

**Free energy of mean-field spin-glass models: Evolution operator and perturbation expansion**V. Janiš,<sup>\*</sup> A. Kauch, and A. Klíč*Institute of Physics, Academy of Sciences of the Czech Republic, Na Slovance 2, CZ-18221 Praha, Czech Republic*

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The full mean-field solution of spin glass models with a continuous order-parameter function is not directly available and approximate schemes must be used to assess its properties. One of the authors recently proposed a representation of the free energy generating this solution via an evolution operator parametrized by attainable values of overlap of magnetizations between different states. Here, we introduce a perturbation expansion for the evolution operator that we use to derive all thermodynamic characteristics via the standard methods of statistical mechanics. We obtain a generic scheme for an approximate calculation of physical quantities of different mean-field spin-glass models at all temperatures. The small expansion parameter is a difference between the continuous order-parameter function and the corresponding order parameter from the solution with one level of replica-symmetry breaking. The first correction beyond the approximation with one level of replica-symmetry breaking is explicitly evaluated in the glassy phase of the Sherrington-Kirkpatrick model.

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**I. INTRODUCTION**

Strongly disordered and frustrated spin systems, the prominent examples of which are spin glasses, display a complicated low-temperature behavior. Due to frustration, the standard homogeneous long-range orders are precluded and replaced by orderings with peculiar properties. First of all, the low-temperature phase shows everywhere ergodicity breaking that cannot be removed or circumvented by the application of external fields or measurable sources.<sup>1</sup> The free energy of such systems manifests in low temperatures a complex landscape with almost degenerate metastable states the description of which demands advanced mathematical tools.<sup>2</sup> The usual way to handle the low-temperature behavior of spin glasses and other frustrated systems is the replica trick transforming static, quenched averaging to a dynamical, annealed one.<sup>3</sup> The limit to zero of the replication index (number of replicas) in physical quantities does not, however, work as a simple perturbation (loop) expansion as initially introduced. One has to use nonperturbative techniques to reach a thermodynamically consistent solution in the limit of zero replicas. Parisi found a way how to break the symmetry of the order parameters in the replicated space of a mean-field Sherrington-Kirkpatrick model.<sup>4</sup> The dependence of the thermodynamic equilibrium state of spin glasses on the replication index and the structure of the order parameters in the replicated phase space reflects volatility of the thermodynamic limit to boundary conditions or the initial state used for the equilibration process. This is demonstrated by the concept of real replicas used to reconstruct homogeneity of the free energy in mean-field spin-glass models.<sup>5</sup>

A thermodynamically consistent mean-field solution of spin glass models with the full, continuous replica-symmetry breaking (RSB) cannot be reached directly but only via either an iterative scheme or a perturbation expansion. The most direct way to approach the solution with a continuous replica-symmetry breaking is to use discrete hierarchies of mathematical replicas. It is, however, almost impossible to go quantitatively beyond a two-level (2RSB) solution.<sup>4</sup> Moreover, approximations with finite numbers of replica hierarchies replace the continuous order-parameter function

by a set of delta functions from which one cannot deduce a detailed structure of the distribution of the equilibrium order parameters. Alternatively, one can expand the full solution near the critical transition point to the spin-glass phase.<sup>6–8</sup> Presently, the most advanced construction of the solution with continuous RSB is a high-order perturbation expansion of a solution of the Parisi nonlinear differential equation resolved numerically by means of a pseudospectral code and Padé approximants.<sup>9</sup> These expansions are applicable only to continuous transitions and to temperatures not too far below the critical point.

Only a few attempts have been undertaken to determine the structure of the solution with full continuous RSB at very low temperatures of the Sherrington-Kirkpatrick model. Iterative solutions with a high number of replica hierarchies were proposed either with the aid of the renormalization group<sup>10</sup> or an expansion around the spherical model,<sup>11</sup> or the replica-symmetric solution.<sup>12</sup> They all try to assess the impact of the continuous order-parameter function and make conclusions on low-temperature properties of the Sherrington-Kirkpatrick model. These approaches rely on the replica trick and a smooth transition from the replica-symmetric to the full RSB solution.

There are generalized spin-glass models with a structure of the phase space of the order parameters differing from that of the Ising spin glass. The mean-field solutions of the Potts glass,<sup>13</sup>  $p$ -spin glass,<sup>14</sup> or quadrupolar glass<sup>15</sup> show intervals of temperature where a first step toward the Parisi solution in the replica trick, the so-called one-level replica symmetry breaking (1RSB), is locally stable and for some parameters, the transition to the glassy phase is discontinuous. Except for an asymptotic expansion near the continuous transition to the glassy phase of the Potts model,<sup>16</sup> there are no other approaches that could describe the coexistence and transitions between the 1RSB and continuous RSB solutions in these models, particularly at low temperatures.

The aim of this paper is to develop an approximate scheme that would be able to interpolate between the 1RSB and the full continuous RSB solutions in general mean-field spin-glass models. We start with an explicit representation of the free energy with a continuous order-parameter function via an evolution operator from Ref. 17 where we rescale its variables

so that the zero-temperature limit is directly accessible. To allow for the limit to zero temperature, we will have to peel off the part corresponding to the 1RSB solution from the full free energy. We then obtain a suitable form of free energy that will allow us to introduce a perturbation expansion around the 1RSB solution in powers of the continuous order-parameter function, or better, of its correction to the respective order parameter from the 1RSB solution. The expansion of the free energy from Ref. 17 in powers of the continuous order-parameter function leads to a series of approximate free energies from which we derive directly approximate expressions for all thermodynamic quantities such as entropy, specific heat, or magnetic susceptibility without the necessity to refer to the replica representation. Such a succession of approximations can be viewed upon as an alternative to the hierarchy of free energies with finite-many hierarchies of replica-symmetry breaking ( $K$ RSB). We evaluate the expansion explicitly in the first order and determine thermodynamic properties of the solution of the Ising spin glass without external magnetic field. We assess reliability and accuracy of the approximation by comparing its results with the high-order expansion around the critical temperature to the glassy phase of Ref. 9 and with Monte Carlo simulations.

We show in Sec. II how a free-energy functional containing the 1RSB order parameters together with a continuous order-parameter function can be derived from the limit of free energies with finite-many hierarchies of replica generations breaking the replica symmetry. We introduce the appropriate scaling so that the limit to zero temperature can explicitly be performed. Stationarity equations for the limiting free energy without replicas are derived in Sec. III. The idea and relations needed to construct a perturbation expansion around the 1RSB solution in powers of the continuous order-parameter function are presented in Sec. IV. The lowest-order approximation in closed form is derived in Sec. V. Numerical results from the first order of the perturbation expansion are presented and compared with other methods in Sec. VI. We summarize the salient properties of the approximate construction in the final Sec. VII.

## II. GENERATING FREE-ENERGY FUNCTIONAL

We cannot avoid introducing replicas and hierarchies of the replicated phase space if we want to describe thermodynamic equilibrium of glassy systems. Independently, whether we do it via the replica trick and the replica symmetry breaking scheme or via a successive use of real replicas of the phase space to enforce thermodynamic homogeneity, we end up with the same result when the replication index is analytically continued to real numbers. The result is a series of free energy densities labeled by a number  $K$  of replica hierarchies used. Free energy with  $K$  replica hierarchies uses  $2K + 1$  order parameters. We have  $\Delta\chi_l, m_l$  with  $l = 1, \dots, K$ , where  $\Delta\chi_l$  stands for the overlap susceptibility between the original spins and those from the  $l$ th hierarchical level and  $m_l$  is a replication index connected with the  $l$ th replica hierarchy. The remaining parameter  $q$  is the averaged square of the local magnetization after  $K$  replications. This free energy can be represented

explicitly as<sup>5</sup>

$$\begin{aligned} f^K(q, \Delta\chi_1, \dots, \Delta\chi_K; m_1, \dots, m_K) \\ = -\frac{1}{\beta} \ln 2 + \frac{\beta}{4} \sum_{l=1}^K m_l \Delta\chi_l \left[ 2 \left( q + \sum_{i=l+1}^K \Delta\chi_i \right) + \Delta\chi_l \right] \\ - \frac{\beta}{4} \left( 1 - q - \sum_{l=1}^K \Delta\chi_l \right)^2 - \frac{1}{\beta} \int_{-\infty}^{\infty} \mathcal{D}\eta \ln Z_K, \end{aligned} \quad (1a)$$

where we used a sequence of partition functions

$$Z_l = \left( \int_{-\infty}^{\infty} \mathcal{D}\lambda_l Z_{l-1}^{m_l} \right)^{1/m_l}. \quad (1b)$$

The initial partition sum for the Ising spin glass reads  $Z_0 = \cosh[\beta(h + \eta\sqrt{q} + \sum_{l=1}^K \lambda_l \sqrt{\Delta\chi_l})]$ . It is the partition sum of the original spin model affected by the interaction with the replicated spins represented via fluctuating Gaussian fields  $\lambda_1, \dots, \lambda_K$ . We denote the Gaussian normalized differential  $\mathcal{D}\lambda \equiv d\lambda e^{-\lambda^2/2} / \sqrt{2\pi}$ .

The physical interpretation of free energy  $f^K$  as discussed in Ref. 17 is as follows. We take the partition sum of the original model  $Z_0$ , replicate it  $m_1$  times and average over the spins from the first replica hierarchy. We downscale the result by a power  $1/m_1$  to keep the correct spin normalization. The new effective partition function  $Z_1$  is replicated  $m_2$  times and averaged over the spins from the second level of the replicated spins. After downscaling, we go on with higher replica levels. If a partition sum  $Z_l$  is thermodynamically homogeneous, it becomes independent of the scaling parameter  $m_{l+1}$ . This just happens if  $\Delta\chi_{l+1} = 0$ . When not, we have to go to a higher level of replica-symmetry breaking. Stationarity conditions then only minimize deviations from the global homogeneity and make free energy  $f_K$  at least locally homogenous, that is, with respect to infinitesimal variations of the replication indices. The full RSB solution with a continuous order-parameter function is then obtained in the limit  $K \rightarrow \infty$  with  $\Delta\chi_l \sim 1/K$ .

Before we proceed to the continuous limit,  $K \rightarrow \infty$ , we rescale the order parameters so that to simplify the explicit temperature dependence and to allow for a straightforward limit to low and eventually zero temperature. The explicit limit to zero temperature can be performed if we rescale the replication indices  $m_l \rightarrow \mu_l = \beta m_l$  and set apart the first overlap susceptibility and the replication index that we denote  $\chi_0, \mu_0$ , respectively. The rescaled hierarchical free energy is

$$\begin{aligned} f^{K+1}(q_0, \chi_0, \mu_0; \Delta\chi_1, \dots, \Delta\chi_K; \mu_1, \dots, \mu_K) \\ = -\frac{1}{\beta} \ln 2 - \frac{\beta}{4} \left( 1 - q_0 - \chi_0 - \sum_{l=1}^K \Delta\chi_l \right)^2 \\ + \frac{1}{4} \mu_0 \chi_0 \left[ 2 \left( q_0 + \sum_{l=1}^K \Delta\chi_l \right) + \chi_0 \right] \\ + \frac{1}{4} \sum_{l=1}^K \mu_l \Delta\chi_l \left[ 2 \left( q_0 + \sum_{i=l+1}^K \Delta\chi_i \right) + \Delta\chi_l \right] \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{\mu_K} \left( \ln \left[ \left\langle \dots \langle Z_0(h_\eta + \lambda\sqrt{\chi_0}) \right. \right. \right. \\
 & \left. \left. \left. + \Lambda_K \right)^{\mu_0/\beta} \right]_{\lambda}^{\mu_1/\mu_0} \dots \right]_{\lambda_K}^{\mu_K/\mu_{K-1}} \right)_{\eta}, \quad (2)
 \end{aligned}$$

where we denote  $h_\eta \equiv h + \eta\sqrt{q_0}$ ,  $\Lambda_K = \sum_{l=1}^K \lambda_l \sqrt{\Delta\chi_l}$ ,  $\langle X(\lambda_l) \rangle_{\lambda_l} = \int_{-\infty}^{\infty} \mathcal{D}\lambda_l X(\lambda_l)$ , and relabel  $q \rightarrow q_0$ .

The asymptotic expansion near the critical temperature in Refs. 6 and 7 proves that in the continuous limit  $K \rightarrow \infty$  we indeed have  $\Delta\chi_l = O(K^{-1})$  and  $\Delta\mu_l = \mu_l - \mu_{l+1} = O(K^{-1})$ . We denote  $X = K^{-1} \sum_{l=1}^K \Delta\chi_l$  and  $x = K^{-1} \sum_{l=1}^n \Delta\chi_l$  for  $x \in [n/K, (n+1)/K]$ . Then the overlap susceptibilities  $\Delta\chi_l$  are no longer order parameters in the continuous limit. They only form an index set for the continuously distributed replication indices  $\mu(x)$ . The generating free-energy functional for the solution with continuous RSB can be represented as a functional:<sup>17</sup>

$$\begin{aligned}
 & f(q_0, \chi_0, \mu_0; X, \mu(x)) \\
 & = -\frac{1}{\beta} \ln 2 - \frac{\beta}{4} (1 - q_0 - \chi_0 - X)^2 \\
 & \quad + \frac{1}{4} \mu_0 \chi_0 [2(q_0 + X) + \chi_0] + \frac{1}{2} \int_0^X dx \mu(x) \\
 & \quad \times [q_0 + X - x] - \langle g_\mu(X, h_\eta) \rangle_\eta. \quad (3)
 \end{aligned}$$

The interacting part of the energy  $g_\mu(x, h)$  obeys a Parisi-like nonlinear differential equation,

$$\frac{\partial g_\mu(x, h)}{\partial x} = \frac{1}{2} \left[ \frac{\partial^2 g_\mu(x, h)}{\partial h^2} + \mu(x) \left( \frac{\partial g_\mu(x, h)}{\partial h} \right)^2 \right], \quad (4)$$

as can be determined from the limit  $K \rightarrow \infty$  in the same manner as in Ref. 18. Its solution can conveniently be represented as<sup>17</sup>

$$\begin{aligned}
 g_\mu(x, h) & = \mathbb{E}_0(h; x, 0) \circ [g_\mu(h)] \\
 & \equiv \mathbb{T}_y \exp \left\{ \frac{1}{2} \int_0^x dy \left[ \partial_{\bar{h}}^2 + \mu(y) g'_\mu(y; h + \bar{h}) \partial_{\bar{h}} \right] \right\} \\
 & \quad \times g_\mu(h + \bar{h})|_{\bar{h}=0}, \quad (5)
 \end{aligned}$$

where we used prime to denote the derivative with respect to the magnetic field  $h$ ,  $g'_\mu(y, h) \equiv \partial_h g_\mu(y, h)$  and introduced a “time-ordering” operator  $\mathbb{T}_y$  ordering products of  $y$ -dependent noncommuting operators from left to right in the  $y$ -decreasing succession. The time-ordered exponential is then defined as a power series of multiple integrals:

$$\begin{aligned}
 & \mathbb{T}_y \exp \left[ \int_a^b dy \widehat{O}(y) \right] \\
 & \equiv 1 + \sum_{n=1}^{\infty} \int_a^b dy_1 \int_a^{y_1} dy_2 \dots \int_0^{y_{n-1}} dy_n \widehat{O}(y_1) \dots \widehat{O}(y_n).
 \end{aligned}$$

Time-ordering operators are a standard tool in representing quantum many-body perturbation expansions.

The initial interacting free energy being propagated by the evolution operator  $\mathbb{E}_0(h; X, 0)$  is the interacting part of the

1RSB free energy:

$$g_\mu(h) = \frac{1}{\mu_0} \ln \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2} \{ \cosh[\beta(h + \phi\sqrt{\chi_0})] \}^{\mu_0/\beta}. \quad (6)$$

To complete the expression for the free energy with continuous RSB, we have to add an equation for function  $g'_\mu(x, h)$ . From the definition of the evolution operator  $\mathbb{E}_0$ , we obtain directly

$$\begin{aligned}
 \frac{\partial g_\mu(x, h)}{\partial h} & = \mathbb{E}_0(h; x, 0) \circ [g'_\mu(h)] + \frac{1}{2} \int_0^x dy \mu(y) \mathbb{E}_0(h; x, y) \\
 & \quad \circ [g'_\mu(y, h) \partial_h g'_\mu(y, h)]. \quad (7a)
 \end{aligned}$$

The solution to this integral equation can be represented via the fundamental evolution operator for this theory with a shifted  $\mathbb{T}$ -ordered exponential:

$$\begin{aligned}
 g'_\mu(x, h) & = \mathbb{E}(h; x, 0) \circ [g'_\mu(h)] \\
 & \equiv \mathbb{T}_y \exp \left\{ \int_0^x dy \left[ \frac{1}{2} \partial_{\bar{h}}^2 \right. \right. \\
 & \quad \left. \left. + \mu(y) g'_\mu(x, h + \bar{h}) \partial_{\bar{h}} \right] \right\} g'_\mu(h + \bar{h})|_{\bar{h}=0}. \quad (7b)
 \end{aligned}$$

The equilibrium state is a stationary solution of free energy  $f(q_0, \chi_0, \mu_0; X, \mu(x))$  from Eq. (3). It is invariant with respect to infinitesimal variations of the scalar order parameters  $q_0, \chi_0, \mu_0, X$  and the continuous order-parameter function  $\mu(x)$  for  $x \in [0, X]$ . We can relate our variables with the standard Parisi representation of the continuous order-parameter function  $q_P(x_P)$  via a transformation  $x_P \rightarrow \beta^{-1}[\mu_0 + \mu(X - x)]$  and  $q_P(x_P) \rightarrow q_0 + \chi_0 + x$ .

### III. STATIONARITY EQUATIONS

The advantage of the representation of the free energy with a continuous RSB from Eqs. (3)–(7) is that we can derive the stationarity equations in the standard way used in statistical mechanics. Evaluating first the derivatives with respect to  $q_0$ ,  $\chi_0$ , and  $\mu_0$ , we obtain

$$q_0 = \langle g'_\mu(X, h_\eta)^2 \rangle_\eta, \quad (8a)$$

$$\begin{aligned}
 (\mu_0 - \beta)\chi_0 + \beta & = \langle \mathbb{E}(h_\eta; X, 0) \circ [g''_\mu(h)] \rangle_\eta \\
 & \quad + \beta \langle \mathbb{E}(h_\eta; X, 0) \circ [g'_\mu(h_\eta)^2] \rangle_\eta, \quad (8b)
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\mu_0}{4} \chi_0 [2(q_0 + X) + \chi_0] \\
 & = \frac{1}{\beta} \langle \mathbb{E}(h_\eta; X, 0) \circ [\langle \rho_\mu(h_\eta, \lambda\sqrt{\chi_0}) \ln \cosh[\beta(h_\eta \\
 & \quad + \lambda\sqrt{\chi_0})]_{\lambda} \rangle_\eta - \frac{1}{\mu_0} \langle \mathbb{E}(h_\eta; X, 0) \\
 & \quad \circ [\ln \cosh^{\mu_0/\beta}[\beta(h_\eta + \lambda\sqrt{\chi_0})]_{\lambda}] \rangle_\eta], \quad (8c)
 \end{aligned}$$

where we denote

$$\rho_\mu(h, \lambda\sqrt{\chi_0}) = \frac{\cosh[\beta(h + \lambda\sqrt{\chi_0})]^{\mu_0/\beta}}{\langle \cosh[\beta(h + \lambda\sqrt{\chi_0})]^{\mu_0/\beta} \rangle_\lambda}. \quad (9)$$

We already mentioned that the individual overlap susceptibilities  $\Delta\chi_l$  are no longer variational parameters in the free energy

with continuous RSB. But their sum  $X = K^{-1} \sum_l \Delta \chi_l$  is. A stationarity equation for this parameter reads

$$X = \langle \mathbb{E}(h_\eta; X, 0) \circ [g'_\mu(h_\eta)^2] \rangle_\eta - \langle g'_\mu(X, h_\eta)^2 \rangle_\eta. \quad (10)$$

Finally, we have to use the stationarity condition for the free energy from Eq. (3) with respect to infinitesimal variations of the order-parameter function  $\mu(x)$ . It is easy to find that it can be expressed as

$$x = \langle \mathbb{E}(h_\eta; X, 0) \circ [g'_\mu(h_\eta)^2] \rangle_\eta - \langle \mathbb{E}(h_\eta; X, x) \circ [g'_\mu(x, h_\eta)^2] \rangle_\eta, \quad (11)$$

valid for any  $x \in [0, X]$ . Note that Eq. (11) for  $x = 0$  is trivial and for  $x = X$  coincides with Eq. (10). Hence only equations for  $0 < x < X$  bring new information for the determination of  $\mu(x)$ .

There is no explicit dependence on  $\mu(x)$  in Eq. (11). It is hence difficult to use it for the determination of this order-parameter function. Since this equation holds for any  $x$ , also its derivative with respect to  $x$  must equally hold. The derivative reads

$$1 = \langle \mathbb{E}(h_\eta; X, x) \circ [g''_\mu(x, h_\eta)^2] \rangle_\eta, \quad (12)$$

which is just a condition for marginal stability of the solution with continuous RSB.<sup>17</sup> We recall that the prime stands for the derivative with respect to the magnetic field.

Still, we did not yet obtain an explicit expression for the function  $\mu(x)$ . Yet another differentiation is needed to obtain an explicit expression for the order-parameter function being

$$\mu(x) = \frac{\langle \mathbb{E}(h_\eta; X, x) \circ [g'''_\mu(x, h_\eta)^2] \rangle_\eta}{2 \langle \mathbb{E}(h_\eta; X, x) \circ [g''_\mu(x, h_\eta)^3] \rangle_\eta}. \quad (13)$$

This equation will prove suitable for approximate evaluations. It is also possible to use another derivative and to obtain an explicit equation for  $\dot{\mu}(x) = d\mu(x)/dx$  that corresponds to  $-dx_P/dq_P$  in the Parisi notation and is interpreted as the probability distribution of pure states.<sup>19</sup> We obtain for this derivative,

$$\begin{aligned} 2\dot{\mu}(x) \langle \mathbb{E}(h_\eta; X, x) \circ [g''_\mu(x, h_\eta)^3] \rangle_\eta \\ = - \langle \mathbb{E}(h_\eta; X, x) \circ [g_\mu^{(iv)}(x, h_\eta)^2] \rangle_\eta + 12\mu(x) \langle \mathbb{E}(h_\eta; X, x) \\ \circ [g''_\mu(x, h_\eta) g'''_\mu(x, h_\eta)^2] \rangle_\eta - 6\mu(x)^2 \langle \mathbb{E}(h_\eta; X, x) \\ \circ [g''_\mu(x, h_\eta)^4] \rangle_\eta. \end{aligned} \quad (14)$$

We see that the second term on the right-hand side has the opposite sign from the others and hence we cannot generally guarantee negativity of  $\dot{\mu}(x)$ .

To complete the stationarity equations, we add explicit expressions for the derivatives of the initial interacting free

energy  $g_\mu(h)$ . With the above introduced notation we have

$$g'_\mu(h) = \langle \rho_\mu(h, \lambda\sqrt{\chi_0}) t(h_\lambda) \rangle_\lambda, \quad (15a)$$

$$\begin{aligned} g''_\mu(h) = \beta \langle \rho_\mu(h, \lambda\sqrt{\chi_0}) [1 - t(h_\lambda)^2] \rangle_\lambda \\ + \mu_0 \langle [\rho_\mu(h, \lambda\sqrt{\chi_0}) t(h_\lambda)]^2 \rangle_\lambda - \langle \rho_\mu(h, \lambda\sqrt{\chi_0}) t(h_\lambda) \rangle_\lambda^2, \end{aligned} \quad (15b)$$

$$\begin{aligned} g'''_\mu(h) = 2 \langle \rho_\mu(h, \lambda\sqrt{\chi_0}) t(h_\lambda) \rangle_\lambda [\mu_0^2 \langle \rho_\mu(h, \lambda\sqrt{\chi_0}) t(h_\lambda) \rangle_\lambda^2 - \beta^2] \\ + (2\beta^2 - 3\beta\mu_0 + \mu_0^2) \langle \rho_\mu(h, \lambda\sqrt{\chi_0}) t(h_\lambda)^3 \rangle_\lambda + 3\mu_0(\beta \\ - \mu_0) \langle \rho_\mu(h, \lambda\sqrt{\chi_0}) t(h_\lambda) \rangle_\lambda \langle \rho_\mu(h, \lambda\sqrt{\chi_0}) t(h_\lambda)^2 \rangle_\lambda. \end{aligned} \quad (15c)$$

We denote  $t(h_\lambda) \equiv \tanh[\beta(h + \lambda\sqrt{\chi_0})]$ . Combining the above equations and the definitions for the derivatives of free energy  $g_\mu$ , we can transform the equation for the overlap susceptibility  $\chi_0$  to

$$\begin{aligned} \chi_0 = \langle \mathbb{E}(h_\eta; X, 0) \circ [\langle \rho_\mu(h_\eta, \lambda\sqrt{\chi_0}) t(h_{\lambda, \eta}) \rangle_\lambda] \rangle_\eta \\ - \langle \mathbb{E}(h_\eta; X, 0) \circ [\langle \rho_\mu(h_\eta, \lambda\sqrt{\chi_0}) t(h_{\lambda, \eta}) \rangle_\lambda^2] \rangle_\eta, \end{aligned} \quad (16)$$

where we abbreviate  $h_{\eta, \lambda} = h + \eta\sqrt{q_0} + \lambda\sqrt{\chi_0}$ .

The stationarity equations fully determine the equilibrium states of the mean-field free energy with a continuous order-parameter function. These equations are not solvable in their full exact form, being a consequence of inability to solve the Parisi nonlinear partial differential equation (4). Before we resort to approximations, we can use the above representation of the free energy and derive exact representations for the equilibrium values of interesting physical quantities. We can do that in the standard way of statistical mechanics without referring to replicas and the discrete representations used in the derivation of the final form of the free energy with a continuous order-parameter function.

We first evaluate the homogeneous magnetic susceptibility. If we use the condition for marginal stability, Eq. (12), we obtain

$$\begin{aligned} \chi_T = \langle g''(X, h_\eta) \rangle_\eta \\ = \beta(1 - q_0 - \chi_0 - X) + \mu_0 \chi_0 + \int_0^X dx \mu(x). \end{aligned} \quad (17)$$

It was argued that  $\chi_T = 1$  in the glassy phase.<sup>20</sup> This cannot be deduced from this exact representation without further reasoning.

As a next interesting thermodynamic quantity we evaluate entropy. For this purpose, we have to calculate the temperature derivative of the initial free energy  $g_\mu$ . With the above notation, we easily derive

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$$\frac{\partial g_\mu(h_\eta)}{\partial T} = \langle \rho_\mu(h_\eta, \lambda\sqrt{\chi_0}) \ln \cosh[\beta(h_{\eta, \lambda})] \rangle_\lambda - \beta \chi_0 [\beta + (\mu_0 - \beta) \langle \rho_\mu(h_\eta, \lambda\sqrt{\chi_0}) t(h_{\eta, \lambda}) \rangle_\lambda] - \beta h_\eta \langle \rho_\mu(h_\eta, \lambda\sqrt{\chi_0}) t(h_{\eta, \lambda}) \rangle_\lambda. \quad (18)$$

Using again the condition for marginal stability and the equations for the equilibrium values of the scalar order parameters we end up with an expression for entropy:

$$\begin{aligned} S(h, T) = - \frac{\partial f(h, T)}{\partial T} = \ln 2 + \beta \chi_0 (\beta - \mu_0) (X + q_0 + \chi_0) + \langle \mathbb{E}(h_\eta; X, 0) \circ [\langle \rho_\mu(h_\eta, \lambda\sqrt{\chi_0}) \ln \cosh[\beta(h_{\eta, \lambda})] \rangle_\lambda] \rangle_\eta \\ - \beta \langle \mathbb{E}(h_\eta; X, 0) \circ [h \langle \rho_\mu(h_\eta, \lambda\sqrt{\chi_0}) t(h_{\eta, \lambda}) \rangle_\lambda] \rangle_\eta - \beta^2 \chi_0 - \beta q_0 \chi_T - \frac{\beta^2}{4} (1 - q_0 - \chi_0 - X)^2. \end{aligned} \quad (19)$$



The last four terms on the right-hand side of Eq. (19) have negative sign and are potentially dangerous for turning the entropy negative at low temperatures.

#### IV. PERTURBATION EXPANSION

Since we cannot solve the stationarity equations for the order parameters of the free energy with continuous RSB, we resort to approximations. We decomposed the full free energy into the zeroth order, corresponding to the 1RSB state, and a correction depending on the continuous order-parameter function  $\mu(x)$ . It is natural to formulate a perturbation expansion around the 1RSB solution in powers of function  $\mu(x)$ . We can do it either directly by using functional derivatives and functionals of  $\mu(x)$  or by introducing an interpolation parameter  $\xi \in (0, 1)$  with which we rescale function  $\mu(x)$  and expand all quantities in  $\xi$ . We choose the first way.

The fundamental quantity to be expanded is the evolution operator

$$\begin{aligned} \mathbb{E}(h_\eta; X, 0) &= \mathbb{T}_x \exp \left\{ \int_0^X dx \left[ \frac{1}{2} \partial_{\bar{h}}^2 + \mu(x) g'_\mu(x, h_\eta + \bar{h}) \partial_{\bar{h}} \right] \right\} \Big|_{\bar{h}=0}. \end{aligned} \quad (20)$$

We have an explicit dependence of this operator on function  $\mu(x)$  and an indirect one via the scalar order parameters  $q_0, \chi_0, \mu_0, X$ , and function  $g'_\mu(x, h_\eta)$ . To determine the complete dependence of the evolution operator on  $\mu(x)$ , we must first evaluate the derivatives with respect to these parameters.

Dependence on the upper bound  $X$  is specific and leads to

$$\frac{\partial \mathbb{E}(h; X, 0)}{\partial X} = \left[ \frac{1}{2} \partial_{\bar{h}}^2 + \mu(X) g'_\mu(X, h) \partial_{\bar{h}} \right] \mathbb{E}(h; X, 0). \quad (21)$$

$$\begin{aligned} \mathbb{E}^{(n+1)}(h; a, b) &= \mathbb{E}^{(n)}(h; a, b) + \int_b^a dx \mu(x) \left\{ \mathbb{E}^{(n)}(h; a, x) g'_\mu(x, h + \bar{h}) \partial_{\bar{h}} \mathbb{E}^{(n)}(h; x, b) + \delta_{a,x} \frac{\delta X}{\delta \mu(x)} \left( \frac{1}{2} \partial_{\bar{h}}^2 + \mu(X) g'_\mu(X, h) \partial_{\bar{h}} \right) \right. \\ &\quad \times \mathbb{E}^{(n)}(h; X, b) + \int_0^X dy \mu(y) \mathbb{E}^{(n-1)}(h; a, x) \left[ \sum_i \frac{\delta p_i}{\delta \mu(y)} \frac{\partial g'_\mu(x, h + \bar{h})}{\partial p_i} + \theta(x - y) \frac{\delta g'_\mu(x, h + \bar{h})}{\delta \mu(y)} \right] \partial_{\bar{h}} \\ &\quad \left. \times \mathbb{E}^{(n-1)}(h; x, b) \right\}^{(n)}, \end{aligned} \quad (24)$$

where  $p_i \in \{q_0, \chi_0, \mu_0\}$  and  $[F[\mu(x)]]^{(n)}$  means that only the  $n$ th power of function  $\mu(x)$  from functional  $F[\mu(x)]$  is taken into account. That is

$$\begin{aligned} [F[\mu(x)]]^{(n)} &= \int dx_1 \dots dx_n \mu(x_1) \dots \mu(x_n) \\ &\quad \times \frac{\delta^n F[\mu(x)]}{\delta \mu(x_1) \dots \delta \mu(x_n)} \Big|_{\mu(x)=0}. \end{aligned}$$

The initial evolution operator is

$$\mathbb{E}^{(0)}(h; a, b) = \exp \left[ \frac{1}{2} (a - b) \partial_{\bar{h}}^2 \right] \quad (25)$$

Partial derivatives with respect to all other scalar parameters have the same generic representation

$$\begin{aligned} \frac{\partial \mathbb{E}(h; a, b)}{\partial p} &= \int_b^a dx \mu(x) \mathbb{E}(h; a, x) \\ &\quad \times \frac{\partial g'_\mu(x, h + \bar{h})}{\partial p} \partial_{\bar{h}} \mathbb{E}(h; x, b). \end{aligned} \quad (22a)$$

From the defining equation for function  $g'_\mu$ , Eq. (7b), we obtain

$$\begin{aligned} \frac{\partial g'_\mu(x, h)}{\partial p} &= \mathbb{E}(h; x, 0) \circ \left[ \frac{\partial g'_\mu(h)}{\partial p} \right] + \int_0^x dy \mu(y) \mathbb{E}(h; x, y) \\ &\quad \circ \left[ g''_\mu(y, h) \frac{\partial g'_\mu(y, h)}{\partial p} \right]. \end{aligned} \quad (22b)$$

We do the same with the functional derivative with respect to  $\mu(x)$ ,

$$\begin{aligned} \frac{\delta \mathbb{E}(h; a, b)}{\delta \mu(x)} &= \mathbb{E}(h; a, x) \circ [g'_\mu(x, h) \partial_h \mathbb{E}(h; x, b)] + \int_x^a dy \mu(y) \\ &\quad \times \mathbb{E}(h; a, y) \circ \left[ \frac{\delta g'_\mu(y, h)}{\delta \mu(x)} \partial_h \mathbb{E}(h; y, b) \right] \end{aligned} \quad (23a)$$

and

$$\begin{aligned} \frac{\delta g'_\mu(x, h)}{\delta \mu(y)} &= \mathbb{E}(h; x, y) \circ [g'_\mu(y, h) g''_\mu(y, h)] + \int_y^x du \mu(u) \\ &\quad \times \mathbb{E}(h; x, u) \circ \left[ \frac{\delta g'_\mu(u, h)}{\delta \mu(y)} g''_\mu(y, h) \right]. \end{aligned} \quad (23b)$$

Putting all the derivatives together we can set a basic equation for an iterative determination of the next approximation to the full evolution operator. Knowing the  $n$ th and  $(n-1)$ th orders of this operator  $\mathbb{E}^{(n)}(h; a, b)$  and  $\mathbb{E}^{(n-1)}(h; a, b)$ , the next approximation then is

and we set  $\mathbb{E}^{(-1)} = 0$  to comply with Eq. (24) for  $n = 0$ . In this way, a formal power expansion in the continuous order-parameter function is exhaustively defined.

#### V. LOWEST-ORDER APPROXIMATION

The zeroth-order approximation is the 1RSB free energy with the evolution operator approximated by Eq. (25). This operator can be transformed to a Gaussian integral. To make the expressions for the equations within this approximation

as compact as possible we introduce the following generic notation:

$$E_x[f(h)] = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} e^{-\phi^2/2} f(\beta(h + \phi\sqrt{x})), \quad (26a)$$

$$\langle \rho_\mu(h) f(h) \rangle_{\chi_0} = \frac{E_{\chi_0}[\cosh(h)^{\mu_0/\beta} f(h)]}{E_{\chi_0}[\cosh(h)^{\mu_0/\beta}]}. \quad (26b)$$

The zeroth-order approximation leads to equations where it is convenient to replace the parameter  $q_0$  with a new variable  $Y = X + q_0$ . Replacing the exact evolution operator  $\mathbb{E}$  by the approximate one from Eq. (25) in Eq. (8), we obtain

$$Y = E_Y[\langle \rho_\mu(h) t(h) \rangle_{\chi_0}^2], \quad (27a)$$

$$\chi_0 = E_Y[\langle \rho_\mu(h) t(h)^2 \rangle_{\chi_0}] - E_Y[\langle \rho_\mu(h) t(h) \rangle_{\chi_0}^2]. \quad (27b)$$

Replication index  $\mu_0$  is determined from

$$\begin{aligned} \frac{1}{4}\beta\chi_0\mu_0(2Y + \chi_0) &= E_Y[\langle \rho_\mu(h) \ln \cosh(h) \rangle_{\chi_0}] \\ &\quad - \frac{\beta}{\mu_0} E_Y[\ln E_{\chi_0}[\cosh(h)^{\mu_0/\beta}]]. \end{aligned} \quad (27c)$$

The equation for parameter  $X$  follows from Eqs. (8a) and (27):

$$X = E_Y[\langle \rho_\mu(h) t(h) \rangle_{\chi_0}^2] - E_{(Y-X)}[E_X[\langle \rho_\mu(h) t(h) \rangle_{\chi_0}^2]]. \quad (28)$$

Finally, the continuous order-parameter function from Eq. (13) in the lowest-order approximation is

$$\mu(x) = \frac{E_{(Y-x)}[E_x[g_\mu'''(h)]^2]}{2E_{(Y-x)}[E_x[g_\mu''(h)]^3]}. \quad (29)$$

Equations (27) determine the order parameters  $\chi_0, \mu_0, Y$  of a 1RSB state. It can be seen from a generating free energy

$$\begin{aligned} f_0(\chi_0, \mu_0, Y) &= -\frac{\beta}{4}(1 - \chi_0 - Y)^2 + \frac{1}{4}\mu_0\chi_0(2Y + \chi_0) \\ &\quad - \frac{1}{\beta} \ln 2 - \frac{1}{\mu_0} E_Y[\ln E_{\chi_0}[\cosh(h)^{\mu_0/\beta}]] \end{aligned} \quad (30)$$

to which these equations define stationarity points. Free energy  $f_0$  is just the free energy from Eq. (3) with the evolution operator from Eq. (25).

The parameter  $X$ , which does not appear in the free energy, Eq. (30), is the first iteration for the length of the interval on which the continuous order-parameter function lives. If  $X > 0$ , then a solution with a continuous RSB exists. Note that Eq. (28) has two solutions:  $X = 0$  and  $X = Y$  for  $h = 0$ . We can always take the latter one as the starting point for the perturbation expansion in which the Sherrington-Kirkpatrick (SK) solution

is then completely circumvented ( $q_{SK} = 0$ ). It means that we always can construct a solution with continuous replica-symmetry breaking for  $h = 0$  independently of whether an equilibrium state with finite-many replica hierarchies is locally stable or not. In an applied magnetic field, a nonzero parameter  $X$  generally exists if an instability condition is satisfied:

$$E_Y[g_\mu''(h)^2] > 1 \quad (31)$$

with  $g_\mu''(h)$  from Eq. (15b) and parameters  $Y, \chi_0$ , and  $\mu_0$  being solutions of Eqs. (27). It is sufficient if condition (31) is satisfied for any of the solutions of Eq. (27), that is, either the paramagnetic, replica-symmetric, or one-level replica-symmetry-breaking solution.

Parameters  $Y, \chi_0, \mu_0$  determined from Eq. (27) define a stationarity point of the free energy from Eq. (30) with one level of replica symmetry breaking. Parameter  $X$  from Eq. (28) and function  $\mu(x)$  from Eq. (29) go beyond the 1RSB solution and represent the leading-order correction to the 1RSB approximation towards a solution with continuous RSB. Parameter  $X$  determines an interval  $[0, X]$  on which the continuous order-parameter function  $\mu(x)$  is defined. Both quantities are determined within the 1RSB approximation.

We must go to the next iteration  $\mathbb{E}^{(1)}$  of the evolution operator to obtain corrections to the 1RSB results. We find from Eq. (24),

$$\begin{aligned} &\langle \mathbb{E}^{(1)}(h_\eta; X, 0) \circ f(h_\eta) \rangle_\eta \\ &= E_Y[f(h)] + \int_0^X dx \mu(x) \left\{ \frac{1}{2} \frac{\delta X}{\delta \mu(x)} E_Y[f''(h)] \right. \\ &\quad \left. + E_{(Y-x)}[E_x[g_\mu'(h)][E_x[f'(h)]] \right\}. \end{aligned} \quad (32)$$

To make the representation in Eq. (32) explicit, we need to determine the functional derivative  $\delta X / \delta \mu(x)$ . To this purpose we add a first-order correction to the 1RSB free energy, which is

$$\Delta f_1 = \frac{1}{2} \int_0^X dx \mu(x) \{ Y - x - E_{(Y-x)}[E_x[g_\mu'(h)]^2] \}. \quad (33)$$

If we now use free energy  $f_1 = f_0 + \Delta f_1$  as a generating functional for parameters  $\chi_0, \mu_0, Y, X$  with  $\mu(x)$  as an external source, the stationarity equations of this free energy define the scalar order parameters with their first correction due to the continuous order-parameter function  $\mu(x)$  obtained from Eq. (29) evaluated within the 1RSB approximation.

If we want to evaluate the first correction to function  $\mu(x)$  from Eq. (29), we have to go further and to calculate the free energy to the second order in  $\mu(x)$ . The second correction to the 1RSB free energy reads

$$\begin{aligned} \Delta f_2 &= - \int_0^X dx \int_0^x dy \mu(x) \mu(y) E_{(Y-x)}[E_x[g_\mu'(h)] \\ &\quad \times E_{(x-y)}[E_y[g_\mu'(h)] E_y[g_\mu''(h)]]]. \end{aligned} \quad (34)$$

If we now add both corrections to the 1RSB free energy, we obtain a new free energy, exact to second order in  $\mu(x)$ , that

has the following representation:

$$\begin{aligned}
f_2(\chi_0, \mu_0, X, Y; \mu(x)) &= f_0 + \Delta f_1 + \Delta f_2 = -\frac{\beta}{4}(1 - \chi_0 - Y)^2 + \frac{1}{4}\mu_0\chi_0(2Y + \chi_0) - \frac{1}{\mu_0}E_Y[\ln E_{\chi_0}[(2 \cosh(h))^{\mu_0/\beta}]] \\
&+ \frac{1}{2}\int_0^X dx \mu(x)\{Y - x - E_{(Y-x)}[E_x[g'_\mu(h)]^2]\} - \int_0^X dx \mu(x)\int_0^x dy \mu(y)E_{(Y-x)}[E_x[g'_\mu(h)]E_{(x-y)} \\
&\times [E_y[g'_\mu(h)]E_y[g''_\mu(h)]]]. \tag{35}
\end{aligned}$$

To derive the correction to  $\mu(x)$  from Eq. (29), we proceed in the same way as in the exact case. Setting zero the first variation of free energy  $f_2$  with respect to  $\mu(x)$  leads to

$$\begin{aligned}
\frac{1}{2}\{Y - x - E_{(Y-x)}[E_x[g'_\mu(h)]^2]\} &= \int_0^x dy \mu(y)E_{(Y-x)}[E_x[g'_\mu(h)]E_{(x-y)}[E_y[g'_\mu(h)]E_y[g''_\mu(h)]] \\
&+ \int_x^X dy \mu(y)E_{(Y-y)}[E_y[g'_\mu(h)]E_{(y-x)}[E_x[g'_\mu(h)]E_x[g''_\mu(h)]]]. \tag{36}
\end{aligned}$$

Its derivative with respect to  $x$  results in

$$\begin{aligned}
1 &= E_{(Y-x)}[E_x[g''_\mu(h)]^2] + 2\int_0^x dy \mu(y)E_{(Y-x)}[E_x[g''_\mu(h)]E_{(x-y)}[E_y[g''_\mu(h)]^2 + E_y[g'_\mu(h)]E_y[g'''_\mu(h)]] \\
&+ 2\int_x^X dy \mu(y)E_{(Y-y)}[E_y[g'_\mu(h)]E_{(y-x)}[E_x[g''_\mu(h)]E_x[g'''_\mu(h)]]], \tag{37}
\end{aligned}$$

and the second derivative leads to an equation from which we obtain the desired correction to the order-parameter function:

$$\begin{aligned}
2\mu(x)E_{(Y-x)}[E_x[g''_\mu(h)]^3] &= E_{(Y-x)}[E_x[g'''_\mu(h)]^2] + 2\int_0^x dy \mu(y)E_{(Y-x)}[E_x[g'''_\mu(h)]E_{(x-y)}[3E_y[g''_\mu(h)]E_y[g'''_\mu(h)]] \\
&+ E_y[g'_\mu(h)]E_y[g_\mu^{(iv)}(h)]] + 2\int_x^X dy \mu(y)E_{(Y-y)}[E_y[g'_\mu(h)]E_{(y-x)}[E_x[g'''_\mu(h)]E_x[g_\mu^{(iv)}(h)]]]. \tag{38}
\end{aligned}$$

Knowing the corrections to all the order parameters, free energy of the 1RSB solution, and to the evolution operator, we can evaluate corrections also to other physical quantities, in particular, magnetic susceptibility  $\chi_T$  and entropy  $S(T)$  by applying our approximation to the exact formulas, Eqs. (17) and (19).

Approximating all physical quantities by expanding the evolution operator in the power series from Eq. (24) in the exact equations for their equilibrium values makes the approximate theory only approximately thermodynamically consistent. That is, the entropy calculated from Eq. (19) with an approximate evolution operator  $\mathbb{E}^{(1)}$  does not coincide with the entropy calculated from the temperature derivative of free energy  $f_1 = f_0 + \Delta f_1$ , since function  $\mu(x)$  is treated in the latter as an external source that depends via Eq. (29) on temperature and, consequently, the two definitions do not coincide. It means that thermodynamic consistency is obeyed by approximate quantities only to one order lower than that chosen in the evolution operator. This deficiency can be removed if we had a free energy being stationary also with respect to infinitesimal fluctuations of function  $\mu(x)$ . Such a free energy is  $f_2$  from Eq. (35). It can be treated as a generating functional for all its variables, including  $\mu(x)$ . The continuous order-parameter function can no longer be treated as a perturbation but is rather determined self-consistently with other scalar order parameters from Eq. (36). Such an approximation would be fully thermodynamically consistent and exact up to the second order in  $\mu(x)$  for all physical quantities.

## VI. RESULTS

The starting point of the presented perturbation expansion, the zeroth-order approximation, is a solution with one level of replica-symmetry breaking (1RSB), the free energy of which with its order parameters is given in Eq. (30). The replica symmetric solution is part of this approximation if we put  $\chi_0 = 0$  and neglect the stationarity equation for this parameter. Alternatively, due to degeneracy of the solution,<sup>5</sup> we can choose  $\mu_0 = \beta$ . Then, free energy  $f_0$  from Eq. (30) becomes independent of susceptibility  $\chi_0$ , which also leads to the replica-symmetric solution. The two ways to reproduce the replica-symmetric solution correspond to different identifications of the Sherrington-Kirkpatrick parameter  $q_{SK}$  with either  $\chi_0$  or  $Y$  in our notation. In the former case, we have  $\chi_0 = 0$  and  $Y = q_{SK} > 0$ , while in the latter  $Y = 0, \mu_0 = 0$ , and  $\chi_0 = q_{SK} > 0$  for  $T < T_c = 1$ . The full 1RSB solution is obtained so that parameters  $\chi_0$  and  $Y$  are calculated self-consistently for each value of  $\mu_0$ . The latter parameter is then determined from the local maximum of free energy  $f_0$ . Without magnetic field we have another degeneracy in Eq. (28) allowing us to choose  $X = Y$ . We use the values of parameters  $Y, \chi_0$ , and  $\mu_0$  from Eq. (27) to determine the continuous order-parameter function  $\mu(x)$  from Eq. (29). In this way, we completed the starting approximation to which we can evaluate corrections in powers of  $\mu(x)$ .

We calculated explicitly only linear corrections in function  $\mu(x)$  to the 1RSB results. In the perturbation expansion around

the 1RSB solution for  $h = 0$ , we obtain  $X = Y$  in all orders. Corrections to the parameters from the 1RSB solution are calculated from stationarity equations for free energy  $f_1 = f_0 + \Delta f_1$  from Eq. (33) with respect to infinitesimal variations of  $Y, \chi_0$ , and  $\mu_0$  and with an external source  $\mu(x)$  determined from Eq. (29) with its parameters fixed at the 1RSB values. Or, equivalently, one can use the evolution operator  $\mathbb{E}^{(1)}$  from Eq. (32) in the exact equations for the order parameters from the exact free energy. Although all the formulas can be evaluated for arbitrary temperature, it is numerically more demanding to reach temperatures  $T < 0.2T_c$ , since computations beyond double precision are required. The precise low-temperature asymptotics  $T \rightarrow 0$  can more effectively be determined by starting with  $T = 0$  and using a low-temperature expansion. Such an approach will be presented in a separate publication.

For comparison, we also applied the expansion in the continuous order-parameter to the replica-symmetric solution. Due to the degeneracy in the identification of the SK solution, we must choose the one that is more unstable. It appears that the SK solution with  $\chi_0 = q_{SK}$  leads to  $Y = X = 0$  in all orders and we generate no corrections to the replica-symmetric solution. The other limit to the SK solution with  $\chi_0 = 0$  leads to  $X = Y > 0, q_0 = 0$  in all orders of the perturbation expansion. It means that an expansion around the SK solution coincides with an expansion around the paramagnetic solution with no SK order parameter.

Simultaneously with the perturbation expansion, we used Monte Carlo simulations on fully connected graphs with up to  $N = 512$  spins with different numbers of Monte Carlo sweeps. Usually we used  $4096 \times 256$  equilibrating steps followed by around one million Monte Carlo steps from which we registered the data for averaging after each 16–256 steps for each configuration of the exchange couplings out of  $6 \times 1024$  random selections.<sup>21</sup>

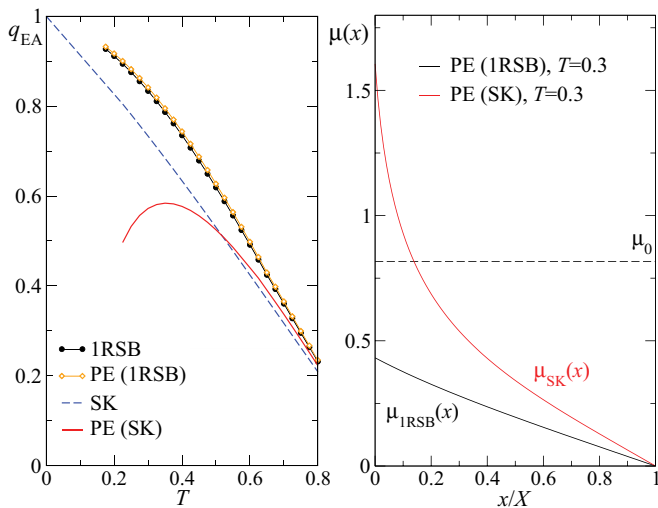


FIG. 1. (Color online) Edwards-Anderson order parameter  $q_{EA}$  (left panel) and the continuous order-parameter function  $\mu(x)$  (right panel) calculated using perturbation expansions (PE) around the 1RSB and replica-symmetric (SK) solutions. Parameter  $\mu_0$  of 1RSB is plotted for reference.

We plotted in Fig. 1 the Edwards-Anderson (EA) order parameter, being in our notation  $q_{EA} = \chi_0 + Y$ , calculated in the replica-symmetric approximation, 1RSB solution and from the first-order of the perturbation expansion (PE) around either of these solutions. We can see that the perturbation expansion in the first order gives almost no correction to the 1RSB approximation, while there are tangible changes due to the perturbation expansion when applied to the SK solution. Further, at lower temperatures ( $T \approx 0.4$ ), the first-order correction to parameter  $X$  that equals  $q_{SK}$  in the lowest order starts to decrease and wrongly downturns the slope of the EA parameter. The reason for this unreliable behavior of the first correction to the SK solution is the value of the “small parameter”  $\mu(x)$  used. The right panel of Fig. 1 displays  $\mu(x)$  at  $T = 0.3$  calculated in the 1RSB and SK solutions. The function from the SK solution is significantly higher, making thus the perturbation expansion around the SK (paramagnetic) solution much less reliable than the one around the 1RSB solution. It is understandable, since the role of the continuous order-parameter function  $\mu(x)$  is partly overtaken by the scalar value  $\mu_0$  in the 1RSB approximation. Unlike function  $\mu(x)$ , parameter  $\mu_0$  is treated nonperturbatively.

To check reliability of the perturbation expansion we compared the Edwards-Anderson order parameter in Fig. 2, left panel, with the one obtained from Monte Carlo simulations and the high-order perturbation expansion (HPE) of Crisanti and Rizzo.<sup>9</sup> We can see that there is not a big difference between the perturbation expansion around the 1RSB solution and Monte Carlo simulations and the expansion of Crisanti and Rizzo for temperatures  $T > 0.3$ . As one expects, a better precision in lower temperatures demands inclusion of higher orders of the expansion. Out of curiosity, we compared (right panel) the Edwards-Anderson parameter from Monte Carlo

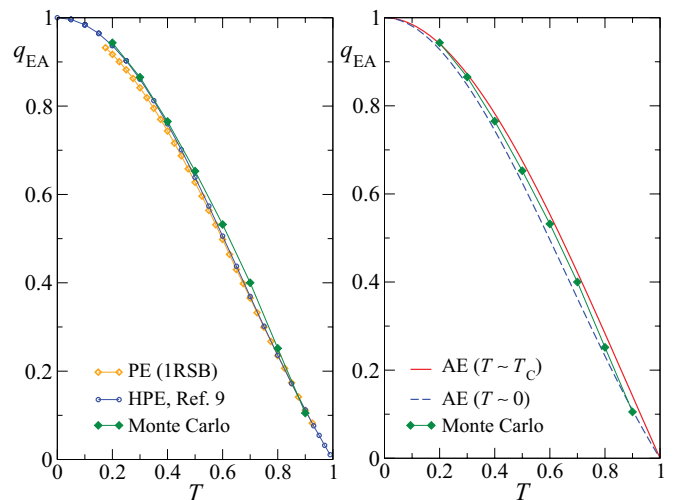


FIG. 2. (Color online) Temperature dependence of the Edwards-Anderson order parameter resulting from different approximations. (Left) Edwards-Anderson parameter from the 1RSB solution with first-order correction [PE (1RSB)], compared with the solution of Ref. 9 (HPE) and Monte Carlo simulations. (Right) Monte Carlo result compared with extrapolations of asymptotic expansions around the transition temperature  $T_c = 1$  (solid line) and zero temperature (dashed line).



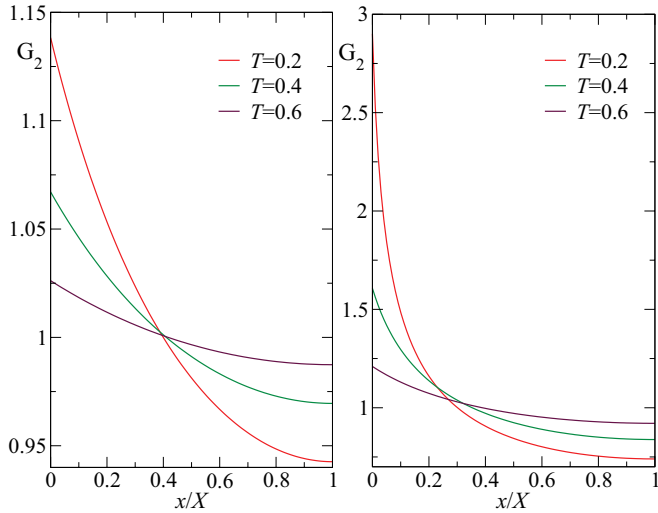


FIG. 3. (Color online) Stability condition, right-hand side of Eq. (12), denoted by  $G_2$ , for the expansion around 1RSB (left) and around SK (right). The curves from top to bottom at  $x = 0$  correspond to temperatures  $T = 0.2, 0.4$ , and  $0.6$ , respectively.

simulations with low orders of two asymptotic expansions, one, high-temperature, around the transition temperature and the other, low-temperature, around zero temperature. The former expansion<sup>6,22</sup> is  $q_{EA} \doteq 1 - 2T^2 + T^3$ , while the latter to the same order is<sup>9</sup>  $q_{EA} \doteq 1 - 1.6T^2 + 0.6T^3$ . We see that the Monte Carlo data are surprisingly well reproduced by the expansions in the respective regions and do not differ much from them in the whole glassy phase. It is worth noting that the expansions are only asymptotic and higher terms destroy this surprising reconstruction of the Monte Carlo Edwards-Anderson parameter.

An important property of the full Parisi solution is the condition of (marginal) stability, Eq. (12), that may be used as one of criteria of reliability and consistency of approximations. We plotted in Fig. 3 the right-hand side of Eq. (12), denoted by  $G_2$ , for the first-order expansion around the 1RSB solution (left panel) and the SK solution (right panel) and different temperatures. Both approximate solutions worsen their reliability with lowering the temperature but the one around the SK solution shows much larger deviations from the stability value  $G_2 = 1$ . The expansion around the 1RSB solution, on the other hand, does not violate the marginal stability significantly for temperatures  $T > 0.2$  and gives a good estimate of the behavior of the full solution with continuous replica-symmetry breaking. At very low temperatures  $T < 0.2$ , the deviation from marginal stability will increase within the lowest-order approximation. We expect, however, that a self-consistent theory determined by free energy  $f_2$  from Eq. (35), where the continuous order-parameter function is treated variationally, will improve significantly towards marginal stability.

Failure of the SK solution to produce physical results at low temperatures can be demonstrated on entropy. In Fig. 4, we plotted entropy (left panel) and free energy (right panel) calculated in the SK, 1RSB, and the first-order expansion around 1RSB. We can see that even though there is not a big difference in free energy between the 1RSB and the expansion around it, an improvement in the entropy at low temperatures is

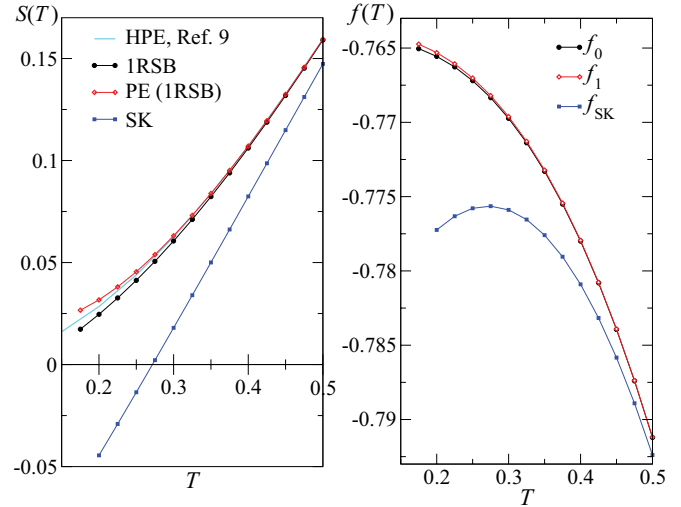


FIG. 4. (Color online) Temperature dependence of entropy (left) calculated in 1RSB, with first correction to it, and the replica-symmetric solution. We also used the data for entropy from Ref. 9. (Right) Free energy of the SK solution ( $f_{SK}$ ), 1RSB ( $f_0$ ), and the first correction to 1RSB ( $f_1$ ).

tangible and goes beyond the high-order expansion of Ref. 9. We know that the solution with 1RSB leads to a negative value of entropy  $S(T) \sim -0.01$  at  $T = 0$ , hence the exact curve must lie above it. Improvement of the perturbation expansion upon the 1RSB solution at low temperatures can also be demonstrated on magnetic susceptibility  $\chi_T$  from Eq. (17) plotted in Fig. 5, left panel. It is expected to stay fixed at the value  $\chi_T = 1$  in the whole low-temperature phase.<sup>20</sup> We also plotted temperature dependence of the internal energy  $u = f - TS$  and compared the result with Monte Carlo simulations (right panel).

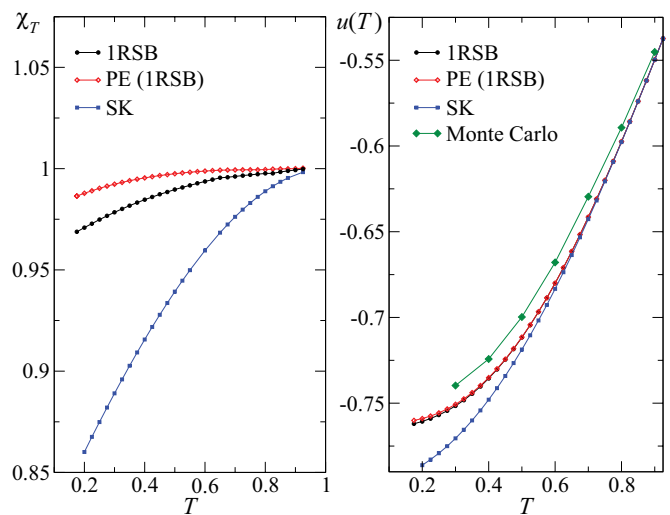


FIG. 5. (Color online) Temperature dependence of magnetic susceptibility  $\chi_T$  (left panel) and internal energy  $u$  (right panel) calculated in 1RSB scheme, with first correction to it and the replica symmetric solution. Monte Carlo result for the internal energy is shown for comparison.

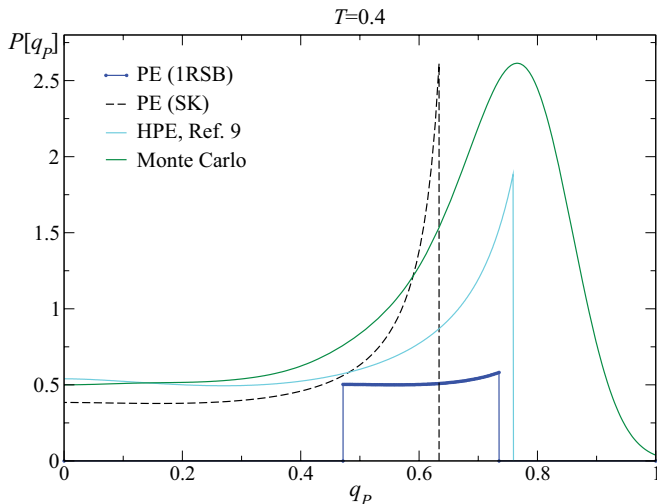


FIG. 6. (Color online) Derivative of the Parisi order parameter  $dx_P/dq_P = P(q_P)$  corresponding in our scheme to  $-\beta^{-1}\dot{\mu}(x)$  is plotted as a function of  $q_P$  in the Parisi notation for  $T = 0.4$ . We also added curves from the Monte Carlo calculation and from Ref. 9.

The continuous order-parameter function is not measurable and one cannot assess the quality of approximations on this function. According to the interpretation of the derivative  $dx_P/dq_P$  in the Parisi notation<sup>19</sup> [in our approach,  $-\beta^{-1}d\mu(x)/dx$ ], we can compare this function with the probability distribution of overlap magnetizations  $P(q) = N^{-1} \sum_{s,s'} P(s)P(s')\delta(q^{ss'} - q)$  accessible via Monte Carlo simulations.<sup>22</sup> We plotted this function at  $T = 0.4$  in Fig. 6. We added also the result from Ref. 9 and the perturbation expansion around the SK solution.

Each of the approximations shows a maximum at  $q_P = q_{EA}$ . Except for the perturbation expansion around the SK solution, all approximations show the maximum for almost the same Edwards-Anderson parameter. The function calculated from the expansion around 1RSB differs from the others in two aspects. First, it leads to a continuous function only between  $\chi_0 > 0$  and  $q_{EA}$ . Second, the slope of  $P(q_P)$  with which its maximum is reached at the upper end,  $q_{EA}$ , is more shallow. Function  $P(q_P)$  in the expansion around 1RSB is not well defined at the end points, it has different limiting values from left and right. The lower end of the defining interval for  $P(q_P)$  is at  $q_P = \chi_0 \approx 0.47$ . There are indications that higher orders of the expansion around the 1RSB solution push this initial value  $\chi_0 \rightarrow 0$  at all temperatures. Parameter  $\chi_0$  in the 1RSB solution decreases also with increasing temperature and tends to zero when  $T \rightarrow T_c = 1$  with a constant function  $P(q_P) \rightarrow 2$ , the exact result of the asymptotic expansion around the critical temperature.<sup>6</sup> A nonzero value of  $\chi_0$  is, however, indispensable in approximate treatments if we want to reach the limit to zero temperature.

A more important feature, rather than the position of the lower bound of the definition domain of  $P(q_P)$ , is the height of its step there. The expansion around 1RSB leads to  $P(\chi_0) \approx 0.502$  and this result is not much affected by higher orders of the expansion. It reproduces well the value of Monte Carlo simulations with  $P(0) \approx 0.50$ . The results from

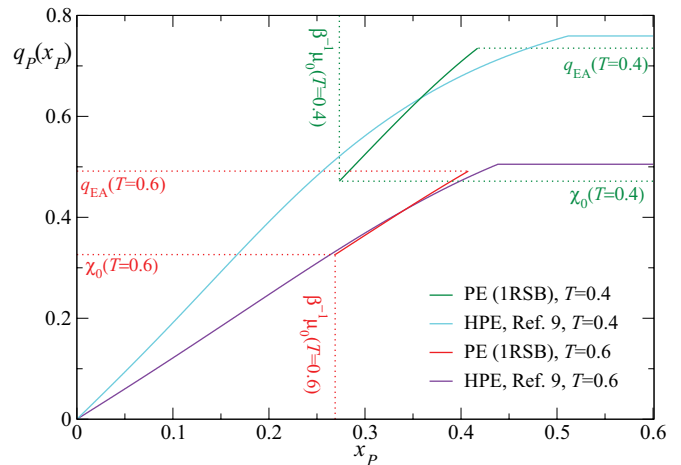


FIG. 7. (Color online) Parisi order parameters:  $q_P(x_P)$  as obtained in Ref. 9 at two different temperatures,  $T = 0.4$  (upper curves) and  $T = 0.6$  (lower curves), compared with the first-order correction to 1RSB of the presented perturbation expansion. Parameters  $\chi_0$ ,  $\beta^{-1}\mu_0$ , and  $q_{EA}$  are marked by dotted lines.

the expansions around SK and from Ref. 9, being  $P(0) \approx 0.38$  and  $0.54$ , respectively, do not do that well. The expansion from Ref. 9 overshoots the Monte Carlo initial value and  $P(q_P)$  then decreases for small values of  $q_P$ , displays a shallow minimum at  $q_m \approx 0.28$  with  $P(q_m) \approx 0.49$  before it starts to steeply grow up to reach the end point. Such a behavior is observed neither in the Monte Carlo data nor in the present perturbation expansion.

The differences in the behavior of distribution function  $P(q_P)$  between the perturbation expansions from Ref. 9 and around 1RSB are more transparent in Fig. 7 where we plotted function  $q_P(x_P)$  for two different temperatures,  $T = 0.6$  and  $0.4$ . Function  $q_P(x_P)$  in the latter expansion remains zero up to  $\beta^{-1}\mu_0$  at which it jumps to  $q_P = \chi_0$ . It means that the expansion around 1RSB does not allow for metastable states with averaged squared magnetization smaller than  $\chi_0$ . Both functions almost linearly increase up to saturation, where  $q_P = q_{EA}$ . We can see that although there is not a remarkable difference in the estimate of the Edwards-Anderson parameter, the difference between the two approaches in the value of  $x_P$  at which  $q_P = q_{EA}$  increases with lowering the temperature.

## VII. CONCLUSIONS

Complexity and low accessibility of reliable mean-field approximations in spin-glass models lies in the continuous order-parameter function one has to introduce to reach a thermodynamically consistent solution for all temperatures. This function enters the Parisi nonlinear differential equation (4) determining the equilibrium interacting free energy. This equation is unsolvable and one must resort to approximations. There is, however, no apparent way to systematically iterate the full solution. The usual way is to use the replica trick and approximations with finite numbers of replica hierarchies. Although one improves in this way upon thermodynamic consistency, one does not learn about the actual ordering in the equilibrium state and the phase space of the order parameters.

It is hence important to have an approximate scheme addressing directly the continuous order-parameter function in the whole glassy phase, including zero temperature, where the thermodynamic inconsistency of discrete approaches is most pronounced.

Here we proposed a construction of a free energy containing the order parameters of a solution with one hierarchy of replica-symmetry breaking together with a continuous order-parameter function. The continuous function enters the Parisi nonlinear differential equation determining the interacting part of the free energy functional. We showed that to reach explicit formulas in the limit to zero temperature ( $\beta \rightarrow \infty$ ) it is necessary to use the 1RSB free energy as the starting point for a perturbation expansion in powers of the continuous order-parameter function. The 1RSB solution not only allows for the explicit limit to zero temperature but also justifies reliability of low-order approximations to rather low temperatures, which is not the case if we apply the same expansion to the replica-symmetric solution, being equivalent to an expansion around the paramagnetic state.

We derived explicit equations for all the variational parameters including the continuous variational function of the generating free-energy functional. We also represented relevant physical quantities without the necessity to refer to replicas and the replica trick. We showed how to construct iteratively systematic approximations to all quantities by expanding them in powers of the continuous-order parameter function beyond the 1RSB representations. We presented a closed form of an approximation containing first corrections in the expansion parameter to all quantities of interest. Comparison with Monte Carlo simulations and the high-order asymptotic expansion of Ref. 9 for the Sherrington-Kirkpatrick model at zero magnetic field proves that even the lowest approximation produces

reliable results and offers a qualitative picture of the behavior of the continuous order-parameter function in the whole glassy phase.

There are two promising directions of the application of the construction developed here. First, one can investigate the zero-temperature properties and the behavior of physical quantities in the asymptotic limit  $T \rightarrow 0$  such as entropy and magnetic susceptibility. Zero-temperature limit demands, however, a different numerical approach to solve the stationarity equations and will be investigated in a separate publication. Second, one can use this approximate scheme in generalized spin-glass models where one expects first-order phase transitions with regions where a 1RSB solution is locally stable and coexists with another one with continuous replica-symmetry breaking. There are presently no approaches allowing to study quantitatively such situations.

Last but not least, we found an explicit free energy with a continuous order-parameter function that is fully thermodynamically consistent and exact to first two orders of this function. It may replace the Parisi solution for which there is no explicit or closed-form representation. Free energy (35) is stationary in the equilibrium state with respect to all its variational parameters, including the order-parameter function  $\mu(x)$ . Unlike representation (3) generating the Parisi solution, the free energy  $f_2$  from Eq. (35) determines  $\mu(x)$  from a solvable linear integral equation, see Eq. (36). Such a free energy can be understood as a solvable Landau functional for the continuous order-parameter function and may shed more light on the behavior of solutions with continuous replica-symmetry breaking. It can serve as a viable improvement upon the 1RSB approximation interpolating between the states with discrete and continuous replica-symmetry breaking in mean-field spin-glass models.

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