such expressions involving the dot product of three-dimensional vectors, this dot product is taken with the flat-space metric,

$$\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2 \equiv \epsilon_{1i}\epsilon_{2j}\eta^{ij}. \tag{4.17}$$

The variation in the full Weyl-gauge term in the action is just the sum of the terms above

$$\frac{g_4^2}{\gamma}\delta S_1 = \int dz\sqrt{-g}[\delta S_{11} + \delta S_{12} + \delta S_{13}]. \tag{4.18}$$

There is, of course, another term that contributes to the three-point function, which comes from the interaction of the metric perturbation with the stress tensor of the gauge field. This evaluates as

$$g_4^2\delta S_2 = \int\sqrt{-g}dz\left[\frac{1}{2}F_{\mu\nu}F_{\rho\sigma}\eta^{\mu\alpha}h_{\alpha\beta}\eta^{\beta\rho}\eta^{\nu\sigma}\right]z^6. \tag{4.19}$$

Note that the conditions (4.6) and (4.7) mean we can drop the term that comes from the variation in $\sqrt{-g}$. We also have an overall minus sign because $\delta g^{\mu\nu} = -g^{\mu\rho}h_{\rho\sigma}\delta g^{\sigma\nu}$. The overall factor of z^6 comes from the four inverse metric factors, but it is important to remember that one needs to include the $\frac{1}{z^2}$ in $h_{\alpha\beta}$ from (4.13).

We can write

$$\begin{aligned} g_4^2\delta S_2 = -\int dz\{[&(\boldsymbol{k}_2\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{k}_3\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_3)\\ &- (\boldsymbol{k}_2\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\epsilon}_2\cdot\boldsymbol{k}_3)\\ &- (\boldsymbol{k}_3\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{k}_2)\\ &+ (\boldsymbol{k}_2\cdot\boldsymbol{k}_3)(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_1)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\epsilon}_1)]R_1R_2R_3\\ &- (\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)R_1\dot{R}_2\dot{R}_3\}. \end{aligned}\tag{4.20}$$

Note that we have regained a minus sign from the two factors of i that get pulled down in the differentiation, although this does not occur in the last term above where we have a z derivative instead. Also note that all factors of z are gone when we account for the $\sqrt{-g}$ and the factor of $\frac{1}{z^2}$ in $h_{\alpha\beta}$ from (4.13).

As a final step in evaluating the three-point function, we now need to perform the radial integrals in (4.18) and (4.19). First, let us perform the radial integrals in (4.18). Note that, once we account for the fact that $\sqrt{-g} = \frac{1}{z^4}$, there is an overall factor of z^2 in every radial integral. These are

$$\int z^2\ddot{R}_1R_2R_3dz = \frac{2|\boldsymbol{k}_1|^2(2|\boldsymbol{k}_1| - |\boldsymbol{k}_2| - |\boldsymbol{k}_3|)}{(|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + |\boldsymbol{k}_3|)^4}, \tag{4.21}$$

$$\int z^2\dot{R}_1\dot{R}_2R_3dz = \frac{6|\boldsymbol{k}_1|^2|\boldsymbol{k}_2|}{(|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + |\boldsymbol{k}_3|)^4}, \tag{4.22}$$

$$\int z^2\dot{R}_1\dot{R}_2R_3dz = \frac{6|\boldsymbol{k}_1|^2|\boldsymbol{k}_3|}{(|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + |\boldsymbol{k}_3|)^4}, \tag{4.23}$$

$$\int z^2R_1\dot{R}_2\dot{R}_3dz = \frac{2|\boldsymbol{k}_2||\boldsymbol{k}_3|(4|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + |\boldsymbol{k}_3|)}{(|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + |\boldsymbol{k}_3|)^4}. \tag{4.24}$$

Now, we turn to the radial integrals in (4.19). These are

$$\int R_1R_2R_3dz = \frac{2|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + |\boldsymbol{k}_3|}{(|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + |\boldsymbol{k}_3|)^2}, \tag{4.25}$$

$$\int R_1\dot{R}_2\dot{R}_3\,dz = |\boldsymbol{k}_2||\boldsymbol{k}_3|\frac{2|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + |\boldsymbol{k}_3|}{(|\boldsymbol{k}_1| + |\boldsymbol{k}_2| + |\boldsymbol{k}_3|)^2}. \tag{4.26}$$

All these integrals are convergent.

B. Final bulk answers in the spinor-helicity formalism

The expressions for the bulk action and the radial integrals above, in principle, give us all the information we need about the boundary correlator. However, to extract some physics from this, it is convenient to choose various helicities for the stress tensor and the currents and then to write down the answer in the spinor-helicity formalism outlined above.

We only need to consider the following three choices of helicities:

(1) Both the currents and the stress tensor have negative helicities.

(2) The stress tensor and one current have negative helicities, but the other current has positive helicity.

(3) The stress tensor has positive helicity, and the two currents have negative helicities.

All other possibilities can be obtained from these ones by permuting the two currents and/or using parity.

The use of the spinor-helicity formalism considerably simplifies the algebraic expressions involved in the answers. The reader, who is interested in the algebra that enters this simplification, should consult the accompanying MATHEMATICA file in the Supplemental Material.[23] Here, we simply present the final answers.

For the case where all the helicities are negative, we have the following expression:

$$\mathcal{K}_{\rm ads}^{---}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = -24\gamma\frac{\langle\lambda_2,\lambda_1\rangle^2\langle\lambda_3,\lambda_1\rangle^2|\boldsymbol{k}_1|}{g_4^2E^4}, \tag{4.27}$$

where we have defined

$$E \equiv |\boldsymbol{k}_1| + |\boldsymbol{k}_2| + |\boldsymbol{k}_3|. \tag{4.28}$$

It is natural for this expression (considered as an analytic function of E) to have a pole at $E = 0$, and in fact, the residue at this pole is related to the four-dimensional flat-space amplitude of a graviton and two gluons as pointed out in Ref. 24. We also note that the usual gravitational interaction does not contribute to this helicity combination at all and the entire combination comes from the Weyl interaction.

When the stress tensor and the first current insertion are dotted with negative helicity polarization vectors and the second current is dotted with a positive helicity polarization vector, we find

$$\mathcal{K}_{\rm ads}^{--+} = -\frac{\langle\lambda_2,\lambda_1\rangle^4(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|-|\boldsymbol{k}_1|)^2(2|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)}{2\langle\lambda_3,\lambda_2\rangle^2|\boldsymbol{k}_1|^2g_4^2E^2}. \tag{4.29}$$

In this case, we find that the Weyl interaction does not contribute to this helicity combination, whereas, the usual gravitational interaction does.

Finally, we come to the the case where the stress tensor has positive helicity and the two currents have negative helicities. For this correlator, we have

$$\mathcal{K}_{\rm ads}^{+--} = 0. \tag{4.30}$$

Neither the Weyl nor the gravitational interaction contribute to this helicity combination!

It is useful to check that these answers, indeed, have the expected behavior under scaling. Recall that the stress tensor has dimension 3, and the two conserved currents have dimension 2 each. Fourier transforming the three-position variables gives us a dimension of -9 of which the momentum space δ function that we have suppressed above soaks up -3. So, we expect the net dimension in momentum space to be 1, which is true for all the expressions above.

The spinor-helicity formalism only makes the Lorentz group on the boundary manifest. It is possible to check that these answers also satisfy the constraints of special conformal transformations as indicated in Ref. 18, but this is a slightly more-involved calculation.

V. MATCHING THE ANSWERS

In this section, we match the answers of the CFT computations of Sec. III (and Appendix B) with the AdS answers of Sec. IV. This will allow us to determine the values of physical parameters in the bulk, that would reproduce the free answers.

A. Scalars

$--+$ *Helicity*. Let us start with (3.8), which we can write as

$$\begin{aligned}\frac{1}{N_s}\mathcal{K}_s^{--+} &= \frac{\langle\lambda_2,\lambda_1\rangle^4(-|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^2}{32\langle\lambda_3,\lambda_2\rangle^2|\boldsymbol{k}_1|^2|\boldsymbol{k}_2||\boldsymbol{k}_3|(|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^2}\\ &\quad\times[(|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^2(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)\\ &\quad-2(2|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)|\boldsymbol{k}_2||\boldsymbol{k}_3|]\\ &= \frac{1}{N_s}\widetilde{\mathcal{K}}_s^{--+} + \mathcal{C}_s^{--+}.\end{aligned} \tag{5.1}$$

Here,

$$\widetilde{\mathcal{K}}_s^{--+} = -N_s\frac{\langle\lambda_2,\lambda_1\rangle^4}{\langle\lambda_3,\lambda_2\rangle^2}\frac{(-|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^2(2|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)}{16|\boldsymbol{k}_1|^2(|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^2} \tag{5.2}$$

has exactly the same functional form as the answer obtained from the AdS calculation in (4.29), and we have defined

$$\mathcal{C}_s^{--+} = \frac{\langle\lambda_2,\lambda_1\rangle^4(-|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^2(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)}{32\langle\lambda_3,\lambda_2\rangle^2|\boldsymbol{k}_1|^2|\boldsymbol{k}_2||\boldsymbol{k}_3|}. \tag{5.3}$$

We now show that (5.3) is purely a contact term. In fact, we can write

$$\mathcal{C}_s^{--+} = \frac{1}{8}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|). \tag{5.4}$$

To check the equivalence of (5.4) and (5.3), we note that

$$\begin{aligned}&\frac{-1}{8}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)\\ &= \frac{-1}{32|\boldsymbol{k}_1|^2|\boldsymbol{k}_2||\boldsymbol{k}_3|}\langle\lambda_1,\lambda_2\rangle^2[\lambda_1,\bar{\lambda}_3]^2(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)\\ &= \frac{1}{32|\boldsymbol{k}_1|^2|\boldsymbol{k}_2||\boldsymbol{k}_3|}\frac{\langle\lambda_1,\lambda_2\rangle^4\langle\bar{\lambda}_2,\bar{\lambda}_3\rangle^2}{E^2}(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)\\ &= \frac{1}{32|\boldsymbol{k}_1|^2|\boldsymbol{k}_2||\boldsymbol{k}_3|}\frac{\langle\lambda_1,\lambda_2\rangle^4}{\langle\lambda_2,\lambda_3\rangle^2}\\ &\quad\times(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|-|\boldsymbol{k}_1|)^2(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|),\end{aligned} \tag{5.5}$$

where the last line is manifestly the same as (5.3).

However, we can write (5.4) as

$$\mathcal{C}_s^{--+} = \frac{1}{32}\epsilon_{1i_1j_1}\epsilon_{2i_2}\epsilon_{3i_3}[\eta^{i_1i_2}\eta^{j_1i_3}(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)]. \tag{5.6}$$

The term in the square brackets is the bare correlator before contracting with the polarization vectors, and this is the term we should Fourier transform to position space. In position space, this is evidently a contact term.

The general rule is that a term that is analytic in two of the momenta yields a contact term when Fourier transformed to position space. In this case, we notice, for example, that, after adding the overall momentum conserving δ function, we have

$$\begin{aligned}&\int|\boldsymbol{k}_2|\delta(\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_1)e^{i\sum\boldsymbol{k}_m\cdot\boldsymbol{x}_m}\prod d^3\boldsymbol{k}_m\\ &= (2\pi)^3\delta(x_1-x_3)\int|\boldsymbol{k}_2|e^{i\boldsymbol{k}_2\cdot(\boldsymbol{x}_2-\boldsymbol{x}_3)}d^3\boldsymbol{k}_2.\end{aligned} \tag{5.7}$$

Contact terms in correlators are very subtle since they depend on the precise definition of the correlator and on the regulator used to compute it. Although they might have physical significance under some circumstances, in this paper, we just drop these additional δ function terms and work with $\tilde{\mathcal{K}}_s^{--+}$ instead of $\mathcal{K}_s^{--+}$.

$+--$ *Helicity*. It turns out that the free answer (3.6) is entirely a contact term in this case! We note that

$$\begin{aligned}
&\frac{1}{4}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{k}_2)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{k}_3)(\boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_3)(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)\\
&\quad=\frac{1}{32|\boldsymbol{k}_1|^2|\boldsymbol{k}_2||\boldsymbol{k}_3|}\langle\lambda_2,\lambda_3\rangle^2\langle\bar\lambda_1,\bar\lambda_2\rangle\langle\bar\lambda_1,\bar\lambda_3\rangle\\
&\qquad\times[\bar\lambda_1,\lambda_2][\bar\lambda_1,\lambda_3](|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)\\
&\quad=\frac{1}{32}\langle\lambda_2,\lambda_3\rangle^4(|\boldsymbol{k}_1|+|\boldsymbol{k}_3|-|\boldsymbol{k}_2|)^2(|\boldsymbol{k}_1|+|\boldsymbol{k}_2|-|\boldsymbol{k}_3|)^2\\
&\qquad\times(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)=\frac{1}{N_s}\mathcal{K}_s^{+--}.
\end{aligned}\tag{5.8}$$

This is consistent with the fact that both the Weyl interaction and the ordinary gravitational interaction yield 0 in the AdS calculation (4.30). For notational consistency, we can set

$$\tilde{\mathcal{K}}_s^{+--}=0,\qquad \mathcal{C}_s^{+--}=\frac{1}{N_s}\mathcal{K}_s^{+--}.\tag{5.9}$$

$---$ *Helicity*. Turning finally to (3.7), we see that this expression can be written as

$$\frac{1}{N_s}\mathcal{K}_s^{---}=\frac{1}{N_s}\tilde{\mathcal{K}}_s^{---}+\mathcal{C}_s^{---},\tag{5.10}$$

where

$$\tilde{\mathcal{K}}_s^{---}=N_s\frac{\langle\lambda_2,\lambda_1\rangle^2\langle\lambda_3,\lambda_1\rangle^2}{4}\left(\frac{|\boldsymbol{k}_1|}{(|\boldsymbol{k}_1|+|\boldsymbol{k}_2|+|\boldsymbol{k}_3|)^4}\right)\tag{5.11}$$

has exactly the same functional form as the AdS answer (4.27) and the contact term $\mathcal{C}_s^{---}$ is as follows:

$$\mathcal{C}_s^{---}=\frac{-1}{8}(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_3)(|\boldsymbol{k}_2|+|\boldsymbol{k}_3|).\tag{5.12}$$

1. Value of γ

Our final task is to find the value of γ for free scalars. To make the normalization of the two-point functions drop out, we can simply consider the ratio,

$$\frac{\tilde{\mathcal{K}}_s^{---}}{\tilde{\mathcal{K}}_s^{--+}}=-12\gamma_s\frac{\mathcal{K}_{\rm ads}^{---}}{\mathcal{K}_{\rm ads}^{--+}}.\tag{5.13}$$

Since the two ratios above should be equal, we find that we should set

$$\gamma_s=-\frac{1}{12},\tag{5.14}$$

where we have added a subscript to distinguish it from the value for free fermions that we find below.

We end by pointing out a very interesting feature of answers (4.27), (4.30), and (4.29): There is no term where the ordinary interaction and the Weyl interaction contribute simultaneously. If we had a term where the two interactions contributed simultaneously, we could have fixed γ by looking at the functional form of the answer. However, γ appears as a simple ratio of two answers, and so, one needs to be extremely careful in determining all the signs and numerical prefactors in the expressions for the various three-point functions correctly.

2. Value of G_N

We can also set the value of G_N from our calculations. Although G_N does not appear in the three-point computations above, it does appear in the computation of the two-point function for the stress tensor from the bulk using the action (C22). If we write the results for the two-point functions in Appendices B and C 5 as

$$\begin{aligned}
\epsilon_{1,i_1}\epsilon_{1,i_2}\epsilon_{2,i_3}\epsilon_{2,i_4}\langle T_s^{i_1i_2}(\boldsymbol{k})T_s^{i_3i_4}(-\boldsymbol{k})\rangle&=C_{T,s}|\boldsymbol{k}|^3(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)^2,\\
\epsilon_{1,i_1}\epsilon_{2,i_2}\langle J_s^{i_1}J_s^{i_2}\rangle&=-C_{J,s}|\boldsymbol{k}|(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2),
\end{aligned}\tag{5.15}$$

then we should demand that the normalization-independent quantities be equal

$$\begin{aligned}
\frac{1}{\sqrt{C_{T,s}}C_{J,s}}\tilde{\mathcal{K}}_s^{---}&=\frac{1}{\sqrt{C_{T,\rm ads}}C_{J,\rm ads}}\mathcal{K}_{\rm ads}^{---},\\
\frac{1}{\sqrt{C_{T,s}}C_{J,s}}\tilde{\mathcal{K}}_s^{--+}&=\frac{1}{\sqrt{C_{T,\rm ads}}C_{J,\rm ads}}\mathcal{K}_{\rm ads}^{--+}.
\end{aligned}\tag{5.16}$$

We have, from the results for two-point functions,

$$\begin{aligned}
C_{T,s}&=\frac{N_s}{128},\quad C_{J,s}=\frac{N_s}{16},\\
C_{T,\rm ads}&=\frac{1}{\pi G_{N,s}},\quad C_{J,\rm ads}=\frac{1}{g_{4,s}^2}.
\end{aligned}\tag{5.17}$$

This leads to the scalar contribution,

$$\frac{1}{G_{N,s}}=\frac{\pi N_s}{512L^2},\tag{5.18}$$

where we have reinstated the dimensionful factor of the AdS radius.

Note that, with this choice, the quantities $C_{T,s}$ and $C_{T,\rm ads}$ do not agree and this is a sign of the fact that, with our conventions, the stress tensor of the bulk theory is normalized differently from that of the boundary theory. This, in turn, results from our choice of Z above (C25). This choice was made to yield a particularly simple graviton bulk-to-boundary propagator, and to get $C_{T,s}$ to match with $C_{T,\rm ads}$, we should have chosen $Z=\frac{-d}{4\pi G_N}$, which is twice the choice that we have currently made.

3. Value of g_4^2

Note that $g_{4,s}^2$ does not appear in the quantities (5.16) at all since it cancels between the three-point and the two-point functions. However, we can choose a value by demanding that the two-point functions of the currents be equal in the bulk and the boundary. Imposing,

$$C_{J,\rm ads}=C_{J,s},\tag{5.19}$$

we can set

$$g_{4,s}^2=\frac{16}{N_s}.\tag{5.20}$$

B. Fermions

The analysis for the fermionic answers is almost identical, so we do not repeat it in detail here. However, with a little work (see the MATHEMATICA file in the Supplemental Material[21]), we find that we can write

$$\begin{aligned}\frac{1}{N_f}\mathcal{K}_f^{---} &= \frac{1}{N_f}\widetilde{\mathcal{K}}_f^{---} + \mathcal{C}_f^{---},\\ \frac{1}{N_f}\mathcal{K}_f^{--+} &= \frac{1}{N_f}\widetilde{\mathcal{K}}_f^{--+} + \mathcal{C}_f^{--+},\\ \frac{1}{N_f}\mathcal{K}_f^{+--} &= \frac{1}{N_f}\widetilde{\mathcal{K}}_f^{+--} + \mathcal{C}_f^{+--},\end{aligned} \tag{5.21}$$

where

$$\begin{aligned}&\widetilde{\mathcal{K}}_f^{---} = -\widetilde{\mathcal{K}}_s^{---}, \quad \widetilde{\mathcal{K}}_f^{--+} = \widetilde{\mathcal{K}}_s^{--+},\\ &\widetilde{\mathcal{K}}_f^{+--} = \widetilde{\mathcal{K}}_s^{+--} = 0,\end{aligned} \tag{5.22}$$

and the analytic remainders are as follows:

$$\mathcal{C}_f^{---} = \frac{1}{2}\mathcal{C}_s^{---}, \quad \mathcal{C}_f^{--+} = \frac{1}{2}\mathcal{C}_s^{--+}, \quad \mathcal{C}_f^{+--} = \frac{1}{2}\mathcal{C}_s^{+--}, \tag{5.23}$$

which are half those of the scalar case above.

Thus, we immediately see that, for free fermions, we have

$$\gamma_f = -\gamma_s = \frac{1}{12}. \tag{5.24}$$

A standard computation of the fermion two-point functions shows that, as for the scalars, we now have

$$\begin{aligned}g_{4,f}^2 &= \frac{16}{N_f},\\ \frac{1}{G_{N,f}} &= \frac{\pi N_f}{512L^2}.\end{aligned} \tag{5.25}$$

Of course, the CFT only has a single G_N, which is simply $1/G_N = 1/G_{N,f} + 1/G_{N,s}$ at this order in $1/N_F$.

C. Topological current

To obtain the value of γ for the topological current, we do not need to perform any additional work. The analysis for the topological current proceeds in the following sequence of steps:

(1) First, we can ignore the third line of (3.15), which includes terms, such as η^{v2j1}, η^{v2i1}, etc., since they are analytic in two of the momenta. This leaves us with the terms involving $\mathcal{K}_s$ and $\mathcal{K}_f$.

(2) Instead of contracting $\mathcal{K}_f$ and $\mathcal{K}_s$ with the polarization vectors $\boldsymbol{\epsilon}_2$ and $\boldsymbol{\epsilon}_3$, we instead need to contract them with the vectors $\boldsymbol{\epsilon}_2 \times \boldsymbol{k}_2/|\boldsymbol{k}_2|$ and $\boldsymbol{\epsilon}_3 \times \boldsymbol{k}_3/|\boldsymbol{k}_3|$. (We need to be careful because we are in Lorentzian space, and the ordinary rules for the cross product take us from two vectors with lowered indices to a vector with a raised index.)

(3) However, this returns the original polarization vectors, up to a sign that depends on the helicity. In particular,

$$\boldsymbol{\epsilon}_2 \times \boldsymbol{k}_2/|\boldsymbol{k}_2| = h_2\boldsymbol{\epsilon}_2, \tag{5.26}$$

where h_2 is the helicity of current 2. A similar formula holds for current 3.

(4) Therefore, we get the same amplitudes as earlier, up to a sign that is 1 if both currents have the same helicity and -1 if the currents have opposite helicities.

This chain of argument immediately yields

$$\gamma_t = \frac{(N_s - N_f)}{12(N_s + N_f)}. \tag{5.27}$$

VI. POSITION SPACE CORRELATORS AND ENERGY FLUX

In this section, we provide an alternate route to fix the value of γ using the three-point functions $\langle TJJ\rangle$ in position space. In particular, we extend the calculation of energy flux in Ref. 6 to arbitrary space-time dimensions d, and by comparing it with the holographic results, we relate γ to the parameters in the three-point correlator of a general CFT obtained by Osborn and Petkou[12] and Erdmenger and Osborn.[25] The latter parameters are known for free CFTs, and so, we obtain an alternate derivation of the $N_F \to \infty$ limits of γ_s and γ_f, consistent with our previous results.

In a CFT, we consider a thought experiment in which a localized disturbance or state is created by the insertion of a conserved vector current $(\boldsymbol{\epsilon}\cdot\boldsymbol{J})$, where $\boldsymbol{\epsilon}$ is a fixed spatial polarization vector. We assume that this local disturbance injects a fixed energy E and the system evolves in time. Now, we can place calorimeters at long distances and further study the anisotropic distribution of energy. In this experiment, a particular quantity, that is, the energy flux escaping to the null infinity, will take a very simple form. If the direction of the null infinity is given by the unit vector $\boldsymbol{n}$, the energy flux collected by the calorimeter is given by

$$\begin{aligned}\langle\mathcal{E}(\boldsymbol{n})\rangle &= \frac{\langle 0|(\boldsymbol{\epsilon}^*\cdot\boldsymbol{J}^\dagger)\mathcal{E}(\boldsymbol{n})(\boldsymbol{J}\cdot\boldsymbol{\epsilon})|0\rangle}{\langle 0|(\boldsymbol{\epsilon}^*\cdot\boldsymbol{J}^\dagger)(\boldsymbol{J}\cdot\boldsymbol{\epsilon})|0\rangle}\\ &= \frac{E}{\Omega_d}\left[1 + \mathcal{A}\left(\frac{|\boldsymbol{\epsilon}\cdot\boldsymbol{n}|^2}{|\boldsymbol{\epsilon}|^2} - \frac{1}{d-1}\right)\right].\end{aligned} \tag{6.1}$$

This form of the energy flux is completely fixed by the energy conservation and $O(d-1)$ symmetry of the construction. Here, $\mathcal{E}(\boldsymbol{n})$ is the energy flux operator, to be introduced shortly in (6.2). The total energy injected by the perturbation is E, and $\Omega_d = 2\pi^{(d-1)/2}/\Gamma(\frac{d-1}{2})$ is the area of the unit $(d-2)$ sphere. Furthermore, $\mathcal{A}$ is a constant which characterizes the CFT. As pointed out after Eq. (1.6), the three-point function $\langle TJJ\rangle$ in real space is completely determined by C_J, C_T, and an additional constant. The coefficient $\mathcal{A}$ is related to this additional constant, and in holography, it is related to the coupling constant γ in (1.7). In this section, we will find $\mathcal{A}$ through field theory and holographic calculations, and by comparing the results, we will fix γ for free scalar and fermionic field theories. First, we begin with the three-point function $\langle TJJ\rangle$ in position space, which is specified by Osborn and Petkou[12] and calculate energy density (6.1) for CFTs.

A. $\mathcal{A}$ in CFTs

To set up the calculations on the field-theory side, we work with a Minkowski metric with a "mostly positive" signature. In our thought experiment, we place the calorimeter at a long distance along the x^1 direction and, hence, the unit vector $n^i = \delta^i_1$. To measure the energy along the null infinity, it is convenient to use the light-cone coordinates, which we define as $x^{\pm} = x^0 \pm x^1$. Then, the energy flux operator is given by[6,14]

$$\mathcal{E}(\boldsymbol{x}_1,\boldsymbol{n}) = \int dx_1^- \left[\lim_{x_1^+ \to \infty} \left(\frac{x_1^+ - x_1^-}{2} \right)^{d-2} T_{--}(x_1^+,x_1^-) \right], \tag{6.2}$$

where T_{--} is the component of the stress-energy tensor. Now, to fix $\mathcal{A}$, it is sufficient to calculate the energy one-point function for a state created by the operator $(\boldsymbol{J}\cdot\boldsymbol{\epsilon})$, which appears in the numerator of (6.1). So, the calculation will boil down to using the expression for three-point function $\langle J_i^\dagger(\boldsymbol{x}_2)T_{--}(\boldsymbol{x}_1,\boldsymbol{n})J_j(\boldsymbol{x}_3)\rangle$ and performing various integrations. We can simplify these integrations by using symmetries of the construction. In the correlations $\langle J_i^\dagger(\boldsymbol{x}_2)T_{--}(\boldsymbol{x}_1,\boldsymbol{n})J_j(\boldsymbol{x}_3)\rangle$, we can use translation invariance to set $\boldsymbol{x}_3 = 0$. By aligning the calorimeter along $n^i = \delta_1^i$, we have also fixed $\boldsymbol{x}_1 = \{x_1^0, x_1^1, 0, \ldots\}$. With these simplifications, we will only need to integrate over the coordinates $\boldsymbol{x}_2 = \boldsymbol{x} = \{x^0, x^1, x^2, \ldots\}$. We further choose the spatial polarization vector $\boldsymbol{\epsilon}$ to be $\boldsymbol{\epsilon} = \{\epsilon^0,\epsilon^1,\epsilon^2,\epsilon^3,\ldots\} = \{0,\cos\theta,\sin\theta,0,\ldots\}$. In this notation, we clearly have $|\boldsymbol{\epsilon}\cdot\boldsymbol{n}| = \cos\theta$, and the numerator of (6.1) takes the following form:

$$\begin{aligned} f(E) = & \int dx^+ dx^- e^{iE[(x^+ + x^-)/2]} \int d^{d-2}x \\ & \times \int dx_1^- \left[\lim_{x_1^+ \to \infty} \left(\frac{x_1^+ - x_1^-}{2} \right)^{d-2} \right. \\ & \left. \times \epsilon^i \epsilon^j \langle J_i(\boldsymbol{x}) T_{--}(x_1^+, x_1^-) J_j(0)\rangle \right], \end{aligned} \tag{6.3}$$

where now, i,j take the values $\{+,-,2\}$. Now, we use the three-point correlator $\langle TJJ\rangle$ in position space to evaluate (6.3). As discussed in Refs. 12 and 25, using the conformal symmetry and Ward identities, the form of the three-point functions in d-dimensional CFTs can be fixed at

$$\langle T_{ij}(\boldsymbol{x}_1)J_k(\boldsymbol{x}_2)J_l(\boldsymbol{x}_3)\rangle = \frac{t_{ijmn}(\boldsymbol{X}_{23})g^{mp}g^{nq}I_{kp}(\boldsymbol{x}_{21})I_{lq}(\boldsymbol{x}_{31})}{|\boldsymbol{x}_{12}|^d|\boldsymbol{x}_{13}|^d|\boldsymbol{x}_{23}|^{d-2}}, \tag{6.4}$$

where

$$\boldsymbol{x}_{12} = \boldsymbol{x}_1 - \boldsymbol{x}_2, \quad \boldsymbol{X}_{12} = \frac{\boldsymbol{x}_{13}}{|\boldsymbol{x}_{13}|^2} - \frac{\boldsymbol{x}_{23}}{|\boldsymbol{x}_{23}|^2},$$
$$\text{and} \quad \hat{X}_i = \frac{X_i}{\sqrt{|\boldsymbol{X}|^2}}. \tag{6.5}$$

Furthermore, we have

$$\begin{aligned} I_{ij}(\boldsymbol{x}) &= g_{ij} - 2\frac{x_i x_j}{|\boldsymbol{x}|^2}, \\ t_{ijmn}(\boldsymbol{X}) &= \hat{a}h_{ij}^1(\hat{\boldsymbol{X}})g_{mn} + \hat{b}h_{ij}^1(\hat{\boldsymbol{X}})h_{mn}^1(\hat{\boldsymbol{X}}) \\ &\quad + \hat{c}h_{ijmn}^2(\hat{\boldsymbol{X}}) + \hat{e}h_{ijmn}^3(\hat{\boldsymbol{X}}), \end{aligned} \tag{6.6}$$

$$\begin{aligned} h_{ij}^1(\hat{\boldsymbol{X}}) &= \hat{X}_i\hat{X}_j - \frac{1}{d}g_{ij}, \\ h_{ijmn}^2(\hat{\boldsymbol{X}}) &= \hat{X}_i\hat{X}_m g_{jn} + \{i \leftrightarrow j, m \leftrightarrow n\} \\ &\quad - \frac{4}{d}\hat{X}_i\hat{X}_j g_{mn} - \frac{4}{d}\hat{X}_m\hat{X}_n g_{ij} + \frac{4}{d^2}g_{ij}g_{mn}, \\ h_{ijmn}^3(\hat{\boldsymbol{X}}) &= g_{im}g_{jn} + g_{in}g_{jm} - \frac{2}{d}g_{ij}g_{mn}. \end{aligned} \tag{6.7}$$

In the above expression, $\{i \leftrightarrow j, m \leftrightarrow n\}$ represents three terms that we get by permuting the indices. Moreover, in (6.6), all the coefficients with "hats" are not independent, and we have the following relations between them:

$$d\hat{a} - 2\hat{b} + 2(d-2)\hat{c} = 0, \quad \hat{b} - d(d-2)\hat{e} = 0. \tag{6.8}$$

Now, to evaluate (6.3), it is convenient to assume that the space time is even dimensional. This assumption allows us to use the residue theorem to evaluate certain integrals when we are performing the calculation for arbitrary d. However, our final results are insensitive to the parity of the space-time dimension, and in the end, we can analytically continue the results to odd space-time dimensions. Now, for even d, we go through the following steps to compute (6.3):

(1) First, we use (6.4) to find the form of $\langle J_i(\boldsymbol{x})T_{--}(\boldsymbol{x}_1)J_j(0)\rangle$.

(2) We take the limit $x_1^+ \to \infty$ to get

$$K_{i--j} = \lim_{x_1^+ \to \infty} \left(\frac{x_1^+ - x_1^-}{2} \right)^{d-2} \langle J_i(\boldsymbol{x})T_{--}(\boldsymbol{x}_1)J_j(0)\rangle. \tag{6.9}$$

(3) Next, we integrate over x_1^-. For that, we time order the operators using the following $i\epsilon$ prescription: $x_1^0 \to x_1^0 - i\epsilon$ and $x^0 \to x^0 - 2i\epsilon$.

(4) We use standard results to integrate over the $(d-2)$ spatial dimensions orthogonal to $x^{\pm}$. While going through this step for different i,j's in (6.3), we find that some of the integrals are divergent. This is just an artifact of performing the integrations along the directions orthogonal to $x^{\pm}$ before integrating over $x^{\pm}$. We do so to simplify the integrations for arbitrary d, and to fix these spurious divergences, we use the techniques of dimensional regularization. At this step, we perform the integration over $(d-2-\kappa)$ spatial dimensions instead of $(d-2)$, and in the final result, we take the limit $\kappa \to 0$. So here, we actually calculate

$$\int d^{d-2-\kappa}x \int dx_1^- K_{i--j}. \tag{6.10}$$

(5) Now, we perform the integration over x^- and x^+,

$$\int dx^- dx^+ e^{i(E/2)x^-} e^{i(E/2)x^+} \int d^{d-2-\kappa}x \int dx_1^- K_{i--j}\,. \tag{6.11}$$

In the contour integrations at this step, we close the loop from above because only then do the integrations converge.

(6) Finally, we take the limit $\kappa \to 0$ to get a finite result,

$$\begin{aligned} Q_{i--j} = \lim_{\kappa\to 0} & \int dx^- dx^+ e^{i(E/2)x^-} e^{i(E/2)x^+} \\ & \times \int d^{d-2-\kappa}x \int dx_1^- K_{i--j}. \end{aligned} \tag{6.12}$$

We repeat the above steps for all the values of i and j in (6.3). Details of these calculations can be found in the attached MATHEMATICA program in the Supplemental Material,[26] and we find that

$$Q_{----} = \frac{(d-2)\{(d+1)[2d\hat{a}+(d-2)\hat{b}+4(d-2)\hat{c}]+2d(d+2)\hat{e}\}\pi^{d/2+2}}{2^{d-1}(d+2)\Gamma\left(\frac{d+2}{2}\right)^3}\left(\frac{E}{2}\right)^{d-1},$$

$$Q_{---+} = -\frac{d\{(d-2)(d+1)\hat{b}+d[d\hat{a}+2(d-2)\hat{c}]\}\pi^{d/2+2}}{2^{d}\Gamma\left(\frac{d}{2}+2\right)\Gamma\left(\frac{d}{2}+1\right)^2}\left(\frac{E}{2}\right)^{d-1},$$

$$Q_{---2} = 0,$$

$$Q_{+---} = -\frac{\left[[d\hat{a}+2(d-2)\hat{c}]\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}+1\right)+(d+1)\hat{b}\Gamma\left(\frac{d}{2}\right)^2\right]\pi^{d/2+2}}{2^{d-2}\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}+2\right)\Gamma\left(\frac{d}{2}\right)^2\Gamma\left(\frac{d}{2}+1\right)}\left(\frac{E}{2}\right)^{d-1},$$

$$Q_{+--+} = \frac{d(d-1)\hat{b}\pi^{d/2+2}}{2^{d-1}\Gamma\left(\frac{d}{2}+1\right)^3}\left(\frac{E}{2}\right)^{d-1}, \quad Q_{+--2}=0, \quad Q_{2---}=0, \quad Q_{2--+}=0,$$

$$Q_{2--2} = -\frac{(d\hat{a}-4\hat{c})\pi^{d/2+2}}{2^{d-3}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}+1\right)^2}\left(\frac{E}{2}\right)^{d-1}.$$

Using these values for Q_{ijkl} and relations (6.8) in (6.3), we find that the energy flux for arbitrary d becomes

$$\langle\mathcal{E}(\boldsymbol{n})\rangle = \frac{E}{\Omega_d}\left[1-\frac{(d-1)[d(d-2)\hat{e}-\hat{c}]}{(d-2)(\hat{e}+\hat{c})}\left(\cos^2\theta-\frac{1}{d-1}\right)\right]. \tag{6.13}$$

Note that the two-point function in the denominator of (6.1) does not have any angular dependence, and it fixes the normalization of higher point functions. Now, we can easily read off the value of $\mathcal{A}$ from (6.13) and can find it to be consistent with results for $d=4$ in Ref. 6.

In Ref. 12, Osborn and Petkou have further studied the position three-point functions for the specific conformal field-theory (1.1). By calculating the collinear three-point functions $\langle TJJ\rangle$ for free scalar and free fermions, they have found the ratio of the coefficients $\hat{c}$ and $\hat{e}$ to be

$$\left(\frac{\hat{e}}{\hat{c}}\right)_s = \frac{1}{d-2} \quad \text{and} \quad \left(\frac{\hat{e}}{\hat{c}}\right)_f = 0. \tag{6.14}$$

These can be further used to find the value of $\mathcal{A}$ in scalar and fermionic conformal field theories to be

$$\mathcal{A}_s = d-1 \quad \text{and} \quad \mathcal{A}_f = -\frac{d-1}{d-2}. \tag{6.15}$$

In the next section, we show how $\mathcal{A}$ is related to the coupling constant γ in action (1.7) for $d=3$. Then, these results are compared with the CFT results (6.15) to fix γ for free field theories.

B. $\mathcal{A}$ from holography and matching the results

The holographic computation of $\mathcal{A}$ for $d=4$ was first performed in Ref. 6 and then was extended to $d=3$ in Ref. 4. These calculations can be easily generalized to arbitrary dimensions, and we find that

$$\mathcal{A} = -4d(d-1)\gamma\,. \tag{6.16}$$

A quick overview of the holographic computation is as follows. According to the AdS-CFT dictionary, the computation of the expectation value of the energy flux in the boundary theory, for a state created by a conserved vector current, boils down to calculating the three-point function between two photons and a graviton. To compute such a three-point function in the bulk gravity (1.7), we need to introduce appropriate metric fluctuations and two gauge-field perturbations in the $(d+1)$-dimensional AdS background. These fluctuations couple to the stress-energy tensor and vector current insertions T_{ij} and J_i on the boundary, and one needs to evaluate their on-shell contribution for the action (1.7) as performed in Sec. IV. The bulk action has two terms. We find that the first term only contributes to the angle-independent component of (6.1) and the second term introduces the anisotropy in the flux distribution. Hence, merely by comparing the contributions from both of the terms, we can easily extract the coefficient $\mathcal{A}$. For more details of this calculation, interested readers can refer to Appendix D of Refs. 6 and 15.

Now, we match the field theory and holographic calculations from Eqs. (6.15) and (6.16) for $d=3$ to find the following values of γ for free scalars and fermions,

$$\gamma_s = -\frac{1}{12} \quad \text{and} \quad \gamma_f = \frac{1}{12}, \tag{6.17}$$

which, indeed, are consistent with the momentum space calculations in (5.14) and (5.24) in the limit $N_F=\infty$.

VII. CONCLUSIONS

The primary results of this paper are the values of γ in Eq. (1.8) for the conserved currents of the $(2+1)$-dimensional CFTs defined in (1.1). Here, γ is defined as a parameter controlling the structure of the zero-temperature three-point correlator $\langle TJJ\rangle$ between the stress-energy tensor and the conserved current. Osborn and Petkou[12] specified the general form of the $\langle TJJ\rangle$ correlator, and γ was exactly connected to their parametrization in Sec. VI. However, γ also appears in the holographic representation of the CFT on AdS_4 and is the coupling constant determining a four-derivative term in a gradient expansion of the effective action: See Eq. (1.7). The latter connection endows γ with much greater physical importance: It determines the structure of a variety of dynamical properties of charge transport at nonzero temperatures, both equilibrium and nonequilibrium. The holographic formulation also leads to the bound $|\gamma| \leqslant 1/12$.[4]

The action (1.7) was derived in Ref. 4 as the most general four-derivative holographic theory in an effective field-theory framework, which is expressed in terms of the gauge flux $F_{\mu\nu}$ and the metric tensor. Furthermore, in Appendix A, it is explicitly shown that the holographic computation of three-point function $\langle TJJ\rangle$ is independent of the choice of four-derivative terms in the action if we reparametrize the couplings properly. In a string-theory context, higher-derivative interaction terms in this action are suppressed by the ratio of the string length scale over the curvature scale of the background geometry. For our perspective, we are viewing the effective field theory as one in which loop corrections are already included in the values of the couplings and, so, is to be evaluated only at tree level. Corrections to our analysis arise from six- (and higher-) derivative terms, and we have not established that such terms are quantitatively small. However, it is encouraging to note that the four-derivative corrections to the two-derivative conductivity were smaller than 33% at all frequencies, and this was, in turn, related to the bound on γ[4]. Also reassuring is the fact that the holographically obtained bound $|\gamma| \leqslant 1/12$ coincides with the exact bound obtained from the CFT methods in Sec. VI.

It will be interesting to push this phenomenological approach by augmenting the action (1.7) by other fields, which are holographic duals of other primary operators of the CFT.[7,27] The most important of these is the mass term $|z_a|^2$ in (1.1), which tunes the CFT away from the critical point at $T=0$. Here, we are assuming we are at the CFT critical point at $T=0$, and so, such a relevant perturbation is not present in the underlying theory at $T=0$; the structure of the interactions in the CFT ensures that there is no change in $\langle |z_a|^2\rangle$ at $T>0$.[28] In the holographic theory, $|z_a|^2$ is represented by a scalar dilaton field Φ. This can influence charge transport by an additional term $\sim \Phi F_{\mu\nu}F^{\mu\nu}$ in (1.7). Such a Φ does not have an expectation value in the AdS_4 theory at $T=0$ and does not acquire one at $T>0$ in the absence of external sources. In the linear-response computation of the conductivity from such an augmented action, the $\sim \Phi F_{\mu\nu}F^{\mu\nu}$ term only influences the conductivity at the one-loop level in the bulk theory, so it need not be included in our tree-level treatment of the effective theory (1.7). Thus, γ remains as the crucial coupling determining the structure of the charge-transport properties of the CFT as recently noted.[7]

In Refs. 4 and 7, it was shown that γ determined the structure of the universal frequency dependence of the conductivity $\sigma(\omega)$ at nonzero temperatures. For $0<\gamma\leqslant 1/12$, it was found that there was a Drude-like peak at $\omega=0$, followed by an eventual saturation at a constant at large ω. Such a structure appears physically reasonable from our present computation of $\gamma=1/12$ for the free-fermion theory with $N_s=0$: The free-fermion theory has a δ function at zero frequency,[29] and it is expected that this will be broadened to a Drude peak upon including interactions.

In the complementary range of $-1/12\leqslant\gamma<0$, it was found[4,7] that $\sigma(\omega)$ had a "dip" at $\omega=0$, rather than a peak. The value $\gamma=-1/12$ is obtained for the free scalar theory with $N_f=0$. We can understand this dip if we interpret the scalar field in (1.1) as representing a *vortex* degree of freedom near, e.g., a superfluid-insulator quantum phase transition.[29] Particle-vortex duality maps the conductivity to its inverse, and the inverse conductivity then has a Drude-like peak at $\omega=0$. Further evidence for this interpretation comes from our computation of $\gamma_t=1/12$ obtained with $N_f=0$ for the topological current of (1.1). Under particle-vortex duality, the charged particle current in the dual theory maps to the topological current of (1.1), and so, this implies a peak in $\sigma(\omega)$ for the charged particle current.

Further applications include the computation of other dynamical consequences of the value of γ. In a recent paper,[7] it was shown that γ crucially determined the structure of the poles and zeros of the complex conductivity in the lower half of the complex frequency plane. These poles and zeros are associated with quasinormal modes of the holographic theory, and they are expected to be central to an understanding of the thermal dynamics of the CFT. Combined with more-precise computations of the value of γ by the methods of the present paper, these connections open up the possibility for precise predictions for the dynamics of the strongly interacting condensed-matter systems.

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APPENDIX A: HOLOGRAPHIC ACTION AND HIGHER-DERIVATIVE INTERACTIONS

In this Appendix, we discuss the effective holographic action for the gauge field A_μ in bulk gravity and four-derivative corrections. We first recall the discussion in Ref. 4 where the most general four-derivative effective action of holographic theories was presented. Here, we add further details to this argument by analyzing the full parameter space of four-derivative interactions of holographic theories and show that it reduces to (1.7) for the case of linear charge transport in CFTs at zero charge density in $2+1$ dimensions.

As discussed in Ref. 4, generally, the interaction terms in the bulk-gravity action are organized by the number of derivatives. For a gauge field in four-dimensional bulk gravity, one can construct 15 covariant and parity-conserving four-derivative terms from the gauge field, metric, and their derivatives. One can further use integration by parts and identities, such as $\nabla_{[\mu}F_{\nu\sigma]} = 0 = R_{[\mu\nu\sigma]\rho}$ and can reduce the action to have eight independent four-derivative terms,

$$\begin{aligned} I_{vec} = \frac{1}{g_4^2}\int d^4x\sqrt{-g}\bigg[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ L^2[\alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 (F^2)^2 + \alpha_4 F^4 \\ &+ \alpha_5 \nabla^\mu F_{\mu\nu}\nabla^\sigma F_\sigma{}^\nu + \alpha_6 R_{\mu\nu\sigma\rho}F^{\mu\nu}F^{\sigma\rho} \\ &+ \alpha_7 R^{\mu\nu}F_{\mu\sigma}F_\nu{}^\sigma + \alpha_8 R F^2]\bigg], \end{aligned} \tag{A1}$$

where $F^2 = F^{\mu\nu}F_{\mu\nu}$, $F^4 = F^\mu_\nu F^\nu_\sigma F^\sigma_\rho F^\rho_\mu$, and α_i are some unspecified dimensionless constants. In the context of string theory, we can expect these interactions to appear in the low-energy effective action as string-loop or α' corrections to the leading two-derivative action.[30] Hence, these new interactions will be part of a perturbative expansion where the contribution of the higher-derivative terms will be suppressed by powers of string scale over the curvature scale of the background. In this framework, we can also use field redefinitions to set all the coupling constants, excluding α_3, α_4, and α_6 to zero.[31] Now, in the remaining action, α_3 and α_4 interactions contain four powers of field strength. Hence, these terms do not contribute to the three-point function $\langle TJJ\rangle$ and to the linear charge-transport properties of the CFT. So, in our effective field-theory framework, the only relevant terms needed for a phenomenological comparison of charge-transport properties with CFTs are the $\alpha_{6,7,8}$ interactions. In the action (1.7), the contribution of these four-derivative interactions are formulated in terms of the Weyl tensor.

Now, we focus on the relevant terms of the most general four-derivative action (A2), which contribute to the $\langle TJJ\rangle$ correlator,

$$\begin{aligned} I'_{vec} = \frac{1}{\tilde g_4^2}\int d^4x\sqrt{-g}\bigg[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + L^2[\alpha_6 R_{\mu\nu\sigma\rho}F^{\mu\nu}F^{\sigma\rho} \\ &+ \alpha_7 R_{\mu\nu}F^{\mu\sigma}F^\nu{}_\rho + \alpha_8 R F^2]\bigg]. \end{aligned} \tag{A2}$$

Note that, for particular relative values of α_6, α_7, and α_8, this action takes the form of (1.7) using (A3). However, as we show now, even for arbitrary couplings, all the charge-transport properties of this new model are identical to (1.7) for a CFT at zero density. For the action (A2), we can find the equations of motion and can confirm that AdS vacuum and neutral black-hole solutions remain unmodified. Particularly, the black-hole solution satisfies the vacuum Einstein equations $R_{\mu\nu} = -3/L^2 g_{\mu\nu}$. Furthermore, the Riemann curvature tensor $R_{\mu\nu\sigma\rho}$ is related to the Weyl tensor $C_{\mu\nu\sigma\rho}$ by the following relation:

$$R_{\mu\nu\sigma\rho} = C_{\mu\nu\sigma\rho} + g_{\mu[\sigma}R_{\rho]\nu} - g_{\nu[\sigma}R_{\rho]\mu} - \frac{1}{3}R\, g_{\mu[\sigma}g_{\rho]\nu}. \tag{A3}$$

By substituting these relations into the action (A2), we find that the action becomes

$$\begin{aligned} I'_{vec} = \frac{1+8\alpha_6+12\alpha_7+48\alpha_8}{\tilde g_4^2}\int d^4x\sqrt{-g}\bigg(&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{\alpha_6}{1+8\alpha_6+12\alpha_7+48\alpha_8}L^2 C_{\mu\nu\sigma\rho}F^{\mu\nu}F^{\sigma\rho}\bigg). \end{aligned} \tag{A4}$$

Hence, expression (A2) of the action is identical to (1.7) if we define

$$g_4^2 = \frac{\tilde g_4^2}{1+8\alpha_6+12\alpha_7+48\alpha_8} \quad \text{and} \quad \gamma = \frac{\alpha_6}{1+8\alpha_6+12\alpha_7+48\alpha_8}. \tag{A5}$$

This implies that all the charge-transport properties of neutral CFTs in the generalized theory (A2) are the same as that of (1.7) with a proper reparametrization. To further support this argument, we can find the bounds on couplings α_6, α_7, and α_8 by directly applying the procedure in Sec. 5 of Ref. 4 on action (A2). The values of these couplings are constrained by demanding that the CFT dual-to-bulk gravity (A2) respects causality,[5,32] and there are no unstable modes of the vector field.[33] For our four-derivative action at tree level, we find that $|\alpha_6/(1+8\alpha_6+12\alpha_7+48\alpha_8)| \leqslant 1/12$, which is consistent with (A5) and the bound $|\gamma| \leqslant 1/12$. Although, here, we cannot fix the numerical values of couplings α_6, α_7, and α_8, our results in this paper for a CFT at zero density are independent of the choice of four-derivative terms in the action.

APPENDIX B: REVIEW OF THE CFT TWO-POINT CORRELATORS $\langle JJ\rangle$ AND $\langle TT\rangle$

In this Appendix, we derive the current and stress-tensor two-point functions given in Eqs. (1.5) and (1.6) and compute C_J and C_T for the free theory. In momentum space, the two-point function for currents and for the stress tensor is just given by two bubble diagrams: one with two scalar boson propagators and the other with two fermion propagators, respectively. The scalar boson contribution to the current-current correlator reads

$$\epsilon_{1,i_1}\epsilon_{2,i_2}\langle J_s^{i_1}(-\boldsymbol{k})J_s^{i_2}(\boldsymbol{k})\rangle = 4N_s\epsilon_1^{i_1}\epsilon_2^{i_2}\int \frac{P_{i_1}P_{i_2}}{P^2(\boldsymbol{P}+\boldsymbol{k})^2}\frac{d^3P}{8\pi^3}. \tag{B1}$$

Using the identity,

$$\int \frac{P_{i_1}P_{i_2}}{P^2(\boldsymbol{P}+\boldsymbol{k})^2}\frac{d^3P}{8\pi^3} = \left(3\frac{k_{i_1}k_{i_2}}{|\boldsymbol{k}|^2} - \eta_{i_1i_2}\right)\frac{|\boldsymbol{k}|}{64}, \tag{B2}$$

one obtains

$$\epsilon_{1,i_1}\epsilon_{2,i_2}\left\langle J_s^{i_1} J_s^{i_2}\right\rangle = -N_s\frac{|\boldsymbol{k}|}{16}\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2, \tag{B3}$$

in agreement with the uncontracted expression in (1.5). This yields $C_{J,s} = N_s/16$ for the complex scalars. The calculation for free Dirac fermions is similar, and one obtains $C_{J,f} = N_f/16$.

For the two-point function of the stress tensor, the scalar boson bubble can be integrated using the identity,

$$\int \frac{d^3P}{8\pi^3}\frac{P_{i_1}P_{i_2}P_{i_3}P_{i_4}}{P^2(\boldsymbol{P}+\boldsymbol{k})^2} = (\eta_{i_1i_2}\eta_{i_3i_4} + \text{two terms})\frac{|\boldsymbol{k}|^3}{1024} - \left(\frac{k_{i_1}k_{i_2}}{|\boldsymbol{k}|^2}\eta_{i_3i_4} + \text{ five terms}\right)\frac{5|\boldsymbol{k}|^3}{1024} + \frac{35k_{i_1}k_{i_2}k_{i_3}k_{i_4}}{1024|\boldsymbol{k}|}, \tag{B4}$$

resulting in the expression,

$$\epsilon_{1,i_1}\epsilon_{1,i_2}\epsilon_{2,i_3}\epsilon_{2,i_4}\langle T_s^{i_1i_2}(\boldsymbol{k})T_s^{i_3i_4}(-\boldsymbol{k})\rangle = \frac{N_s}{256}2(\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2)^2|\boldsymbol{k}|^3. \tag{B5}$$

Note that this agrees with (1.6) with $C_{T,s} = N_s/128$. An identical computation for the two-point function of the stress tensor for free Dirac fermions gives $C_{T,f} = N_f/128$.

These quantities enable us to fix the values of certain coupling constants in the gravity theory in Sec. V.

APPENDIX C: AdS-CFT CORRELATORS AND TWO-POINT FUNCTIONS

This Appendix provides background on the methods of gauge-gravity duality for readers who are condensed-matter physicists.

The AdS-CFT conjecture states that theories of quantum gravity on $(d+1)$-dimensional anti-de Sitter space (denoted AdS_{d+1}) are dual to d-dimensional conformal field theories that live on the boundary of AdS. The theory in AdS_{d+1} is called the bulk theory, and the theory on the boundary is called the boundary theory. In the version of the correspondence that we will be using here, the bulk theory will live on the Poincaré patch of AdS_{d+1} (the metric for this patch was already described above), whereas, the boundary theory will live on $R^{d-1,1}$.

More precisely, the conjecture states that each field in the bulk corresponds to an operator on the boundary; second, if we perform the path integral in the bulk theory with asymptotic boundary conditions fixed for these fields, then this equals the generating functional of the boundary theory with sources turned on for the corresponding operators.

Such an approach makes sense as long as we can distinguish individual fields in the bulk and there is a corresponding decomposition of the spectrum of operators in the boundary theory in terms of single- and double-trace operators. This decomposition is possible for theories with a large-N expansion, and it is in this regime that the AdS-CFT conjecture has been widely tested.

We now describe this conjecture quantitatively and explain how it may be used to calculate and to compare correlation functions.

1. Prescription

In this section, we describe the prescription for computing correlation functions in AdS-CFT. This follows Ref. 2 with some refinements that were made in Ref. 34. (See Ref. 35 for a review.)

A scalar field of mass squared m^2 in the bulk is dual to an operator of dimension $\Delta = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2}$ on the boundary. If we solve the equations of motion for a free field of this mass, we find that, near the boundary, we can have $\phi \sim z^{d-\Delta}$ and $\phi \sim z^{\Delta}$. The solution that grows at the boundary is called the "non-normalizable" solution, whereas, the other one is called the "normalizable" solution. If we work in Euclidean AdS, then fixing the coefficient of the non-normalizable mode and demanding regularity in the interior automatically fixes the normalizable mode. In Lorentzian AdS, the normalizable mode can be set independently for timelike momenta, but below, we consider those solutions that come from a continuation of the Euclidean solutions.

The original prescription for correlation functions[2] was given for massless fields. For massive fields, we need to be careful about regularization because the non-normalizable mode diverges as we approach the boundary. So, we will cut the AdS space off at $z = \epsilon$ and will consider performing the bulk path integral with the following regularized boundary condition for the scalar field as we approach the boundary:

$$\phi(\boldsymbol{x},z) \underset{z\to\epsilon}{\longrightarrow} \epsilon^{d-\Delta}\phi_0(\boldsymbol{x}). \tag{C1}$$

The idea is to work with this boundary condition and to extract the finite part in ϵ at the end of the calculation. Then, the AdS-CFT prescription is that

$$\int e^{-S}\mathcal{D}\phi\bigg|_{\text{bound}} = \langle e^{\int \phi_0(\boldsymbol{x})O(\boldsymbol{x})d^dx}\rangle_{\text{CFT}}. \tag{C2}$$

Here, the left-hand side is shorthand for the path integral in the bulk performed with the boundary conditions (C1), whereas, the right-hand side is an expectation value in the conformal field theory. Although to lighten the notation, we have chosen a particular coordinate system to represent the boundary conditions (C1), the prescription is independent of this choice.

The original conjecture (C2) was made in specific contexts: For example, one of the best studied examples of the AdS-CFT duality is when the bulk theory is a type-IIB string theory on $\mathrm{AdS}_5 \times S^5$ and the boundary theory is an $\mathcal{N} = 4$ super-Yang-Mills theory. Several other examples are known.

However, where correlation functions are concerned, the prescription (C2) may be examined just as well within effective field theory. This means that we take some effective field theory in the bulk and compute the left-hand side of (C2) at tree level in the bulk. This computation can be used to *define* a generating functional in a CFT for leading order in $\frac{1}{N}$. [As we mentioned above, the prescription (C2) makes sense when we have a perturbative parameter that allows us to differentiate between single- and double-trace operators, and we are using

N as shorthand for this parameter here.] This is because one can show that the quantity obtained this way satisfies all the constraints of conformal invariance and the operator product expansion for leading order in $\frac{1}{N}$ in the boundary theory.

Now, let us turn to the stress tensor and conserved currents. The graviton in the bulk is dual to the stress tensor on the boundary, and a gauge field is dual to conserved currents. Now, consider performing the bulk path integral with the following boundary conditions for the metric and gauge fields:

$$\begin{aligned} g_{zz}(\boldsymbol{x},z) &\underset{z\to 0}{\longrightarrow} \frac{1}{z^2}; \quad g_{zi}(\boldsymbol{x},z) \underset{z\to 0}{\longrightarrow} 0;\\ g_{ij}(\boldsymbol{x},z) &\underset{z\to 0}{\longrightarrow} \frac{1}{z^2}[\eta_{ij} + \chi_{ij}(\boldsymbol{x})],\\ A_z(\boldsymbol{x},z) &\underset{z\to 0}{\longrightarrow} 0; \quad A_i(\boldsymbol{x},z) \underset{z\to 0}{\longrightarrow} \mathbb{V}_i(\boldsymbol{x}). \end{aligned} \tag{C3}$$

Then, the bulk path integral with these boundary conditions is conjectured to be the same as the following generating functional of the conformal field theory:

$$\left\langle \exp\left\{\int [\chi_{ij}(\boldsymbol{x})T^{ij}(\boldsymbol{x}) + \mathbb{V}_i(\boldsymbol{x})j^i(\boldsymbol{x})]d^d\boldsymbol{x}\right\}\right\rangle.$$

2. Scalar two-point function

The simplest setting in which we can test these ideas is to evaluate two-point functions. Consider a free massive scalar with action,

$$S_{\text{bulk}} = -\frac{1}{2}\int \sqrt{-g}[(\partial_\mu\phi)^2 + m^2\phi^2]. \tag{C4}$$

At leading order, we can evaluate the left-hand side of (C2) in the saddle-point approximation. Let us also take

$$\phi_0(\boldsymbol{x}) = \lambda_1 e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}} + \lambda_2 e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}}. \tag{C5}$$

We need to find a solution of the equations of motion,

$$(\Box - m^2)\phi = 0, \tag{C6}$$

that respects (C1).

In fact, it is rather subtle to write down such a solution. The authors of Ref. 34 showed that the correct method was to write down the following solution:

$$\begin{aligned} \phi(\boldsymbol{x},z) = \epsilon^{d-\Delta}\bigg[&\lambda_1\frac{(|\boldsymbol{k}_1|z)^{d/2}K_{\Delta-d/2}(|\boldsymbol{k}_1|z)}{(|\boldsymbol{k}_1|\epsilon)^{d/2}K_{\Delta-d/2}(|\boldsymbol{k}_1|\epsilon)}e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}}\\ &+\lambda_2\frac{(|\boldsymbol{k}_2|z)^{d/2}K_{\Delta-d/2}(|\boldsymbol{k}_2|z)}{(|\boldsymbol{k}_2|\epsilon)^{d/2}K_{\Delta-d/2}(|\boldsymbol{k}_2|\epsilon)}e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}}\bigg], \end{aligned} \tag{C7}$$

where K is the modified Bessel function. Here, we have defined $|\boldsymbol{k_m}|$ to be taken in the Lorentzian metric with a mostly positive signature, i.e., the boundary metric is defined to be $\text{diag}(-1,1,1,\ldots,1)$. For timelike $\boldsymbol{k}$, we should take its norm to have a negative imaginary part; this continues the modified Bessel function K to a Hankel function $H^{(1)}$.

We can superpose solutions of different momenta so that the sum has a δ-function support at a given point; such a solution is called a bulk-to-boundary propagator. If we Fourier transform the bulk-to-boundary propagator, we get a solution of the sort above.

It is very tempting to expand (C7) in powers of ϵ so that we have

$$\begin{aligned} \phi(\boldsymbol{x},z) = \frac{2^{1/2(d-2\Delta)+1}}{\Gamma\left(-\frac{d}{2}+\Delta\right)}&[\lambda_1|\boldsymbol{k}_1|^{\Delta-d/2}z^{d/2}K_{\Delta-d/2}(|\boldsymbol{k}_1|z)e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}}\\ &+\lambda_2|\boldsymbol{k}_2|^{\Delta-d/2}z^{d/2}K_{\Delta-d/2}(|\boldsymbol{k}_2|z)]\\ &+\mathrm{O}(\epsilon^{2\Delta-d}) + \mathrm{O}(\epsilon)\,. \end{aligned} \tag{C8}$$

However, as shown in Ref. 34, we *cannot* discard the subleading terms in ϵ at this stage because there is a second divergence when we evaluate the on-shell action and these subleading terms then contribute to $\mathrm{O}(\epsilon^0)$ in the final answer.

Now, let us compute the two-point function using the prescription above. The on-shell action is divergent if we take $\epsilon \to 0$, so we should perform the calculation with ϵ kept finite and should extract the ϵ^0 term at the end.

In the solution (C7), the on-shell action is simply

$$S_{\text{on-shell}} = \frac{-1}{2}\int \sqrt{-g}z^2\phi\frac{\partial\phi}{\partial z}d^dx\bigg|_{z=\epsilon}. \tag{C9}$$

A short calculation shows that the ϵ^0 term on the right-hand side of (C9) that is bilinear in λ_1 and λ_2 is as follows:

$$\begin{aligned} \frac{\partial^2 S_{\text{on-shell}}}{\partial\lambda_1\partial\lambda_2} = -(2\Delta-d)&\frac{\Gamma\left(\frac{d}{2}+1-\Delta\right)}{\Gamma\left(\Delta-\frac{d}{2}+1\right)}\left(\frac{|\boldsymbol{k}_1|}{2}\right)^{2\Delta-d}\\ &\times\delta(\boldsymbol{k}_1+\boldsymbol{k}_2) + \cdots, \end{aligned} \tag{C10}$$

where the $\cdots$ are higher- and lower-order terms in ϵ. The terms that are divergent as $\epsilon \to 0$ are analytic in the momentum, and so they can be removed by local counterterms.

From the prescription (C2), we can see that this is also the two-point function of the operator O in the conformal field theory,

$$\langle O(\boldsymbol{k}_1)O(\boldsymbol{k}_2)\rangle = C_\Delta|\boldsymbol{k}_1|^{2\Delta-d}\delta(\boldsymbol{k}_1+\boldsymbol{k}_2), \tag{C11}$$

where C_Δ is the numerical constant in (C10). In fact, this is precisely what one expects from conformal invariance for a primary operator of dimension Δ.

3. An alternate prescription

For the stress tensor, and even for scalar fields, at leading order in $\frac{1}{N}$ (i.e., at tree level in the bulk), it is often convenient to replace the prescription (C2) with an equivalent prescription.[36] This prescription simply states that, if we write the metric as

$$g_{\mu\nu} = g^{\text{AdS}}_{\mu\nu} + h_{\mu\nu}, \tag{C12}$$

where $g^{\text{AdS}}_{\mu\nu}$ is the metric (4.1) and consider field configurations that satisfy the asymptotic conditions (C1), then

$$\begin{aligned} &\langle T_{i_1j_1}(\boldsymbol{x}_1)\cdots T_{i_nj_n}(\boldsymbol{x_n})\rangle_{\text{boundary}}\\ &\quad= Z^n\lim_{z_i\to 0} z_1^{2-d}\cdots z_n^{2-d}\big\langle h_{i_1j_1}(\boldsymbol{x}_1,z_1)\cdots h_{i_nj_n}(\boldsymbol{x_n},z_n)\big\rangle_{\text{bulk}}. \end{aligned} \tag{C13}$$

This is the statement that: *Boundary correlators are just boundary values of bulk Green's functions*. Here, Z is a wave-function renormalization factor. At tree level in the bulk, this factor is just a constant as we see below, and so we have written Z^n rather than writing separate factors for each insertion. Z just fixes the overall normalization of operators, and so, at tree level, it is not physically relevant, but we retain it for later convenience. For scalar operators, the analogous prescription is as follows:

$$\langle O(\boldsymbol{x}_1)\cdots O(\boldsymbol{x}_n)\rangle_{\text{boundary}} = Z^n \lim_{z\to 0} z_1^{-\Delta}\cdots z_n^{-\Delta}\langle\phi(\boldsymbol{x}_1,z_1)\cdots\phi(\boldsymbol{x}_n,z_n)\rangle_{\text{bulk}}. \quad \text{(C14)}$$

This is the prescription that we will use to evaluate two-point functions.

4. Scalar two-point function re-derived

To get a feel for this prescription, let us re-derive the result above for the two-point function of scalar operators. The scalar two-point Green's function in the bulk is given by Ref. 37,

$$G(\boldsymbol{x},z,\boldsymbol{x}',z') = \int \frac{d^d\boldsymbol{k}}{(2\pi)^d} G_{\boldsymbol{k}}(z,z')e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} = -\int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\frac{dp^2}{2}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}z^{d/2}J_{\Delta-d/2}(pz)J_{\Delta-d/2}(pz')(z')^{d/2}}{(\boldsymbol{k}^2+p^2-i\epsilon)}. \quad \text{(C15)}$$

We can check that this Green's function obeys: (Note that Ref. 37 defines the Green's function with an additional minus sign on the right-hand side),

$$(\Box - m^2)G(\boldsymbol{x},z,\boldsymbol{x}',z') = \frac{1}{\sqrt{-g}}\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(z-z'). \quad \text{(C16)}$$

In Fourier space, the relation (C16) is simply

$$z^{d+1}\frac{\partial}{\partial z}z^{1-d}\frac{\partial G_k(z,z')}{\partial z} - m^2G_k(z,z') - z^2k^2G_k(z,z') = \delta(z-z')z^{d+1}. \quad \text{(C17)}$$

We can verify that this is satisfied by virtue of the identity,

$$\int pJ_\nu(pz)J_\nu(pz')dp = z^{-1}\delta(z-z'). \quad \text{(C18)}$$

After performing the p integral and transforming to momentum space, we find that the two-point Green's function can be written

$$G(\boldsymbol{k},z_1,z_2) = -(z_1z_2)^{d/2}I_{\Delta-d/2}(|\boldsymbol{k}|z^<)K_{\Delta-d/2}(|\boldsymbol{k}|z^>), \quad \text{(C19)}$$

where $z^< = \min(z_1,z_2)$ and $z^> = \max(z_1,z_2)$. (We have written this as a function of one momentum, rather than two because the two momenta are forced to be equal by momentum conservation.)

With this choice, when we now take the limit where one point goes to the boundary and take $Z = -(2\Delta - d)$, we find

$$Z\lim_{z_1\to 0} z_1^{-\Delta}G(\boldsymbol{k},z_1,z_2) = \frac{2^{(1/2)(d-2\Delta)+1}}{\Gamma\left(-\frac{d}{2}+\Delta\right)}|\boldsymbol{k}|^{\Delta-d/2}z_2^{d/2}K_{\Delta-d/2}(|\boldsymbol{k}|z_2). \quad \text{(C20)}$$

Note that this matches the naive bulk-to-boundary propagator of (C8). We could also use a different value of Z provided that, in calculating higher point functions, we consistently use the bulk-to-boundary propagator that comes from taking the limit above. When we take both points for the boundary, we recover the two-point function of the boundary operator,

$$\langle O(\boldsymbol{k})O(-\boldsymbol{k})\rangle = Z^2\lim_{z_2\to 0}z_2^{-\Delta}\lim_{z_1\to 0}z_1^{-\Delta}G(\boldsymbol{k},z_1,z_2) = -(2\Delta-d)\frac{\Gamma\left(\frac{d}{2}+1-\Delta\right)}{\Gamma\left(\Delta-\frac{d}{2}+1\right)}\left(\frac{|\boldsymbol{k}_1|}{2}\right)^{2\Delta-d} \times \delta(\boldsymbol{k}_1+\boldsymbol{k}_2)+\cdots. \quad \text{(C21)}$$

Here, once again, as we take $z_2 \to 0$, we find a divergent term that is analytic in the momenta and so, a δ function in position space. This is indicated by the $\cdots$, which are unimportant.

Note that this prescription is somewhat more straightforward than evaluating the on-shell action since we do not have to worry about the subleading terms in ϵ in imposing (C1), and so we use it for the stress tensor and conserved currents.

5. Two-point function of the stress tensor and currents

To evaluate the two-point function of the stress tensor using AdS-CFT, we simply need to evaluate the two-point function of the metric fluctuation in AdS. We will consider the Hilbert-Einstein action.

$$S_{\text{grav}} = \frac{-1}{16\pi G_N}\int\sqrt{-g}(R-2\Lambda), \quad \text{(C22)}$$

where Λ is the cosmological constant. We now expand the metric out as

$$g_{\mu\nu} = g_{\mu\nu}^{\text{ads}} + h_{\mu\nu}. \quad \text{(C23)}$$

The propagator in the gauge where we set $h_{zi} = h_{zz} = 0$ is easily evaluated and is found to be as follows:[38]

$$G_{ij,kl}^{\text{grav}}(\boldsymbol{k},z_1,z_2) = 8\pi G_N\int\left[\frac{z_1^{d/2-2}J_{d/2}(pz_1)J_{d/2}(pz_2)(z_2)^{d/2-2}}{(\boldsymbol{k}^2+p^2-i\epsilon)} \times \frac{1}{2}\left(\mathcal{T}_{ik}\mathcal{T}_{jl}+\mathcal{T}_{il}\mathcal{T}_{jk}-\frac{2\mathcal{T}_{ij}\mathcal{T}_{kl}}{d-1}\right)\right]\frac{-dp^2}{2}, \quad \text{(C24)}$$

where $\mathcal{T}_{ij} = \eta_{ij} + k_ik_j/p^2$.

First, let us take the limit $z_1 \to 0$ and take Z in (C13) to be $Z = -\frac{d}{8\pi G_N}$. With this, we see that when we take $z_1 \to 0$,

$$Z \lim_{z_1\to 0} z_1^{2-d} G^{\text{grav}}_{ij,kl}(\boldsymbol{k},z_1,z_2) = \frac{1}{2}\left(\widetilde{\mathcal{T}}_{ik}\widetilde{\mathcal{T}}_{jl} + \widetilde{\mathcal{T}}_{il}\widetilde{\mathcal{T}}_{jk} - \frac{2\widetilde{\mathcal{T}}_{ij}\widetilde{\mathcal{T}}_{kl}}{d-1}\right) \times \left[\frac{2^{-d/2+1}}{\Gamma\left(\frac{d}{2}\right)}|\boldsymbol{k}|^{d/2} z_2^{d/2-2} K_{d/2}(|\boldsymbol{k}|z_2)\right], \tag{C25}$$

where $\widetilde{\mathcal{T}}_{ij} = \eta_{ij} - k_i k_j/|\boldsymbol{k}|^2$.

For $d = 3$, which is the case that we are interested in, this takes on a very simple form

$$Z \lim_{z_1\to 0} z_1^{2-d} G^{\text{grav}}_{ij,kl}(\boldsymbol{k},z_1,z_2) = \frac{1}{2z_2^2}(\widetilde{\mathcal{T}}_{ik}\widetilde{\mathcal{T}}_{jl} + \widetilde{\mathcal{T}}_{il}\widetilde{\mathcal{T}}_{jk} - \widetilde{\mathcal{T}}_{ij}\widetilde{\mathcal{T}}_{kl})e^{-|\boldsymbol{k}|z_2}(1 + |\boldsymbol{k}|z_2). \tag{C26}$$

This is the bulk-to-boundary propagator that we use below.

Taking the limit as $z_2 \to 0$, we now find that

$$\langle T_{ij}(\boldsymbol{k})T_{kl}(-\boldsymbol{k})\rangle = -\frac{1}{8\pi G_N}|\boldsymbol{k}|^d \frac{\Gamma\left(1-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}+1\right)}\frac{d}{2} \times \left(\widetilde{\mathcal{T}}_{ik}\widetilde{\mathcal{T}}_{jl} + \widetilde{\mathcal{T}}_{il}\widetilde{\mathcal{T}}_{jk} - \frac{2\widetilde{\mathcal{T}}_{ij}\widetilde{\mathcal{T}}_{kl}}{d-1}\right). \tag{C27}$$

Let us now specialize for the case where $d = 3$. We now have

$$\langle T_{ij}(\boldsymbol{k})T_{kl}(-\boldsymbol{k})\rangle = \frac{4}{8\pi G_N}|\boldsymbol{k}|^3(\widetilde{\mathcal{T}}_{ik}\widetilde{\mathcal{T}}_{jl} + \widetilde{\mathcal{T}}_{il}\widetilde{\mathcal{T}}_{jk} - \widetilde{\mathcal{T}}_{ij}\widetilde{\mathcal{T}}_{kl}) \quad \text{for } d = 3. \tag{C28}$$

This matches with the answer obtained from the CFT in (1.6).

Similarly, we can obtain the two-point function of currents in the Maxwellian theory in the bulk. (For this, we set $\gamma = 0$ for the moment.) We start with the Maxwell action,

$$S_{\text{gauge}} = \frac{-1}{4g_4^2}\int \sqrt{-g}F_{\mu\nu}F^{\mu\nu}. \tag{C29}$$

The bulk-to-bulk propagator of currents in the "axial gauge" (where we set the z component of the gauge field to 0) is given by

$$G^{\text{axial},ab}_{ij}(\boldsymbol{k},z_1,z_2) = g_4^2\int \frac{-dp^2}{2(2\pi)^d}\left[\frac{(z_1z_2)^{\nu_1}J_{\nu_1}(pz_1)J_{\nu_1}(pz_2)\mathcal{T}_{ij}\delta^{ab}}{(\boldsymbol{k}^2 + p^2 - i\epsilon)}\right], \tag{C30}$$

with $\nu_1 = \frac{d}{2} - 1$. Repeating the process above and now taking $Z = \frac{2-d}{g_4^2}$, we find that, when $z_1 \to 0$, we get

$$Z \lim_{z_1\to 0} z_1^{1-d} G^{\text{axial}}_{ij}(\boldsymbol{k},z_1,z_2) = \frac{2^{(1/2)(2-d)+1}}{\Gamma\left(\frac{d}{2}-1\right)}|\boldsymbol{k}|^{d/2-1}z_2^{d/2-1}K_{d/2-1}(|\boldsymbol{k}|z_2)\widetilde{\mathcal{T}}_{ij}. \tag{C31}$$

For $d = 3$, we simply have

$$Z \lim_{z_1\to 0} z_1^{1-d} G^{\text{axial}}_{ij}(\boldsymbol{k},z_1,z_2) = \widetilde{\mathcal{T}}_{ij}e^{-|\boldsymbol{k}|z_2} \quad \text{for } d = 3. \tag{C32}$$

The two-point function of currents is given by

$$\langle j_i(\boldsymbol{k})j_j(-\boldsymbol{k})\rangle = \frac{1}{g_4^2}(2-d)\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{|\boldsymbol{k}_1|}{2}\right)^{d-2}\widetilde{\mathcal{T}}_{ij}. \tag{C33}$$

For $d = 3$, we have the remarkably simple expression,

$$\langle j_i(\boldsymbol{k})j_j(-\boldsymbol{k})\rangle = -\frac{1}{g_4^2}|\boldsymbol{k}_1|\widetilde{\mathcal{T}}_{ij}, \tag{C34}$$

which agrees with (1.5) and fixes $C_J = 1/g_4^2$.

APPENDIX D: SPINOR -HELICITY FORMALISM

In this Appendix, we review the spinor-helicity formalism for correlation functions in three-dimensional conformal field theories that was described briefly in Sec. II. The spinor-helicity formalism adapted for three-dimensional Lorentzian CFTs is also described in Sec. 2 of Ref. 22.

In our conventions, the boundary metric is Lorentzian and mostly positive. This means that, for two boundary vectors,

$$\boldsymbol{k}\cdot\boldsymbol{k} = (k_1)^2 + (k_2)^2 - (k_0)^2. \tag{D1}$$

We use boldface for vectors but not for their components. We use i, j, etc., for boundary space-time indices and μ, ν, etc., for bulk space-time indices. We use m, n, etc., to index the particle number. Finally, the components of a momentum vector come with a naturally lowered index.

Our σ matrix conventions are the following:

$$\sigma^0_{\alpha\dot\alpha} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma^1_{\alpha\dot\alpha} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma^2_{\alpha\dot\alpha} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma^3_{\alpha\dot\alpha} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{D2}$$

Given a three-momentum $\boldsymbol{k} = (k_0,k_1,k_2)$ as we described in Sec. II, we convert it into spinors using

$$k_{\alpha\dot\alpha} = k_0\sigma^0_{\alpha\dot\alpha} + k_1\sigma^1_{\alpha\dot\alpha} + k_2\sigma^2_{\alpha\dot\alpha} + i|\boldsymbol{k}|\sigma^3_{\alpha\dot\alpha} = \lambda_\alpha\bar\lambda_{\dot\alpha}, \tag{D3}$$

where

$$|\boldsymbol{k}| \equiv \sqrt{\boldsymbol{k}\cdot\boldsymbol{k}} = \sqrt{k_1^2 + k_2^2 - k_0^2}. \tag{D4}$$

If $\boldsymbol{k}$ is spacelike to start with, then the σ^3 component is imaginary.

In components, we have the following expressions for the spinors:

$$\lambda = \left(\sqrt{k_0 + i|\boldsymbol{k}|}, \frac{k_1 + ik_2}{\sqrt{k_0 + i|\boldsymbol{k}|}}\right), \quad \bar\lambda = \left(\sqrt{k_0 + i|\boldsymbol{k}|}, \frac{k_1 - ik_2}{\sqrt{k_0 + i|\boldsymbol{k}|}}\right). \tag{D5}$$

We have the freedom to rescale the spinors by any complex number: $\lambda \to \alpha\lambda, \bar\lambda \to \frac{1}{\alpha}\bar\lambda$ without changing the momentum. If we do this for spinors corresponding to an external particle, then this rescales the polarization vectors, and amplitudes pick up a simple phase.

We can raise and lower spinor indices using the ϵ tensor. We choose the ϵ tensor to be $i\sigma_2$ for both the dotted and the undotted indices. This means that

$$\epsilon^{01} = 1 = -\epsilon^{10}, \tag{D6}$$

and spinor dot products are defined via

$$\begin{aligned} \langle\lambda_1, \lambda_2\rangle &= \epsilon^{\alpha\beta}\lambda_{1\alpha}\lambda_{2\beta} = \lambda_{1\alpha}\lambda_2^{\alpha}, \\ \langle\bar{\lambda}_1, \bar{\lambda}_2\rangle &= \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{1\dot{\alpha}}\bar{\lambda}_{2\dot{\beta}} = \bar{\lambda}_{1\dot{\alpha}}\bar{\lambda}_2^{\dot{\alpha}}. \end{aligned} \tag{D7}$$

In the case of four-dimensional flat-space scattering amplitudes, all expressions can be written in terms of the two kinds of dot products above. However, in our case, we should expect our expressions for CFT$_3$ correlators to only have a manifest $SO(2,1)$ invariance. This means that we might have mixed products between dotted and undotted indices. Such a mixed product extracts the z component of the vector and is performed by contracting with σ^3,

$$2i|\boldsymbol{k}| = (\sigma^3)^{\alpha\dot{\alpha}}k_{\alpha\dot{\alpha}} \equiv [\lambda, \bar{\lambda}]. \tag{D8}$$

The reader should note that we use square brackets only for this mixed product; products of both left- and right-handed spinors are denoted by angular brackets. Second, we note that this mixed dot product is symmetric,

$$[\lambda, \bar{\lambda}] = [\bar{\lambda}, \lambda]. \tag{D9}$$

When we take the dot products of two three-momenta, we have

$$\begin{aligned} \boldsymbol{k}\cdot\boldsymbol{q} \equiv (k_1q_1 + k_2q_2 - k_0q_0) = -\tfrac{1}{2}\big(\langle\lambda_k, \lambda_q\rangle\langle\bar{\lambda}_k, \bar{\lambda}_q\rangle \\ + \tfrac{1}{2}[\lambda_k, \bar{\lambda}_k][\lambda_q, \bar{\lambda}_q]\big). \end{aligned} \tag{D10}$$

Another fact to keep in mind is that,

$$\begin{aligned} \boldsymbol{k}_1 + \boldsymbol{k}_2 &= \boldsymbol{k}_3 \\ \Rightarrow \quad & \lambda_1\bar{\lambda}_1 + \lambda_2\bar{\lambda}_2 = \lambda_3\bar{\lambda}_3 \\ & + \frac{1}{2}([\lambda_1, \bar{\lambda}_1] + [\lambda_2, \bar{\lambda}_2] - [\lambda_3, \bar{\lambda}_3])\sigma^3. \end{aligned} \tag{D11}$$

We also need a way to convert dotted-to-undotted indices. We write

$$\lambda^{\dagger}_{\dot{\alpha}} = \sigma^3_{\alpha\dot{\alpha}}\lambda^{\alpha}, \quad \bar{\lambda}^{\dagger}_{\alpha} = \sigma^3_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}. \tag{D12}$$

This has the property that

$$\langle\, \mu, \lambda^{\dagger}\rangle = [\, \mu, \lambda], \tag{D13}$$

where the quantity on the right-hand side is defined in (D8).

With all this, we can write down polarization vectors for conserved currents. The polarization vectors for a momentum vector $\boldsymbol{k}$ associated with spinors $\lambda,\bar{\lambda}$ are given by

$$\begin{aligned} \epsilon^{+}_{\alpha\dot{\alpha}} &= 2\frac{\bar{\lambda}^{\dagger}_{\alpha}\bar{\lambda}_{\dot{\alpha}}}{[\lambda, \bar{\lambda}]} = \frac{\bar{\lambda}^{\dagger}_{\alpha}\bar{\lambda}_{\dot{\alpha}}}{i|\boldsymbol{k}|}, \\ \epsilon^{-}_{\alpha\dot{\alpha}} &= 2\frac{\lambda_{\alpha}\lambda^{\dagger}_{\dot{\alpha}}}{[\lambda, \bar{\lambda}]} = \frac{\lambda_{\alpha}\lambda^{\dagger}_{\dot{\alpha}}}{i|\boldsymbol{k}|}. \end{aligned} \tag{D14}$$

These vectors are normalized so that

$$\boldsymbol{\epsilon}^{+}\cdot\boldsymbol{\epsilon}^{+} = \boldsymbol{\epsilon}^{-}\cdot\boldsymbol{\epsilon}^{-} = 0, \quad \boldsymbol{\epsilon}^{+}\cdot\boldsymbol{\epsilon}^{-} = 2. \tag{D15}$$

Polarization tensors for the stress tensor are just outer products of these vectors with themselves,

$$e^{\pm}_{ij} = \epsilon^{\pm}_i\epsilon^{\pm}_j. \tag{D16}$$

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