Effective field theories for topological insulators by functional bosonization

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(Received 24 October 2012; published 25 February 2013)

Effective field theories that describe the dynamics of a conserved U(1) current in terms of "hydrodynamic" degrees of freedom of topological phases in condensed matter are discussed in general dimension $D = d + 1$ using the functional bosonization technique. For noninteracting topological insulators (superconductors) with a conserved U(1) charge and characterized by an integer topological invariant [more specifically, they are topological insulators in the complex symmetry classes (class A and AIII), and in the "primary series" of topological insulators, in the eight real symmetry classes], we derive the BF-type topological field theories supplemented with the Chern-Simons (when D is odd) or the θ (when D is even) terms. For topological insulators characterized by a $\mathbb{Z}_2$ topological invariant (the first and second descendants of the primary series), their topological field theories are obtained by dimensional reduction. Building on this effective field theory description for noninteracting topological phases, we also discuss, following the spirit of the parton construction of the fractional quantum Hall effect by Block and Wen, the putative "fractional" topological insulators and their possible effective field theories, and use them to determine the physical properties of these nontrivial quantum phases.

I. INTRODUCTION

Topological phases are fully quantum-mechanical states of matter that are not characterized by spontaneous breaking of a global symmetry of the quantum-mechanical system. While gapped in the bulk, quite often these phases have *gapless* excitations at their boundary, signaling the highly entangled nature of their ground states. Since the discovery of the integer quantum Hall effect (IQHE) and the factional quantum Hall effect (FQHE),[1] the list of topological phases in nature has been expanded, in particular, by the recent discovery of time-reversal symmetric topological insulators in two and three dimensions (2D and 3D) in systems with strong spin-orbit coupling[2–11] and the identification of ^{3}He-B phase as a topological superconductor (actually a superfluid).[12] Unlike the IQHE, the topological character of these topological insulators and superconductors (i.e., the stable gapless edge or surface modes) is protected by time-reversal symmetry (TRS). The presence or absence of a topological distinction among gapped phases for a given set of symmetries and for given spatial dimensions can be studied systematically, and is summarized in the "periodic table" of topological insulators and superconductors for noninteracting fermions.[13–15]

One of the defining properties of topological insulators is their characteristic response to external electromagnetic fields. A classic example for this is the quantum Hall effect (QHE), which is characterized by the nonzero, *quantized* value of the Hall conductance. The three-dimensional time-reversal symmetric topological insulator is characterized by nonzero magnetoelectric polarizablility, which, in the presence of time-reversal symmetry, also takes a quantized value.[11] Correspondingly, these responses are described by a field theory supplemented with a term of topological origin, such as the Chern-Simons (CS) term, or the axion term. For a wide class of topological insulators and superconductors, such topological *response* theories have been studied and carefully classified.[11,16–19] When it exists, such *topologically protected*

response is a powerful way to characterize a topological phase, as it is not limited to noninteracting topological phases.

There is, however, a subtle but profound distinction between a topological phase and a phase with a topologically protected response. For instance, the IQHE is a phase of a (generally interacting) two-dimensional electron gas in a strong magnetic field whose Hall conductance is precisely defined, and protected, by a topological invariant.[20] Similarly, its edge states are also protected in this case by their chiral nature. However, the fractional quantum Hall states are *topological fluids*. In addition to having a topologically protected fractional quantum Hall conductance, these fluids are characterized by having nontrivial ground-state degeneracies on topologically nontrivial surfaces, excitations that carry fractional charge and fractional statistics (Abelian or non-Abelian), and by a set of edge states with generally complex properties. The universal properties of such *topological phases* are given in terms of effective low-energy hydrodynamic topological field theories of these fluids.[21–23]

While powerful, the effective field theories of the response of a system to an external local probe should be distinguished from *internal* or *hydrodynamic* topological field theories describing the global properties and the excitations of topological phases, as in the example of the FQH fluids.[23] Just as fluid dynamics is an efficient description of a collection of a macroscopic number of particles, the dynamics of topological insulators may be well described in terms of collective degrees of freedom, rather than relying on microscopic electrons (quasiparticles). In fact, such a picture was developed for the QHE, and we have a good understanding of a quantum Hall system as a droplet of electron liquid. The hydrodynamic topological field theory description of the quantum Hall droplet is given by the Chern-Simons (CS) gauge theory. Once such effective description of the low-energy physics is established, it is likely to be robust against interactions, and has a wider range of applicability than the noninteracting microscopic system. The purpose of this paper is to develop such a hydrodynamic

effective field theory that is capable of describing collective excitations in topological insulators in general dimension. For previous studies, see, for example, Refs. 24–27.

The CS description of the two-dimensional QHE can be derived, for example, from composite particle theories. In the composite boson theory,[28] one first attaches a unit flux to each electron to make it a boson. For the completely filled Landau level, this will cancel the external magnetic field on average, and thus we have a composite boson system at zero magnetic field, but interacting with a statistical CS gauge field. A subsequent duality transformation delivers the CS theory, which gives us a hydrodynamical description of the filled Landau level. Similarly, by using the composite fermion picture,[29,30] the same CS theory can also be derived.[31]

The flux attachment concept is successfully used to derive the hydrodynamic theories of the Laughlin and Jain states of the FQHE. However, it only works in two dimensions since it is based on the existence of the Chern-Simons gauge theory and a connection with the braiding group. In 2D, the excitations (or vortices) can carry the quantum numbers of representations of the braid group, which allows for fractional statistics. However, the concept of braiding is only meaningful in two spatial dimensions since the braiding of particle world lines is not topologically stable in other spatial dimensions. In addition, for technical reasons, flux attachment is somewhat cumbersome to apply even for 2D lattice systems (see, e.g., Ref. 32). Thus an alternative procedure to derive effective hydrodynamic field theories of topological phases is desirable.

One alternative approach for deriving effective field theories is bosonization. This method is closely related to the well-known boson-fermion correspondence in $1 + 1$ dimensions, which is an exact operator correspondence between a theory of massless (Dirac) fermions and a theory of a massless relativistic scalar field. We will show here that one can use an approach known as "functional bosonization" to derive effective hydrodynamic field theories for general topological phases both in two and three dimensions. Although for spatial dimensions $d > 1$ the functional bosonization approach is not an exact mapping, it is nevertheless useful to derive effective low-energy theories in massive phases, which is what we are after in this work. In particular, in this paper, we will apply functional bosonization to the $(3 + 1)$D topological insulator to derive an effective topological hydrodynamic theory of the form of the BF theory in $3 + 1$ space-time dimensions.

This paper is organized as follows. In Sec. II, we start by applying functional bosonization to the topological insulators in the primary series of the periodic table of topological insulators.[13,15] We derive BF-type topological field theories supplemented with a term of topological origin. Next, the physical picture suggested by functional bosonization is described in Sec. III, making some comparison with flux attachment, and composite particle theories. In Sec. IV, we discuss hydrodynamic topological field theory descriptions of $\mathbb{Z}_2$ topological insulators, including the three-dimensional time-reversal symmetric topological insulators, and the two-dimensional quantum spin Hall effect. They are obtained from the BF-type topological field theories of the primary series by dimensional reduction. N.B. we will use D to represent space-time dimensions and d to represent space dimensions. Thus $D = 2$ is $1 + 1$-dimensional space-time.

In Sec. V, we explore the possibility of topological phases that arise because of strong interactions, e.g., "fractional" topological insulators. Starting from the BF theories for noninteracting topological insulators, and following the parton approach pioneered by Blok and Wen[33,34] in the FQHE by implementing strong correlations as a constraint among fields, we give a derivation of topological field theories that describe strongly interacting topological phases. Finally, we present our conclusions in Sec. VI.

II. FUNCTIONAL BOSONIZATION

A. Functional bosonization

We begin with a summary of functional bosonization. Our starting point is the fermionic partition function in $D = d + 1$ space-time dimensions:

$$Z(A^{\text{ex}}) = \int \mathcal{D}[\bar{\psi}, \psi] \exp(i K_F[\bar{\psi}, \psi, A^{\text{ex}}]), \quad (2.1)$$

where $K_F[\bar{\psi}, \psi, A^{\text{ex}}]$ is the fermionic action describing the (topological) insulator in question, and A^{ex}_μ is an external U(1) gauge field, which couples minimally to the fermion field $\bar{\psi}, \psi$ as a source (background). The correlation functions of the electrical current operator can be obtained from the generating functional $Z(A^{\text{ex}})$ as

$$\langle j^{\mu_1}(x_1) j^{\mu_2}(x_2) \cdots \rangle = \frac{1}{i} \frac{\delta}{\delta A^{\text{ex}}_{\mu_1}(x_1)} \frac{1}{i} \frac{\delta}{\delta A^{\text{ex}}_{\mu_2}(x_2)} \cdots \ln Z(A^{\text{ex}}).$$
$$(2.2)$$

Functional bosonization makes use of the gauge invariance of the action and of the functional integral under a local U(1) gauge transformation: for an a_μ that is pure gauge, the partition function is invariant under the shift $A^{\text{ex}} \to A^{\text{ex}} + a$,

$$Z(A^{\text{ex}} + a) = Z(A^{\text{ex}}). \quad (2.3)$$

Thus one can represent $Z(A^{\text{ex}})$ as

$$Z(A^{\text{ex}}) = \int \mathcal{D}[a]_{\text{pure}} Z(A^{\text{ex}} + a), \quad (2.4)$$

up to some normalization, where $\int \mathcal{D}[a]_{\text{pure}}$ is the path integral over the gauge field a_μ with the condition that it is a pure gauge. Thus, in terms of the field strength $f_{\mu\nu}(a) = \partial_\mu a_\nu - \partial_\nu a_\mu$, the allowed gauge field configurations are required to obey $f_{\mu\nu}(a) = 0$ for all possible pairs of indices μ, ν ($\mu < \nu$).

In the case of a system of fermions on an open manifold with fixed boundary conditions, such as a disk in two dimensions or a ball in three dimensions, this procedure does not change the partition function. This is true independently of the dynamics of the fermions (provided it is gauge-invariant). This is also true for a closed topologically trivial manifold (such as a sphere). On the other hand, in the case of fermions on a closed topologically nontrivial manifold, such as a torus in any dimension, this procedure is equivalent to averaging the partition function over the large gauge transformations on the torus. Thus this is equivalent to averaging the partition function over "twisted" boundary conditions.

The pure gauge condition can be imposed by inserting the δ functional

$$\prod_x \prod_{\mu,\nu,\lambda,\ldots}^{\mu<\nu<\lambda\cdots} \epsilon^{\mu\nu\lambda\cdots\alpha\beta} \delta\{f_{\alpha\beta}(a(x))\} \quad (2.5)$$

in the path integral, where $\epsilon^{\mu\nu\cdots\alpha\beta}$ is the Levi-Civita symbol in D space-time dimensions, and $\prod_{\mu,\nu,\lambda,\ldots}^{\mu<\nu<\lambda\cdots}$ runs over $D(D-1)/2$ independent directions. For example, when $D = 3$ and $D = 4$, the δ functional is given by $\prod_x \prod_{\mu=1}^3 \epsilon^{\mu\nu\lambda} \delta(f_{\nu\lambda}(a))$, and $\prod_x \prod_{\mu,\nu}^{\mu<\nu} \epsilon^{\mu\nu\lambda\sigma} \delta(f_{\lambda\sigma}(a))$, respectively. The δ functional can be exponentiated by introducing an auxiliary rank $(D-2)$ tensor field $b_{\mu\nu\cdots}$ as

$$Z(A^{\text{ex}}) = \int \mathcal{D}[a,b] Z(A^{\text{ex}} + a)$$
$$\times \exp\left[-\frac{i}{2} \int d^D x\, b_{\mu\nu\cdots} \epsilon^{\mu\nu\cdots\alpha\beta} f_{\alpha\beta}(a) \right]. \quad (2.6)$$

Using the invariance of the integration measure under a shift of the gauge fields, $a \to a - A^{\text{ex}}$, we can write the partition function $Z(A^{\text{ex}})$ as

$$Z(A^{\text{ex}}) = \int \mathcal{D}[a,b] Z(a) \exp\left\{ -\frac{i}{2} \int d^D x\, b_{\mu\nu\cdots} \epsilon^{\mu\nu\cdots\alpha\beta} \right.$$
$$\left. \times [f_{\alpha\beta}(a) - f_{\alpha\beta}(A^{\text{ex}})] \right\}. \quad (2.7)$$

From the partition function (2.7), the current correlation functions can be computed as the correlation functions of the tensor field $b_{\mu\nu\cdots}$,

$$\langle j^{\mu_1}(x_1) j^{\mu_2}(x_2) \cdots \rangle$$
$$= \langle \epsilon^{\mu_1\nu_1\lambda_1\cdots} \partial_{\nu_1} b_{\lambda_1\cdots}(x_1) \epsilon^{\mu_2\nu_2\lambda_2\cdots} \partial_{\nu_2} b_{\lambda_2\cdots}(x_2) \cdots \rangle, \quad (2.8)$$

suggesting the correspondence (bosonization rule)

$$j^\mu(x) \Leftrightarrow \epsilon^{\mu\nu\lambda\rho\cdots} \partial_\nu b_{\lambda\rho\cdots}(x), \quad (2.9)$$

where the $(D-2)$-form tensor field $b_{\mu\nu\cdots}$ was introduced as a Lagrange multiplier field. This system has, in fact, a (local) gauge symmetry as its partition function is invariant under

$$b_{\mu\nu\cdots} \to b_{\mu\nu\cdots} + \partial_{\{\mu} \xi_{\nu\lambda\cdots\}} \quad (2.10)$$

where the symbol $\{\cdots\}$ fully antisymmetrizes the indices. Observe also that in Eq. (2.7) the field strength for $b_{\mu\nu\cdots}$, which is the $(D-1)$-form field h defined by $h := db$ (in the differential form notation), does not appear.

A consequence of the bosonization rule (2.9) and the form of the partition function (2.7) is that local magnetic fluxes (or, in general dimension, magnetic flux "tubes") couple to the tensor field b. In addition, in a system with an energy gap, the world lines of fermionic excitations (i.e., the quasiparticle currents) are minimally coupled to the gauge field a_μ.

To complete the bosonization mapping, we need to evaluate the fermionic path integral $Z(a)$. Our discussion so far is quite general and applicable to gapped/gapless, interacting/noninteracting systems in any dimension. However, for a theory of massless fermions (free or interacting), the fermionic path integral $Z(a)$ can only be evaluated exactly in $D = 1 + 1$ dimensions. In general space-time dimension D, the fermion path integral $Z(a)$ is a nonlocal (but gauge

invariant) functional of the gauge field. On the other hand, if the band gap (mass) is finite, an approximate form of the fermionic path integral $Z(a)$ can be obtained using the inverse mass expansion. Below, we will discuss first the case of noninteracting (topological) insulators and we will later extend these results to the interacting cases. The path integral calculation is most conveniently described by choosing the Dirac representative of topological insulators as a microscopic model. The resulting bosonized action is then a sum of local gauge-invariant operators. In particular, the parts of it which are topological do not depend on microscopic details. These topological terms are always marginal and dominate the low-energy physics for $d < 3$, and they also only depend on universal properties of the system. In particular, they contain the information on the topological invariants of the microscopic model.

In the next section, we discuss some technical and subtle points about functional bosonization that might be skipped during a first reading. After this discussion, we move on to discuss what this approach predicts in various dimensions D using several examples.

B. Generalities

The functional bosonization technique is rooted in the well-known Fermi-Bose equivalence in $D = 1 + 1$ dimensions.[35–42] (For reviews, on bosonization see Refs. 32, 43, and 44). It is known that the density-density commutator of a $D = 1 + 1$-dimensional chiral fermion is anomalous in the sense that it develops the so-called Schwinger term, while one would naively expect (for a system with a relativistic spectrum) that the charge and current density operators at different positions (momenta) commute with each other at equal times. This is due to the underlying chiral anomaly. This anomalous commutator allows one to represent the fermion density operator in terms of a boson field. Indeed, if we represent the (normal-ordered) operators for the fermion charge density j_0 and current density j_1 by the two-vector $j_\mu = (j_0, j_1)$, then the anomalous commutator $[j_0(x), j_1(y)] = (i/\pi)\partial_x \delta(x - y)$ implies that the current can be represented in terms of a scalar (Bose) field ϕ, i.e., $j_\mu = (1/\sqrt{\pi})\epsilon_{\mu\nu}\partial^\nu\phi$, which is consistent with the requirement of local conservation of the charge current, $\partial_\mu j^\mu = 0$. In the Abelian case, the resulting effective field theory for the Bose field ϕ turns out to be free (up to a finite renormalization of the Luttinger parameter and the speed of excitations due to interaction effects). In other words, the dynamics of the fermionic system (interacting or not) is fully represented in terms of the dynamics of its conserved currents. In this sense, in one space dimension, bosonization yields an exact hydrodynamic representation of the system. In fact, the full excitation spectra of both systems are identical and so are their partition functions. This exact Fermi-Bose equivalence is a consequence of the kinematic restrictions of one-dimensional motion for systems with a relativistic (linear) dispersion. Using these identities, one can establish a one-to-one mapping between seemingly different systems such as the $D = 2$-dimensional massive Thirring model and the $D = 2$-dimensional sine-Gordon model.[38,39] Such a correspondence also extends to $D = 2$-dimensional theories with non-Abelian symmetries.[45]

In $1 + 1$ space-time dimensions, these bosonization operator identities can also be formulated in terms of the functional integral language. In this (functional) bosonization approach, the fermion determinant of a Dirac operator coupled to gauge fields is computed by means of a local chiral transformation. At the classical level (i.e., in the Lagrangian) in $1 + 1$ dimensions it is possible to (formally) cancel the coupling of the Fermi field to a gauge field $A_\mu(x)$ by means of a local chiral transformation, which acts on the Dirac fermion as $\psi(x) \to \exp[i\phi(x)\gamma_5]\psi(x)$, where $\phi(x)$ is an arbitrary smooth function of space-time coordinates x and γ_5 is a suitable Pauli matrix. In a gauge-invariant theory, the chiral anomaly implies that the integration measure of the fermion path integral is not invariant under the chiral transformation. As a result there is a nontrivial Jacobian associated with the chiral transformation.[46] This Jacobian is computable in terms of the fermion determinant that leads to the bosonized form of the effective action theory, which is a local functional of the Bose field $\phi(x)$. This approach has been applied with great success to several $1 + 1$-dimensional theories of massless fermions.[41] In massive theories, the Fermi-Bose mapping relates the two descriptions as identities valid order by order in a perturbation theory in powers of the fermion mass term.[38,39,47] It also yields an alternative derivation of non-Abelian bosonization yielding the Wess-Zumino-Witten model[45] by means of an exact computation of the fermion determinant.[42]

For $d > 1$, it is not possible, in general, to find an exact mapping in the form of operator identities, between a *local* theory of fermions and a *local* theory of bosons. In practice, for $d > 1$, bosonization reduces to finding an effective field theory in terms of Bose fields. Physically, this is largely due to the fact that the kinematic restrictions of one dimension do not exist for $d > 1$ and, as a result, the spectrum of a local fermionic theory, interacting or not, is not equivalent to the spectrum of a local bosonic theory.[48] The resulting effective field theory is local only in the case of theories with a finite gap in the fermionic spectrum and for energies much smaller than the gap. The prime example of such a theory is the hydrodynamic field theory of the FQHE.

For the case of functional bosonization, the problem is that the resulting fermion determinant for massless fermions coupled to gauge fields is a gauge-invariant but nonlocal functional of the gauge field and, contrary to the case of $D = 2$, cannot be computed in closed form. However, in the case of massive fermions, the contribution of the fermion determinant to the effective action of the gauge fields can be expressed in terms of local gauge-invariant terms (for manifolds without boundary) with coefficients that involve powers of the inverse of the mass, i.e., a gradient expansion. This approach naturally is only meaningful for massive theories. In the case of massive Dirac fermions in $D = 2 + 1$ dimensions, it has been shown[51–56] that the correlation functions of the conserved fermionic currents are the same as the correlation functions of a *dual* topological Chern-Simons gauge theory in the low-energy regime (only!). A mapping of the correlators of other fermion bilinears can also be determined but, again, only in the low-energy and long-wavelength regime.

Below, we will show that functional bosonization can be used to derive hydrodynamic theories of topological insulators in $d = 1, \ldots, 4$. Specifically, we give the functional

bosonization results for noninteracting topological insulators in $D = d + 1$ with $d = 1, \ldots, 4$, which crucially conserve electromagnetic U(1) charge. These topological phases belong to "the complex symmetry classes" and to "the primary series" of the eight real symmetry classes in the periodic table; by "complex symmetry classes" we refer to symmetry classes A and AIII, while the "primary series" of topological insulators (superconductors) is located on the diagonal of the periodic table, and its members are characterized by an integer $\mathbb{Z}$ topological invariant.

After this, we will discuss field theory descriptions of noninteracting topological phases characterized by $\mathbb{Z}_2$ invariants. These theories can be obtained from "the primary series" by dimensional reduction and can be divided into two different classes: first and second descendants. These phases include the time-reversal symmetric topological insulators in $D = 3 + 1$ (a first descendant) and $D = 2 + 1$ (the quantum spin Hall effect, a second descendant) dimensions. It is also possible to extend our discussion to topological phases (topological superconductors) that conserve non-Abelian currents such as a spin SU(2) current.[57] Such non-Abelian functional bosonization is discussed in Appendix A.

C. Functional Bosonization Examples

1. $D = 1 + 1$ (AIII or BDI)

There are no nontrivial topological insulator states in $D = 1 + 1$ dimensions if we do not require a protected symmetry. That is, topological insulators (in fact, topological phases in general) in $D = 1 + 1$ must be symmetry-protected ones. In the following, we consider band insulators with sublattice (chiral) symmetry. In the Altland-Zirnbauer classification,[58] they belong to symmetry class AIII. In $D = 1 + 1$, topological insulators in symmetry class AIII are characterized by an integer-valued topological invariant, the winding number ν. A canonical example for such systems is polyacetylene.[59]

In addition to the sublattice symmetry, we can further impose time-reversal symmetry, which squares to be either $+1$ or -1 (corresponding to symmetry class BDI or CII). In either case, band insulators in $D = 1 + 1$ dimensions in these symmetry classes are characterized by an integral topological invariant ν, in much the same way as symmetry class AIII in $D = 1 + 1$. Below, we will consider symmetry class BDI in $D = 1 + 1$, which is in the primary series in the periodic table.

The symmetry class BDI can be realized either in terms of complex (Dirac) or real (Majorana) fermions.[17,60,61] In the latter case, instead of imposing a sublattice symmetry, we would impose a reality condition (originating from the fact that we are dealing with Majorana fermions) combined with time-reversal symmetry. The reality condition is equivalent to a charge-conjugation symmetry and when combined with time-reversal plays a role similar to sublattice symmetry. Our discussion below, however, will focus on the case of complex fermions, as functional bosonization takes advantage of the presence of a U(1) global symmetry.

To summarize, we consider (topological) band insulators with U(1) global symmetry (particle number conservation) in symmetry class AIII or BDI in $D = 1 + 1$ dimensions. For

these cases, $Z(a)$ in Eq. (2.7) can be evaluated as

$$\ln Z(a) = \frac{i\theta}{2\pi} \int d^2x \, \epsilon^{\mu\nu} \partial_\mu a_\nu + \cdots . \qquad (2.11)$$

Here, the angle θ is given in terms of the bulk topological invariant, the winding number ν, as $\theta = \nu\pi \mod 2\pi$. In Eq. (2.11), terms with more fields and derivatives such as Maxwell terms are suppressed as they are more irrelevant [indicated by $\cdots$ in Eq. (2.11)].

Thus the bosonized partition function is given by

$$Z(A^{\text{ex}}) = \int \mathcal{D}[a,b] \exp i \int d^D x \mathcal{L} \qquad (2.12)$$

with the effective low-energy Lagrangian density

$$\mathcal{L} = -b\epsilon^{\mu\nu} \partial_\mu (a_\nu - A_\nu^{\text{ex}}) + \frac{\theta}{2\pi} \epsilon^{\mu\nu} \partial_\mu a_\nu + \cdots . \qquad (2.13)$$

In summary, the effective low-energy theory of massive fermions in $1+1$ dimensions is a BF theory (the first term) and a topological term whose coupling constant is the topological invariant $\theta = \pi\nu$, where ν is the winding number. Notice that in $D = 1+1$ dimensions the BF theory involves a vector (gauge) field a_μ and a scalar field b.

2. $D = 2 + 1$ *(A or D)*

In $D = 2+1$, we discuss the bosonization of topological insulators in symmetry classes A and D. They are topological insulators belonging to the primary series in the terminology introduced above. These topological insulators are characterized by an integer topological invariant, Ch, the first Chern number, which is nothing but the Hall conductivity σ_{xy} (in units of e^2/h). The symmetry class A is defined as a set of fermionic quadratic Hamiltonians that possess no discrete symmetry, and the canonical example of the topological insulator in this class is the integer QHE. The symmetry class D is obtained, from symmetry class A, by imposing additional particle-hole symmetry while keeping the electromagnetic U(1) symmetry.[11] This should be distinguished from the superconducting realization of symmetry class D, where there is no conserved U(1) charge whereas the particle-hole "symmetry" simply reflects the fact that fermionic Bogoliubov quasiparticles must satisfy a reality constraint.

For these cases, $Z(a)$ can be computed in the low-energy limit as

$$\ln Z(a) = \frac{i\text{Ch}}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \cdots , \qquad (2.14)$$

where Ch is the integer-valued bulk topological invariant, the first Chern number, of the topological insulator. As in the case of $D = 2$, terms that are less relevant compared the Chern-Simons term, including the Maxwell term, are suppressed by powers of the inverse of the mass (the energy gap). Thus

$$Z(A^{\text{ex}}) = \int \mathcal{D}[a,b] \exp i \int d^D x \mathcal{L} \qquad (2.15)$$

with the effective low-energy Lagrangian density

$$\mathcal{L} = -b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A_\lambda^{\text{ex}}) + \frac{\text{Ch}}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda . \qquad (2.16)$$

Thus, here too, we obtain a BF theory (this time in $2+1$ dimensions) and a topological invariant term, the Chern-Simons action, whose coupling constant (or "level") is the (topological invariant) Chern number of the microscopic band. In $D = 2+1$, the BF term relates two gauge fields, b_μ and a_μ. This effective action has a formal similarity with the hydrodynamic theory of the FQHE. However, in the hydrodynamic theory of the FQHE, the two gauge fields are the hydrodynamic field b_μ and the statistical gauge field a_μ, with the important difference that the Chern-Simons term affects the hydrodynamic field.[23,62] In the present case, the bosonized action (2.16) represents an integer QHE with a quantized Hall conductance $\sigma_{xy} = (e^2/h)\text{Ch}$. Thus this is a theory of the quantized anomalous Hall effect.

3. $D = 3 + 1$ *(AIII or DIII classes)*

In $D = 3+1$, topological insulators in symmetry class AIII are characterized by an integral topological invariant, the three-dimensional integer-valued winding number ν, and are protected by a chiral symmetry. An example of topological insulators in this class can be found in Ref. 63, which discusses a lattice tight-binding model description. This topological insulator is somewhat analogous to the time-reversal symmetric topological insulator in symmetry class AII in that it supports a Dirac fermion surface state, and has a nontrivial axion-electrodynamics response to the external electromagnetic field. The difference is, however, that the latter is characterized by a $\mathbb{Z}_2$ topological invariant, rather than an integer topological invariant. We will discuss the bosonization of the time-reversal symmetric topological insulator in symmetry class AII in a later section.

The topological insulator in symmetry class DIII is similar to the topological insulator in symmetry class AIII above but with the requirement of an additional particle-hole or time-reversal symmetry (which leads to equivalent results since these symmetries, when combined with the sublattice symmetry, give the third symmetry). As in the cases of $D = 2$ and 3 above, this realization of symmetry class DIII in terms of complex fermions [i.e., with conserved U(1) charge] should be distinguished from the superconducting realization of symmetry class DIII.

For these cases, $Z(a)$ can be computed, again, in the low-energy limit, as (see, for example, Ref. 63 for calculations in terms of the Dirac representative)

$$\ln Z(a) = i \int d^4x \left[\frac{\theta}{8\pi^2} \epsilon^{\mu\nu\lambda\rho} \partial_\mu a_\nu \partial_\lambda a_\rho \right.$$
$$\left. - \frac{1}{4\pi g^2} f_{\mu\nu} f^{\mu\nu} + \cdots \right]. \qquad (2.17)$$

Here, the angle θ is related to the winding number as $\theta = \nu\pi \mod 2\pi$.[14] Thus

$$Z(A^{\text{ex}}) = \int \mathcal{D}[a,b] \exp i \int d^D x \mathcal{L} \qquad (2.18)$$

with the effective Lagrangian

$$\mathcal{L} = -b_{\mu\nu} \epsilon^{\mu\nu\lambda\rho} \partial_\lambda (a_\rho - A_\rho^{\text{ex}}) + \frac{\theta}{8\pi^2} \epsilon^{\mu\nu\lambda\rho} \partial_\mu a_\nu \partial_\lambda a_\rho$$
$$- \frac{1}{4\pi g^2} f_{\mu\nu} f^{\mu\nu} + \cdots . \qquad (2.19)$$

In the last line, we have written down the Maxwell term explicitly since it is also marginal in $D = 4$.

4. $D = 4 + 1$ *(A or AII classes)*

Finally, we discuss the bosonization of the topological insulators in $D = 4 + 1$. In fact, the bosonization rule applies to any dimension. One of our motivations to study the case of $D = 4 + 1$ is that the lower-dimensional topological insulators, in particular, the time-reversal symmetric topological insulators in $D = 3 + 1$ and $D = 2 + 1$, are the first and second descendants of the $D = 4 + 1$ topological insulator in class AII.[11,14] In $D = 4 + 1$, topological insulators in symmetry class A and AII are characterized by an integer-valued topological invariant, the second Chern number Ch_2.

For these cases, $Z(a)$ can once again be computed as

$$\ln Z(a) = i \int d^5 x \left[\frac{\mathsf{Ch}_2}{24\pi^2} \epsilon^{\mu\nu\lambda\rho\sigma} a_\mu \partial_\nu a_\lambda \partial_\rho a_\sigma \right.$$
$$\left. - \frac{1}{4\pi g^2} f_{\mu\nu} f^{\mu\nu} + \cdots \right], \qquad (2.20)$$

where the integer Ch_2 is the second Chern number of the topological insulator. Thus

$$Z(A^{\text{ex}}) = \int \mathcal{D}[a,b] \exp i \int d^D x \mathcal{L}, \qquad (2.21)$$

where the effective Lagrangian is now given by

$$\mathcal{L} = -b_{\mu\nu\lambda} \epsilon^{\mu\nu\lambda\rho\sigma} \partial_\rho \left(a_\sigma - A^{\text{ex}}_\sigma \right)$$
$$+ \frac{\mathsf{Ch}_2}{24\pi^2} \epsilon^{\mu\nu\lambda\rho\sigma} a_\mu \partial_\nu a_\lambda \partial_\rho a_\sigma + \cdots , \qquad (2.22)$$

where we have dropped the Maxwell term in the last equation.

5. *Interactions*

We have not included interactions so far, but perturbative effects of weak interactions can easily be taken into account. As an example, consider the local current-current interaction (which is the analog of the Luttinger-Thirring interaction in $D = 1 + 1$ dimensions)

$$-u \int d^D x \, j^\mu j_\mu, \qquad (2.23)$$

where u is the coupling constant. Because of the bosonization rule, this leads to a term of the form

$$-u \int d^D x \, (\epsilon^{\mu\nu\lambda\rho\cdots} \partial_\nu b_{\lambda\rho\cdots})(\epsilon_{\mu\alpha\beta\gamma\cdots} \partial^\alpha b^{\beta\gamma\cdots}). \qquad (2.24)$$

III. PHYSICAL PICTURES FROM FUNCTIONAL BOSONIZATION

In this section, we discuss the physical picture that functional bosonization provides in three different situations with increasing complexity: an example of a topologically trivial insulator, a time-reversal breaking topological insulator (the QHE or quantum anomalous Hall effect) in $D = 2 + 1$ dimensions, and a time-reversal invariant topological insulator in $D = 3 + 1$ dimensions.

A. Topologically trivial insulators

We start our discussion from topologically trivial insulators and we will make a few comments. We will discuss (i) the normalization of the gauge fields when the BF theory description is considered on a compact spatial manifold (ii) functional bosonization and the dual picture of insulators, and (iii) the presence/absence of fermionic excitations in the low-energy spectrum. In fact, these comments are not limited to topologically trivial insulators, and we will continue our discussion of them for nontrivial topological insulators in the later sections.

(i) Functional bosonization gives us a bosonized description of nontopological band insulators in terms of the BF theory *without* a CS or θ term, i.e., the pure BF theory. In $D = 2 + 1$ dimensions, the bosonized Lagrangian of a nontopological band insulator is given by Eq. (2.16) with vanishing Chern number, $\mathsf{Ch} = 0$. On a compact spatial manifold, such as a torus, we should be careful of the normalization of the gauge fields, which must be done relative to their compactification condition. On the torus, where the spatial coordinates (x_1, x_2) are identified periodically as $x_i \equiv x_i + L_i$, the gauge fields α^i_j ($\alpha^1_\mu \equiv b_\mu$ and $\alpha^2_\mu \equiv a_\mu$), in the $\alpha^i_0 = 0$ gauge, are normalized such that they are global gauge-transform equivalent to $\alpha^i_j + 2\pi n_j / L_j$, where $n_{j=1,2}$ are integers. When properly normalized, the BF Lagrangian is given by

$$\mathcal{L} = -\frac{2\mathsf{k}}{4\pi} b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu \left(a_\lambda - A^{\text{ex}}_\lambda \right) \qquad (3.1)$$

with $\mathsf{k} = 1$. With this normalization, the ground-state degeneracy is one (i.e., no ground-state degeneracy).

(ii) The BF theory description of (nontopological) band insulators resembles the dual description of BCS superconductors.[64–66] The bosonized theory of the trivial-band insulator from functional bosonization is naturally related to the dual theory of (band/Mott) insulators.[67,68] Below, let us take a closer look at this by focusing on $D = 2 + 1$, although a similar discussion applies for $D \neq 2 + 1$.

In (2+1)-D superconductors, a defect (vortex) of the condensate $\Delta(\vec{r})$ traps a magnetic field $\nabla \times A_{\text{EM}}(\vec{r})$; the phase $\phi(\vec{r})$ of the superconducting order parameter $\Delta(\vec{r}) = |\Delta(\vec{r})| \exp i\phi(\vec{r})$ is related to the magnetic field as $(2\pi)^{-1} \Phi_0 \oint_{\partial S} \nabla \phi \cdot d\boldsymbol{l} = \int_S (\nabla \times A_{\text{EM}}) \cdot d\boldsymbol{S}$, where Φ_0 is the flux quantum, and if we reinstate h, c, and e, $\Phi_0 = hc/(2e)$; the magnetic flux is localized in the region S.

In the dual description of an insulator, we view the charge density modulation $\delta\rho(\vec{r})$ caused, perhaps, by external doping, as a pointlike "defect." As in the (2+1)d superconductor, we postulate that such a defect traps a fictitious magnetic field $\nabla \times \boldsymbol{a}$:

$$\int_S \delta\rho \, dS = e \int_S (\nabla \times \boldsymbol{a}) \cdot d\boldsymbol{S}, \qquad (3.2)$$

where a_μ is a fictitious gauge field. As will become clear, this gauge field is the same gauge field appearing in Eq. (3.1). As the local charge density modulation is viewed as a vortex, we can postulate the existence of an order parameter χ. The phase of χ, $\chi = |\chi| \exp i\Theta$, is related to the charge density as

$$e \oint_{\partial S} \nabla \Theta \cdot d\boldsymbol{l} = \int_S \delta\rho \, dS. \qquad (3.3)$$

As in the dual theory of superconductors, the above description in terms a_μ and χ can be dualized (in the condensed phase) and can instead be rewritten in terms of two gauge fields a_μ and b_μ, which are the gauge fields appearing in Eq. (3.1). Upon dualization, b_μ couples minimally to the vortex current of χ, $j_V^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu\partial_\lambda\Theta$. This is, in fact, consistent with the bosonization rule $j^\mu \propto \epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda$.

(iii) The bosonic descriptions (in terms of a_μ and χ, or in terms of a_μ and b_μ) do not contain any fermionic excitations even though we started from a gapped fermionic theory (a band insulator). Drawing again an analogy to BCS superconductors, this resembles to the fact that, under duality, a superconductor in a uniform field is equivalent to the insulating phase of the same bosonic theory at uniform charge density.[69–71] In the latter theory, the charges are gapped Cooper pairs, not fermions. For this reason, the bosonic theory only describes arrays of Josephson junctions in which the fermionic excitations do no exist (they are bound on each superconducting grain). The connection between the gapped fermionic theory and the bosonized descriptions is that they share the same topological limit in the form of a BF theory. As we will see in the next section, the bosonized description of a $D = 2 + 1$ dimensional topological insulator is different and supports fermionic excitations (electrons) due to the presence of the Chern-Simons term.

B. Topological insulator in $D = 2 + 1$

Let us now move on to a more nontrivial discussion of a topological insulator. The bosonized Lagrangian of the $D = 2 + 1$ dimensional topological insulator (in symmetry class A or D) is given by Eq. (2.16):

$$
\begin{aligned}
\mathcal{L} &= -\frac{2\mathsf{k}}{4\pi}b_\mu\epsilon^{\mu\nu\lambda}\partial_\nu(a_\lambda - A_\lambda^{\text{ex}}) + \frac{\mathsf{Ch}}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \\
&= \frac{K_{ij}}{4\pi}\alpha_\mu^i\epsilon^{\mu\nu\lambda}\partial_\nu\alpha_\lambda^j + \frac{\mathsf{k}}{2\pi}b_\mu\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda^{\text{ex}},
\end{aligned} \tag{3.4}
$$

where we integrated by parts once, and introduced the following notation: $(\alpha_\mu^1, \alpha_\mu^2) = (b_\mu, a_\mu)$ and

$$
K = \begin{pmatrix} 0 & -\mathsf{k} \\ -\mathsf{k} & \mathsf{Ch} \end{pmatrix}. \tag{3.5}
$$

We have normalized b_μ as before (assuming we are on a compact manifold such as a torus) and $\mathsf{k} = 1$.

This effective theory also makes sense for $\mathsf{k} \neq 1$. However, in this case, this effective hydrodynamic theory cannot be derived from a free-fermion system, even if it is a Chern insulator. In fact, this effective field theory describes a topological fluid. This can be checked by noting that it describes a system with a nontrivial ground-state degeneracy k^g, where g is the number of handles of the surface. In other terms, a theory with $\mathsf{k} \neq 1$ describes a fractionalized topological insulator, e.g., a fractional quantum Hall fluid, as we will see in Sec. V.

The Lagrangian (3.4) is the hydrodynamic effective field theory for $D = 2 + 1$ dimensional topological insulators. Some examples include the QHE realized in a two-dimensional electron gas with Landau levels produced by a uniform magnetic field as well as the Chern-insulator i.e., lattice fermion systems with nonzero Chern number but without

uniform magnetic field, such as the Haldane model.[72] Note that we have arrived at Eq. (2.16) without using composite particle theories (see below for more comments) and that for the Chern-insulator, the flux attachment transformation is highly nontrivial but not impossible.[73–75]

It is readily seen that the bosonized theory reproduces the QHE. From the equations of motion, $\delta S/\delta a_\mu = \delta S/\delta b_\mu = 0$,

$$
\begin{aligned}
-\frac{2\mathsf{k}}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda + \frac{\mathsf{Ch}}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda &= 0, \\
-\epsilon^{\mu\nu\lambda}\partial_\nu(a_\lambda - A_\lambda^{\text{ex}}) &= 0.
\end{aligned} \tag{3.6}
$$

From the definition of the electrical current,

$$
j^\mu := \frac{\delta S}{\delta A_\mu^{\text{ex}}} = \frac{2\mathsf{k}}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda, \tag{3.7}
$$

we then conclude the QHE as

$$
j^\mu = \frac{\mathsf{Ch}}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda^{\text{ex}}. \tag{3.8}
$$

While functional bosonization delivers the two-component CS theory (the BF-CS theory), it is possible to derive the familiar single-component CS theory by integrating over a_μ. If we eliminate a_μ by using the equation of motion, for the simplest case of $\mathsf{k} = \mathsf{Ch} = 1$, we obtain the effective action

$$
\mathcal{L} = -\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu A_\lambda^{\text{ex}}. \tag{3.9}
$$

This is the familiar single-component CS theory (here for the case of the integer QHE). Observe that, as it should, the CS coefficient in front of $\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda$ has the opposite sign as compared to the CS theory of the response, $(\mathsf{Ch}/4\pi)\epsilon^{\mu\nu\lambda}A_\mu^{\text{ex}}\partial_\nu A_\lambda^{\text{ex}}$.

The transition from the BF-Chern Simons-theory (3.4) to the single-component CS theory (3.9) is quite analogous to the derivation of the CS description for the (integer) QHE in terms of the composite boson theory. In the composite boson theory, one introduces a CS theory that attaches a flux (fluxes) to electrons, and the bosonic composite particle (electron + flux), in terms of a dualized language, is expressed by a U(1) gauge field; these two U(1) gauge fields correspond to a_μ and b_μ in functional bosonization. In fact, if we "dualize back" the BF-Chern Simons theory (3.4), we obtain the Chern-Simons-Landau-Ginzburg action, in terms of the CS gauge field a_μ and some bosonic field Φ. Comparing with the composite boson theory, the bosonic Φ could be interpreted as a composite boson field. Thus, in functional bosonization, while we have not used the flux attachment, the resulting effective field theory is quite similar to the composite boson theory of the IQHE.

As is clear from the comparison to the composite boson theory, the hydrodynamic theory (3.9) [and hence Eq. (3.4)] includes a fermionic excitation (electron) due to the presence of the Chern-Simons term. The world lines of point defects (point sources for the vortex current that couples to b_μ), when linked, pick up a π phase (i.e., a fermionic sign) due to the Chern-Simons term. This should be contrasted with the hydrodynamic theory for topologically trivial insulators (3.1), which does not have the Chern-Simons term, and thus no fermionic statistics of its excitations.

C. Topological insulator in $D = 3 + 1$

The bosonized Lagrangian (2.19) describes the $D = 3 + 1$-dimensional (topological) insulator in the symmetry class AIII (DIII). As a possible microscopic realization that conserves a U(1) charge, let us consider the following fermionic action:

$$
\begin{aligned}
&K_F[\psi^\dagger, \psi, A^{\mathrm{ex}}] \\
&= \int dt \sum_{r,r'} \psi^\dagger(r) \big[(i\partial_t + A_0^{\mathrm{ex}})\delta_{r,r'} - \mathcal{H}(r,r',A^{\mathrm{ex}}) \big] \psi(r'),
\end{aligned}
\tag{3.10}
$$

where $\psi(r)/\psi^\dagger(r)$ is the fermion annihilation/creation field operators at site r and $\mathcal{H}(r,r',A^{\mathrm{ex}})$ represents a (tight-binding) Hamiltonian of a chiral topological insulator, minimally coupled to the external electromagnetic U(1) gauge field A_μ^{ex}. [Here, $\psi(r)$ can possibly be multicomponent, but internal indices will be suppressed below.] A single-particle Hamiltonian $\mathcal{H}$ in symmetry class AIII is sublattice symmetric in that it anticommutes with a unitary matrix Γ, $\{\mathcal{H}, \Gamma\} = 0$. Γ is diagonal in sublattice indices and takes values ± 1 for sublattice A and B, respectively. This means, in turn, that under the unitary transformation (particle-hole transformation)

$$
\begin{aligned}
\mathcal{C}\psi(t,r)\mathcal{C}^{-1} &= (-1)^r \psi^\dagger(t,r), \\
(-1)^r &= \begin{cases} +1, & r \in \text{A sublattice} \\ -1, & r \in \text{B sublattice} \end{cases}
\end{aligned}
\tag{3.11}
$$

followed by time reversal

$$
\mathcal{T}\psi(t,r)\mathcal{T}^{-1} = \psi(-t,r), \quad \mathcal{T}i\mathcal{T}^{-1} = -i, \tag{3.12}
$$

the fermion bilinear $\int dt \sum_{r,r'} \psi^\dagger(r)\mathcal{H}(r,r',A^{\mathrm{ex}})\psi(r')$ is left unchanged while the sign of A_0^{ex} is flipped, i.e., $\mathcal{TC}$ sends ($i = 1,2,3$)

$$
\begin{aligned}
\mathcal{TC}: \quad A_0^{\mathrm{ex}}(t,r) &\to -A_0^{\mathrm{ex}}(-t,r), \\
A_i^{\mathrm{ex}}(t,r) &\to +A_i^{\mathrm{ex}}(-t,r),
\end{aligned}
\tag{3.13}
$$

and $\vec{E}^{\mathrm{ex}}(t,r) \to -\vec{E}^{\mathrm{ex}}(-t,r)$, $\vec{B}^{\mathrm{ex}}(t,r) \to +\vec{B}^{\mathrm{ex}}(-t,r)$.

The functional bosonization recipe in Sec. II, when applied to the fermionic action, delivers the hydrodynamic field theory (2.19):

$$
\mathcal{L} = -b_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda(a_\rho - A_\rho^{\mathrm{ex}}) + \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu \partial_\lambda a_\rho, \tag{3.14}
$$

where we have dropped the Maxwell term for simplicity. The θ angle is $\theta = \nu\pi$, where $\nu \in \mathbb{Z}$ is the topological invariant (winding number). This is consistent with chiral symmetry; because of the transformation law of A^{ex} and, similarly, a_μ under $\mathcal{TC}$, θ is mapped to $-\theta$ by $\mathcal{TC}$. With the invariance of the θ term under a shift $\theta \to \theta + 2\pi$, the value of the θ angle allowed by chiral symmetry is an integral multiple of π.

From the BF coupling $b_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda A_\rho^{\mathrm{ex}}$ or from the bosonization rule $j^\mu \propto \epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}$, one also reads off the transformation law

$$
\begin{aligned}
\mathcal{TC}: \quad b_{0i}(t,r) &\to +b_{0i}(-t,r), \quad b_{i0}(t,r) \to +b_{i0}(-t,r), \\
b_{ij}(t,r) &\to -b_{ij}(-t,r),
\end{aligned}
\tag{3.15}
$$

where $i,j = 1,2,3$.

The effective Lagrangian (3.14) obtained from functional bosonization was discussed previously in Refs. 27, 76, 77, and Eq. (3.14) should be compared with the one proposed in Ref. 24:

$$
\begin{aligned}
\mathcal{L} = {}&\frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}a_\mu\partial_\nu b_{\lambda\rho} + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}A_\mu^{\mathrm{ex}}\partial_\nu b_{\lambda\rho} \\
&+ C\epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu \partial_\lambda A_\rho^{\mathrm{ex}},
\end{aligned}
\tag{3.16}
$$

where $C = \pm 1/(8\pi)$. One can check, upon integration over a and b, we reproduce the axion response term

$$
S = \frac{\pm 1}{8\pi}\int dt\,d^3x\,\epsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu^{\mathrm{ex}}\partial_\lambda A_\rho^{\mathrm{ex}}. \tag{3.17}
$$

As compared to Eq. (3.14), the effective action (3.16) does not have the axion term for the a_μ field. However, assuming there is no monopole in a_μ field configurations, with the gauge transformation in Eq. (3.14),

$$
b_{\mu\nu} \to b_{\mu\nu} + \frac{\theta}{16\pi^2}(\partial_\mu a_\nu - \partial_\nu a_\mu), \tag{3.18}
$$

one can transform Eq. (3.14) into Eq. (3.16).

As in the case of $D = 2 + 1$, one can readily verify that the bosonized effective Lagrangian reproduces the physics of the topological insulator in $D = 3 + 1$ dimensions; by eliminating $b_{\mu\nu}$ and a_μ using the equations of motion, $\delta S/\delta a_\mu = \delta S/\delta b_{\mu\nu} = 0$,

$$
\begin{aligned}
\epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} - \frac{1}{4\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\nu(\theta\partial_\lambda a_\rho) &= 0, \\
\epsilon^{\mu\nu\lambda\sigma}\partial_\lambda(a_\sigma - A_\sigma^{\mathrm{ex}}) &= 0,
\end{aligned}
\tag{3.19}
$$

one obtains the axion term for the external gauge field

$$
\frac{1}{8\pi^2}\int d^4x\,\theta\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu^{\mathrm{ex}}\partial_\rho A_\sigma^{\mathrm{ex}}, \tag{3.20}
$$

as expected for the response of the topological insulator in symmetry classes AIII and DIII in $D = 4$. Also, from the bosonization rule $j^\mu \equiv \epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}$, the electrical current is given by

$$
j^\mu = \epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} = \frac{1}{4\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\nu(\theta\partial_\lambda A_\rho^{\mathrm{ex}}). \tag{3.21}
$$

Taking θ to be a step function, which jumps from $\pi \to 0$ at the system's boundary, this equation tells us that surface Hall conductance is $\sigma_{xy} = 1/4\pi = e^2/2h$ where we have reinserted units. Alternatively, when $\theta = \pi = \text{const}$, but A_ρ^{ex} has a monopole configuration, this equation tells us the monopole acquires a charge, which is a manifestation of the Witten effect.[78–80]

To discuss the effect of monopoles, it is convenient to introduce *a monopole gauge field* $f_{\mu\nu}^{\mathrm{M}}(A^{\mathrm{ex}})$ and $f_{\mu\nu}^{\mathrm{M}}(a)$ for A_μ^{ex} and a_μ, respectively.[27,81,82] In the presence of a monopole in a U(1) gauge field A_μ (it can be either A_μ^{ex} or a_μ), the monopole gauge field $f_{\mu\nu}^{\mathrm{M}}(A)$ is a field that is related to the monopole current $j^{\mathrm{M},\mu}$ as

$$
\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\partial_\nu f_{\lambda\rho}^{\mathrm{M}}(A) = j^{\mathrm{M},\mu}. \tag{3.22}
$$

More specifically, it is given by

$$
f_{\mu\nu}^{\mathrm{M}}(A)(x) = \frac{q_m}{2}\epsilon_{\mu\nu\lambda\rho}\delta^{\lambda\rho}(x;S) = q_m\tilde{\delta}_{\mu\nu}(x;S), \tag{3.23}
$$

where q_m is the strength of the monopole, and $\delta_{\lambda\rho}(x; S)$ is singular on the world surface S,

$$\delta^{\mu\nu}(x; S) := \int d\sigma d\tau \left[\frac{\partial X^\mu(\sigma,\tau)}{\partial\sigma} \frac{\partial X^\nu(\sigma,\tau)}{\partial\tau} - (\mu \leftrightarrow \nu) \right]$$
$$\times \delta^{(4)}[x - X(\sigma,\tau)]. \quad (3.24)$$

The surface S is the world sheet of the Dirac string, parameterized by its world-sheet coordinates $X^\mu(\sigma,\tau)$, and bounded by a world line L of the monopole, $\partial S = L$. One verifies

$$\tfrac{1}{2}\epsilon^{\mu\nu\lambda\rho}\partial_\nu\tilde\delta_{\lambda\rho}(x; S) = \delta^\mu(x; L), \quad (3.25)$$

where

$$\delta^\mu(x; L) = \int d\sigma \frac{\partial X^\mu(\sigma,\tau)}{\partial\sigma}\delta^{(4)}[x - X(\sigma)]. \quad (3.26)$$

One then concludes $f_{\mu\nu}^{\mathrm{M}}(A^{\mathrm{ex}})$ is related to the monopole current as Eq. (3.22). It represents the magnetic field inside the infinitely thin solenoid that emanates from the monopole.

The physically observable field strength, $f_{\mu\nu}^{\mathrm{obs}}(A)$, is the difference of the field strength for the *integrable* vector potential A_μ and the monopole gauge field, $f_{\mu\nu}^{\mathrm{obs}}(A) = f_{\mu\nu}(A) - f_{\mu\nu}^{\mathrm{M}}(A)$. The curl of the observable field strength $f_{\mu\nu}^{\mathrm{obs}}(A)$ is nonzero in the presence of a magnetic monopole, $(1/2)\epsilon^{\mu\nu\lambda\rho}\partial_\nu f_{\lambda\rho}^{\mathrm{obs}}(A) = -j^{\mathrm{M},\mu}$, as expected, where we noted the integrability of A_μ, $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)A_\lambda = 0$. While both A_μ and $f_{\mu\nu}^{\mathrm{M}}(A)$ depend on the choice of the world surface S (i.e., for a given Dirac string L, there are many surfaces satisfying $\partial S = L$), $f_{\mu\nu}^{\mathrm{obs}}(A)$ does not. In other words, $f_{\mu\nu}^{\mathrm{obs}}(A)$ is invariant under the *monopole gauge transformation*,

$$f_{\mu\nu}^{\mathrm{M}}(A) \to f_{\mu\nu}^{\mathrm{M}}(A) + \partial_\mu\eta_\nu - \partial_\nu\eta_\mu, \quad A_\mu \to A_\mu + \eta_\mu. \quad (3.27)$$

The partition function, invariant under both the ordinary U(1) gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$ and the monopole gauge transformation (3.27), can be constructed from the Lagrangian

$$\mathcal{L} = -\frac{1}{2}b_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}[f_{\lambda\rho}(a) - f_{\lambda\rho}(A^{\mathrm{ex}})]$$
$$+ \frac{\theta}{32\pi^2}\epsilon^{\mu\nu\lambda\rho}[f_{\mu\nu}(a) - f_{\mu\nu}^{\mathrm{M}}(a)][f_{\lambda\rho}(a) - f_{\lambda\rho}^{\mathrm{M}}(a)]. \quad (3.28)$$

As opposed to the second term (the axion term), the first term (the BF term) in Eq. (3.28) is not manifestly invariant under the monopole gauge transformation. However, given the bosonization rule $\epsilon^{\mu\nu\lambda\rho\sigma}\partial_\nu b_{\rho\sigma} \propto j^\mu$, Dirac's quantization condition of the electric and magnetic charges tells us that the BF term is merely shifted by an integer multiple of 2π under the monopole gauge transformation, and hence the partition function is not affected.

Let us now focus on the case where θ is constant throughout the bulk. We can use the equations of motion $\delta S/\delta a_\mu = \delta S/\delta b_{\mu\nu} = 0$ derived from the effective action (3.28) to find that the electrical current is given in terms of the monopole

current as

$$j^\mu = \epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} = -\frac{\theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\nu f_{\lambda\rho}^{\mathrm{M}}(A^{\mathrm{ex}}) = -\frac{\theta}{4\pi^2}j^{\mathrm{M},\mu}. \quad (3.29)$$

Given that in $D = 2 + 1$ the gauge field a_μ plays the role of attaching a flux (fluxes) to electrons, it is tempting, and perhaps instructive, to view the gauge field a_μ in $D = 3 + 1$ as a proper generalization of the CS "statistical gauge field." From the equation of motion $\epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} = -(\theta/4\pi^2)j^{\mathrm{M},\mu}$, the gauge field a_μ is attaching a monopole to the electron (and as in $D = 2 + 1$, the temporal component a_0 enforces a constraint). If we further choose, in the presence of a vortex line, the world sheet of the Dirac string to be identical to the vortex world sheet, the boundary of the vortex world sheet (the end of the vortex) is the world line of a monopole (i.e., a dyon since $\theta \neq 0$). In this case, Eq. (3.29) gives $b_{\mu\nu} = -(\theta/8\pi^2)f_{\mu\nu}^{\mathrm{M}}(A^{\mathrm{ex}})$ up to a choice of gauge.

IV. DIMENSIONAL REDUCTION

We have derived the effective BF topological field theories for the primary series of the topological insulators in the periodic table. In microscopic fermionic theories, $\mathbb{Z}_2$ topological insulators (such as the time-reversal symmetric $\mathbb{Z}_2$ topological insulator in $D = 3 + 1$ dimensions, and the quantum spin Hall effect in $D = 2 + 1$ dimensions) can be derived as a "descendant" of the primary series by the Kaluza-Klein dimensional reduction.[11] We now discuss the Kaluza-Klein dimensional reduction of the BF type effective action we derived. This is expected to deliver an effective bosonic field theory description for descendant $\mathbb{Z}_2$ topological insulators. Similar dimensional reduction was discussed in Ref. 11 for the topological *response* theories.

Let us start from the $D = 4 + 1$-dimensional BF-CS theory (the "parent" theory):

$$S = -\frac{1}{2\pi}\int d^5x \, b_{\mu\nu\lambda}\epsilon^{\mu\nu\lambda\rho\sigma}\partial_\rho(a_\sigma - A_\sigma^{\mathrm{ex}})$$
$$+ \frac{\mathsf{Ch}_2}{24\pi^2}\int d^5x \, \epsilon^{\mu\nu\lambda\rho\sigma}a_\mu\partial_\nu a_\lambda\partial_\rho a_\sigma + \cdots, \quad (4.1)$$

where $\mu, \nu, \lambda, \ldots = 0, 1, 2, 3, 4$. The topological insulator in $D = 4 + 1$ dimensions is time-reversal symmetric (symmetry class AII) and so is the effective hydrodynamic theory; the action (4.1) is invariant under time-reversal defined by $(i, j, k = 1, \ldots, 4)$

$$\mathcal{T}: \quad a_0(t, r) \to +a_0(-t, r), \quad a_i(t, r) \to -a_i(-t, r),$$
$$b_{0ij}(t, r) \to -b_{0ij}(-t, r), \quad b_{ijk}(t, r) \to +b_{ijk}(-t, r). \quad (4.2)$$

We now expand the fields as

$$\Phi(x_\mu) = \sum_{n_w=-\infty}^{+\infty} e^{i2\pi n_w w/L_w}\Phi(x_i, n_w), \quad (4.3)$$

where Φ represents a field, L_w is the circumference of x^4 direction, and $x_i = (x_0, x_1, x_2, x_3)$ and $x_4 = w$. By "shrinking" the x^4 direction by taking $L_w \to 0$, the "parent" $D = 4 + 1$-dimensional theory is reduced to a descendant theory in

$D = 3 + 1$ dimensions. In doing so, we view each Fourier mode $\Phi(x_i, n_w)$ as a field in $D = 3 + 1$ dimensions (Kaluza-Klein modes). The modes with $n_w \neq 0$ have a gap that grows as we take $L_w \to 0$ and hence, at low energies, only the Fourier modes with $n_w = 0$ are important. [This can be seen in the presence of proper kinetic terms (e.g., the Maxwell term) for a_μ and $b_{\mu\nu\lambda}$, which are not shown explicitly above]. We define

$$
\begin{aligned}
a_w(x_i, n_w = 0) &=: \phi(x_i)/L_w, \\
b_{wij}(x_i, n_w = 0) &=: \tfrac{1}{3} u_{ij}(x_i)/L_w, \\
A_w^{\text{ex}}(x_i, n_w = 0) &=: \theta^{\text{ex}}(x_i)/L_w,
\end{aligned}
\tag{4.4}
$$

$(i, j = 0, \dots, 3)$. The resulting $D = 3 + 1$-dimensional Lagrangian inherits the field content of the parent theory: 1-form gauge fields a_i and A_i^{ex}, a 3-form gauge field b_{ijk}, where the space-time index in $D = 4$ dimensions i, j, k runs from 0 to 3. In addition, as introduced in Eq. (4.4), it also contains two scalar fields, ϕ and θ^{ex}, and one 2-form gauge field u_{ij}. The scalar field θ^{ex} is the background axion field, which satisfies $\theta^{\text{ex}} = \pi (0)$ inside (outside) of the $D = 3 + 1$-dimensional topological insulator. Putting together, the effective topological field theory for the $D = 3 + 1$-dimensional topological insulator is given by the Lagrangian

$$
\begin{aligned}
\mathcal{L} = & -\frac{\epsilon^{\mu\nu\lambda\rho}}{2\pi}(\phi - \theta^{\text{ex}})\partial_\mu b_{\nu\lambda\rho} - \frac{\epsilon^{\mu\nu\lambda\rho}}{2\pi}\left(a_\mu - A_\mu^{\text{ex}}\right)\partial_\nu u_{\lambda\rho} \\
& + \frac{\mathsf{Ch}_2}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\phi \partial_\mu a_\nu \partial_\lambda a_\rho + \cdots,
\end{aligned}
\tag{4.5}
$$

where the space-time index μ, ν, λ now runs from 0 to 4. The last two terms resemble the effective field theory of the $D = 3 + 1$-dimensional topological insulator in the primary series, i.e., the BF theory with the axion term, Eq. (2.19), whereas the first term is absent in Eq. (2.19). The $\mathbb{Z}_2$ nature of the system lies in the restriction on $\mathsf{Ch}_2 \theta^{\text{ex}}$, which is quantized by time-reversal symmetry to be fixed at $\mathsf{Ch}_2 \theta^{\text{ex}} = 2n\pi$ or $(2n + 1)\pi$, where n is an integer. The $\mathbb{Z}_2$ nature is manifest in the fact that the bulk of the material only uniquely determines whether $\mathsf{Ch}_2 \theta^{\text{ex}}$ is an even or odd multiple of π not what the value of n is.[11] If we calculate the electrical current response, we find

$$
\frac{\delta S}{\delta A_\mu^{\text{ex}}} = j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\nu u_{\lambda\rho},
$$

$$
\frac{\delta S}{\delta a_\mu} = -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\nu u_{\lambda\rho} + \frac{2\mathsf{Ch}_2}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\nu(\phi \partial_\lambda a_\rho) = 0,
$$

$$
\frac{\delta S}{\delta b_{\nu\rho}} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\mu(\phi - \theta^{\text{ex}}) = 0,
$$

$$
\frac{\delta S}{\delta u_{\lambda\rho}} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\nu\left(a_\mu - A_\mu^{\text{ex}}\right) = 0,
\tag{4.6}
$$

which gives rise to

$$
j^\mu = \frac{\mathsf{Ch}_2}{4\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\nu\theta^{\text{ex}}\partial_\lambda A_\rho^{\text{ex}},
\tag{4.7}
$$

where we have ignored possible monopole contributions to the current. This response implies that if $\mathsf{Ch}_2 \theta^{\text{ex}} = (2n + 1)\pi$ inside the material and $2n\pi$ outside, then there is a half-integer quantum Hall effect on the surface.

We can continue the reduction down to the second descendant in $D = 2 + 1$ to obtain a hydrodynamic effective

field theory for the quantum spin Hall effect; we separate the $D = 4$-dimensional coordinates (x_0, x_1, x_2, x_3) into the $D = 3$-dimensional ones (x_0, x_1, x_2) and $x_3 \equiv z$. The z direction is compactified on a circle with radius L_z and then we take $L_z \to 0$. The fields can be Fourier decomposed in the z direction, as in Eq. (4.3), and only the Fourier modes with $n_z = 0$ (the momentum quantum number in the z direction) are kept. This second step of dimensional reduction introduces, in addition to two new scalar fields ψ, χ^{ex}, a vector field v_i, and a second 2-form gauge field g_{ij} $(i, j = 0, 1, 2, 3)$. They are defined, from the $D = 3 + 1$-dimensional fields, as

$$
\begin{aligned}
a_z(x_i, n_z) &=: \psi(x_i)/L_z, \\
b_{zij}(x_i, n_z) &=: \tfrac{1}{3} g_{ij}(x_i)/L_z, \\
u_{zi}(x_i, n_z) &=: \tfrac{1}{2} v_i(x_i)/L_z, \\
A_z^{\text{ex}}(x_i, n_z) &=: \chi^{\text{ex}}(x_i)/L_z.
\end{aligned}
\tag{4.8}
$$

The resulting $D = 2 + 1$-dimensional Lagrangian is

$$
\begin{aligned}
\mathcal{L} = & -\frac{\epsilon^{\mu\nu\lambda}}{2\pi}(\phi - \theta^{\text{ex}})\partial_\mu g_{\nu\lambda} - \frac{\epsilon^{\mu\nu\lambda}}{2\pi}(\psi - \chi^{\text{ex}})\partial_\mu u_{\nu\lambda} \\
& - \frac{\epsilon^{\mu\nu\lambda}}{2\pi}\left(a_\mu - A_\mu^{\text{ex}}\right)\partial_\nu v_\lambda + \frac{\mathsf{Ch}_2}{4\pi^2}\epsilon^{\mu\nu\lambda}\phi \partial_\mu \psi \partial_\nu a_\lambda,
\end{aligned}
\tag{4.9}
$$

where the space-time index μ, ν, λ now runs from 0 to 2. We recognize the third term $\epsilon^{\mu\nu\lambda}(a_\mu - A_\mu^{\text{ex}})\partial_\nu v_\lambda$ as the BF coupling in $D = 2 + 1$ dimensions with b_μ replaced by v_μ. The presence of such term is largely expected based upon the phenomenology of the nonchiral (helical) edge modes of the quantum spin Hall effect. However, there are some additional terms, which have not been previously discussed in dynamical gauge theories: the coupling to the real scalar fields ϕ and ψ, which could be combined into a single complex scalar field. We note that this theory, unlike other hydrodynamic theories of the quantum spin Hall effect,[6,83–87] only has a single U(1) gauge invariance [unlike the more conventional U(1) × U(1) for charge and spin]. This is a natural result since in real materials spin is not conserved generically and only the local charge U(1) invariance is preserved.

It is a simple exercise to read off, from the above effective field theories, a response of the system to the external field. For example, for $D = 2 + 1$ dimensions, the definition of the electrical current $j^\mu := \delta S/\delta A_\mu^{\text{ex}}$ together with the equation of motions derived from the effective action (4.9),

$$
\frac{\delta S}{\delta A_\mu^{\text{ex}}} = j^\mu = \frac{\epsilon^{\mu\nu\lambda}}{2\pi}\partial_\nu v_\lambda,
$$

$$
\frac{\delta S}{\delta a_\mu} = -\frac{\epsilon^{\mu\nu\lambda}}{2\pi}\partial_\nu v_\lambda + \frac{\mathsf{Ch}_2}{4\pi^2}\epsilon^{\mu\nu\lambda}\partial_\nu(\phi \partial_\lambda \psi) = 0,
\tag{4.10}
$$

gives rise to

$$
j^\mu = \frac{\mathsf{Ch}_2}{4\pi^2}\epsilon^{\mu\nu\lambda}\partial_\nu(\phi \partial_\lambda \psi).
\tag{4.11}
$$

If we assume that $\epsilon^{\mu\nu}\partial_\mu\partial_\nu\psi = 0$, then we can write this in terms of the external scalar fields as

$$
j^\mu = \frac{\mathsf{Ch}_2}{4\pi^2}\epsilon^{\mu\nu\lambda}\partial_\nu\theta^{\text{ex}}\partial_\lambda\chi^{\text{ex}}.
\tag{4.12}
$$

A similar expression for the electrical current was derived in Ref. 11 in terms of the effective topological response

theory. If we think of the field $\phi\partial_\lambda\psi$ as a velocity field V_λ, then this response implies that there is charge bound to the vorticity of the velocity field $j^0 = (\mathsf{Ch}_2/4\pi^2)\nabla \times \boldsymbol{V}$, similar to the quantum Hall effect where charge is bound to the magnetic flux. A simple example is the case when $\theta^{\mathrm{ex}}(x,y) = \pi\Theta(y)$ and $\chi^{\mathrm{ex}}(x,y) = 2\pi[\Theta(x) - 1/2]$, where θ^{ex} represents a jump in the axion angle at an edge and χ^{ex} represents a magnetic domain wall on the edge. The charge confined to the edge magnetic domain wall is $Q = (e\mathsf{Ch}_2/4\pi^2)\int dx\,dy\,2\pi^2\delta(x)\delta(y) = e\mathsf{Ch}_2/2$. The distinction between integer and half-integer charge, i.e., Ch_2 even or odd, gives an electromagnetic characteristic to determine the $\mathbb{Z}_2$ topological nature of the 2D time-reversal invariant topological insulator.[11] Another consequence of the action is the fermionic mutual statistics between quasiparticles that couple as $j^\mu a_\mu$ and $K^\mu v_\mu$. When Ch_2 vanishes the currents j^μ, K^μ have mutual fermionic statistics but when $\mathsf{Ch}_2 \neq 0$ the current j^μ is shifted such that $\tilde{j}^\mu$ and K^μ carry mutual fermionic statistics where

$$\tilde{j}^\mu = j^\mu - \frac{\mathsf{Ch}_2}{4\pi^2}\epsilon^{\mu\nu\lambda}\partial_\nu(\phi\partial_\lambda\psi). \tag{4.13}$$

V. FRACTIONAL STATES

Our discussions so far concern a bosonized description of noninteracting topological band insulators or weakly interacting topological insulators that are adiabatically connected to a topological band insulator. In these systems, functional bosonization gives rise to the BF topological field theory with the unit level $\mathsf{k} = 1$. In this section, we will discuss the possibility of topological states that can be induced by strong interactions and characterized by, e.g., nonzero ground-state degeneracy. While the existence of strongly interacting topological insulators with topological order is not well established microscopically for $D > 3$, we can use the hydrodynamic bosonization formalism of the preceding sections to explore possible mechanisms that lead to a topological order. One such route is the fractionalization of electrons due to strong correlations. We will implement this through the parton construction a la Blok and Wen,[33,34] to find effective topological field theories.

A. Parton Construction for the Chern Insulator in $D = 2 + 1$

We start with a construction of time-reversal breaking fractional states in $D = 2 + 1$ dimensions by combining functional bosonization and the parton construction. When applied to elections in the (lowest) Landau level, this approach is equivalent to the composite particle (composite boson and composite fermion) theories.[23,28–32,88–93] The functional bosonization is also readily applicable to fractional quantum Hall states formed on a lattice, i.e., "fractional Chern insulators." (See Refs. 94–106 for recent studies on the fractional Chern insulators.)

We focus on the effective field theory for the hierarchy states at $\nu = \mathsf{m}/(\mathsf{mp} + 1)$ in terms of the parton construction. Following Refs. 33 and 34, we first split the electron into $\mathsf{p} + 1$ partons. Here, p is an even integer, and we require partons to obey Fermi statistics. Since the electrons do not split in reality,

we impose a constraint

$$j_\mu^{(i)} = j_\mu^{(j)}, \quad i,j = 1,\ldots,\mathsf{p}+1, \tag{5.1}$$

on the parton densities $j_\mu^{(i)}$. The ith parton carries electric charge e_i, and $e = \sum_i e_i = 1$. We assume that all partons are in independent integer quantum Hall states, but we treat the $i = 1,\ldots,\mathsf{p}$-th flavors and the $i = \mathsf{p} + 1$-th flavor differently. For $i = 1,\ldots,\mathsf{p}$, we assume the filling faction is $\nu^{(i)} = 1$, whereas for $i = \mathsf{p} + 1$, $\nu^{(i)} = \mathsf{m}$. The total filling fraction is

$$\nu = \frac{1}{\mathsf{p} + 1/\mathsf{m}} = \frac{\mathsf{m}}{\mathsf{mp} + 1}. \tag{5.2}$$

Each parton can be bosonized by functional bosonization. For $i = 1,\ldots,\mathsf{p}$-th partons, they can be described by the following $D = 2 + 1$ BF theories with the CS term,

$$\mathcal{L}^{(i)} = \frac{-1}{2\pi}\epsilon b^{(i)}\partial a^{(i)} + \frac{1}{4\pi}\epsilon a^{(i)}\partial a^{(i)} - e_i j^{(i)} \cdot A^{\mathrm{ex}}, \tag{5.3}$$

where the repeated indices i are not summed, and we have introduced short-hand notations $A^{\mathrm{ex}} \cdot j \equiv A_\mu^{\mathrm{ex}} j^\mu$, $\epsilon a\partial a \equiv \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$, etc. On the other hand, for the $i = \mathsf{p} + 1$-st parton, since we have m filled Landau levels, we introduce m separate gauge fields, $b_\mu^{I=1,\ldots,\mathsf{m}}$ and $a_\mu^{I=1,\ldots,\mathsf{m}}$, each representing the condensate in the Ith Landau level. The Lagrangian for the $i = \mathsf{p} + 1$-th parton is

$$\mathcal{L}^{(\mathsf{p}+1)} = \sum_{I=1}^{\mathsf{m}} \left(\frac{-1}{2\pi}\epsilon b^I \partial a^I + \frac{1}{4\pi}\epsilon a^I \partial a^I - e_{\mathsf{p}+1} j^I A^{\mathrm{ex}} \right). \tag{5.4}$$

The total Lagrangian is given by $\mathcal{L} = \sum_{i=1}^{\mathsf{p}} \mathcal{L}^{(i)} + \mathcal{L}^{(\mathsf{p}+1)}$.

With the constraints imposed on the parton densities,

$$\epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda^{(i)} = \epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda^{(\mathsf{p}+1)} = \sum_I \epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda^I, \tag{5.5}$$

($i = 1,\ldots,\mathsf{p}$), the total Lagrangian (after solving the constraints) is given by

$$\begin{aligned} \mathcal{L} = &\frac{-1}{2\pi}\sum_I \epsilon b^I \partial a^I + \frac{-1}{2\pi}\sum_{I,i} \epsilon b^I \partial a^{(i)} \\ &+ \frac{1}{4\pi}\sum_I \epsilon a^I \partial a^I + \frac{1}{4\pi}\sum_i \epsilon a^{(i)} \partial a^{(i)} \\ &- \frac{e}{2\pi}\sum_I \epsilon \partial b^I A^{\mathrm{ex}}. \end{aligned} \tag{5.6}$$

This can be written more compactly as

$$\mathcal{L} = \sum_{i,j} \frac{\tilde{K}_{ij}}{4\pi}\epsilon\alpha^i\partial\alpha^j - \sum_i \frac{e}{2\pi}q_i\epsilon\partial\alpha^i A^{\mathrm{ex}}, \tag{5.7}$$

where $\alpha = (b^I, a^I, a^{(i)})$, the charge vector q_i is given as $q_i = 1$ for the first m entries, whereas $q_i = 0$ otherwise, and the K matrix is

$$\tilde{K} = \begin{pmatrix} 0 & -I_\mathsf{m} & -J_{\mathsf{m},\mathsf{p}} \\ -I_\mathsf{m} & I_\mathsf{m} & 0 \\ -J_{\mathsf{p},\mathsf{m}} & 0 & I_\mathsf{p} \end{pmatrix}, \tag{5.8}$$

where I_m is an $m \times m$ identity matrix and $J_{m,p}$ is the $m \times p$ matrix with all matrix elements equal to 1.

The topological order encoded in the $\tilde{K}$-matrix CS theory (5.7) can equivalently be described by a multicomponent CS theory with fewer components. For example, when $m = 1$ and $p = 2$, the filling fraction is $\nu = 1/3$ and

$$\tilde{K} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{5.9}$$

The ground-state degeneracy can be read off from $|\mathrm{Det}\, \tilde{K}| = 3$, as expected for the Laughlin state at $\nu = 1/3$. We can integrate out the $a^{(i)}$ one by one using the equations of motion to arrive at the more familiar form of the K matrix; with the equations of motion $\delta S/\delta a_\mu^{(i)} = -(2\pi)^{-1}\epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda + (2\pi)^{-1}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^{(i)} = 0$, the Lagrangian

$$\mathcal{L} = \frac{-1}{2\pi}\epsilon b\partial a + \frac{-1}{2\pi}\sum_{i=1}^{p}\epsilon b\partial a^{(i)}$$
$$+ \frac{1}{4\pi}\epsilon a\partial a + \frac{1}{4\pi}\sum_{i=1}^{p}\epsilon a^{(i)}\partial a^{(i)} - \frac{e}{2\pi}\epsilon\partial bA^{\mathrm{ex}} \tag{5.10}$$

can be reduced to

$$\mathcal{L} \rightarrow \frac{-1}{2\pi}\epsilon b\partial a + \frac{1}{4\pi}\epsilon a\partial a$$
$$+ \frac{-p}{2\pi}\epsilon b\partial a^{(p)} + \frac{p}{4\pi}\epsilon a^{(p)}\partial a^{(p)}. - \frac{e}{2\pi}\epsilon\partial bA^{\mathrm{ex}}. \tag{5.11}$$

This is identical to the effective action obtained from the flux attachment composite particle approach, where b_μ plays the role of the vortex gauge field whereas a_μ is the statistical CS gauge field that transmutes the statistics of the electrons.

B. Parton construction for topological insulators in $D = 3 + 1$

We can formally repeat the parton construction in $D = 3 + 1$. As before, we postulate that electrons are fractionalized, consist of $p + 1$ partons, and each parton is in its topological insulator phase. For each parton, we can apply functional bosonization to derive its hydrodynamic theory. Solving the constraints among parton densities, we will arrive at multicomponent BF theories with a topological term (e.g., axion or θ term in $D = 3 + 1$). In this section, we assume partons are in a $D = 3 + 1$-dimensional topological insulator phase in symmetry class AIII or DIII characterized by an integer topological invariant. See Refs. 107–109 for previous studies of time-reversal symmetric fractional topological insulators in $D = 3 + 1$ in terms of the parton construction.

We thus write down the following Lagrangian $\mathcal{L} = \sum_{i=1}^{p}\mathcal{L}^{(i)} + \mathcal{L}^{(p+1)}$ for partons [see Eq. (3.14)]:

$$\mathcal{L}^{(i)} = -\epsilon^{\mu\nu\lambda\rho}b_{\mu\nu}^{(i)}\partial_\lambda a_\rho^{(i)} + \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu^{(i)}\partial_\lambda a_\rho^{(i)} - e_i j_\mu^{(i)}A^\mu, \tag{5.12}$$

for $i = 1, \ldots p$-th flavors, and

$$\mathcal{L}^{(p+1)} = -\epsilon^{\mu\nu\lambda\rho}\sum_{I=1}^{m}b_{\mu\nu}^{I}\partial_\lambda a_\rho^{I} + \frac{\theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\sum_{I=1}^{m}\partial_\mu a_\nu^{I}\partial_\lambda a_\rho^{I}$$
$$- e_{p+1}\sum_{I=1}^{m}j_\mu^{I}A^\mu, \tag{5.13}$$

for the $p + 1$-th flavor. Here, $e = \sum_i e_i = 1$, the parton densities are written in terms of the two-form gauge fields $b_{\mu\nu}^{(i)}$ and $b_{\mu\nu}^{I}$ as $j^{I,\mu} = \epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}^{I}$, etc., and are subject to the constraint

$$\epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}^{(i)} = \epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}^{(p+1)} = \sum_{I=1}^{m}\epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}^{I}, \tag{5.14}$$

($i = 1, \ldots, p$). Solving the constraint, following similar steps to those leading to Eq. (5.6), the resulting effective field theory is

$$\mathcal{L} = -\sum_I \epsilon^{\mu\nu\lambda\rho}b_{\mu\nu}^{I}\partial_\lambda\left(a_\rho^{I} + \sum_i a_\rho^{(i)}\right)$$
$$+ \frac{\theta}{8\pi^2}\sum_I \epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu^{I}\partial_\lambda a_\rho^{I} + \frac{\theta}{8\pi^2}\sum_i \epsilon^{\mu\nu\lambda\rho}\partial_\mu a_\nu^{(i)}\partial_\lambda a_\rho^{(i)}$$
$$- e\sum_I \epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}^{I}A_\mu^{\mathrm{ex}}. \tag{5.15}$$

For a simple case where $m = 1$ and $p = k - 1$, introducing $b_{\mu\nu} := b_{\mu\nu}^{I=1}$ and labeling the k gauge fields as $\alpha_\mu^{a=1,2,\ldots,k} = (a_\mu^{(1)}, a_\mu^{(2)}, \ldots, a_\mu^{I=1})$, the effective Lagrangian is given by

$$\mathcal{L} = -\epsilon^{\mu\nu\lambda\rho}b_{\mu\nu}\partial_\lambda\sum_{a=1}^{k}\alpha_\rho^{a} + \frac{\theta}{8\pi^2}\sum_{a=1}^{k}\epsilon^{\mu\nu\lambda\rho}\partial_\mu\alpha_\nu^{a}\partial_\lambda\alpha_\rho^{a}$$
$$- e\epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}A_\mu^{\mathrm{ex}}. \tag{5.16}$$

As before in the case of $D = 2 + 1$, we can eliminate α_μ^a one by one. [The following steps should be compared with Eqs. (5.10) and (5.11) in the $D = 2 + 1$-dimensional case.] From $\delta S/\delta\alpha_\mu^a = \delta S/\delta b_{\mu\nu} = 0$,

$$\epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} - \frac{1}{4\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\nu\left(\theta\partial_\lambda\alpha_\rho^a\right) = 0,$$
$$\epsilon^{\mu\nu\lambda\sigma}\partial_\lambda\left(\sum_a\alpha_\sigma^a - A_\sigma^{\mathrm{ex}}\right) = 0. \tag{5.17}$$

If we assume $\theta = \mathrm{const}$ and neglect monopole configurations, the first equation gives $\epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} = 0$. However, if we take into account monopoles, we claim, from the first equation,

$$b_{\lambda\rho} = \frac{1}{4\pi^2}\theta\partial_\lambda\alpha_\rho^a, \quad a = 1, \ldots, k. \tag{5.18}$$

Then, $\alpha^1 = \alpha^2 = \cdots = \alpha^k \equiv \alpha$, and hence the Lagrangian, after eliminating α_μ^a, is

$$\mathcal{L} = -k\epsilon^{\mu\nu\lambda\rho}b_{\mu\nu}\partial_\lambda\alpha_\rho + \frac{k\theta}{8\pi^2}\epsilon^{\mu\nu\lambda\rho}\partial_\mu\alpha_\nu\partial_\lambda\alpha_\rho$$
$$- e\epsilon^{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho}A_\mu^{\mathrm{ex}}. \tag{5.19}$$

Further integrating over α_μ by using $\delta S/\delta b_{\mu\nu} = 0$,

$$\epsilon^{\mu\nu\lambda\sigma}\partial_\lambda\left(k\alpha_\sigma - A_\sigma^{\mathrm{ex}}\right) = 0, \tag{5.20}$$

we then arrive at

$$\mathcal{L} = \frac{\theta}{8\pi^2 \mathsf{k}} \epsilon^{\mu\nu\lambda\rho} \partial_\mu A_\nu^{\text{ex}} \partial_\lambda A_\rho^{\text{ex}}. \qquad (5.21)$$

[This step should be compared with Eq. (3.20).]

C. Parton construction for the quantum spin Hall insulator in $D = 2 + 1$

There are two natural ways to apply the parton construction for $D = 2 + 1$ time-reversal invariant insulators: (i) create the necessary copies of the gauge fields in $D = 4 + 1$ and then perform dimensional reduction twice or (ii) perform dimensional reduction from a nonfractionalized theory in $D = 4 + 1$ and then create replicas of the relevant fields. We will take the former approach. Our result, in fact, matches what would be found by taking Eq. (5.21) and simply performing dimensional reduction on A_μ^{ex}.

We begin with Eq. (4.9) and add replicas for the $\phi, \psi, a_\mu, g_{\mu\nu}, u_{\mu\nu}$, and v_μ fields to find the Lagrangians

$$\mathcal{L}^{(i)} = -\frac{\epsilon^{\mu\nu\lambda}}{2\pi}(\phi^{(i)} - e_i\theta^{\text{ex}})\partial_\mu g_{\nu\lambda}^{(i)} - \frac{\epsilon^{\mu\nu\lambda}}{2\pi}(\psi^{(i)} - e_i\chi^{\text{ex}})\partial_\mu u_{\nu\lambda}^{(i)}$$
$$- \frac{\epsilon^{\mu\nu\lambda}}{2\pi}(a_\mu^{(i)} - e_i A_\mu^{\text{ex}})\partial_\nu v_\lambda^{(i)} + \frac{\epsilon^{\mu\nu\lambda}}{4\pi^2}\phi^{(i)}\partial_\mu \psi^{(i)}\partial_\nu a_\lambda^{(i)}, \qquad (5.22)$$

where $i = 1, \ldots, \mathsf{p}$. Note that we have crucially included the e_i charges in front of the scalar fields $\theta^{\text{ex}}, \chi^{\text{ex}}$ as they arise from the dimensionally reduced $A_w^{\text{ex}}, A_z^{\text{ex}}$ fields, respectively, and would enter the $D = 4 + 1$ Lagrangian with the corresponding charge. For the $\mathsf{p} + 1$-st flavor, we have

$$\mathcal{L}^{(\mathsf{p}+1)} = -\sum_{I=1}^{\mathsf{m}} \frac{\epsilon^{\mu\nu\lambda}}{2\pi}(\phi^I - e_I\theta^{\text{ex}})\partial_\mu g_{\nu\lambda}^I$$
$$- \sum_{I=1}^{\mathsf{m}} \frac{\epsilon^{\mu\nu\lambda}}{2\pi}(\psi^I - e_I\chi^{\text{ex}})\partial_\mu u_{\nu\lambda}^I$$
$$- \sum_{I=1}^{\mathsf{m}} \epsilon^{\mu\nu\lambda}(a_\mu^I - e_I A_\mu^{\text{ex}})\partial_\nu v_\lambda^I$$
$$+ \sum_{I=1}^{\mathsf{m}} \frac{\epsilon^{\mu\nu\lambda}}{4\pi^2}\phi^I\partial_\mu \psi^I\partial_\nu a_\lambda^I. \qquad (5.23)$$

For the simple case when $\mathsf{m} = 1, \mathsf{p} = \mathsf{k} - 1$ the constraint for the parton densities identifies all of the $v_\mu^{(i)} \equiv v^i, g_{\mu\nu}^{(i)} \equiv g_{\mu\nu}$, and $u_{\mu\nu}^{(i)} = u_{\mu\nu}$. We can then use the equation of motion

$$\frac{\delta S}{\delta a_\mu^{(i)}} = \epsilon^{\mu\nu\lambda}\left[\frac{1}{4\pi^2}\partial_\nu(\phi^{(i)}\partial_\lambda \psi^{(i)}) - \frac{1}{2\pi}\partial_\nu v_\lambda\right] = 0 \quad (5.24)$$

to eliminate $a_\mu^{(i)}$ and arrive at

$$\mathcal{L} = \frac{\epsilon^{\mu\nu\lambda}}{2\pi}\left[\left(\theta^{\text{ex}} - \sum_{i=1}^{\mathsf{k}} \phi^{(i)}\right)\partial_\mu g_{\nu\lambda} + \left(\chi^{\text{ex}} - \sum_{i=1}^{\mathsf{k}} \psi^{(i)}\right)\partial_\mu u_{\nu\lambda}\right]$$
$$+ \frac{\mathsf{k}}{4\pi^2}\epsilon^{\mu\nu\lambda}A_\mu^{\text{ex}}\partial_\nu \Omega_\lambda, \qquad (5.25)$$

where $\Omega_\lambda = \phi^{(i)}\partial_\lambda \psi^{(i)}$ is the same for all i from the equation of motion $\delta S/\delta a_\mu^{(i)} = 0$. Using $\delta S/\delta \phi^{(i)} = 0$ and $\delta S/\delta \psi^{(i)} = 0$,

we can show that $\psi^{(i)} \equiv \psi$ and $\phi^{(i)} \equiv \phi$ are identical for all i, respectively. Eliminating ψ and ϕ, we finally arrive at

$$\mathcal{L} = \frac{1}{4\pi^2 \mathsf{k}} \epsilon^{\mu\nu\lambda} A_\mu^{\text{ex}} \partial_\nu(\theta^{\text{ex}}\partial_\lambda \chi^{\text{ex}}). \qquad (5.26)$$

Physically, this term implies that on the edge of a fractional quantum spin Hall system there will be fractional multiples of $e/2$ charge, i.e., $Q = e/(2\mathsf{k})$ on an antiphase magnetic domain wall.

VI. DISCUSSION

The effective Chern-Simons field theory approach is one of the most successful theoretical frameworks of the $D = 2 + 1$-fractional quantum Hall effect. In this paper, we aimed to extend this type of hydrodynamic formulation to a broader class of noninteracting, as well as interacting, topological insulators, in arbitrary dimensions with a suitable set of discrete symmetries.

We close with a few relevant comments. First, our approach relies crucially on the presence of a U(1) gauge symmetry. While we have focused on topological insulators with the electromagnetic charge U(1) symmetry, there are topological phases that preserve non-Abelian symmetry [such as spin-singlet topological superconductors in $D = 2 + 1$ and $3 + 1$ that preserve spin SU(2) rotation symmetry]. Our approach can easily be extended to such situations (see Appendix A). There are also various topological phases that do not conserve any quantum numbers, except possibly energy and momentum. In particular, these include topological superconductors in symmetry class D ($D = 2$) and DIII ($D = 3$). For such topological superconductors, Ref. 110 proposed a BF-type topological field theory with fermionic degrees of freedom.

Second, we note that in various situations, it may be useful to initially consider more U(1) charges than are actually conserved. For example, in the quantum spin Hall effect, while the electromagnetic U(1) symmetry is strictly conserved, spin rotation symmetry is not. It is, however, still a useful starting point to consider a system with U(1) symmetries associated with both charge and spin rotations around, for example, the z axis. The system is then invariant under an expanded U(1) × U(1) symmetry. Functional bosonization in this case then gives rise to a BF theory with Chern-Simons terms. This is, in fact, the usual approach and gives rise to doubled Chern-Simons theories with U(1) × U(1) symmetry [more generally, with U(1)^N × U(1)^N symmetry]. Ultimately, however, the quantum spin Hall effect does not depend crucially on the presence of the additional U(1) spin symmetry; its stability is guaranteed by a $\mathbb{Z}_2$ topological invariant, which has nothing to do with the presence of the S_z spin rotation symmetry. Starting from the BF theory with U(1) × U(1) symmetries, the "unwanted" S_z conservation can be broken by introducing monopole processes or possibly the Higgs mechanism. The former was discussed for the BF theory describing a BCS superconductor in Ref. 66, which has a $\mathbb{Z}_2$ topological order. It would be interesting to see if one started from a U(1) × U(1) theory and implemented either symmetry breaking mechanism if one would generate a hydrodynamic field theory that is equivalent with what we constructed in this article by only using the U(1) charge symmetry.

Finally, due to the lack of microscopic realizations of fractional topological insulators (in particular, in $D > 2 + 1$), the usefulness of our effective field theory approach is not yet entirely clear. It is, however, interesting to speculate on the possible mechanism of such states. In the functional bosonization scheme, we need to "raise the level" of the BF term, as it is this term that controls the ground-state degeneracy. In this paper, we have explored the parton construction. Another possible way to raising the level would be to use a Higgs field. This mechanism was discussed in the fractional quantum Hall effect, in particular, in non-Abelian fractional quantum Hall states. Together with functional bosonization, this method may allow us to discuss the same kinds of fractional states obtained by the parton construction as well as different kinds of states.[111,112]

ACKNOWLEDGMENTS

APPENDIX A: NON-ABELIAN FUNCTIONAL BOSONIZATION

Here, we describe the functional bosonization for non-Abelian currents. We start from the generating functional $Z(A_\mu^{\rm ex})$ for the correlation functions of non-Abelian currents, where $A_\mu^{\rm ex}$ is a Lie-algebra valued external field. We rewrite $Z(A_\mu^{\rm ex})$ as

$$Z(A_\mu^{\rm ex}) = \int \mathcal{D}[a_\mu] \prod_x \prod_\mu \delta(a_\mu - A_\mu^{\rm ex}) Z(a_\mu). \quad (A1)$$

The δ functional can be written as

$$\prod_x \prod_\mu \delta(a_\mu - A_\mu^{\rm ex})$$
$$= \Delta_{\rm FP}(a) \prod_x \prod_{\mu,\nu,\lambda,\dots}^{\mu<\nu<\lambda\cdots} \epsilon_{\mu\nu\lambda\cdots\alpha\beta} \prod_a \delta[f_{\alpha\beta}^a(a) - f_{\alpha\beta}^a(A^{\rm ex})], \quad (A2)$$

where $f_{\alpha\beta}(a) = \partial_\alpha a_\beta - \partial_\beta a_\alpha + [a_\alpha, a_\beta]$, $\prod_a$ runs over the Lie algebra index, and $\prod_{\mu,\nu}$ runs over $n = D(D-1)/2$ independent directions. The Jacobian $\Delta_{\rm FP}(a)$ can be expressed in terms of a functional integral as follows. By integrating over $A_\mu^{\rm ex}$ in Eq. (A2),

$$1 = \int \mathcal{D}[A_\mu] \prod_x \prod_\mu \delta(a_\mu - A_\mu)$$
$$= \Delta_{\rm FP}(a) \int \mathcal{D}[A_\mu] \prod_x \prod_{\mu,\nu,\rho,\dots}^{\mu<\nu<\rho\cdots} \epsilon_{\mu\nu\rho\cdots}$$
$$\times \prod_a \delta[f_{\alpha\beta}^a(a) - f_{\alpha\beta}^a(A)]. \quad (A3)$$

(Here we denote $A_\mu^{\rm ex} = A_\mu$ to lighten notations). Consider

$$(*) := \int \mathcal{D}[A] \prod_x \prod_{\alpha,\beta}^{\alpha<\beta} \delta[f_{\alpha\beta}^a(a) - f_{\alpha\beta}^a(A)]. \quad (A4)$$

We expand $f_{\alpha\beta}(A)$ as

$$f_{\alpha\beta}(A) = f_{\alpha\beta}(a) + D_\alpha(a)\delta A_\beta - D_\beta(a)\delta A_\alpha, \quad (A5)$$

where $A = a + \delta A$. Then,

$$(*) = \int \mathcal{D}[\delta A_\mu] \prod_x \prod_{\alpha,\beta}^{\alpha<\beta} \delta[D_\alpha^a(a)\delta A_\beta - D_\beta^a(a)\delta A_\alpha]. \quad (A6)$$

This can be rewritten with a bosonic auxiliary field $\beta_{\alpha\beta}$ as

$$(*) = \int \mathcal{D}[\beta_{\alpha\beta}, \delta A_\mu] \exp i \int d^D x$$
$$\times \sum_{\alpha<\beta} \beta_{\alpha\beta}^a [D_\alpha^a(a)\delta A_\beta - D_\beta^a(a)\delta A_\alpha], \quad (A7)$$

where $\beta_{\alpha\beta} = -\beta_{\beta\alpha}$ and $\beta_{\alpha\beta} = 0$ for $\beta = \alpha$. Thus

$$\Delta_{\rm FP}(a)^{-1} = \int \mathcal{D}[\beta_{\alpha\beta}, \delta A_\mu] \exp i \int d^D x \, \beta_{\alpha\beta}^a D_\alpha^a(a)\delta A_\beta. \quad (A8)$$

The functional determinant can then be written, in terms of fermionic ghosts $\bar{c}_{\alpha\beta}$ and c_μ,

$$\Delta_{\rm FP}(a) = \int \mathcal{D}[\bar{c}_{\alpha\beta}, c_\mu] \exp i \int d^D x \, \bar{c}_{\alpha\beta}^a D_\alpha(a) c_\beta^a. \quad (A9)$$

Then, the generating functional can be written as

$$Z(A^{\rm ex}) = \int \mathcal{D}[a, b, \bar{c}, c] \exp i \, S,$$
$$S = \int d^D x \, {\rm tr} \{ \bar{c}_{\mu\nu\dots} \epsilon^{\mu\nu\cdots\alpha\beta} D_\alpha(a) c_\beta$$
$$+ b_{\mu\nu\dots} \epsilon^{\mu\nu\cdots\alpha\beta} [f_{\alpha\beta}(a) - f_{\alpha\beta}(A^{\rm ex})] + L(a) \}, \quad (A10)$$

where we introduced $L(a)$ by $Z(a) = \exp i \int d^D x \, {\rm tr} \, L(a)$.

Let us now introduce auxiliary fields

$$S = \int d^D x \, {\rm tr} \{ \bar{c}_{\mu\nu\dots} \epsilon^{\mu\nu\cdots\alpha\beta} D_\alpha(a) c_\beta$$
$$+ b_{\mu\nu\dots} \epsilon^{\mu\nu\cdots\alpha\beta} [f_{\alpha\beta}(a) - f_{\alpha\beta}(A^{\rm ex})]$$
$$+ L(a - h) - i(lh^\mu h_\mu - 2\bar{\chi}h^\mu c_\mu) \}. \quad (A11)$$

Here, l and $\bar{\chi}$ are bosonic and fermionic scalars, respectively, and h_μ is a bosonic Lie-algebra valued field, $h_\mu = h_\mu^a t_a$. The integration over l sets $h^\mu h_\mu = 0 \Rightarrow h_\mu = 0$ and then we go back to the original action. The action is invariant under the following BRST transformation:

$$\delta a_\mu = c_\mu, \quad \delta h_\mu = c_\mu, \quad \delta\bar{\chi} = l, \quad \delta l = 0,$$
$$\delta c_\mu = 0, \quad \delta\bar{c}_{\mu\nu\dots} = -2b_{\mu\nu\dots}, \quad \delta b_{\mu\nu\dots} = 0. \quad (A12)$$

It is possible to write the action as

$$iS = \delta G,$$
$${\rm where} \quad G = -\frac{i}{2} \int d^D x \, {\rm tr} [\bar{c}_{\mu\nu\dots} \epsilon^{\mu\nu\cdots\alpha\beta} f_{\alpha\beta}(a)]. \quad (A13)$$

APPENDIX B: QUANTIZATION OF THE BF THEORY WITH THE CHERN-SIMONS AND AXION TERM

From the bosonized action, we can read off, e.g., the current-current correlation functions. If we take the BF-CS theory (3.4), however, we do not have charge fluctuations in the bulk of the system; with only terms of topological origin (such as BF and CS terms), there is no dynamics. This situation corresponds to the limit where the band gap (mass) is taken to be infinitely large. To see charge fluctuations, we need either to have other terms than BF and CS terms, or to introduce a boundary to a system. The former means that we keep the band gap finite, or consider electron-electron interactions. Let us thus consider the Lagrangian with the Maxwell term

$$\mathcal{L} = -\frac{2\mathsf{k}}{4\pi} b_\mu \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda - A_\lambda) + \frac{\mathsf{Ch}}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$
$$- \frac{1}{4\pi g^2} F_{\mu\nu} F^{\mu\nu}, \tag{B1}$$

where g^2 is a coupling constant, A_μ is the *dynamical* electromagnetic U(1) gauge field and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. [A_μ is not a static source as in Eq. (3.4)]. Given the equation of motion $\delta S/\delta b_\mu = 0 \Rightarrow F_{\mu\nu}(A) = f_{\mu\nu}(a)$, the Maxwell term $-(4\pi g^2)^{-1} F_{\mu\nu} F^{\mu\nu}$ gives rise to the Maxwell term for a_μ, $-(4\pi g^2)^{-1} f_{\mu\nu} f^{\mu\nu}$. Alternatively, integrating over A_μ gives rise to a current-current interaction $\propto j^\mu(x) K_{\mu\nu}(x - y) j^\nu(y)$ where according to the bosonization rule, $j^\mu \propto \epsilon^{\mu\nu\lambda} \partial_\nu b_\lambda$.

The BF-CS-Maxwell theory (B1) can be canonically quantized by going through the Dirac quantization procedure. Some nonzero canonical commutation relations are given by

$$\left[b_i(\vec{x}), b_j(\vec{x}') \right] = \frac{-i 2\pi \mathsf{Ch}}{\mathsf{k}^2} \epsilon^{ij} \delta^{(2)}(\vec{x} - \vec{x}'),$$
$$\left[a_i(\vec{x}), \pi_a^j(\vec{x}') \right] = \left[A_i(\vec{x}), \pi_A^j(\vec{x}') \right] = i \delta_i^j \delta^{(2)}(\vec{x} - \vec{x}'), \tag{B2}$$

where $\vec{x} = (x^1, x^2)$ is the spatial coordinates, i, j run over the spatial components of the coordinates $i, j = x, y$, and π_a^i and π_A^i are the canonical momentum for a_i and A_i and are given by

$$\pi_a^i = -\frac{2\mathsf{k}}{4\pi} \epsilon^{ij} b_j + \frac{\mathsf{Ch}}{4\pi} \epsilon^{ij} a_j,$$
$$\pi_A^i = +\frac{2\mathsf{k}}{4\pi} \epsilon^{ij} b_j + \frac{1}{\pi g^2} E_i, \tag{B3}$$

where $E_i = \partial_0 A_i - \partial_i A_0$. The canonical commutation relations between current operators are given by

$$[j^0(\vec{x}), j^0(\vec{x}')] = 0, \quad [j^0(\vec{x}), j^i(\vec{x}')] = \frac{-i\mathsf{Ch}^2 g^2}{4\pi} \partial_i \delta^{(2)}(\vec{x} - \vec{x}'),$$
$$[j^i(\vec{x}), j^j(\vec{x}')] = \frac{-i\mathsf{Ch}^3 g^2}{8\pi} \epsilon^{ij} \delta^{(2)}(\vec{x} - \vec{x}'). \tag{B4}$$

Similarly, in $D = (3 + 1)$ dimensions, one can use the bosonized action (3.14) to read off the current-current commutation relations in the bulk, once we allow charges to fluctuate, by adding an Maxwell term, say. Let us consider the Lagrangian,

$$\mathcal{L} = -b_{\mu\nu} \epsilon^{\mu\nu\lambda\rho} \partial_\lambda (a_\rho - A_\rho) + \frac{\theta(x)}{8\pi^2} \epsilon^{\mu\nu\lambda\rho} \partial_\mu a_\nu \partial_\lambda a_\rho$$
$$- \frac{1}{4\pi g^2} F_{\mu\nu} F^{\mu\nu}, \tag{B5}$$

where as before A_μ is a dynamical U(1) gauge field.

The BF-axion-Maxwell theory (B5) can be canonically quantized by going through the Dirac quantization procedure. As one may infer from the (half-integral) quantum Hall effect at the surface of $D = 3 + 1$-dimensional topological insulators, the current-current commutators are "anomalous" only at the location where $\theta(x)$ changes, i.e., at an interface between topologically trivial and topologically nontrivial insulators. For simplicity, we assume that $\theta(x)$ depends only on the x^3 (z) component of the spatial coordinates. Some nonzero canonical commutation relations are given by

$$[b_{m3}(\vec{x}), b_{n3}(\vec{x}')] = \frac{-i(\partial_3 \theta)}{16\pi^2} \epsilon^{mn} \delta^{(3)}(\vec{x} - \vec{x}'),$$
$$\left[a_m(\vec{x}), \pi_a^n(\vec{x}') \right] = \left[A_m(\vec{x}), \pi_A^n(\vec{x}') \right] = i \delta_m^n \delta^{(3)}(\vec{x} - \vec{x}'), \tag{B6}$$

where m, n run over the spatial components of the coordinates $m, n = x, y$, and π_a^i and π_A^i are the canonical momentum for a_i and A_i and are given by

$$\pi_a^i = -\epsilon^{ijk} b_{jk} + \frac{\theta}{4\pi^2} \epsilon^{ijk} \partial_j a_k,$$
$$\pi_A^i = +\epsilon^{ijk} b_{jk} + \frac{1}{\pi g^2} E_i. \tag{B7}$$

The canonical commutation relations between current operators are given by

$$[j^0(\vec{x}), j^0(\vec{x}')] = 0,$$

$$[j^0(\vec{x}), j^m(\vec{x}')] = \frac{-i(\partial_3 \theta)^2 g^2}{16\pi^3} \partial_m \delta^{(3)}(\vec{x} - \vec{x}'), \tag{B8}$$

$$[j^m(\vec{x}), j^n(\vec{x}')] = \frac{-i(\partial_3 \theta)^3 g^2}{64\pi^4} \epsilon^{mn} \delta^{(3)}(\vec{x} - \vec{x}') \qquad .$$

This is exactly the same as the quantum Hall effect in the topological insulator in $2 + 1$ dimensions, see Eq. (B4), but with the integer Ch replaced by $\partial_3 \theta$.

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