Quantum entanglement and topological order in hole-doped valence-bond solid states

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We present a detailed analysis of topological properties of the valence-bond solid (VBS) states doped with fermionic holes. As concrete examples, we consider the supersymmetric extension of the SU(2) and the SO(5) VBS states, dubbed UOSp(1|2) and UOSp(1|4) supersymmetric VBS states, respectively. Specifically, we investigate the string-order parameters and the entanglement spectra of these states to find that, even when the parent states (bosonic VBS states) do not support the string order, they recover it when holes are doped and the fermionic sector appears in the entanglement spectrum. These peculiar properties are discussed in light of the symmetry-protected topological order. To this end, we characterize a few typical classes of symmetry-protected topological orders in terms of supermatrix product states (SMPS). From this, we see that the topological order in the bulk manifests itself in the transformation properties of the SMPS in question and thereby affects the structure of the entanglement spectrum. Then, we explicitly relate the existence of the string order and the structure of the entanglement spectrum to explain the recovery and the stabilization of the string order in the supersymmetric systems.

I. INTRODUCTION

The valence-bond solid (VBS) states had been originally introduced by Affleck, Kennedy, Lieb, and Tasaki[1] to build explicit model ground states which realize the properties of the generic integer-spin antiferromagnetic spin chains conjectured by Haldane.[2] Quite unexpectedly, on top of the properties already anticipated from other analyses (e.g., quantum-disordered ground state with short-range spin correlations, gapped triplet spin excitations, etc.), these states exhibit many striking features such as the emergent boundary excitations (edge states)[3] and the existence of hidden string order.[4,5] In the case of spin-1 systems, it has been argued[6] that the hidden topological (string) order is a consequence of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking occurring in the system after applying the nonlocal unitary transformation. The idea of nonlocal hidden order and edge states has been to some extent generalized[7–10] to other values of integer spin S although the hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is never broken[8] in the case of even S. Through these studies, it has been recognized that there are some differences[8,9] in the ground-state properties according to the parity of S. Nevertheless, by analogy with the quantum Hall systems,[11] the ground state of generic integer-spin antiferromagnetic chains, including the original VBS state and its higher-spin generalizations,[12] characterized by certain kinds of nonlocal correlations and emergent edge states, has been called "topological" in a rough sense.

Recent development in quantum-information-theoretic approaches to quantum many-body problems enables us to extract information on the bulk topological order from the entanglement properties of the *ground-state* wave function.[13–15] The topological states in one-dimensional (1D) spin systems have been reconsidered[16,17] from the modern point of view and the precise meaning of the topological Haldane phase has been clarified. In these studies, the string-order parameters and the edge states, which in general are not robust against small perturbations, are replaced by more robust objects (i.e., the structure of the entanglement spectrum or the structure of tensor network). In particular, it has been shown in Ref. 17

that the existence of (at least one of) the discrete symmetries (time reversal, link inversion, and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry) divides all states of matter in 1D into two categories: topologically nontrivial ones and the rest. Generic odd-integer-S spin chains belong to the former, while even-S chains to the latter. The hallmark of the topological phase protected by the above discrete symmetries is that all entanglement levels are evenfold degenerate. In this formulation, the difference between odd S and even S is naturally understood in terms of the entanglement structure. It should also be mentioned that the topological phases of one-dimensional gapped spin systems have been classified by group cohomology,[18,19] and the detailed analyses based on the Lie-group symmetries are reported in Ref. 20.

In this paper, we present an exhaustive discussion about the effects of coexisting bosonic and fermionic degrees of freedom on (symmetry-protected) topological phases in 1D. Clearly, this kind of question is motivated in part by hole doping in the Haldane-gap systems.[21–23] In order to incorporate the coexisting bosons and fermions, for mathematical convenience, we use supersymmetry (SUSY), which relates bosons carrying integer spins and fermions with half-odd-integer spins. Several "topological phases" with SUSY have been found so far in, e.g.,. quantum Hall systems,[24] VBS states,[25] and ultracold-atom systems.[26] However, the precise characterization of these SUSY topological phases has not been obtained so far and it would be quite useful to investigate symmetry-protected topological order in model SUSY systems from the entanglement point of view.

As the model SUSY states, we consider a class of supersymmetric VBS (SVBS) states defined by the Schwinger operator consisting of $2K$ bosons which represent the bosonic degrees of freedom at each site (e.g., localized integer spins) and N fermions which correspond to doped fermionic holes (with K and N being integers). This class is interesting since it includes the SVBS states investigated in Refs. 25 and 27 as well as the SUSY extension of the SO(5) VBS state and the Sp(N) VBS state introduced respectively in Refs. 28 and 29. The (S)VBS states are rare examples where we can study nontrivial topological properties even in 1D, and most of the calculations

can be done without relying on any approximation. Taking advantage of such properties of the SVBS states, we uncover the roles of SUSY in topological phases in 1D.

The generalized hidden string order in the SVBS states[25] has been investigated already in the previous work[27] by the authors. In contrast to what is known for the bosonic counterpart (the spin-S VBS state[12]), the symptom of the nontrivial topological order has been observed in the analysis of the string order even for the even-integer superspin. To be more precise, even when the string order vanishes, it revives upon the hole doping; this might suggest the existence of topological order in the SVBS states *regardless* of the parity of bulk superspin $\mathcal{S}$. In order for the better understanding of this phenomenon, we first characterize symmetry-protected topological orders in SUSY systems in the language of entanglement. To this end, we use the supermatrix product state (SMPS) formalism to generalize the arguments of Ref. 17 and derive the relation between topological order in the bulk and the entanglement structure. The SMPS formalism further enables us to obtain the explicit relation between the entanglement spectrum and the string-order parameters, and thereby to clarify why the hidden string order revives after doping.

As has been emphasized in the previous work,[27] in spite of its name, the SMPS formalism does not assume any particular form of SUSY. In fact, we do not need even postulate *exact* SUSY and the only prerequisite is that the local Hilbert space is made up of the bosonic part and the fermionic one. In view of the ability of (S)MPS in approximating any gapped states in 1D with arbitrary precision,[30,31] our results are applicable to a wider class of 1D systems with some kind of relation between bosons and fermions.

The organization of this paper is as follows. In Sec. II, we introduce a class of UOSp($N|2K$)-invariant SVBS states ($2K$ being the number of boson species and N for fermions) with arbitrary superspins using the Schwinger operator. We then construct the explicit SMPS representation for $(N,K) = (1,1)$ [UOSp(1|2)] and (1,2) [UOSp(1|4)] and summarize several important properties of these states. As the first step toward the investigation of topological order, we explicitly evaluate the string-order parameters in the above two types of SVBS states for different values of superspins in Sec. III. There we find that the revival of the string order already observed for UOSp(1|2) in Ref. 27 occurs in other SUSY cases as well. In Sec. IV, the entanglement spectrum of these SVBS states (in the limit of infinite-size systems) is derived and typical features of the spectrum are discussed. In order to understand the results obtained in the previous section and characterize symmetry-protected topological order in 1D SUSY systems, we generalize the argument of Ref. 17 to SUSY systems in Sec. V and relate the structure of the entanglement spectrum and the bulk topological order. Finally, the relationship between the degeneracy of the entanglement spectrum and nonvanishing string-order parameters is clarified in Sec. VI by using the (S)MPS formalism. Section VII is devoted to summary and discussions.

II. SVBS STATES AND SMPS FORMALISM

In this section, we briefly describe how the standard MPS formalism is generalized to the cases with SUSY. Let us begin with constructing the MPS of the spin-M (M: integer) SU(2) valence-bond solid (VBS) state[12] starting from its representation in terms of the SU(2) Schwinger operators $\phi = (b^{1\dagger}, b^{2\dagger})^{\mathrm{t}}$:

$$|\text{VBS}\rangle^{(M)} = \prod_j \left(b_j^{1\dagger} b_{j+1}^{2}{}^{\dagger} - b_j^{2\dagger} b_{j+1}^{1}{}^{\dagger} \right)^M |\text{vac}\rangle$$

$$= \prod_j \left(\phi_j^{\mathrm{t}} \, i\sigma_2 \phi_{j+1} \right)^M |\text{vac}\rangle, \qquad (1)$$

where the metric [or, the SU(2) charge conjugation matrix]

$$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (2)$$

has been used to form a maximally entangled (singlet) pair between the sites j and $j+1$. Therefore, by construction, the VBS state is SU(2) invariant and represents a spin-isotropic state.

A. General idea

The standard construction of the VBS-type of states[32] starts by preparing two auxiliary degrees of freedom on each site of the lattice. Then, the (bosonic) VBS state is constructed first by creating singlets between pairs of those auxiliary objects on adjacent sites and then by projecting the tensor product of the two auxiliary objects on each site onto the desired physical Hilbert space.

The SVBS states are introduced by including the states with one or more fermionic holes into the above bosonic Hilbert space. Mathematically, we replace the usual Lie-group symmetry [e.g., SU(2)] with that of the super Lie group UOSp($N|2K$) corresponding to $2K$ bosonic degrees of freedom and N fermionic ones [for a review of super Lie groups, see, for instance, Ref. 33, and for UOSp($N|2K$), Ref. 34]. Specifically, the SVBS states with UOSp($N|2K$) symmetry are defined as

$$|\text{SVBS}(N|2K)\rangle^{(M)} = \prod_{\langle i,j \rangle} (\psi_i^{\mathrm{t}} \mathcal{R}_{N|2K} \psi_j)^M |\text{vac}\rangle, \qquad (3)$$

where ψ stands for the UOSp($N|2K$) Schwinger operator

$$\psi = (b^{1\dagger}, b^{2\dagger}, \ldots, b^{2K\dagger}, f^{1\dagger}, \ldots, f^{N\dagger})^{\mathrm{t}}. \qquad (4)$$

The $2K$ bosons $b^{\sigma\dagger}$ ($\sigma = 1, 2, \ldots, 2K$) and the N fermions f^μ ($\mu = 1, 2, \ldots, N$) satisfy the commutation relations $[b^\sigma, b^{\tau\dagger}] = \delta^{\sigma\tau}$, $\{f^\mu, f^{\nu\dagger}\} = \delta^{\mu\nu}$, $[b^\sigma, f^\mu] = [b^\sigma, f^{\mu\dagger}] = 0$. The matrix $\mathcal{R}_{N|2K}$ signifies the UOSp($N|2K$)-invariant matrix

$$\mathcal{R}_{N|2K} = \begin{pmatrix} J_{2K} & 0 \\ 0 & -1_N \end{pmatrix}, \qquad (5)$$

where the USp($2K$)-invariant $2K \times 2K$ antisymmetric matrix J_{2K} is defined using the Pauli matrix σ_2 as

$$J_{2K} = \begin{pmatrix} i\sigma_2 & & & 0 \\ & i\sigma_2 & & \\ & & \ddots & \\ 0 & & & i\sigma_2 \end{pmatrix} \qquad (6)$$

and 1_N denotes the N-dimensional identity matrix. By using the above equations, it is straightforward to show that the product of spinors $\psi_i^t \mathcal{R}_{N|2K} \psi_j$ is singlet under UOSp($N|2K$).

As the number of fermion species N corresponds to that of SUSY in the system, hereafter we call the SVBS states defined by (3) and (4) the UOSp($N|2K$) SVBS states. In this paper, we give the detailed discussions for the two $N = 1$ cases, specifically $(K, N) = (1, 1)$ and $(K, N) = (2, 1)$, in which the following isomorphisms between the orthogonal groups and the unitary symplectic groups hold: SO(3) $\simeq$ USp(2)/$\mathbb{Z}_2$ ($K = 1$), SO(5) $\simeq$ UOSp(4)/$\mathbb{Z}_2$ ($K = 2$). For UOSp($N|2$) ($K = 1$), the metric matrix is given by

$$\mathcal{R}_{N|2} = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -1_N \end{pmatrix}, \tag{7}$$

and for UOSp($N|4$) ($K = 2$) by

$$\mathcal{R}_{N|4} = \begin{pmatrix} i\sigma_2 & 0 & 0 \\ 0 & i\sigma_2 & 0 \\ 0 & 0 & -1_N \end{pmatrix}. \tag{8}$$

The particle number at each site is related to the superspin $\mathcal{S}$ via

$$2\mathcal{S} = \sum_{\alpha=1}^{2K+N} \psi^{\alpha\dagger}\psi^\alpha = \sum_{\sigma=1}^{2K} b^{\sigma\dagger}b^\sigma + \sum_{\mu=1}^{N} f^{\mu\dagger}f^\mu = zM, \tag{9}$$

where z is the lattice-coordination number ($z = 2$ in one dimension). Throughout this paper, we reserve the symbol $\mathcal{S}$ for superspin and use S for the bosonic spin. Since $\sum_\mu f^{\mu\dagger}f^\mu$ takes either 0 or 1, the possible values of SU(2) spin, which is equal to the half of the number of bosons at each site, are

$$S = \frac{1}{2}\sum_{\sigma=1}^{2K} b^{\sigma\dagger}b^\sigma$$
$$= \frac{1}{2}zM, \frac{1}{2}zM - \frac{1}{2}, \frac{1}{2}zM - 1, \dots, \frac{1}{2}zM - \frac{1}{2}N. \tag{10}$$

(If $N \geqslant zM$, it is implied that the above sequence terminates at $S = 0$.) One may find that the inclusion of SUSY introduces, as well as the states with the spin magnitude $zM/2$ which exist already in the SU(2) case, those with spin smaller by $\frac{1}{2}$. In what follows, we consider the one-dimensional cases (i.e., $z = 2$) unless otherwise stated.

For the 1D chain ($z = 2$), the above sequence reads as

$$S = M, \quad M - \frac{1}{2}, \quad M - 1, \dots, M - \frac{1}{2}N, \tag{11}$$

and correspondingly the emergent edge spin takes the following values:

$$s = \frac{1}{2}M, \quad \frac{1}{2}M - \frac{1}{2}, \quad \frac{1}{2}M - 1, \dots, \frac{1}{2}M - \frac{1}{2}N. \tag{12}$$

(Again, if $N \geqslant M$, the above sequence is understood as to stop at $s = 0$.) The dimension of the physical Hilbert space at each site constructed in this way is given by the sum of the one of each bosonic Hilbert space with a fixed boson number ($2\mathcal{S} - n$):

$$d_{\mathcal{S}}(N|2K) = \sum_{n=0}^{N} \binom{2K + 2\mathcal{S} - n - 1}{2K - 1}. \tag{13}$$

It should be noted here that the Schwinger-operator construction presented here does not cover all the possible VBS-type states with UOSp($N|2K$) symmetry. In fact, there is an important class of VBS states[35] which is a SUSY generalization of a series of SO($2n + 1$)-invariant and USp($2K$)-invariant states considered, respectively, in Refs. 36,37 and 29. However, most of the conclusions obtained here hold for those models as well.

The UOSp($N|2K$) SVBS state (3) may be rewritten as

$$|\text{SVBS}(N|2K)\rangle^{(M)} = \prod_i \left(\psi_i^t \mathcal{R}_{N|2K}\psi_{i+1}\right)^M |\text{vac}\rangle$$
$$\equiv \prod_i \left(\Psi_i^t R_{N|2K}^{(M)}\Psi_{i+1}\right)|\text{vac}\rangle, \tag{14}$$

where Ψ_i is a graded fully symmetric representation of UOSp($N|2K$) of the order M and $R_{N|2K}^{(M)}$ is the metric for this representation.[34] Another equivalent form (a matrix-product form)[27] may be useful for practical purposes:

$$|\text{SVBS}(N|2K)\rangle^{(M)} = \mathcal{A}_1 \mathcal{A}_2, \dots, \mathcal{A}_L, \tag{15}$$

where the matrix $\mathcal{A}_i$ is defined as

$$\mathcal{A}_i \equiv R_{N|2K}^{(M)}\Psi_i\Psi_i^t|\text{vac}\rangle_i. \tag{16}$$

B. UOSp(1|2) SVBS states

Let us begin with the simplest case[25,27] $(N, K) = (1, 1)$. The graded Schwinger operator is given by

$$\psi_i = \left(b_i^{1\dagger}, b_i^{2\dagger}, f_i^\dagger\right)^t \equiv \left(a_i^\dagger, b_i^\dagger, f_i^\dagger\right)^t, \tag{17}$$

and the corresponding SVBS state, which we call the UOSp(1|2) SVBS state (precisely, this is the one dubbed *type-I* in Ref. 27), is given by

$$|\text{SVBS}(1|2)\rangle^{(M)} = \prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger - r f_i^\dagger f_{i+1}^\dagger)^M |\text{vac}\rangle, \tag{18}$$

where we have added the fermion doping parameters r by hand. However, such a parameter may be absorbed in the redefinition of the normalization of fermions ($f^\dagger \mapsto f^\dagger/\sqrt{r}$, $f \mapsto \sqrt{r}f$) and the SVBS states possess SUSY even for finite values of the parameter r.

1. $\mathcal{S} = 1$

Let us consider the superspin $\mathcal{S} = 1$ case. Since $\mathcal{S}$ is related to the number M of SUSY valence bonds through (9), the case $M = 1$ of Eq. (18) corresponds to $\mathcal{S} = 1$.

The SVBS state on a finite open chain specified its edge states α and β, respectively, on the sites 1 and L:

$$|\text{SVBS}(1|2)\rangle_{\alpha\beta}^{(1)} = (\mathcal{R}_{1|2}\psi_1)^\alpha \prod_{i=1}^{L-1} \left(\psi_i^t \mathcal{R}_{1|2}\psi_{i+1}\right)\psi_L^\beta|\text{vac}\rangle, \tag{19}$$

where $\psi_j^t = (a_j^\dagger, b_j^\dagger, \sqrt{r}f_j^\dagger)$ and the UOSp(1|2) metric $\mathcal{R}_{1|2}$ is defined in (7). The state $|\text{SVBS-I}\rangle_{\alpha\beta}^{(M=1)}$ can be expressed as a product of the matrices $\mathcal{A}_i^{(1)}$ defined on each site:

$$|\text{SVBS}(1|2)\rangle_{\alpha\beta}^{(1)} = \left(\mathcal{A}_1^{(1)}\mathcal{A}_2^{(1)}, \dots, \mathcal{A}_L^{(1)}\right)_{\alpha\beta}, \tag{20}$$

where $\mathcal{A}_j^{(1)}$ is given by

$$\mathcal{A}_j^{(1)} = \mathcal{R}_I^{(2)} \psi_j \psi_j^{\text{t}} |\text{vac}\rangle_j = \begin{pmatrix} |0\rangle_j & \sqrt{2}|-1\rangle_j & \sqrt{r}|-1/2\rangle_j \\ -\sqrt{2}|1\rangle_j & -|0\rangle_j & -\sqrt{r}|1/2\rangle_j \\ -\sqrt{r}|1/2\rangle_j & -\sqrt{r}|-1/2\rangle_j & 0 \end{pmatrix} = \sum_{a=-1,0,1} A(a)|a\rangle + \sum_{\sigma=-1/2,1/2} A(\sigma)|\sigma\rangle, \quad (21)$$

with

$$A(1) = \begin{pmatrix} 0 & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(-1) = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A(1/2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sqrt{r} \\ -\sqrt{r} & 0 & 0 \end{pmatrix}, \quad A(-1/2) = \begin{pmatrix} 0 & 0 & \sqrt{r} \\ 0 & 0 & 0 \\ 0 & -\sqrt{r} & 0 \end{pmatrix}. \tag{22}$$

The five basis states corresponding to the $\mathcal{S} = 1$ irreducible representation (denoted by **5**) are given by

$$|1\rangle = \frac{1}{\sqrt{2}} a^{\dagger 2} |\text{vac}\rangle, |0\rangle = a^\dagger b^\dagger |\text{vac}\rangle, |-1\rangle = \frac{1}{\sqrt{2}} b^{\dagger 2} |\text{vac}\rangle,$$

$$|1/2\rangle = a^\dagger f^\dagger |\text{vac}\rangle, |-1/2\rangle = b^\dagger f^\dagger |\text{vac}\rangle, \tag{23}$$

where $|\text{vac}\rangle$ is the vacuum of both the boson and the fermion: $a|\text{vac}\rangle = b|\text{vac}\rangle = f|\text{vac}\rangle = 0$. The first three states correspond to the spin-1 (**3**) representation of SU(2), and the second two states constitute **2** with spin-$\frac{1}{2}$.

The parent Hamiltonian of the state (19) is constructed[25,27] in such a way that the local Hamiltonian $h_{j,j+1}$ acting on the bond $(j, j+1)$ annihilates all the nine states appearing in the product $\mathcal{A}_j \mathcal{A}_{j+1}$. Therefore, the ground state on a finite open chain is ninefold degenerate with respect to the matrix indices. Since the ψ_j and ψ_j^{t} represent the two auxiliary degrees of freedom at the site j, the above ninefold degeneracy reflects the existence of the three edge degrees of freedom on both edges of an open chain:

$$|\uparrow\rangle\rangle = a^\dagger |\text{vac}\rangle, |\downarrow\rangle\rangle = b^\dagger |\text{vac}\rangle, |0\rangle\rangle = f^\dagger |\text{vac}\rangle. \tag{24}$$

As the *doping parameter* r is changed, the state (19) interpolates between the two well-known states: at $r \to 0$, $|\text{SVBS}(1|2)\rangle^{(1)}$ is reduced to the original VBS state[1] $|\text{VBS}\rangle$:

$$|\text{SVBS}(1|2)\rangle^{(1)} \to |\text{VBS}\rangle^{(1)} = \prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger)|\text{vac}\rangle, \tag{25}$$

while, at $r \to \infty$, $|\text{SVBS-I}\rangle$ is reduced to the Majumdar-Ghosh (MG) dimer state[38] $|\text{MG}\rangle$:

$$|\text{SVBS}(1|2)\rangle^{(1)} \to \prod_i f_i^\dagger |\text{MG}\rangle, \tag{26}$$

where

$$|\text{MG}\rangle = \left(\prod_{i:\text{even}} - \prod_{i:\text{odd}}\right)(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger)|\text{vac}\rangle. \tag{27}$$

In the discussion of the entanglement spectra (Sec. IV), we will see in the two limits the entanglement entropy nicely interpolates between that of the VBS state and the MG state.

2. Higher $\mathcal{S}$

It is easy to generalize the above strategy to the cases with general superspin $\mathcal{S}$. In Ref. 27, the expression of the $\mathcal{A}$ matrix for superspin-$\mathcal{S}$ type-I SVBS state is given as

$$\mathcal{A}_{ab}^{(\mathcal{S})}(j) = \mathcal{F}_a^{\text{L}}(a_j^\dagger, b_j^\dagger, f_j^\dagger) \mathcal{F}_b^{\text{R}}(a_j^\dagger, b_j^\dagger, f_j^\dagger)|\text{vac}\rangle_j, \tag{28}$$

where the $\mathcal{S}$th-order polynomials $\mathcal{F}_a^{\text{L}}$ and $\mathcal{F}_b^{\text{R}}$ are defined in Eqs. (C3 a) and (C3 b) of Ref. 27. The above expression may be readily rewritten into the standard form (16):

$$\mathcal{A}_{ab}^{(\mathcal{S})}(j) = R_{1|2}^{(\mathcal{S})} \Psi_j \Psi_j^{\text{t}} |\text{vac}\rangle_j, \tag{29a}$$

where

$$(\Psi_j)_a \equiv \mathcal{F}_a^{\text{R}}(a_j^\dagger, b_j^\dagger, f_j^\dagger) \quad (1 \leqslant a \leqslant 2\mathcal{S} + 1),$$

$$(R_{1|2}^{(\mathcal{S})})_{ab} \equiv \begin{cases} (-1)^{a-1}\delta_{b,(\mathcal{S}+2)-a} & (1 \leqslant a,b \leqslant \mathcal{S}+1), \\ (-1)^{\mathcal{S}-(a-1)}\delta_{b,(3\mathcal{S}+3)-a} & (\mathcal{S}+2 \leqslant a,b \leqslant 2\mathcal{S}+1). \end{cases} \tag{29b}$$

C. UOSp(1|4) SVBS states

Now, we proceed to the case $(N, K) = (1, 2)$ (one fermion species and four bosonic). For UOSp(1|4), the graded Schwinger operator is given as

$$\psi = (b^{1\dagger}, b^{2\dagger}, b^{3\dagger}, b^{4\dagger}, \sqrt{r} f^\dagger)^{\text{t}}. \tag{30}$$

These five operators correspond to the five-dimensional representation (**5**) of UOSp(1|4); the first four $(b^{1\dagger}, b^{2\dagger}, b^{3\dagger}, b^{4\dagger})$, respectively, create the four bosonic states

$$|1\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle, \quad |2\rangle = |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle,$$

$$|3\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \quad |4\rangle = |-\tfrac{1}{2}, \tfrac{1}{2}\rangle, \tag{31}$$

which are already contained in the spinor representation of SO(5), and the last one $f^\dagger$ creates the fermionic state $|5\rangle = |f\rangle$. We prepare z copies of **5**s to construct the physical Hilbert space at each site of the lattice with the coordination number z and, according to which representation is chosen from the tensor product of z **5**s, we can obtain several different types of MPSs. For instance, since a pair of **5**s is

decomposed as

$$\mathbf{5} \otimes \mathbf{5} \sim \mathbf{1} \oplus \mathbf{10} \oplus \mathbf{14}, \tag{32}$$

two different SVBS states (**10** and **14**) are obtained in one dimension ($z = 2$).

Following the general method described in Sec. II A, one can construct the following UOSp(1|4) SVBS state:

$$
\begin{aligned}
&|\text{SVBS}(1|4)\rangle^{(M)} \\
&= \prod_{\langle i,j \rangle} \left(\psi_i^{\text{t}} \mathcal{R}_{1|4} \psi_j \right)^M |\text{vac}\rangle \\
&= \prod_{\langle i,j \rangle} \left(b_i^{1\dagger} b_j^{2\dagger} - b_i^{2\dagger} b_j^{1\dagger} + b_i^{3\dagger} b_j^{4\dagger} - b_i^{4\dagger} b_j^{3\dagger} - r f_i^{\dagger} f_j^{\dagger} \right)^M |\text{vac}\rangle,
\end{aligned}
\tag{33}
$$

where the summation is taken over the nearest-neighbor pairs $\langle i,j \rangle$ and r denotes a real parameter varying from 0 to ∞. The state has the same structure as the UOSp(1|2) SVBS state except for the metric $\mathcal{R}_{1|4}$ defined in (5) or (8). The superspin $\mathcal{S}$ in this state is given as

$$2\mathcal{S} = \sum_{\sigma=1}^{4} b_i^{\sigma\dagger} b_i^{\sigma} + f_i^{\dagger} f_i = zM. \tag{34}$$

The dimension of the local physical Hilbert space (i.e., the size of the representation $\mathcal{S}$) (13) reads as, for $(N, K) = (1, 2)$,

$$
\begin{aligned}
d_{\mathcal{S}}(1|4) &= \binom{2\mathcal{S} + 3}{3} + \binom{2\mathcal{S} + 2}{3} \\
&= \frac{(4\mathcal{S} + 3)(2\mathcal{S} + 1)(\mathcal{S} + 1)}{3}.
\end{aligned}
\tag{35}
$$

In the following, we consider the one-dimensional case ($z = 2$) with $M = 1 (= \mathcal{S})$ where the SO(5) spin magnitude takes the following two values:

$$S_i = \frac{1}{2} \sum_{\sigma=1}^{4} b_i^{\sigma\dagger} b_i^{\sigma} = M, M - \frac{1}{2} \tag{36}$$

and $d_1(1|4) = 14$.

On a finite one-dimensional chain, the UOSp(1|4) SVBS state (33) may be written as

$$
\begin{aligned}
|\text{SVBS(T)}\rangle_{\alpha_{\text{L}}, \alpha_{\text{R}}} &= \{\mathcal{R}_{1|4} \psi_1\}_{\alpha_{\text{L}}} \prod_{j=1}^{L-1} \left(\psi_j^{\text{t}} \mathcal{R}_{1|4} \psi_{j+1} \right) \{\psi_L^{\text{t}}\}_{\alpha_{\text{R}}} |\text{vac}\rangle \\
&= \left(\mathcal{A}_1^{(\text{T})} \mathcal{A}_2^{(\text{T})}, \ldots, \mathcal{A}_L^{(\text{T})} \right)_{\alpha_{\text{L}}, \alpha_{\text{R}}},
\end{aligned}
\tag{37}
$$

where $\mathcal{R}_{1|4}$ is given by (8) with $N = 1$. The matrix $\mathcal{A}$ is defined by

$$
\begin{aligned}
\mathcal{A}^{(\text{T})} = \mathcal{R}_{1|4} \psi \psi^{\text{t}} &=
\begin{pmatrix}
|1,2\rangle & \sqrt{2}|2,2\rangle & |2,3\rangle & |2,4\rangle & \sqrt{r}|2,f\rangle \\
-\sqrt{2}|1,1\rangle & -|1,2\rangle & -|1,3\rangle & -|1,4\rangle & -\sqrt{r}|1,f\rangle \\
|1,4\rangle & |2,4\rangle & |3,4\rangle & \sqrt{2}|4,4\rangle & \sqrt{r}|4,f\rangle \\
-|1,3\rangle & -|2,3\rangle & -\sqrt{2}|3,3\rangle & -|3,4\rangle & -\sqrt{r}|3,f\rangle \\
-\sqrt{r}|1,f\rangle & -\sqrt{r}|2,f\rangle & -\sqrt{r}|3,f\rangle & -\sqrt{r}|4,f\rangle & 0
\end{pmatrix} \\
&\equiv \sum_{\sigma \leqslant \tau=1}^{4} A_{\text{T}}^{(\text{B})}(\sigma, \tau) |\sigma, \tau\rangle + \sum_{\sigma=1}^{4} A_{\text{T}}^{(\text{F})}(\sigma) |\sigma, f\rangle,
\end{aligned}
$$

where the $D = 14$ basis states are given in terms of the graded Schwinger operators in (30) as ($\sigma, \tau = 1, 2, 3, 4$)

$$
\begin{aligned}
|\sigma, \sigma\rangle &\equiv \frac{1}{\sqrt{2}} (b^{\sigma\dagger})^2 |\text{vac}\rangle, \\
|\sigma, \tau\rangle &\equiv b^{\sigma\dagger} b^{\tau\dagger} |\text{vac}\rangle \quad (\sigma < \tau), \\
|\sigma, f\rangle &\equiv b^{\sigma\dagger} f^{\dagger} |\text{vac}\rangle.
\end{aligned}
\tag{38}
$$

The expressions of the 14 matrices $A(\sigma, \tau)$ and $A(\sigma)$ are given in Appendix A 1.

Since the Schwinger operators are used, it is obvious that the physical Hilbert space thus constructed is the $\mathcal{S} = 1$ (i.e. **14**) fully symmetric representation in the tensor-product decomposition (32):

$$(\mathbf{5} \otimes \mathbf{5})_{\text{fully sym.}} = \mathbf{14} \xrightarrow{\text{SO(5)}} \mathbf{10} \oplus \mathbf{4}, \tag{39}$$

where "$\rightarrow$" denotes the decomposition into the SO(5) irreducible representations. As in the case of UOSp(1|2) $[(N, K) = (1, 1)]$, the physical Hilbert space contains two irreducible representations of SO(5): the spinor (**4**) and the

adjoint (**10**) representations. Since all the 14 bases correspond to the components of the rank-2 symmetric tensor made of the two constituent spinors (**5**), we call the MPS thus constructed *tensor type* and use the suffix "T".

A remark is in order here about other possible MPSs. In fact, as has been mentioned before, another important MPS is obtained[35] if we use the 10-dimensional antisymmetric representation (vector representation; hence the MPS may be called vector type) instead of the 14-dimensional one:

$$(\mathbf{5} \otimes \mathbf{5})_{\text{antisym.}} = \mathbf{10} \xrightarrow{\text{SO(5)}} \mathbf{5} \oplus \mathbf{4} \oplus \mathbf{1}. \tag{40}$$

The MPS obtained in this way is a direct generalization of the SO(5)-invariant MPS considered in Refs. 36 and 37. The details of this class of MPS will be reported elsewhere.[35]

1. Limiting cases

Now, let us consider the two important limiting cases $r \to 0$ and $r \to \infty$. In the limit $r = 0$, the UOSp(1|4) SVBS

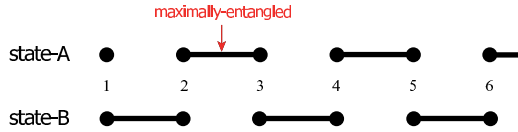FIG. 1. (Color online) The $r \to \infty$ limit of $\mathcal{S} = 1$ SVBS state. Filled circles denote the bosonic qubits [$S = \frac{1}{2}$ spins for UOSp(1|2) and four-dimensional SO(5) spinors for UOSp(1|4)]. On a chain with even number of sites, the MPS is block diagonal with the (1,1) block $\mathcal{B}_{1,1}$ and the (2,2) block $\mathcal{B}_{2,2}$ corresponding to states A and B, respectively.

states (33) or (37) reduce to the following VBS states:

$$|\text{VBS}\rangle = \prod_{\langle i,j \rangle} \left(b_i^{1\dagger} b_j^{2\dagger} - b_i^{2\dagger} b_j^{1\dagger} + b_i^{3\dagger} b_j^{4\dagger} - b_i^{4\dagger} b_j^{3\dagger} \right)^M |\text{vac}\rangle, \tag{41}$$

dubbed *bosonic SO(5) VBS state* in Ref. 28.

In the other limit $r \to \infty$, the dominant part of $\mathcal{A}^{(\mathrm{T})}$ reads as (after dropping factors proportional to $\sqrt{r}$)

$$\mathcal{A}_\infty^{(\mathrm{T})}(j) = \begin{pmatrix} 0 & 0 & 0 & 0 & |2\rangle_j \\ 0 & 0 & 0 & 0 & -|1\rangle_j \\ 0 & 0 & 0 & 0 & |4\rangle_j \\ 0 & 0 & 0 & 0 & -|3\rangle_j \\ -|1\rangle_j & -|2\rangle_j & -|3\rangle_j & -|4\rangle_j & 0 \end{pmatrix}. \tag{42}$$

Then, the two-site MPS $\mathcal{A}_\infty^{(\mathrm{T})}(j)\mathcal{A}_\infty^{(\mathrm{T})}(j+1)$ takes the following block-diagonal form:

$$\mathcal{A}_\infty^{(\mathrm{T})}(j)\mathcal{A}_\infty^{(\mathrm{T})}(j+1) = \pm \begin{pmatrix} \mathcal{B}_{1,1}(j,j+1) & 0 \\ 0 & \mathcal{B}_{2,2}(j,j+1) \end{pmatrix}, \tag{43}$$

where $|1\rangle, \ldots, |4\rangle$ are defined in Eq. (31) and the (2,2) block is the SO(5) singlet made up of two spinors:

$$\begin{aligned} &\mathcal{B}_{2,2}(j,j+1) \\ &= |1\rangle_j |2\rangle_{j+1} - |2\rangle_j |1\rangle_{j+1} + |3\rangle_j |4\rangle_{j+1} - |4\rangle_j |3\rangle_{j+1}. \end{aligned} \tag{44}$$

When the 4×4 matrix $\mathcal{B}_{1,1}(j,j+1)$ is multiplied by $\mathcal{B}_{1,1}(j+2,j+3)$ from the right, a new SO(5) singlet is inserted at the bond $(j+1,j+2)$. Therefore, one sees that the string of $\mathcal{A}_\infty^{(\mathrm{T})}$ represents an SO(5) generalization of the Majumdar-Ghosh valence-bond crystal[38] (see Fig. 1). The vector-type UOSp(1|4) SVBS state mentioned above shares the same property.[35]

III. STRING ORDER

One of the striking features of these VBS states is the existence of nonlocal order called *string order*. In the usual spin systems, it is known[6] that the string order is a manifestation of the spontaneous $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry breaking in the ground state.
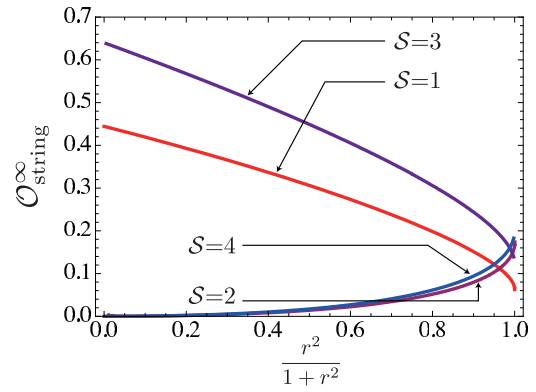

FIG. 2. (Color online) The string-order parameter $\mathcal{O}_{\text{string}}^\infty$ for several values of superspin $\mathcal{S}$ plotted as a function of r (Ref. 27). Note that $\mathcal{O}_{\text{string}}^\infty(r=0) = 0$ for even $\mathcal{S}$ corresponding to the vanishing of string-order parameter for even S.

A. UOSp(1|2) SVBS states

In the case of the usual (pure) spin systems, the string-order parameters are defined by the infinite-distance limit of the string correlation functions[4]

$$O_{\text{string}}^z \equiv \lim_{n \nearrow \infty} \left\langle S_j^z \exp\left[i\pi \sum_{k=j}^{j+n-1} S_k^z \right] S_{j+n}^z \right\rangle, \tag{45a}$$

$$O_{\text{string}}^x \equiv \lim_{n \nearrow \infty} \left\langle S_j^x \exp\left[i\pi \sum_{k=j+1}^{j+n} S_k^x \right] S_{j+n}^x \right\rangle. \tag{45b}$$

It is straightforward to generalize the string-order parameters to the case with SUSY by replacing the spin operators S^a to their $(4\mathcal{S}+1)$-dimensional expressions. For superspin $\mathcal{S} = 1$, it is given by[27] [$O_{\text{string}}^x = O_{\text{string}}^z$ by SU(2) symmetry]

$$O_{\text{string}}^{x,z}(r) = \frac{4\{r^4 + 14r^2 + 18 + 2(r^2+3)\sqrt{8r^2+9}\}}{(8r^2+9)(\sqrt{8r^2+9}+3)^2}. \tag{46}$$

In the limit $r \to 0$, the above string expression reproduces the well-known value[6] $\frac{4}{9}$ (perfect string correlation). In the opposite limit $r \nearrow \infty$, the string-order parameter $\mathcal{O}_{\text{string}}^\infty$ approaches to a finite value $\frac{1}{16}$, which implies that the string order survives in the $r \nearrow \infty$ limit. This agrees with the fact that the spin-1 Haldane state is adiabatically connected to the spin-$\frac{1}{2}$ dimer state.[39]

One can readily generalize the above results to the higher-$\mathcal{S}$ cases,[25] which are SUSY analogs of the higher-spin (bosonic) VBS state introduced in Ref. 12. In the original spin-S VBS states ($r=0$), the string-order parameters have been investigated[8,9] and it has been concluded that they vanish for even integer S. In contrast, for finite values of the doping parameter r, the string-order parameters revive[27] due to the existence of SUSY (see Fig. 2). This interesting behavior will be discussed in Sec. VI in light of symmetry-protected topological order.

B. UOSp(1|4) SVBS states

In Ref. 37, it has been pointed out that the idea of hidden symmetry breaking[6] and the associated string-order

parameters[4] can be generalized to a class of models with higher symmetry $SO(2n+1)$ by using the 2^n-dimensional spinor representation as the auxiliary Hilbert space.

The four string-order parameters for the $SO(5)$ $(n = 2)$ VBS state are defined[37] by analogy with their $SU(2)$ cousin:

$$O_{\text{string}}^{ab} \equiv \lim_{n \nearrow \infty} \left\langle L_j^{ab} \exp \left[i\pi \sum_{k=j}^{j+n-1} L_k^{ab} \right] L_{j+n}^{ab} \right\rangle \quad (47)$$

[$L^{ab} = -L^{ba}$ are the $SO(5)$ generators]. The set of integers (a,b) (with $a,b = 1,2,3,4,5$) labels the 10 generators and we may choose, e.g., $(a,b) = (1,2), (2,5), (3,4)$, and $(4,5)$.

Since, by the $SO(5)$ symmetry, the string-order parameters are independent of the $SO(5)$ indices a,b, we can assume $(a,b) = (1,2)$ without a loss of generality. In Ref. 37, it has been argued that the string order of the $SO(5)$ VBS state is a consequence of the hidden $(\mathbb{Z}_2 \times \mathbb{Z}_2)^2$ symmetry breaking. In the original $SU(2)$ case, we pick up a pair $\{S^z, S^x\}$ and the two commuting $\mathbb{Z}_2$'s are generated by $e^{i\pi S^x}$ and $e^{i\pi S^z}$, the former of which plays the role of the flipping operator of S^z. In the case $SO(5)$, we have two [=rank of $SO(5)$] such pairs (e.g., $\{L^{12}, L^{25}\}$ and $\{L^{34}, L^{45}\}$) and this is why the square of $\mathbb{Z}_2 \times \mathbb{Z}_2$ appears. Similarly, as we already know that the generalized string order exists[27] in the $UOSp(1|2)$ SVBS state, we can expect finite string order in the case of $UOSp(1|4)$ as well by considering two pairs of string-order parameters.

First, we set $r = 0$ and consider the $SO(5)$ limit. By plotting the eigenvalues of local (L^{12}, L^{34}) appearing in the string (37) of $\mathcal{A}^{(\text{T})}$, one can easily see[28] that both L_{12} and L_{34} exhibit a kind of hidden antiferromagnetic order which is essentially the same as that observed[4] in the $S = 1$ VBS state. In fact, the string-order parameter (47) for $(a,b) = (1,2)$ [$(a,b) = (3,4)$] removes the effects of the randomly inserted zeros in the L^{12} (L^{34}) configuration to pick up the hidden antiferromagnetic order.

The generalization of Eq. (47) to the $UOSp(1|4)$ SVBS state with arbitrary superspin $\mathcal{S}$ is straightforward; for $\mathcal{S} = 1$, the bosonic generators L^{ab} are replaced by the 14-dimensional matrices (the explicit forms of them are not very important). The MPS formalism enables us to obtain the following result:

$$O_{\text{string}}^{ab} = \frac{\left\{ 4r^2 + 3 \left(\sqrt{16r^2 + 25} + 5 \right) \right\}^2}{(16r^2 + 25) \left(\sqrt{16r^2 + 25} + 5 \right)^2}$$

$$\rightarrow \begin{cases} \frac{9}{25} & (r \rightarrow 0), \\ \frac{1}{16} & (r \rightarrow \infty). \end{cases} \quad (48)$$

In order to highlight qualitatively different behaviors with respect to the superspin $\mathcal{S}$, we plot the result in Fig. 3 together with that of the superspin-2 case

$$O_{\text{string}}^{ab} = \frac{49(7 - \sqrt{40r^2 + 49})^2}{400(40r^2 + 49)}. \quad (49)$$

From this plot, one can clearly see that, for finite doping, *both* the $\mathcal{S} = 1$ and 2 states are topological, while the latter is nontopological (i.e., non-Haldane) at $r = 0$ (see also Fig. 2). The limiting value $\frac{1}{16}$ is equal to the string order of the $\mathcal{S} = 1$ $UOSp(1|2)$ SVBS at $r \rightarrow \infty$. Similar results have been obtained[35] for the vector-type MPS mentioned in Sec. II C.
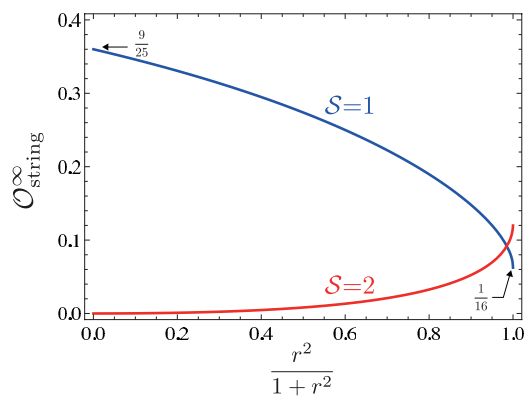

FIG. 3. (Color online) The infinite-distance limit of the string correlation function for $UOSp(1|4)$ states. At $r = 0$, the string order reproduces the known result (Ref. 28) $9/25 = 0.36$. Also plotted is the string order of the $\mathcal{S} = 2$ ($M = 2$) state. As in the $UOSp(1|2)$ case, the string order vanishes at $r = 0$ and revives after doping.

IV. ENTANGLEMENT SPECTRA OF SVBS STATES

In the pioneering paper, Li and Haldane[15] argued that the entanglement spectrum, which is obtained by taking logarithm of the Schmidt eigenvalues (or, the eigenvalues of the reduced density matrix) of the ground-state wave function, might be the fingerprint of the physical edge states that reflect the topological order in the bulk. Specifically, the entanglement levels below the entanglement gap reflect the structure of the physical edge excitations.[15,40] Later, the entanglement spectrum has been proven useful in uncovering the bulk topological properties in a variety of systems (e.g., quantum Hall systems,[40–42] topological insulators,[43–45] and spin chains[17,46]) only by looking at their ground-state wave functions. Since entanglement cut creates point boundaries in one dimension, we may expect that the discrete level structure of the entanglement spectrum reflects the bulk topological order.

In order to carry out the explicit calculation of the Schmidt coefficients (or, entanglement spectrum), we adopt the SMPS formalism introduced in our previous paper.[27] One of the biggest merits of using the SMPS formalism is that the Schmidt decomposition, which is the essential step of the calculation, is *almost* done already when we write the SMPS expression. Therefore, all we have to do is to rewrite the SMPS into the form of the Schmidt decomposition by using the singular-value decomposition.[47,48] However, when the (S)MPSs with different edge states are asymptotically orthogonal to each other in the infinite-size limit [this is the case in all (S)MPSs discussed below], the entanglement spectrum is most easily obtained from the (infinite-size) norms for different edge states:

$$\lambda_\alpha = \lim_{j, L-j, L \nearrow \infty} \sqrt{\frac{\mathcal{N}_j(\alpha_L, \alpha) \mathcal{N}_{L-j}(\alpha, \alpha_R)}{\mathcal{N}_L(\alpha_L, \alpha_R)}}, \quad (50)$$

where $\mathcal{N}_j$ is the squared norm of the MPS on a length-j system

$$\mathcal{N}_j(\alpha, \beta) \equiv |(\mathcal{A}_1 \mathcal{A}_2, \ldots, \mathcal{A}_j)_{\alpha,\beta}|^2. \quad (51)$$
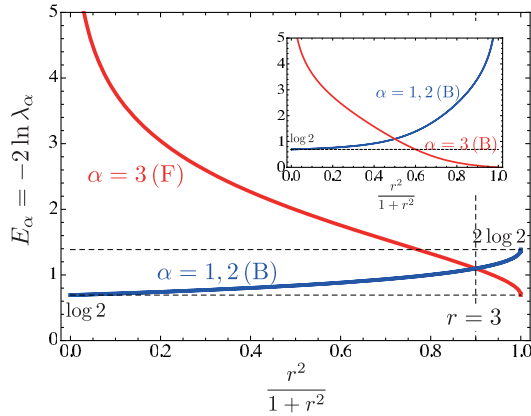
FIG. 4. (Color online) The behavior of entanglement spectrum of the $\mathcal{S} = 1$ UOSp(1|2) SVBS state (the inset is for the bosonic-pair VBS state). "B" and "F" denote bosonic and fermionic parts of the spectrum, respectively.

A. UOSp(1|2) SVBS states

1. $\mathcal{S} = 1$

By utilizing the SMPS, the Schmidt coefficients of the SVBS infinite chain are readily derived as

$$\lambda_B{}^2 \equiv \lambda_1{}^2 = \lambda_2{}^2 = \frac{1}{4} + \frac{3}{4\sqrt{9 + 8r^2}}, \quad (52a)$$

$$\lambda_F{}^2 \equiv \lambda_3{}^2 = \frac{1}{2} - \frac{3}{2\sqrt{9 + 8r^2}}, \quad (52b)$$

which are shown in Fig. 4, and the corresponding entanglement entropy

$$S_{EE} = -\sum_\alpha \lambda_\alpha{}^2 \ln_2 \lambda_\alpha{}^2 \quad (53)$$

is also depicted in Fig. 5. From the entanglement spectra, we find that the bosonic and the fermionic sectors exhibit distinct behaviors. As mentioned in Sec. IV A, the SVBS chain interpolates the original VBS chain ($r = 0$) and the MG dimer chain ($r \to \infty$). Then, we expect the entanglement entropy of the SVBS chain also reduces that of VBS at $r = 0$, and that of

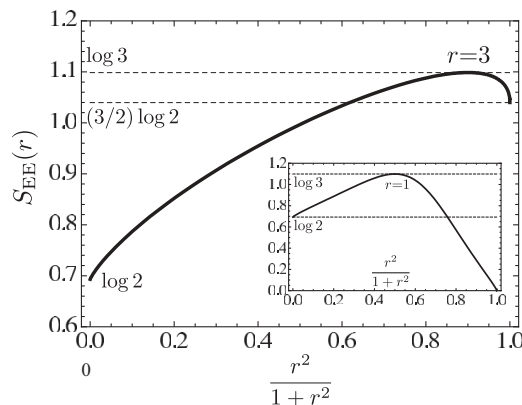FIG. 5. The behavior of the entanglement entropy of the $\mathcal{S} = 1$ UOSp(1|2) SVBS state. (The inset is for the bosonic-pair VBS state.)

MG at $r \to \infty$. Indeed, in such two limits, the entanglement entropy gives those of the VBS and MG dimer chains:

$$\lim_{r \to 0} S_{EE}(r) = \ln 2, \quad \lim_{r \to \infty} S_{EE}(r) = \frac{3}{2}\ln 2. \quad (54)$$

The states are maximally entangled when

$$\lambda_1{}^2 = \lambda_2{}^2 = \lambda_3{}^2 = 1/3 (\text{at } r = 3), \quad (55)$$

where the entanglement entropy takes the maximal value $S_{EE}^{(max)} = \ln 3$. In contrast to the usual bosonic VBS states,[49–51] the entanglement entropy S_{EE} of the SVBS states differs from what is expected from the dimension of the MPS matrices (i.e., bond dimension); they attain the maximal entanglement *only* at a particular value of the doping parameter r, which is different from the position of the maximal entanglement of the corresponding maximally entangled pairs (for more details, see the Supplemental Material[52]).

The "level crossing point" ($r = 3$) between the bosonic and the fermionic spectra generally does not imply a quantum phase transition, in the sense that divergence of physical quantities, e.g., spin-spin correlation length, does not occur at the point. The (open) $\mathcal{S} = 1$ SVBS chain accommodates $\mathcal{S} = \frac{1}{2}$ superspins at the edges, *i.e.*, the number of the edge degrees of freedom is 3 corresponding to $a^\dagger |vac\rangle$, $b^\dagger |vac\rangle$, and $f^\dagger |vac\rangle$. Therefore, as has been found[50,51] in the usual bosonic VBS states, one sees that the entanglement entropy is bounded by the logarithm of the number of the edge degrees of freedom. However, here is one remarkable point; since the parameter r controls the contributions of the bosonic and the fermionic degrees of freedom, one might expect that the entanglement is maximal at $r = 1$ where they appear with equal amplitudes [indeed, this is the case for a system of two $\mathcal{S} = \frac{1}{2}$ superqubits (see Ref. 52)]. Contrary to this naive expectation, the explicit calculation indicates that the maximally entangled point is located at $r = 3$ due to many-body effect of SUSY. Note still in the bosonic many-body case, the entanglement is maximal at $r = 1$ (see the inset in Fig. 5).

To see a property peculiar to the SUSY states, let us introduce a "boson-pair VBS state"

$$|\text{b-p VBS}\rangle = \prod_j (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger - r c_j^\dagger c_{j+1}^\dagger)|vac\rangle, \quad (56)$$

where $c_i^\dagger$ denotes the creation operator of bosonic holes that satisfies $[c_i, c_j^\dagger] = \delta_{ij}$ and $a_j^\dagger a_j + b_j^\dagger b_j + c_j^\dagger c_j = 2$. The new state $|\text{b-p VBS}\rangle$ derived simply by replacing the fermionic operator $f^\dagger$ in the SVBS state (18) with bosonic one $c^\dagger$ neither has the inversion symmetry with respect to the center of a link (*link inversion*) nor has the UOSp(1|2) symmetry. More importantly, The entanglement spectrum is plotted in the inset of Fig. 4. As in the $\mathcal{S} = 1$ SVBS state, the boson-pair VBS chain has three Schmidt eigenvalues, two of which are doubly degenerate and the other is nondegenerate. On the other hand, the entanglement entropy (see the inset of Fig. 5) exhibits a different asymptotic behavior for $r \to \infty$ since $|\text{b-p VBS}\rangle$ reduces, in the limit $r \to \infty$, to the product state $\prod_j c_j^\dagger |vac\rangle$, while the SUSY version $|\text{SVBS}(1|2)\rangle$ still retains finite entanglement due to SUSY.

2. $\mathcal{S} = 2$

Next, we proceed to the $\mathcal{S} = 2$ SVBS chain. The bulk superspin is $\mathcal{S} = 2$ which consists of SU(2) $S = 2$ and $S = \frac{3}{2}$ spins. Therefore, we have five Schmidt coefficients, three of which (bosonic part) come from SU(2) $S = 1$ and the remaining two (fermionic part) come from SU(2) $S = \frac{1}{2}$. The Schmidt coefficients are calculated as

$$\lambda_{\mathrm{B}}{}^2 \equiv \lambda_1{}^2 = \lambda_2{}^2 = \lambda_3{}^2 = \frac{1}{6} + \frac{5(4 + \sqrt{25 + 24r^2})}{6(25 + 24r^2 + 4\sqrt{25 + 24r^2})},$$
(57a)

$$\lambda_{\mathrm{F}}{}^2 \equiv \lambda_4{}^2 = \lambda_5{}^2 = \frac{1}{4} - \frac{5(4 + \sqrt{25 + 24r^2})}{4(25 + 24r^2 + 4\sqrt{25 + 24r^2})}.$$
(57b)

The bosonic part is triply degenerate as in the case of original $S = 2$ VBS chain, while the fermionic part, which newly appeared in the SUSY case, is doubly degenerate. Such double degeneracy is a fingerprint of a symmetry-protected topological (Haldane) phase in 1D.[17] In the absence of fermionic holes ($r = 0$), the fermionic part of the spectrum is infinitely higher lying (see Fig. 6) and the entanglement of the system is completely determined only by the bosonic part which does not show the signature of the Haldane phase.

In the SUSY case, on the other hand, the fermionic levels appear above the finite entanglement gap and there always exists doubly degeneracy in the Schmidt coefficients which accounts for the topological stability of the SVBS state regardless of the parity of the bulk superspin $\mathcal{S}$. We will revisit this in Sec. V. As shown in Fig. 6, the five Schmidt coefficients take the same value $\frac{1}{5}$ at $r = 5$, and the asymptotic behaviors of the entanglement entropy are

$$\lim_{r \to 0} S_{\mathrm{EE}}(r) = \ln 3, \quad \lim_{r \to \infty} S_{\mathrm{EE}}(r) = \ln 2 + \frac{1}{2}\ln 6. \quad (58)$$

Thus, at $r \to \infty$, the $\mathcal{S} = 2$ SVBS state supports the finite entanglement entropy and does not reduce to a simple product state as in the $\mathcal{S} = 1$ SVBS chain.
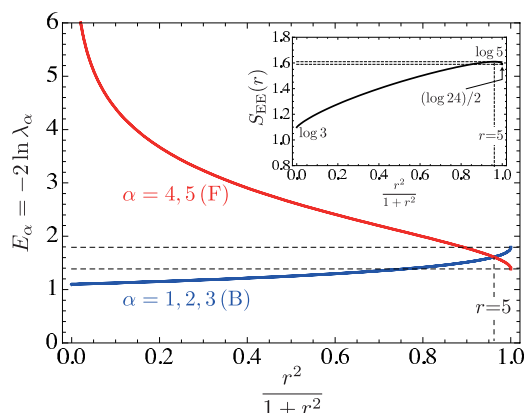

FIG. 6. (Color online) The entanglement spectrum and the entanglement entropy (inset) of the $\mathcal{S} = 2$ UOSp(1|2) SVBS chain. "B" and "F" denote bosonic and fermionic parts of the spectrum, respectively.
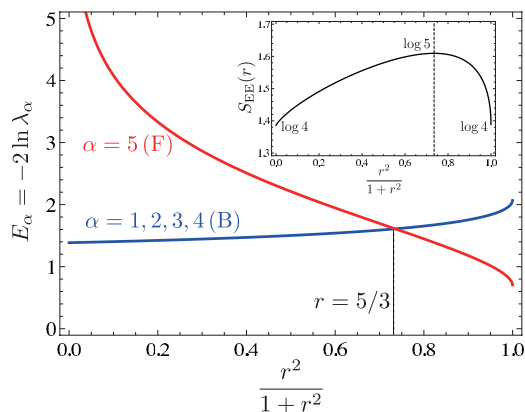

FIG. 7. (Color online) The entanglement spectrum of the UOSp(1|4) SVBS state (37). The entanglement entropy of the same state is shown in the inset.

B. UOSp(1|4) SVBS states

In the case of UOSp(1|4) $[(N,K) = (1,2)]$, we obtain the entanglement spectrum of the MPS (33) as

$$[\lambda_\sigma(r)]^2 = \frac{1}{8} + \frac{5}{8\sqrt{16r^2 + 25}} \quad (\sigma = 1,2,3,4),$$
$$[\lambda_5(r)]^2 = \frac{1}{2} - \frac{5}{2\sqrt{16r^2 + 25}},$$
(59)

which are plotted in Fig. 7 together with the corresponding entanglement entropy. The bosonic part of the spectrum is quadratically degenerate, while the fermionic part is nondegenerate. In both cases, the entanglement entropy $S_{\mathrm{EE}}(r)$ takes its maximal value $\ln 5$ at intermediate value of $r = \frac{5}{3}$ where all the five Schmidt coefficients coincide. The entanglement entropy $S_{\mathrm{EE}}(r)$ exhibits the following asymptotic behaviors:

$$\lim_{r \to 0} S_{\mathrm{EE}}(r) = \lim_{r \to \infty} S_{\mathrm{EE}}(r) = \ln 4 . \quad (60)$$

If we had a boson $b^{5\dagger}$ instead of the fermion $f^\dagger$ in (33) as in the boson-pair VBS state equation (56), entanglement would vanish in the limit $r \to \infty$. Therefore, the existence of finite entanglement even in the $r \to \infty$ limit may be attributed to the fermionic property of the holes.

Here, it should be emphasized that all the limiting behaviors (54), (58), and (60) can be understood from the viewpoint of the edge states; basically, the limiting value of $S_{\mathrm{EE}}(r)$ is determined solely by information of the irreducible representation which describes the emergent edge states. In fact, the general formulas (B9) and (B12) given in Appendix B reproduce the above results.

V. SUPERSYMMETRY-PROTECTED TOPOLOGICAL ORDER

In this section, we show that a family of SVBS states $|\mathrm{SVBS}(1|2K)\rangle$ exhibits the generalized topological order which will be characterized below. Our argument is a SUSY generalization of the one presented in Ref. 17. In the following arguments, we utilize the SMPS formalism. The SMPS formalism itself is defined independent of the super Lie-group symmetries, and is a general formalism to treat a system

of boson-fermion mixture whose ground state is represented by a supermatrix. Therefore, although we mainly discuss the SVBS states which have specific underlying particular super Lie-group symmetries, the following arguments apply to any boson-fermion–mixture systems.

Before going into the detail, we first characterize the symmetry operation (both unitary and antiunitary) within the framework of MPS.[53] The MPS $\mathcal{A}_1\mathcal{A}_2\ldots$ is said to be invariant under the (anti)unitary operation if the transformed state $\mathcal{A}_1'\mathcal{A}_2'\ldots$ coincides with the original one up to an overall phase. Then, it can be shown[53] that the invariance of a pure MPS is equivalent to the existence of a D-dimensional (D being the size of the MPS matrix A) unitary matrix U which satisfies

$$A'(m) = e^{i\theta} U^\dagger A(m) U. \tag{61}$$

The phase θ is not universal and depends, in general, on the symmetry operation in question.

The (c-number) unitary matrix U in (61) may be postulated as

$$U_I = \begin{pmatrix} U_B & 0 \\ 0 & U_F \end{pmatrix}, \tag{62}$$

where U_B and U_F are unitary matrices that act on the two bosonic subspaces having different fermion numbers. The reason for choosing the above form may be seen as follows. First, we note that Eq. (61) implies that the MPS transforms like

$$|\Psi\rangle \mapsto \mathrm{str}(U^\dagger \mathcal{A}_1 \mathcal{A}_2 \ldots \mathcal{A}_{2n+1} U), \tag{63}$$

where *supertrace* is defined as

$$\mathrm{str}\begin{pmatrix} A_B^{(1)} & A_F^{(1)} \\ A_F^{(2)} & A_B^{(2)} \end{pmatrix} = \mathrm{tr}A_B^{(1)} - \mathrm{tr}A_B^{(2)}. \tag{64}$$

While in the case of bosonic MPS, this, combined with $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, immediately implies $\mathcal{I}|\Psi\rangle \propto |\Psi\rangle$, the relation $\mathrm{str}(AB) = \mathrm{str}(BA)$ holds only when A and B are supermatrices (that contain the Grassmann-odd blocks in their off-diagonal parts). In fact, if A and B were merely the c-number matrices, A and B, in general, would not commute inside $\mathrm{str}(\ldots)$: $\mathrm{str}(AB) \neq \mathrm{str}(BA)$. To satisfy $\mathrm{str}(AB) = \mathrm{str}(BA)$ only with c-number matrices, either A or B is forbidden to have c-number components in the off-diagonal blocks.

Physically, the above relation states that the original symmetry operation (acting on the physical Hilbert space on each site) "fractionalizes" into the ones (U and $U^\dagger$) which act on the edge states on both ends of the system.

In what follows, we parametrize the $A(m)$ matrices in terms of the $D \times D$ matrices $[\Lambda, \{\Gamma(m)\}]$ as $A(m) = \Gamma(m)\Lambda$. The diagonal matrix Λ contains the Schmidt eigenvalues in its diagonal elements $[\mathrm{tr}(\Lambda^2) = 1]$ and commutes with the unitary matrix: $[\Lambda, U] = 0$. In what follows, we use the symbol Γ for the MPS A matrices in the *canonical* form.[47]

Then, the Γ matrices satisfy the condition for the canonical MPS on infinite-size systems[48]

$$\sum_m \Gamma^\dagger(m)\Lambda^2\Gamma(m) = \mathbf{1}_D. \tag{65}$$

(For more details about the properties of U, see Appendix A.) In terms of these Γ matrices, Eq. (61) reads as

$$\Gamma'(m) = e^{i\theta} U^\dagger \Gamma(m) U. \tag{66}$$

Now, let us determine the properties of U satisfying the above equation for specific symmetry operations.

A. Inversion symmetry

A matrix product state on a circle is given by

$$|\Psi\rangle = \mathrm{str}(\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_{2n+1}), \tag{67}$$

where "str" denotes the supertrace. By the inversion with respect to a given link, the state is transformed as

$$\mathcal{I}|\Psi\rangle = \mathrm{str}(\mathcal{A}_{2n+1}\ldots\mathcal{A}_2\mathcal{A}_1). \tag{68}$$

Here, we use the property of the supertrace $\mathrm{str}(M_1M_2) = \mathrm{str}[(M_1M_2)^{\mathrm{st}}] = \mathrm{str}(M_2^{\mathrm{st}}M_1^{\mathrm{st}})$ to rewrite the above as

$$\mathcal{I}|\Psi\rangle = \mathrm{str}(\mathcal{A}_1^{\mathrm{st}}\mathcal{A}_2^{\mathrm{st}}\ldots\mathcal{A}_{2n+1}^{\mathrm{st}}), \tag{69}$$

where *supertransposition* "st" is defined as

$$\begin{pmatrix} M_1 & N_1 \\ N_2 & M_2 \end{pmatrix}^{\mathrm{st}} \equiv \begin{pmatrix} M_1^{\mathrm{t}} & N_2^{\mathrm{t}} \\ -N_1^{\mathrm{t}} & M_2^{\mathrm{t}} \end{pmatrix}. \tag{70}$$

Therefore, the link inversion $\mathcal{I}$ amounts, in terms of $\mathcal{A}$, to

$$\mathcal{A}_i \xrightarrow{\mathcal{I}} \mathcal{A}_i^{\mathrm{st}}. \tag{71}$$

If we write

$$\mathcal{A}_i = \sum_m \Lambda\Gamma(m)|m\rangle_i, \tag{72}$$

we see that $\mathcal{I}$ acts on $\Gamma(m)$ as

$$\Gamma(m) \xrightarrow{\mathcal{I}} \Gamma'(m) = \Gamma(m)^{\mathrm{st}}. \tag{73}$$

Here, m labels both bosonic and fermionic components and $\Gamma(m)$ are given by

$$\begin{aligned} \Gamma(m) &= \begin{pmatrix} M_1(m) & 0 \\ 0 & M_2(m) \end{pmatrix} \quad (m\text{: bosonic}), \\ \Gamma(m) &= \begin{pmatrix} 0 & N_1(m) \\ N_2(m) & 0 \end{pmatrix} \quad (m\text{: fermionic}). \end{aligned} \tag{74}$$

Originally, M_1, M_2, N_1, and N_2 are all *c-number* coefficient matrices. However, for practical reasons, it is often convenient to assume that the basis states are commuting and take into account the anticommuting properties of the fermionic states by supermatrices.

If $\mathcal{I}$ leaves the MPS invariant up to a phase, the general relation[53] (66) implies that there exists a unitary matrix U_I satisfying

$$\Gamma(m)^{\mathrm{st}} = e^{i\theta_I} U_I^\dagger \Gamma(m) U_I. \tag{75}$$

In fact, we can prove that θ can take the only two values, 0 and π, namely,

$$U_I^\dagger \Gamma(m) U_I = \pm\Gamma(m)^{\mathrm{st}}. \tag{76}$$

For later convenience, we introduce the following diagonal matrix having the same block-diagonal structure as U_I:

$$P \equiv \begin{pmatrix} \mathbf{1}_{\mathrm{B}} & 0 \\ 0 & -\mathbf{1}_{\mathrm{F}} \end{pmatrix} \quad (U_I P = P U_I). \tag{77}$$

Then, the fact that the link inversion squares to unity leads to an important conclusion that U_I is a "symmetric" or "antisymmetric" unitary matrix

$$U_I^{\mathrm{t}} = \pm P U_I. \tag{78}$$

The appearance of P is closely related to the property of supertransposition:

$$(A^{\mathrm{st}})^{\mathrm{st}} = P A P. \tag{79}$$

We give the outline of the proof in the Appendix C.

By computing the determinant of the above, one can show that either fermionic (when the sign $+$ occurs) or bosonic ($-$) sector has evenfold degeneracy in each entanglement level, which we will use as the fingerprint of the SUSY-protected topological order.

B. Time-reversal symmetry

Before discussing the properties of SMPS under time reversal, let us define the time-reversal operation in the SUSY case. Under the time-reversal transformation $\mathcal{T}$, the spin is transformed as

$$S_a \xrightarrow{\mathcal{T}} -S_a. \tag{80}$$

In the usual matrix representation, the above relation can be expressed as

$$S_a \rightarrow -S_a = (e^{i\pi S_y} K) S_a (K e^{-i\pi S_y}) = R_{ab}^y(\pi) S_b^*, \tag{81}$$

where K is the complex conjugation operator and $R^y(\pi)$ represents the π rotation around the y axis:

$$R^y(\pi) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{82}$$

As in the usual case, time-reversal operation is defined as

$$\begin{aligned} S_a &\xrightarrow{\mathcal{T}} (e^{i\pi S_y} K) S_a (K e^{-i\pi S_y}) = -S_a, \\ S_\sigma &\xrightarrow{\mathcal{T}} (e^{i\pi S_y} K) S_\sigma (K e^{-i\pi S_y}) = \epsilon_{\sigma\tau} S_\tau, \end{aligned} \tag{83}$$

where UOSp(1|2) superspin matrices S_a ($a = x,y,z$) and S_σ ($\sigma = \theta_1, \theta_2$) are defined as

$$S_a = \frac{1}{2} \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}, \quad S_\sigma = \frac{1}{2} \begin{pmatrix} 0 & \tau_\sigma \\ -(i\sigma_2\tau_\sigma)^{\mathrm{t}} & 0 \end{pmatrix}, \tag{84}$$

with the Pauli matrices σ_a and $\tau_1 = (1,0)^{\mathrm{t}}$ and $\tau_2 = (0,1)^{\mathrm{t}}$. The fermionic generators S_σ have the off-diagonal blocks which transform as different irreducible representations of SU(2) and act as spin-$\frac{1}{2}$ raising and lowering matrices. In the Schwinger operator representation, S_σ are explicitly given by $S_{\theta_1} = \frac{1}{2}(a^\dagger f + f^\dagger b)$, $S_{\theta_2} = \frac{1}{2}(b^\dagger f - f^\dagger a)$. Under the time-reversal transformation, the SU(2) spinor states are interchanged: $|\uparrow\rangle = a^\dagger|0\rangle \rightarrow |\downarrow\rangle = b^\dagger|0\rangle$, $|\downarrow\rangle = b^\dagger|0\rangle \rightarrow -|\uparrow\rangle = -a^\dagger|0\rangle$,

and the spinless fermion state remains the same: $f^\dagger|0\rangle \rightarrow f^\dagger|0\rangle$. This implies that the time-reversal transformation of S_σ is given by (83). Then, we have $\mathcal{T}^2 S_\sigma = -S_\sigma$, so the relation $\mathcal{T}^2 = -1$ for half-integer spins appears for the "fermionic spins."

In fact, for integer superspins, $\mathcal{T}$ satisfies[54]

$$\mathcal{T}^2 = \mathcal{P}, \quad (\mathcal{P})_{mn} = \delta_{mn}(-1)^{F(n)}, \tag{85}$$

where $\mathcal{P}$ acting on the physical Hilbert space is analogous to P in Eq. (77) acting on the auxiliary space and, due to the fermion-number operator $F(n)$ [$F(n) = 0$ or $F(n) = 1$ when n labels the bosonic or fermionic variables], $(-1)^{F(n)}$ gives a minus sign for the fermionic sector of the (physical) Hilbert space.

Using the above properties, one can readily see that the time-reversal operation transforms $\Gamma(m)$ as

$$\Gamma(m) \xrightarrow{\mathcal{T}} \Gamma(m)' = \sum_n R_{mn}^y(\pi) \Gamma(n)^*. \tag{86}$$

Then, time-reversal invariance of the SMPS means that there exists a unitary U_T such that[53]

$$\sum_n R_{mn}^y(\pi) \Gamma^*(n) = e^{i\theta_T} U_T^\dagger \Gamma(m) U_T, \tag{87}$$

with $e^{i\theta_T} = \pm 1$. The property $\mathcal{T}^2 = \mathcal{P}$ (for integer superspin) requires that the unitary matrix U_T should satisfy

$$U_T^{\mathrm{t}} = \pm P U_T. \tag{88}$$

Since this is exactly the same as Eq. (78) for the link inversion, a similar conclusion is drawn about the entanglement spectrum.

C. $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry[6] in the original bosonic case is generated by the two commuting π rotations around x and z axes. However, the symmetry around each axis alone does not directly imply the double degeneracy of the entanglement spectrum. Rather, it has been shown[17] that their combination leads to a nontrivial conclusion concerning the entanglement spectrum. In the following, we show that an analogous symmetry leads to a similar conclusion even in the presence of SUSY.

The π rotation around the x (z) axis $\hat{u}_x(\pi)$ [$\hat{u}_z(\pi)$] acts on SMPS as

$$\Gamma(m) \xrightarrow{\hat{u}_a(\pi)} \Gamma(m)' = \sum_n R_{mn}^a(\pi) \Gamma(n) \quad (a = x,z), \tag{89}$$

where $R_{mn}^a(\pi)$ is the $(4\mathcal{S} + 1)$-dimensional rotation matrix of UOSp(1|2) [see, e.g., Eq. (C20)]. The right-hand side is equivalent to the action of a unitary matrix U_a (Ref. 53):

$$\sum_n R_{mn}^a(\pi) \Gamma(n) = e^{i\theta_a} U_a^\dagger \Gamma(m) U_a \quad (a = x,z). \tag{90}$$

Then, the property $(R^a)^2 = \mathcal{P}$ implies the following:

$$e^{2i\theta_x} = 1 \Rightarrow e^{i\theta_x} = \pm 1, \quad U_a P U_a = e^{i\phi_a} \mathbf{1}. \tag{91}$$

The phase factor $e^{i\phi_a}$ can be absorbed in the definition of U_a and we may assume $U_a^\dagger = P U_a$ ($a = x,z$) hereafter.

On the other hand, for the combination of the rotations $\hat{u}_x(\pi)$ and $\hat{u}_z(\pi)$, we obtain (see Appendix C 3 for detail)

$$e^{i\theta_{xz}} = \pm 1, \quad (U_z P U_x)(U_z^\dagger U_x^\dagger) = e^{i\phi_{xz}} \mathbf{1}. \quad (92)$$

By using $U_a^\dagger = P U_a$ obtained above, one can show $e^{i\phi_{xz}} = \pm 1$ and the following exchange property:

$$U_x U_z = \pm P U_z U_x. \quad (93)$$

In terms of the block components $U_{a,\mathrm{B}}$ and $U_{a,\mathrm{F}}$, this reads as

$$U_{x,\mathrm{B}} U_{z,\mathrm{B}} = \pm U_{z,\mathrm{B}} U_{x,\mathrm{B}}, \quad U_{x,\mathrm{F}} U_{z,\mathrm{F}} = \mp U_{z,\mathrm{F}} U_{x,\mathrm{F}}, \quad (94)$$

which immediately implies the same degenerate structure of the entanglement spectrum as in the two previous cases.

D. $(\mathbb{Z}_2 \times \mathbb{Z}_2)^2$ symmetry in UOSp(1|4) SVBS

Now let us discuss the entanglement spectrum in the systems with SO(5) symmetry and its SUSY generalization UOSp(1|4). Inversion symmetry acts independently of the internal symmetry and leads to exactly the same conclusion as above. The crucial difference from the SU(2) case is the existence of $(\mathbb{Z}_2 \times \mathbb{Z}_2)^2$ symmetry[37] in a class of the SO(5) VBS states.[55] Specifically, the group $(\mathbb{Z}_2 \times \mathbb{Z}_2)^2$ consists of the following 16 elements:

$$\overbrace{[1, R^{12}(\pi)] \times [1, R^{25}(\pi)]}^{\mathbb{Z}_2 \times \mathbb{Z}_2} \times \overbrace{[1, R^{34}(\pi)] \times [1, R^{45}(\pi)]}^{\mathbb{Z}_2 \times \mathbb{Z}_2}, \quad (95)$$

with $R^{ab}(\pi) \equiv \exp(i\pi\sigma_{ab})$ [σ_{ab}: SO(5) generators]. The four-fold degeneracy of the entanglement spectra of the SO(5) VBS states has been discussed[56] from the viewpoint of $(\mathbb{Z}_2 \times \mathbb{Z}_2)^2$ symmetry.

It is straightforward to generalize the above symmetry to the UOSp(1|4) case; now, the matrices $R^{ab}(\pi)$ satisfying $R^{ab}(\pi)^2 = \mathbf{1}$ are replaced by the block-diagonal matrices of the form[57]

$$R^{ab}(\pi) = \begin{pmatrix} R_{ab}^{(\mathrm{B})} & 0 \\ 0 & R_{ab}^{(\mathrm{F})} \end{pmatrix}. \quad (96)$$

For instance, in the superspin-1 UOSp(1|4) SVBS state discussed in Sec. II C, $R_{ab}^{(\mathrm{B})}$ and $R_{ab}^{(\mathrm{F})}$ are given by $R^{ab}(\pi)$ in the adjoint (**10**) and the spinor (**4**) representations of SO(5), respectively. It is easy to show that the above matrices satisfy

$$R^{ab}(\pi) R^{ab}(\pi) = \mathcal{P}_{4|10} (\text{no sum for } a \text{ and } b), \quad (97a)$$

$$\begin{aligned} R^{12}(\pi) R^{25}(\pi) &= \mathcal{P}_{4|10} R^{25}(\pi) R^{12}(\pi), \\ R^{34}(\pi) R^{45}(\pi) &= \mathcal{P}_{4|10} R^{45}(\pi) R^{34}(\pi), \end{aligned} \quad (97b)$$

$$R^{25}(\pi) R^{45}(\pi) = \mathcal{P}_{4|10} R^{45}(\pi) R^{25}(\pi), \quad (97c)$$

$$\begin{aligned} R^{12}(\pi) R^{34}(\pi) &= R^{34}(\pi) R^{12}(\pi), \\ R^{12}(\pi) R^{45}(\pi) &= R^{45}(\pi) R^{12}(\pi), \\ R^{25}(\pi) R^{34}(\pi) &= R^{34}(\pi) R^{25}(\pi), \end{aligned} \quad (97d)$$

with

$$\mathcal{P}_{4|10} \equiv \begin{pmatrix} 1_{10} & 0 \\ 0 & -1_4 \end{pmatrix}. \quad (98)$$

Now, we can apply the argument in Sec. V C since we have the same exchange relations (97a) and (97b) as before. Then, we immediately conclude that there exist two sets of the corresponding unitary matrices $\{U_{12}, U_{25}\}$ and $\{U_{34}, U_{45}\}$ satisfying

$$\sum_n [R^{ab}(\pi)]_{mn} \Gamma(n) = e^{i\theta_{ab}} U_{ab}^\dagger \Gamma(m) U_{ab}, \quad U_{ab}^\dagger = P U_{ab},$$

$$U_{12} U_{25} = \pm P U_{25} U_{12}, \quad U_{34} U_{45} = \pm P U_{45} U_{34}, \quad (99)$$

where the matrix P is defined in Eq. (77). Note that the same sign should be chosen for the two exchange relations above by the SO(5) symmetry.

The role of the unitary transformation U_{ab} is clear. First we note that, as in the SO(5) case, the following two are mutually commuting generators of the same block-diagonal form as $R^{ab}(\pi)$ [Eq. (96)]:

$$L^{ab} = \begin{pmatrix} \sigma_{ab}^{(\mathrm{B})} & 0 \\ 0 & \sigma_{ab}^{(\mathrm{F})} \end{pmatrix} \quad (100)$$

and can be used as the weight of UOSp(1|4). Since R^{25} and R^{45} act on the weight (L^{12}, L^{34}) as

$$\begin{aligned} R^{25\dagger} L^{12} R^{25} &= -L^{12}, \quad R^{45\dagger} L^{12} R^{45} = L^{12}, \\ R^{25\dagger} L^{34} R^{25} &= L^{34}, \quad R^{45\dagger} L^{34} R^{45} = -L^{34}, \end{aligned} \quad (101)$$

it is legitimate to assume that the algebra is represented in the product space $V_1 \otimes V_2$ where V_1 and V_2, respectively, correspond to $\{U_{12}, U_{25}\}$ and $\{U_{34}, U_{45}\}$. For instance, the two unitary operations U_{25} and U_{45} actually mean

$$U_{25} \otimes \mathbf{1}, \quad \mathbf{1} \otimes U_{45}, \quad (U_{25} \otimes \mathbf{1})(\mathbf{1} \otimes U_{45}) = U_{25} \otimes U_{45},$$

$$(\mathbf{1} \otimes U_{45})(U_{25} \otimes \mathbf{1}) = (P U_{25}) \otimes U_{45}. \quad (102)$$

Now, we use the fact that V_1 and V_2 should always have even-dimensional sectors $V_1^{(\mathrm{e})}$ and $V_2^{(\mathrm{e})}$ [they have the same dimensions by the SO(5) symmetry] to show that the dimension of $V_1^{(\mathrm{e})} \otimes V_2^{(\mathrm{e})}$ should be an integer-multiple of four. This explains the existence of the fourfold-degenerate entanglement level in the UOSp(1|4) SVBS states (see also the argument in Appendix C 4).

VI. RELATIONS BETWEEN STRING-ORDER PARAMETER AND TOPOLOGICAL ORDER

Later, the use of the string-order parameters in detecting the Haldane phase was criticized[16] since they are well defined only in a restricted class of models and fail to capture the robustness of the Haldane phase as a symmetry-protected topological phase (see Refs. 58 and 59 for the attempts at alternative order parameters). Now, a natural question arises: Under what conditions do the string-order parameters (45a) and (45b) correctly capture the topological nature of the Haldane phase? In the following, we will uncover the explicit relationship between the string order and the topological order to answer to this question.

A. String-order parameters in MPS framework

Let us first consider the structure of the string-order parameters (45a) and (45b) from the MPS point of view.[9,53] In evaluating them using MPS, the following matrices are necessary:

$$[T^a]_{\bar{\alpha},\alpha;\bar{\beta},\beta} \equiv \sum_{m,n=1}^{d} [A^*(m)]_{\bar{\alpha},\bar{\beta}} [A(n)]_{\alpha,\beta} \langle m|S^a|n\rangle,$$

$$[T_{\text{string}}]_{\bar{\alpha},\alpha;\bar{\beta},\beta} \equiv \sum_{m,n=1}^{d} [A^*(m)]_{\bar{\alpha},\bar{\beta}} [A(n)]_{\alpha,\beta} \langle m|e^{i\pi S^a}|n\rangle,$$

$$[T^a_{\text{string}}]_{\bar{\alpha},\alpha;\bar{\beta},\beta} \equiv \sum_{m,n=1}^{d} [A^*(m)]_{\bar{\alpha},\bar{\beta}} [A(n)]_{\alpha,\beta} \langle m|S^a e^{i\pi S^a}|n\rangle$$

$$(a = x,z) \quad (103)$$

as well as the usual transfer matrix. For instance, the MPS expression of the string-order parameter $\mathcal{O}^z_{\text{string}}$ (for an open chain) reads as

$$O^z_{\text{string}} \equiv \left\langle S^z_j \exp\left[i\pi \sum_{k=j}^{j+n-1} S^z_k\right] S^z_{j+n} \right\rangle$$
$$= T^{N_L} T^z_{\text{string}} (T_{\text{string}})^{n-1} T^z T^{N_R}, \quad (104)$$

where we have omitted the denominator necessary to normalize the MPS. The two parts T^{N_L} ($N_L = j - 1$) and T^{N_R} ($N_R = L - n - j$) are straightforward; for the canonical MPS, they reduce, in the infinite-size limit, to

$$[T^{N_L}]_{\bar{\alpha}_L,\alpha_L;\bar{\beta},\beta} \xrightarrow{N_L \nearrow \infty} \delta_{\bar{\alpha}_L,\alpha_L} \delta_{\bar{\beta},\beta},$$
$$[T^{N_R}]_{\bar{\alpha},\alpha;\bar{\beta}_R,\beta_R} \xrightarrow{N_R \nearrow \infty} \delta_{\bar{\alpha},\alpha} \delta_{\bar{\beta}_R,\beta_R}. \quad (105)$$

The boundary-dependent factors $\delta_{\bar{\alpha}_L,\alpha_L}$ and $\delta_{\bar{\beta}_R,\beta_R}$ are canceled by those coming from the denominator. Therefore, all we have to compute is the infinite-distance limit ($n \nearrow \infty$) of the following quantity:

$$\sum_{\alpha,\beta} [T^z_{\text{string}} (T_{\text{string}})^{n-1} T^z]_{\alpha,\alpha;\beta,\beta}. \quad (106)$$

B. String-order parameters and entanglement spectrum

Now, we show that the existence of nonvanishing string-order parameters serves as the *sufficient condition* for the symmetry-protected topological order discussed in the previous section. Let us begin with the simpler case of the usual VBS states. Since we are interested in the long-distance limit $|i - j| \nearrow \infty$, we need to know the asymptotic behavior of the string $(T_{\text{string}})^{|i-j|}$. To this end, we can borrow the results of Ref. 53 (Theorem 2); according to the theorem, the MPS should be invariant under both of the π rotations

$$\hat{u}_x = \otimes_j e^{-i\pi S^x_j}, \quad \hat{u}_z = \otimes_j e^{-i\pi S^z_j} \quad (107)$$

in order for the string $(T_{\text{string}})^{|i-j|}$ not to vanish in the long-distance limit. Then, Lemma 1 of Ref. 53 guarantees that there exists a pair of unitary matrices U_x and U_z which are unique

and satisfy

$$\sum_{n=1}^{d} R^{(S)}_a(\pi)_{mn} A(n) = e^{i\theta_a} U^\dagger_a A(m) U_a \quad (a = x,z; \ e^{i\theta_a} = \pm 1),$$

$$(108)$$
$$(U_a)^2 = \mathbf{1}, \quad U_x U_z = \pm U_z U_x,$$

where the two sign choices are *independent*. The above exchange property between U_x and U_z has a very important implication to the structure of the entanglement spectrum:[17]

$$\det\{(U_x U_z)_\lambda\} = \det\{(U_x)_\lambda\} \det\{(U_z)_\lambda\}$$
$$= (\pm 1)^{d_\lambda} \det\{(U_z U_x)_\lambda\}$$
$$= (\pm 1)^{d_\lambda} \det\{(U_x)_\lambda\} \det\{(U_z)_\lambda\} \quad (\neq 0).$$
$$(109)$$

Therefore, the degree of degeneracy d_λ of each entanglement level λ should be even when U_x and U_z are anticommuting. Typically, this happens in the VBS states with *odd integer S*.

Now, we show that when the string-order parameters are nonvanishing $\mathcal{O}^{z,x}_{\text{string}} \neq 0$, the minus sign realizes (i.e., U_x and U_z anticommute) in Eq. (109) and the entanglement spectrum has the degenerate structure. To this end, we investigate Eq. (106). First of all, the invariance of the MPS under $\hat{u}_{x,z}$ implies that the string part $(T_{\text{string}})^{n-1}$ reduces essentially to a phase $(e^{i\theta_a})^{n-1} = (\pm 1)^{n-1}$. This is a direct consequence of Theorem 2 of Ref. 53 and is easily understood since the overlap $\langle \Psi | \hat{u}_a | \Psi \rangle = (T_{\text{string}})^L$ vanishes otherwise. The price to pay is the boundary factors appearing at the two endpoints of the string correlation functions (see Fig. 8):

$$\sum_{\alpha,\beta} \left\{ T^z_{\text{string}} \left(\sum_{n=1}^{D^2} \mathbf{V}^{(u)}_{R,n} \mathbf{V}^{(u)}_{L,n} \right) (T_{\text{string}})^{n-1} T^z \right\}_{\alpha,\alpha;\beta,\beta}$$
$$\xrightarrow{|i-j| \nearrow \infty} \sum_{\alpha,\beta} \left\{ (T^z_{\text{string}} \mathbf{V}^{(u)}_{R,1})(\mathbf{V}^{(u)}_{L,1} T^z) \right\}_{\alpha,\alpha;\beta,\beta}$$
$$= \sum_{\alpha,\beta} \left\{ (T^z_{\text{string}} \{\mathbf{1} \otimes U^\dagger_a\} \mathbf{1})(\mathbf{1} \{\mathbf{1} \otimes U_a\} T^z) \right\}_{\alpha,\alpha;\beta,\beta}, \quad (110)$$

where $\mathbf{V}^{(u)}_{L/R,n}$ denotes the left (L) and the right (R) eigenvectors of T_{string}.
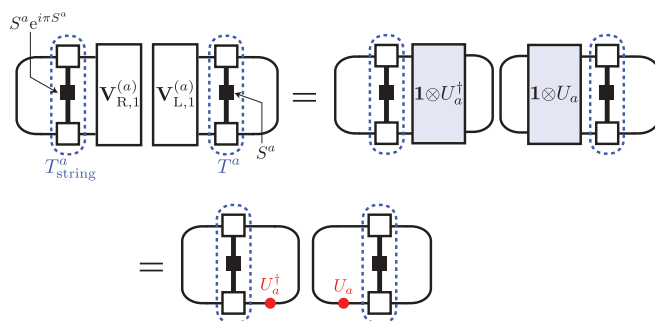

FIG. 8. (Color online) Diagrammatic representation of the main part of string correlation function $\{(T^z_{\text{string}} \mathbf{V}^{(u)}_{R,1})(\mathbf{V}^{(u)}_{L,1} T^z)\}$.
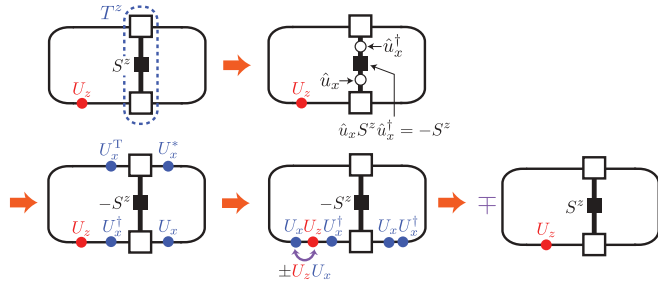
FIG. 9. (Color online) Rewriting the boundary factor (for $a = z$) using $\hat{u}_x$. When U_x and U_z are anticommuting, the minus sign coming from $\hat{u}_x S^z \hat{u}_x^\dagger = -S^z$ is canceled and an overall plus sign is recovered.

To see whether the boundary factors are nonvanishing or not, we consider the right-boundary factor ($\mathbf{1}\{\mathbf{1}\otimes U_z\} T^z$) of $\mathcal{O}_{\text{string}}^z$ (i.e., $a = z$). First, we rewrite it by using (see the second figure of Fig. 9)

$$S^z = \hat{u}_x^\dagger \hat{u}_x S^z \hat{u}_x^\dagger \hat{u}_x = \hat{u}_x^\dagger (-S^z) \hat{u}_x \quad (\hat{u}_x = \otimes_k e^{-i\pi S^x}). \quad (111)$$

The unitary operators $\hat{u}_x^\dagger$ and $\hat{u}_x$ appearing on both sides of $-S^z$ can be absorbed into the MPS matrices by using Eq. (108) (the third figure of Fig. 9). By rearranging the unitary matrices $U_x U_z$ (the fourth figure of Fig. 9), we arrive at the expression

$$\begin{aligned} \mathbf{1}\{\mathbf{1}\otimes U_z\} T^z &= \mathbf{1}\left\{\mathbf{1}\otimes(U_x U_z U_x^\dagger)\right\}(-T^z) \\ &= \mathbf{1}\left\{\mathbf{1}\otimes(\pm U_z U_x U_x^\dagger)\right\}(-T^z) \\ &= \mp\mathbf{1}\{\mathbf{1}\otimes U_z\} T^z. \end{aligned} \quad (112)$$

Therefore, we see that the boundary factors, and hence the string-order parameter itself, vanish when U_x and U_z are commuting (as, e.g., in the even-S VBS states). On the other hand, if both of the string-order parameters are finite, this immediately implies that the ground-state MPS is not only invariant under the two π rotations[53] $\hat{u}_x$ and $\hat{u}_z$, but also has the adjoint $U_{x,z}$ matrices satisfying

$$U_x U_z = -U_z U_x. \quad (113)$$

By the argument in Ref. 17, the ground state is topologically nontrivial in the sense that each entanglement level is evenfold degenerate. Therefore, the finiteness of the pair of string-order parameters $\mathcal{O}_{\text{string}}^{x,z}$ is the *sufficient* condition for the topological phase. It is crucial that *both* O_{string}^x and O_{string}^z are nonzero for the existence of the topological order. For instance, one can construct a solvable spin-1 model[60] which exhibits a kind of "hidden order" similar to the one in the VBS model and has[61] $O_{\text{string}}^x = 0$ and $O_{\text{string}}^z \neq 0$. In fact, in this case, the two entanglement eigenvalues are no longer degenerate and the state is not topological.

C. Case of SMPS

Basically, we follow the same line of arguments to show that finite string correlation implies the topological phase. The only difference is that now we have the P matrix (77) in the key equation (113):

$$U_x U_z = \pm P U_x U_z. \quad (114)$$

Correspondingly, the last step (see Fig. 9) in evaluating the boundary factor is modified. Specifically, instead of Eq. (112),
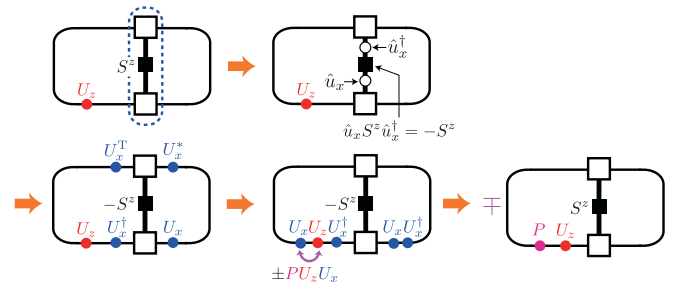

FIG. 10. (Color online) Rewriting the boundary factor (for $a = z$) using $\hat{u}_x$. When U_x and U_z are anticommuting, the minus sign coming from $\hat{u}_x S^z \hat{u}_x^\dagger = -S^z$ is canceled and an overall plus sign is recovered. Note that an extra P matrix appears in the SUSY case.

we have (see Fig. 10)

$$\mathbf{1}\{\mathbf{1}\otimes U_z\} T^z = \mathbf{1}\{\mathbf{1}\otimes(U_x U_z U_x^\dagger)\}(-T^z) = \mp\mathbf{1}\{\mathbf{1}\otimes P U_z\} T^z. \quad (115)$$

Therefore, one of the two components (bosonic and fermionic) vanishes just by symmetry:

$$\quad (116)$$

Therefore, if the two string-order parameters are both nonvanishing, either the bosonic or the fermionic sector exhibits the degenerate structure mentioned in Sec. V and the ground state is topologically nontrivial.

Now, it is straightforward to generalize the above argument to the case of UOSp(1|4) to show that when *all* the four string-order parameters

$$\mathcal{O}_{\text{string}}^{ab} \equiv \lim_{|i-j|\nearrow\infty}\left\langle L_i^{ab} \exp\left[i\pi \sum_{k=i}^{j-1} L_k^{ab}\right] L_j^{ab}\right\rangle \quad (117)$$

[where $(a,b) = (1,2)$, $(2,5)$, $(3,4)$, and $(4,5)$, and L_{ab} are the SO(5) generators] are nonzero, $2^2\times$(integer)-fold degeneracy occurs in some (bosonic or fermionic) sectors of the entanglement spectrum.

VII. SUMMARY AND DISCUSSIONS

We investigated the effects of doped fermionic holes on the topological phases in quantum antiferromagnets. To this end, we first introduced a family of SVBS states which may be thought of as the hole-doped version of the usual (bosonic) VBS states, e.g., spin-S SU(2) states, the SO(5) and the Sp(N) VBS states. One of the standard ways of looking at the topological properties in these states is to investigate the string-order parameters. We explicitly evaluated the behaviors of the string-order parameters of the UOSp(1|2) and the UOSp(1|4) SVBS states for various values of superspin $\mathcal{S}$,

and found that even when the string-order parameters vanish identically in the absence of doping, they revive immediately after holes are introduced in the system. This might suggest that the doped holes change the property of the ground state and thereby stabilize the topological phase.

To better understand the nature of the states, we calculated the entanglement spectrum. Basically, the spectrum consists of the bosonic and the fermionic sectors; at zero doping $r = 0$, the fermionic sector is separated from the bosonic sector, which constitutes the low-"energy" part of the spectrum, by an infinitely large entanglement gap. Upon doping, the fermionic sector starts participating in the entanglement. The point is that the existence of supersymmetry allows the coexistence of the two sectors having different entanglement structures. In addition to that, the entanglement spectra in the SUSY systems exhibit the following salient features: (i) In contrast to naive expectation, the SUSY entanglement spectra for the bosonic and the fermionic sectors do not coincide with each other at $r = 1$, as a consequence of SUSY many-body effect. (ii) In the two extreme limits of the doping parameter, $r \to 0$ and ∞, the entanglement spectra of the SVBS states indeed reproduce those of the original bosonic VBS state and the Majumdar-Ghosh–type states, respectively.

On the basis of the observations made for the particular states [UOSp(1|2) SVBS and UOSp(1|4) SVBS], we characterized, with the help of the SMPS formalism, the symmetry-protected topological orders in the SUSY systems in terms of the entanglement spectrum. According to the results, there always exists a topologically protected sector (whose degenerate structure depends on the symmetry of the SMPS in question) in the spectrum of the SUSY systems. Also, by using the SMPS formalism, we clarified an intimate connection between the finiteness of the string-order parameters and the degenerate structure of the entanglement spectra; the finite string order is the *sufficient* condition for the degeneracy in the entanglement spectrum, which is the fingerprint of the (topological) Haldane state in the bulk. These explain the revival of the string order upon doping.

The above remarkable features can be understood in light of the SUSY edge-state picture. Intuitively, the degenerate structure can be understood by the existence of fictitious "edge" superspins that appear at the entanglement cut of the chain. When the bulk system has superspin $\mathcal{S}$, two superspins $\mathcal{S}/2$s, which consist of the SU(2) spin $\mathcal{S}/2$ and its superpartner $\mathcal{S}/2 - 1/2$, emerge at the edges:

$$\mathcal{S}/2 \overset{\text{SUSY}}{\longleftrightarrow} \mathcal{S}/2 - 1/2. \tag{118}$$

Then, there always exist half-odd-integer spins at the edges *regardless of the parity of the bulk superspin* since SUSY, being the symmetry that relates the state with integer spin and that with half-odd-integer spin, guarantees the coexistence of both. Such half-odd-integer edge spins bring the evenfold degeneracy to the entanglement spectrum of the UOSp(1|2)-symmetric systems. Therefore, if we have a topological phase (e.g., Haldane phase) characterized by the above type of degenerate structures in the entanglement spectrum, it exists for *all* values of superspin $\mathcal{S}$. A similar argument applies, with due modification, to cases with other types of SUSY. In this

sense, one may say that SUSY plays a unique role in stabilizing the topological phases of matter in 1D.

Since our study presented here is restricted to a particular class of VBS states with SUSY, one obvious future direction would be to extend it to more generic models. The argument for symmetry-protected topological orders presented in this paper can be generally applied to *any* system whose ground-state wave function is given by the (S)MPS states. Thus, it would be interesting to see, for instance, the robustness of the Haldane phase in the SUSY Heisenberg model with respect to the parity of the bulk superspin $\mathcal{S}$. This might highlight the unique behavior of SUSY topological phases in comparison to the bosonic counterparts studied in Ref. 17.

Another future direction is the generalization to higher dimensions. In higher dimensions, the SVBS states generally interpolate between the bosonic VBS states and the resonating-valence-bond (RVB) type of states,[62,63] where the wave function is given by the summation over all possible dimer coverings of singlet [i.e., $(a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger)$] bonds (in 1D, we have the Majumdar-Ghosh valence-bond crystals). The latter is well known to have nontrivial topological properties[63] and it would be interesting to study the change in the entanglement properties and the edge-state structure as the doping is varied by using the techniques of projected entangled pair states (PEPS).[64]

Application to other topologically nontrivial states of matter, such as quantum Hall states or various topological states in cold-atom systems, is even more interesting. For instance, the SUSY-extended Laughlin wave function, which has a close analogy with the SVBS states studied here, interpolates between different quantum Hall ground states, such as the Laughlin states and the Moore-Read Pfaffian states. In this respect, as the SVBS states in 1D provided a unifying way of deriving the entanglement spectra of the (bosonic) VBS state and the MG dimer state, the study of the entanglement spectra of the SUSY Laughlin wave function will naturally give a unifying understanding of the entanglement structure of various quantum Hall ground states.

Note added. Finally, we would like to comment on the recent work on the nonlocal order parameters for the symmetry-protected topological order. When completing this paper, we became aware of a recent preprint by Pollmann and Turner (Ref. 59) which also discusses the string-order parameter from the entanglement point of view. Although some of the conclusions obtained there overlap with ours, the main goal there is to go beyond the string-order parameter and is different from that of this paper.

APPENDIX A: *A* MATRICES FOR UOSp(1|4) SVBS STATES

1. Superspin-1 SVBS

The 14 5×5 *A* matrices for the $\mathcal{S} = 1$ SVBS state discussed in Sec. II C are explicitly given as

$$A(1,1) = -A(2,2)^{\mathrm{t}} = -\sqrt{2}\begin{pmatrix} \sigma_- & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A(3,3) = -A(4,4)^{\mathrm{t}} = -\sqrt{2}\begin{pmatrix} 0_2 & 0 & 0 \\ 0 & \sigma_- & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A(1,2) = \begin{pmatrix} \sigma_3 & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A(1,3) = -A(2,4)^{\mathrm{t}} = -\begin{pmatrix} 0_2 & \sigma_- & 0 \\ \sigma_- & 0_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A1a}$$

$$A(1,4) = A(2,3)^{\mathrm{t}} = \frac{1}{2}\begin{pmatrix} 0_2 & -1_2 + \sigma_3 & 0 \\ 1_2 + \sigma_3 & 0_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A(3,4) = \begin{pmatrix} 0_2 & 0 & 0 \\ 0 & \sigma_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A(1) = A(2)^{\mathrm{st}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{r} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\sqrt{r} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(3) = A(4)^{\mathrm{st}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{r} \\ 0 & 0 & -\sqrt{r} & 0 & 0 \end{pmatrix}, \tag{A1b}$$

where the symbols "t" and "st" denote the transposition and supertransposition (70), respectively. They can be represented by linear combinations of the UOSp(1|4) generators.

2. Properties

As has been discussed in Sec. V A, the link-inversion symmetry is implemented in the SMPS as

$$\mathcal{I}: A(m) \mapsto A(m)^{\mathrm{st}}, \tag{A2}$$

or to write the bosonic and the fermionic components separately

$$\mathcal{I}: A(\sigma,\tau) \mapsto A(\sigma,\tau)^{\mathrm{t}}, \quad A(\sigma) \mapsto A(\sigma)^{\mathrm{st}}. \tag{A3}$$

Then, it can be shown

$$A(m)^{\mathrm{st}} = \mathcal{W}^{\dagger} A(m) \mathcal{W}, \tag{A4}$$

where

$$\mathcal{W} = \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix} \tag{A5}$$

with

$$W = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}. \tag{A6}$$

APPENDIX B: EDGE STATES AND GENERAL ASYMPTOTIC BEHAVIOR OF ENTANGLEMENT

The asymptotic behaviors (54), (58), and (60) can be understood from a more general point of view. Let us consider the UOSp(1|2K) SVBS state with bulk superspin $\mathcal{S}$. The UOSp(1|2K) SVBS has $N = 1$ supersymmetry, and consists of one bosonic sector and one fermionic sector. For the bulk superspin $\mathcal{S}$, the emergent superspin-$\mathcal{S}/2$ objects appear at the edges and the UOSp(1|2K) SVBS state accommodates the graded fully symmetric representation[34] at each edge:

$$|m_1, m_2, \ldots, m_{2K}\rangle$$
$$= \frac{1}{\sqrt{m_1! m_2! \ldots m_{2K}!}} (b_1^{\dagger})^{m_1} (b_2^{\dagger})^{m_2} \ldots (b_{2K}^{\dagger})^{m_{2K}} |\mathrm{vac}\rangle, \tag{B1a}$$

$$|n_1, n_2, \ldots, n_{2K}\rangle$$
$$= \frac{1}{\sqrt{n_1! n_2! \ldots n_{2K}!}} (b_1^{\dagger})^{n_1} (b_2^{\dagger})^{n_2} \ldots (b_{2K}^{\dagger})^{n_{2K}} f^{\dagger} |\mathrm{vac}\rangle, \tag{B1b}$$

with $m_1 + m_2 + \cdots + m_{2K} = n_1 + n_2 + \cdots + n_{2K} + 1 = \mathcal{S}$. Then, the number of the bosonic and fermionic states on each edge are, respectively, given by

$$D_{\mathrm{B}} = \binom{\mathcal{S} + 2K - 1}{\mathcal{S}} = \frac{(\mathcal{S} + 2K - 1)!}{(2K - 1)! \mathcal{S}!}, \tag{B2a}$$

$$D_{\mathrm{F}} = \binom{\mathcal{S} + 2K - 2}{\mathcal{S} - 1} = \frac{(\mathcal{S} + 2K - 2)!}{(2K - 1)!(\mathcal{S} - 1)!}. \tag{B2b}$$

[The bosonic degrees of freedom coincide with the fully symmetric representation of USp($2K$).[29]] For instance, for the UOSp(1|2) ($K = 1$) SVBS state, we have

$$D_{\mathrm{B}} = \mathcal{S} + 1, \quad D_{\mathrm{F}} = \mathcal{S}, \tag{B3}$$

while for the UOSp(1|4) ($K = 2$) SVBS state,

$$D_{\mathrm{B}} = \frac{1}{6}(\mathcal{S} + 1)(\mathcal{S} + 2)(\mathcal{S} + 3),$$
$$D_{\mathrm{F}} = \frac{1}{6}\mathcal{S}(\mathcal{S} + 1)(\mathcal{S} + 2). \tag{B4}$$

In the infinite chain limit, the spin degrees of freedom are equivalent:

$$\lambda_1{}^2 = \lambda_2{}^2 = \ldots = \lambda_{D_{\mathrm{B}}}{}^2 \equiv \lambda_{\mathrm{B}}{}^2,$$
$$\lambda_{D_{\mathrm{B}}+1}{}^2 = \lambda_{D_{\mathrm{B}}+2}{}^2 = \ldots = \lambda_{D_{\mathrm{B}}+D_{\mathrm{F}}}{}^2 \equiv \lambda_{\mathrm{F}}{}^2, \tag{B5}$$
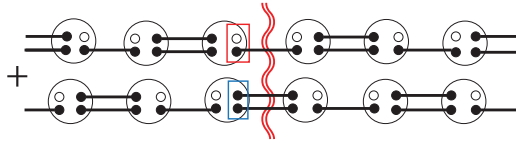
FIG. 11. (Color online) The $r \to \infty$ limit of the bulk superspin $\mathcal{S} = 2$ UOSp(1|4) SVBS state is given by the superposition of the two partially dimerized states related to each other by one-site translation. When we make an entanglement cut at an arbitrary bond (shown by wavy lines), we always have two different kinds of sections: one with 4 "fermionic" edge states (upper) and the one with 10 "bosonic" edge states (lower). These two different sections respectively yield fourfold- and tenfold-degenerate entanglement levels.

and the normalization condition of the Schmidt coefficients $\sum_{\alpha=1}^{D_B+D_F} |\lambda_\alpha|^2 = 1$ is rewritten as

$$D_B \cdot \lambda_B{}^2 + D_F \cdot \lambda_F{}^2 = 1. \tag{B6}$$

Then, the entanglement entropy is expressed as

$$S_{EE}(r) = -\sum_{\alpha=1}^{D_B} |\lambda_\alpha|^2 \ln|\lambda_\alpha|^2 - \sum_{\alpha=1}^{D_F} |\lambda_{D_B+\alpha}|^2 \ln|\lambda_{D_B+\alpha}|^2$$
$$= -D_B |\lambda_B|^2 \ln|\lambda_B|^2 - D_F |\lambda_F|^2 \ln|\lambda_F|^2. \tag{B7}$$

At $r = 0$, only the Schmidt coefficients of boson sector survive and Eq. (B6) implies

$$\lambda_B{}^2 = \frac{1}{D_B}, \quad \lambda_F{}^2 = 0, \tag{B8}$$

and hence

$$\lim_{r \to 0} S_{EE}(r) = \ln D_B. \tag{B9}$$

Thus, the entanglement entropy of the spin S original VBS states is reproduced.

On the other hand, in the limit $r \to \infty$, the SVBS states reduce to the (partially) dimerized states (see Fig. 11). In the upper state in Fig. 11, the fermionic edge states appear, while in the lower the edge states are bosonic. Since both cases appear with equal weights, the sum of the Schmidt coefficients for the bosonic sector and that for the fermionic sector should be equal:

$$\sum_{\alpha=1}^{D_B} \lambda_\alpha{}^2 = \sum_{\alpha=1}^{D_F} \lambda_{D_B+\alpha}{}^2 = 1/2. \tag{B10}$$

Therefore, we have

$$\lambda_B{}^2 = \frac{1}{2D_B}, \quad \lambda_F{}^2 = \frac{1}{2D_F} \tag{B11a}$$

for $r \to \infty$, and the corresponding entanglement entropy is derived as

$$\lim_{r \to \infty} S_{EE}(r) = \ln(2\sqrt{D_B D_F}) \tag{B12}$$

with D_B and D_F given by Eq. (B4). Thus, from the entanglement point of view, the role of SUSY is twofold. First, it necessitates two different Schmidt eigenvalues corresponding to the $N = 1$ SUSY. Second, it enables the system to support finite entanglement even in the limit $r \to \infty$.

For the superspin-$\mathcal{S}$ UOSp(1|2) SVBS states,[27] the entanglement entropy behaves as

$$\lim_{r \to 0} S_{EE}(r) = \ln(\mathcal{S} + 1), \tag{B13a}$$

$$\lim_{r \to \infty} S_{EE}(r) = \ln 2 + \frac{1}{2} \ln[\mathcal{S}(\mathcal{S} + 1)], \tag{B13b}$$

which, for $\mathcal{S} = 1$ and 2, reproduces the results (54) and (58). For the superspin-$\mathcal{S}$ UOSp(1|4) SVBS states, on the other hand,

$$\lim_{r = 0} S_{EE}(r) = \ln(\mathcal{S} + 1)(\mathcal{S} + 2)(\mathcal{S} + 3) - \ln 6, \tag{B14a}$$

$$\lim_{r \to \infty} S_{EE}(r) = -\ln 3 + \ln(\mathcal{S} + 1)(\mathcal{S} + 2) + \frac{1}{2} \ln \mathcal{S}(\mathcal{S} + 3). \tag{B14b}$$

Setting $\mathcal{S} = 1$, we reproduce the previous result (60).

APPENDIX C: PROOFS

In this appendix, we outline the proofs of the important relations (78), (88), and (93). For later convenience, we derive a useful property of pure canonical MPSs. Suppose that we have a pure MPS whose canonical form is characterized by the MPS data[32,48] (Λ, Γ) and that it satisfies the following relation for some unitary matrix U:

$$\Gamma(m) = e^{i\theta_U} U^\dagger \Gamma(m) U. \tag{C1}$$

Since the MPS is canonical, the following holds:

$$\sum_m \Gamma^\dagger(m) \Lambda^2 \Gamma(m) = \mathbf{1}_D. \tag{C2}$$

Physically, it states that the D^2-dimensional vector $\mathbf{V}_L^{(0)}$,

$$(\mathbf{V}_L^{(0)})_{a;b} \equiv \delta_{ab} \quad (1 \leqslant a, b \leqslant D), \tag{C3}$$

is the dominant left eigenvector of the left transfer matrix

$$(T_L)_{\bar{a},a;\bar{b},b} \equiv \sum_m [\Lambda \Gamma^*(m)]_{\bar{a}\bar{b}} [\Lambda \Gamma(m)]_{ab}. \tag{C4}$$

Plugging $\Gamma^\dagger(m) = e^{-i\theta_U} U^\dagger \Gamma^\dagger(m) U$ into (C2), we obtain

$$e^{-i\theta_U} \sum_m U^\dagger \Gamma^\dagger(m) U \Lambda^2 \Gamma(m) = \mathbf{1}_D \tag{C5}$$

or equivalently

$$\sum_m \Gamma^\dagger(m) \Lambda U \Lambda \Gamma(m) = e^{i\theta_U} U. \tag{C6}$$

This implies that the unitary matrix

$$U_{\bar{b}b} = \sum_a \{\mathbf{1} \otimes U\}_{aa;\bar{b}b} \equiv \sum_a \delta_{a\bar{b}} U_{ab}, \tag{C7a}$$

when viewed as a D^2-dimensional vector, is the left eigenvector of T_L with the eigenvalue $e^{i\theta_U}$:

$$U T_L = e^{i\theta_U} U. \tag{C7b}$$

Since, by assumption of canonical MPS, $\mathbf{1}_D$ is the unique left eigenvector with the eigenvalue $|\lambda| = 1$, we conclude

$$e^{i\theta_U} = 1, \quad U = e^{i\phi}\mathbf{1}_D. \tag{C8}$$

Since in deriving the above, we have only assumed that the (infinite-system) MPS in question is pure and takes the canonical form, (C8) holds for any MPS (including SMPS) satisfying the assumption.

1. Inversion symmetry

We use the property $\mathcal{I}^2 = 1$ to derive the important property (78) of the adjoint U_I matrix. Applying supertransposition st on (75) and using $(A^{\text{st}})^{\text{st}} = PAP$, we obtain

$$\Gamma(m) = e^{2i\theta_I}(U_I P U_I^*)^\dagger \Gamma(m)(U_I P U_I^*). \tag{C9}$$

Postulate U is the block-diagonal matrix

$$U = \begin{pmatrix} U_{\text{B}} & 0 \\ 0 & U_{\text{F}} \end{pmatrix}. \tag{C10}$$

By Eqs. (C1) and (C8), this implies that the $D \times D$ matrix $(U_I P U_I^*)$ should be equal (up to an overall phase) to the unit matrix

$$(U_I P U_I^*) = e^{i\Phi_I}\mathbf{1}_D. \tag{C11}$$

After multiplying U_I^{t} from the right and making transposition, we deduce

$$U_I = e^{-2i\Phi_I}P^2 U_I = e^{-2i\Phi_I}U_I \Leftrightarrow e^{-i\Phi_I} = \pm 1. \tag{C12}$$

Therefore, we obtain Eq. (78):

$$U_I^{\text{t}} = \pm P U_I. \tag{C13}$$

It is interesting to calculate U_I for superspin-S UOSp(1|2) SVBS states. For the $\mathcal{S} = 1$ SVBS state, U is identified as

$$U_I = \mathcal{R}_{1|2} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{C14}$$

which satisfies

$$U_I^\dagger \Gamma(m) U_I = -\Gamma(m)^{\text{st}}, \quad U_I^{\text{t}} = -P U_I. \tag{C15}$$

For the $\mathcal{S} = 2$ SVBS state, U is identified as

$$U_I = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{C16}$$

and $\Gamma(m)$ satisfy the relation

$$U_I^\dagger \Gamma(m) U_I = +\Gamma(m)^{\text{st}}, \tag{C17a}$$

$$U_I^{\text{t}} = P U_I. \tag{C17b}$$

For the $\mathcal{S} = 1$ UOSp(1|4) SVBS state, we use the relations given in Appendix A 1 to show that

$$\Gamma(m)^{\text{st}} = U_I^\dagger \Gamma(m) U_I, \quad U_I^{\text{t}} = -P U_I \tag{C18}$$

with $U_I = \mathcal{W}$ defined in Eq. (A5). This is consistent with the existence of the fourfold-degenerate entanglement level in this state (see Fig. 7).

2. Time-reversal symmetry

If the MPS is invariant under time reversal, the Γ matrices satisfy[53]

$$\sum_n R_{mn}^y(\pi) \Gamma^*(n) = e^{i\theta_T} U_T^\dagger \Gamma(m) U_T, \tag{C19}$$

where the rotation matrix $R_{mn}^y(\pi)$ takes the block-diagonal form

$$R^y(\pi) = \begin{pmatrix} R_{\mathcal{S}}^y(\pi) & \mathbf{0} \\ \mathbf{0} & R_{\mathcal{S}-1/2}^y(\pi) \end{pmatrix} \tag{C20}$$

with $R_{\mathcal{S}}^y(\pi)$ and $R_{\mathcal{S}-1/2}^y(\pi)$ being the ordinary rotation matrices for spin-$\mathcal{S}$ and $(\mathcal{S} - 1/2)$, respectively. Since $\mathcal{T}^2 = \mathcal{P}$ [see Eq. (85)],

$$\begin{aligned}
(-1)^{F(l)}\Gamma(l) &= \sum_{m=1}^d R_{lm}^y \left\{ \sum_{n=1}^d R_{mn}^y \Gamma^*(n) \right\}^* \\
&= \sum_{m=1}^d R_{lm}^y \left\{ e^{-i\theta_T} U_T^{\text{t}} \Gamma^*(m) U_T^* \right\} \\
&= \{U_T U_T^*\}^\dagger \Gamma(l)\{U_T U_T^*\}
\end{aligned} \tag{C21}$$

or equivalently

$$\Gamma(l) = \{U_T U_T^*\}^\dagger (-1)^{F(l)} \Gamma(l)\{U_T U_T^*\}. \tag{C22}$$

By using the property

$$(-1)^{F(l)}\Gamma(l) = P\Gamma(l)P, \tag{C23}$$

Eq. (C22) may be rewritten as

$$\Gamma(l) = \{U_T P U_T^*\}^\dagger \Gamma(l)\{U_T P U_T^*\}. \tag{C24}$$

Now, we can apply Eqs. (C1) and (C8) to conclude

$$U_T^{\text{t}} = \pm P U_T. \tag{C25}$$

For $\mathcal{S} = 1$ UOSp(1|2) SVBS state, with $\Gamma(1) = \mathcal{A}(1)$, $\Gamma(2) = \mathcal{A}(0)$, $\Gamma(3) = \mathcal{A}(-1)$, $\Gamma(4) = \mathcal{A}(1/2)$, $\Gamma(5) = \mathcal{A}(-1/2)$ (22), and

$$U_T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R^y(\pi) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{C26}$$

we have

$$\sum_n R^y_{mn}(\pi)\Gamma^*(n) = U_T^\dagger \Gamma(m) U_T \qquad (C27)$$

and

$$U_T^{\mathrm{t}} = -P U_T. \qquad (C28)$$

For $\mathcal{S} = 2$ UOSp(1|2) SVBS state, with

$$U_T = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad (C29)$$

$$R^y(\pi) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

we have

$$\sum_n R^y_{mn}(\pi)\Gamma^*(n) = +U_T^\dagger \Gamma(m) U_T \qquad (C30)$$

and

$$U_T^{\mathrm{t}} = +P U_T. \qquad (C31)$$

3. $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

Finally, consider the π rotation around the x and the z axes,

$$\Gamma(m) \to \sum_n R^a_{mn}(\pi)\Gamma(n) \quad (a = x,z). \qquad (C32)$$

Instead of $(R^a)^2 = \mathbf{1}$ in the bosonic case, R^a in the SUSY case satisfies $(R^a)^2 = \mathcal{P}$ [$(\mathcal{P})_{mn} = \delta_{mn}(-1)^{F(n)}$]. Therefore, the use of the terminology $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is not precise. However, to underline the connection to its bosonic counterpart, we use the terminology in the SUSY cases as well.

When the MPS has such a symmetry, we have[53]

$$\sum_n R^a_{mn}(\pi)\Gamma(n) = e^{i\theta_a} U_a^\dagger \Gamma(m) U_a \quad (a = x,z) \quad (C33)$$

for some block-diagonal unitary matrix

$$U_a = \begin{pmatrix} U_{a,\mathrm{B}} & 0 \\ 0 & U_{a,\mathrm{F}} \end{pmatrix}. \qquad (C34)$$

Now, let us consider what (C33) implies. We begin by $(R^a)^2 = \mathcal{P}$ (valid for integer superspin S):

$$(\mathcal{P})_{nn}\Gamma(n) = P\Gamma(n)P$$
$$= e^{i\theta_a} \sum_m R^a_{mn}(\pi) U_a^\dagger \Gamma(m) U_a$$
$$= e^{2i\theta_a}(U_a^\dagger)^2 \Gamma(n) U_a^2, \qquad (C35)$$

which, after Ps are rearranged, reads as

$$\Gamma(n) = e^{2i\theta_a}(U_a P U_a)^\dagger \Gamma(n)(U_a P U_a), \qquad (C36)$$

implying

$$(U_a P U_a) = e^{i\phi_a}\mathbf{1}_D . \qquad (C37)$$

The phase $e^{i\phi_a}$ can be absorbed in the definition of U_a and we have

$$(U_a P U_a) = \mathbf{1}_D \Leftrightarrow U_a^\dagger = P U_a. \qquad (C38)$$

Next, we consider the product of the two rotations R^x and R^z. In the case of SUSY, they obey the following exchange relation:

$$R^x R^z = \mathcal{P} R^z R^x. \qquad (C39)$$

When combined with Eq. (C33), this translates into the following relation for Γ:

$$(U_x U_z)^\dagger \Gamma(m)(U_x U_z) = (U_z P U_x)^\dagger \Gamma(m)(U_z P U_x). \qquad (C40)$$

After rearranging the U's, we arrive at the form to which Eqs. (C1) and (C8) are applicable:

$$\Gamma(m) = (U_z P U_x U_z^\dagger U_x^\dagger)^\dagger \Gamma(m)(U_z P U_x U_z^\dagger U_x^\dagger). \qquad (C41)$$

Therefore, we have

$$U_z P U_x U_z^\dagger U_x^\dagger = e^{i\phi_{xz}}\mathbf{1}_D \qquad (C42)$$

with $e^{i\phi_{xz}} = \pm 1$. The resulting equation

$$U_x U_z = \pm P U_z U_x \qquad (C43)$$

or

$$U_{x,1}U_{z,1} = \pm U_{z,1}U_{x,1}, \quad U_{x,2}U_{z,2} = \mp U_{z,2}U_{x,2} \quad (C44)$$

implies the degenerate structure of the entanglement spectrum.

Let us calculate U matrices for superspin-$\mathcal{S}$ UOSp(1|2) SVBS states. For odd $\mathcal{S}$, they assume the following form:

$$U_a^{(\mathcal{S})} = -i\begin{pmatrix} R^a_{\mathcal{S}/2}(\pi) & \mathbf{0} \\ \mathbf{0} & R^a_{(\mathcal{S}-1)/2}(\pi) \end{pmatrix}, \qquad (C45a)$$

which satisfy

$$U_x U_z = -P U_z U_x. \qquad (C45b)$$

Therefore, the degenerate spectrum appears in the bosonic sector.

For even $\mathcal{S}$, on the other hand, they are given by

$$U_a^{(\mathcal{S})} = \begin{pmatrix} R^a_{\mathcal{S}/2}(\pi) & \mathbf{0} \\ \mathbf{0} & R^a_{(\mathcal{S}-1)/2}(\pi) \end{pmatrix}, \qquad (C46a)$$

satisfying

$$U_x U_z = +P U_z U_x, \qquad (C46b)$$

which implies that the fermionic spectrum exhibits the degenerate structure.

4. $(\mathbb{Z}_2 \times \mathbb{Z}_2)^2$ symmetry

In this appendix, we summarize some useful relations concerning the A matrices of the UOSp(1|4) $\mathcal{S}=1$ SVBS states given in Appendix A 1. The invariance of the MPS under $R^{ab}(\pi)$ defined in Eq. (96) implies[53] the existence of the 5×5 unitary matrices U_{ab} satisfying

$$\sum_{n=1}^{14} [R^{ab}(\pi)]_{mn} A(n) = +U_{ab}^\dagger A(m) U_{ab}. \tag{C47}$$

Specifically, U_{ab} are given by

$$U_{12} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & i \end{pmatrix},$$

$$U_{25} = \begin{pmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \end{pmatrix}, \tag{C48a}$$

$$U_{34} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & i \end{pmatrix},$$

$$U_{45} = \begin{pmatrix} 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \end{pmatrix}. \tag{C48b}$$

It is easy to check that these matrices satisfy

$$(U_{12})^2 = (U_{25})^2 = (U_{34})^2 = (U_{45})^2 = \mathcal{P}_{1|4},$$
$$U_{12}U_{25} = -\mathcal{P}_{1|4}U_{25}U_{12}, \quad U_{34}U_{45} = -\mathcal{P}_{1|4}U_{45}U_{34},$$
$$U_{25}U_{45} = -\mathcal{P}_{1|4}U_{45}U_{25},$$
$$U_{12}U_{34} = U_{34}U_{12}, \quad U_{12}U_{45} = U_{45}U_{12}, \quad U_{25}U_{34} = U_{34}U_{25}, \tag{C49}$$

where

$$\mathcal{P}_{1|4} = \begin{pmatrix} 1_4 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C50}$$

By the general argument in Sec. V D, one concludes that in some sectors *all* the entanglement levels are four×(integer)-fold degenerate as is seen in Fig. 7.

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