Random evolution approach to universal conductance statistics

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It is shown that the Dorokhov-Mello-Pereyra-Kumar equation can be solved by propagating an ordinary stochastic differential equation. Such a random evolution approach allows any transport statistics to be easily calculated from the ballistic to localization regime for an arbitrary number of channels. As an example, a disordered wire with reflectionless contacts is considered. The conductance distribution, transmission channel density, and shot-noise suppression are fully analyzed.

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Quantum transport through a disordered wire is one of the fundamental problems in mesoscopic physics. At low enough temperatures, electrons can travel over a distance of the order of the localization length ξ without undergoing inelastic scattering. The conductance statistics in such phase-coherent conductors displays many interesting features. The most famous example is universal conductance fluctuations (UCFs):[1] The amplitude of the conductance variation from sample to sample is of the order of e^2/h and it does not depend on the size of the sample, its material properties, and average conductance.

Since the early experimental observations of UCFs in normal metals,[2] a large body of work has been devoted to understanding the basic physics of mesoscopic conductors. The universality in the conductance statistics suggested that it could be described by a relatively simple Hamiltonian and led to the development of a random-matrix theory (RMT) of quantum transport.[3–6] In this approach one considers a wire with N conducting channels and calculates the $N \times N$ transmission matrix t based on the flux conservation and symmetry of the system with respect to time reversal. All the physical characteristics can be evaluated once the eigenvalues $T_1, T_2, \ldots, T_N$ of the product $tt^\dagger$ are known. For example, the conductance (in units of $G_0 = \frac{2e^2}{h}$) is given by the transmission coefficient $g = \mathrm{Tr}(tt^\dagger) = \sum_n T_n$. Introducing another set of variables $\lambda_n = \frac{1-T_n}{T_n}$, one can show that the probability distribution $P(\boldsymbol{\lambda}, L)$ as a function of length L of the disordered wire satisfies the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation[4,6]

$$\frac{\partial P}{\partial L} = \frac{2}{\xi} \sum_{i=1}^{N} \frac{\partial}{\partial \lambda_i} \left[\lambda_i (1+\lambda_i) J \frac{\partial}{\partial \lambda_i} \frac{P}{J} \right], \tag{1}$$

where $J = \Pi_{i<j} |\lambda_i - \lambda_j|^\beta$. The symmetry index is $\beta = 1$ for systems with time-reversal symmetry, $\beta = 2$ if the time-reversal symmetry is broken, and $\beta = 4$ if the time-reversal symmetry is conserved but the spin-rotational symmetry is broken.

The DMPK equation is the starting point to study the transition from a metallic to insulating regime, weak localization, effects of point contacts or tunnel barriers, statistics of metal-superconductor junctions, and shot-noise reduction. An exact solution of the DMPK equation is only known for $\beta = 2$.[7] Most of the results have been obtained in the large-N limit for the metallic and deep localization regimes (see, e.g., Ref. 6 and references therein). The conductance probability distribution $P(g)$ at small N was calculated by a Monte Carlo technique based on mapping Eq. (1) to a Schrödinger equation for N interacting fermions.[8] An asymptotic analytical form of $P(\boldsymbol{\lambda}, L)$ for arbitrary N was found in Ref. 9 and used in Ref. 10 to study the transmission channel density $\rho(T) \equiv \langle \sum_n \delta(T - T_n) \rangle$.

In spite of this progress, the crossover between the metallic and localization regimes and the statistical properties of systems with a finite number of propagating modes are still not well understood. Experimental observation of the anomalous conductance distribution in gold wires was reported in Ref. 11. Numerical calculations of $P(g)$ showed asymmetric broad distributions which agreed with the one-side log-normal behavior obtained from the maximum-entropy ansatz.[8,12,13] These results concur with the critical conductance distribution at an integer quantum Hall transition,[14,15] suggesting that anomalous distribution may be independent on the microscopic details. Transport studies in the framework of the Anderson model also indicate that $P(g)$ in quasi-one-dimensional conductors differ quantitatively from the distributions in systems with stronger disorder.[16,17] At the same time, recent measurements of the transmission channel density in nanoscale contacts show that the transport statistics even at small g may be unexpectedly close to RMT predictions in the universal diffusive limit $1 \ll g \ll N$.[18]

Strong conductance fluctuations have been observed also in semiconductor nanowire heterostructures.[19] The conductance in random wires with a moderate number of channels has been calculated with the effective mass approximation,[20] tight-binding models,[21–24] and an *ab initio* approach.[25] These studies display a very similar behavior: var(g) reaches a maximum value close to the classical result $2/15$ (Refs. 3 and 5) at L of the order of the mean free path l_e and shows the same universal trend at larger L, which can be characterized by a single length parameter l_e (or ξ). Its energy dependence correlates with the band structure of the pristine wire,[21,26–28] suggesting that the transport statistics can be explained by the N-channel model.

In this paper we present a general scheme for solving the DMPK equation with time-reversal symmetry in a wide range of its parameters. Our method is based on the observation that Eq. (1) is equivalent to a random evolution defined by a N-dimensional Riccati-type matrix equation

$$\frac{du}{dx} = uH^\dagger u - H, \tag{2}$$

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with a symmetric initial condition $u^T(0) = u(0)$ and δ-correlated stochastic coefficients

$$H_{ij} = H_{ji}, \quad \langle H \rangle = 0, \quad \langle H_{ij}(x) H_{kl}(x') \rangle = 0, \tag{3}$$

$$\langle H_{ij}(x) H^*_{kl}(x') \rangle = D(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta(x - x'), \tag{4}$$

where $\langle \cdots \rangle$ stands for averaging over realization.

To establish this connection, we introduce the Hermitian matrix $\mathcal{T} = 1 - uu^\dagger$ and show that the Fokker-Planck equation for its eigenvalues coincides with Eq. (1). We note that, since the propagation in Eq. (2) preserves the symmetry, $u^T = u$ and $\mathcal{T} u = u \mathcal{T}^T$ for any realization of H. Let $F(t_1, t_2, \ldots) \equiv F(\boldsymbol{t})$ be an arbitrary real-valued function of $t_n \equiv \mathrm{Tr}\, \mathcal{T}^n$. From Eq. (2) we obtain

$$\frac{d}{dx}\langle F \rangle = \sum_n n \left\langle \frac{\partial F}{\partial t_n} \mathrm{Tr}(\mathcal{T}^n u H^\dagger) \right\rangle + \text{c.c.} \tag{5}$$

The averaging is performed as usual by substituting all the functions of $u(x)$ on the right hand side by the corresponding formal integral solutions $\int^x \ldots H(x')dx'$ and taking advantage of the δ correlator in Eq. (4) to eliminate the integrals. We obtain various terms in the form $\langle \ldots \mathrm{Tr}(\mathcal{T}^n)\mathrm{Tr}(\mathcal{T}^m) \rangle$ or $\langle \ldots \mathrm{Tr}[(u^\dagger \mathcal{T}^n)(\mathcal{T}^m u)^T] \rangle$, where $\ldots$ stands for the first or second partial derivatives of F. Because of the symmetry of u, the latter expression can be reduced to $\langle \ldots (t_{n+m} - t_{n+m+1}) \rangle$ and we arrive at

$$\begin{aligned} \frac{d}{dx}\langle F \rangle = D \Bigg\langle & \sum_n (n(n-1)t_n - n^2 t_{n+1}) \frac{\partial F}{\partial t_n} \\ & + 2 \sum_{nm} nm(t_{n+m} - t_{n+m+1}) \frac{\partial^2 F}{\partial t_n \partial t_m} \\ & + \sum_n n \sum_{m=1}^{n-1} t_{n-m} t_m \frac{\partial F}{\partial t_n} - \sum_n n \sum_{m=1}^{n} t_{n-m+1} t_m \frac{\partial F}{\partial t_n} \Bigg\rangle. \end{aligned} \tag{6}$$

Equation (6) is equivalent to a Fokker-Planck equation for the probability distribution on the infinite set of $\{t_n\}$. These are not independent and can be expressed as $t_n(\boldsymbol{\lambda}) \equiv \sum_{i=1}^N (1 + \lambda_i)^{-n}$ in terms of N variables λ_i. Then, one can easily check that a solution of Eq. (6) is given by

$$\langle F(\boldsymbol{t}) \rangle_x = \int d\boldsymbol{\lambda} F(\boldsymbol{t}(\boldsymbol{\lambda})) P(\boldsymbol{\lambda}, D\xi x), \tag{7}$$

where P satisfies Eq. (1) with $\beta = 1$. This result shows that any statistics of the transmission eigenvalues in the DMPK equation can be calculated by averaging the corresponding function of the eigenvalues of $\mathcal{T}$ (i.e., $uu^\dagger$) in Eq. (2) over realizations of H.

The Riccati-type equation appears naturally in the framework of quantum scattering theory. As shown in Ref. 29, the problem of wave transmission through a random media can be generally reduced to a initial-value problem and the reflection coefficient satisfies a stochastic Riccati equation relative to the size of the disordered region. In the present case, Eq. (2) can be shown to describe the evolution of the reflection matrix in a continuous $2N \times 2N$ matrix model of N channels with unit group velocity and the off-diagonal backscattering term given by Eqs. (3) and (4). $H^T = H$ then becomes a necessary condition to ensure the S-matrix symmetry (a part of which is $u^T = u$) in the absence of magnetic field. The kinetic mean free path in this model $l_e = \frac{1}{2D(N+1)}$ and Eq. (7) gives $\xi = 2l_e(N + 1)$, in agreement with RMT.[6] In this connection, we should mention other microscopic models,[30,31] which provide an underlying stochastic Hamiltonian for the DMPK equation. However, to our best knowledge, an equivalent stochastic dynamics has never been used for solving Eq. (1).

In the rest of this paper, we apply the random evolution approach to calculate the most important statistical characteristics in the crossover and localization regimes. All the results are presented in terms of the dimensionless length $s \equiv 2L/\xi$. In practice we use a discrete analog of the δ-correlated noise and rescale H,x in order to obtain unit propagation steps. The numerical solution of the matrix Riccati equation is very stable and, unlike the transfer-matrix method, the dynamical range problem[32] does not arise in this case. Physical solutions of Eq. (2) remain uniformly bounded, $|u_{ij}| < 1$, and can be safely propagated over large distances for eventually any N. Each realization starts from $u(0) = 0$, which corresponds to the initial probability distribution $\Pi\delta(\lambda_i)$ in the DMPK equation (a piece of disordered wire of length L between two ideal leads). Other initial conditions are also possible but not considered here. The simulations proceed as follows. (1) Set $u = 0$. (2) Generate $N(N + 1)$ independent random numbers $\{x_{\nu\mu}, y_{\nu\mu}\}$, $\nu \leqslant \mu$ from $N(0,\Delta)$ and construct a symmetric $N \times N$ matrix, $H : H_{\nu\nu} = (x_{\nu\nu} + iy_{\nu\nu})$, $H_{\nu\mu} = H_{\mu\nu} = \frac{1}{\sqrt{2}}(x_{\nu\mu} + iy_{\nu\mu})$, $\nu < \mu$. (3) Propagate Eq. (2) over the unit x interval. (4) Repeat steps (2) and (3) M times and obtain a realization of $\{T_n\}$ at $s = 2M\Delta$ (as well as at all the intermediate $s < 2M\Delta$). Repeat steps (1)–(4) N_s times to accumulate statistics and calculate desired distributions or average values. The choice of the discretization parameter Δ depends on the problem. $\Delta \sim 0.01$ is found to be small enough to mimic the random evolution in the diffusion regime. Smaller Δ may be needed at $s > 2$. In the calculations below we used $\Delta \sim 10^{-2}$–10^{-3} and $N_s \sim 10^5$–10^6.

The conductance density distribution. Figure 1 shows the evolution of the density distribution $P(g)$ in a wire with $N = 5$ propagating channels from a Gaussian in the quasiballistic regime to the one-side log-normal distribution in the crossover regime. The distributions at selected $\langle g \rangle$ values in the insets reproduce the previous results of the Monte Carlo simulation for $N = 6$.[8] Our results show that the system "forgets" the initial conditions at $\langle g \rangle \sim 1$ (Thouless criterion) and the density distribution $P(g)$ (at given $\langle g \rangle$) becomes nearly independent on N. Note, however, that the L dependence of $\langle g \rangle$ may be still quite different in this regime.

The variance of the conductance var(g) as a function of s is shown in Fig. 2 for different numbers of propagated channels. For comparison we also show var(g) calculated from the exact probability distribution[33] for $N = 1$ and the asymptotic approximation[9] for $N = 2$–5. The latter demonstrates an unexpectedly good accuracy far beyond the metallic regime, which provides support for the recent calculations of the transmission channel density.[10] The inset in Fig. 2 shows on an enlarged scale the crossover regime where var(g) approaches the universal value 2/15 in the large-N limit. The straight line

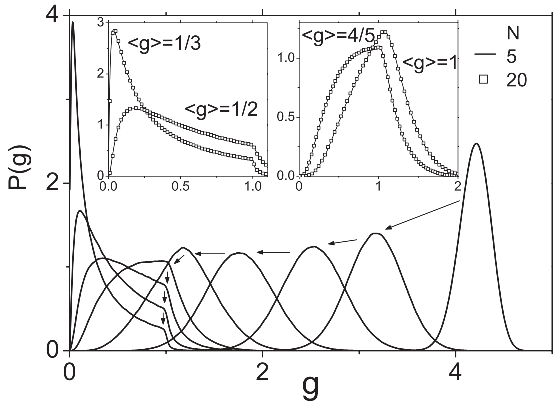

FIG. 1. The probability distribution $P(g)$ in a wire with $N = 5$ channels at various s. The arrows show the direction of increasing s from the quasiballistic ($s = 0.06$, $\langle g\rangle \approx 4.2$) to weak localization ($s = 3.2$, $\langle g\rangle \approx 0.28$) regime. Insets: $P(g)$ at four selected $\langle g\rangle$ values in the crossover regime. The distributions for $N = 5$ (solid lines) and $N = 20$ (open squares) coincide within a few percent.

represents the universal asymptote $\frac{2}{15} - \frac{4}{315}s$ obtained from the nonlinear sigma model.[35]

The transmission channel density $\rho(T)$ carries more detailed information and allows any linear statistics $\sum_{i=1}^{N} f(T_i)$ on the transmission eigenvalues $\{T_i\}$ to be evaluated. In the universal diffusive regime, RMT predicts the bimodal distribution

$$\rho(T) = \frac{\langle g\rangle}{2T\sqrt{1-T}}, \qquad (8)$$

with an appropriate cutoff at small T to ensure the normalization $\int_0^1 \rho(T)dT = N$. One of the consequences of Eq. (8) is the $1/3$ shot-noise suppression, which is another striking universal property of the diffusive conductors.[36] Our calculations show that a close-to-universal regime can be actually realized in a thin wire with a small number of conducting channels. Figure 3 shows the evolution of the normalized density $\tilde{\rho} \equiv \rho/\langle g\rangle$ from a quasiballistic density peak around $T = 1$ at $s = 0.1$ to the bimodal distribution (red thicker line) in the crossover regime. This behavior correlates with Fig. 2: $\tilde{\rho}$ approaches the bimodal distribution at the same $s \approx s_{max}$ where the variance of the conductance reaches a maximum value. Figure 3(d) also demonstrates that the channel density at s_{max} becomes very close to the bimodal distribution even at small $N \sim 5$–10. $\tilde{\rho}$ "dwells" in the vicinity of the universal distribution within a finite s interval $\sim[s_{max},1]$ and at larger s starts changing towards developing a localization peak around $T = 0$ (not shown).

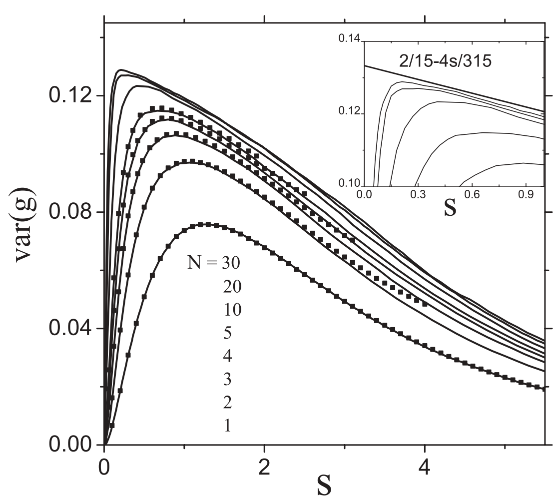

FIG. 2. The variance of the conductance var(g). The number of channels varies from $N = 1$ (the lowest curve) to $N = 30$ (the top curve). var(g) calculated from the exact (Ref. 33) ($N = 1$) and asymptotic (Ref. 9) ($N = 2$–5) probability distributions are shown for comparison (black squares). Inset: var(g) is approaching the universal asymptote $\frac{2}{15} - \frac{4s}{315}$ in the large-N limit.

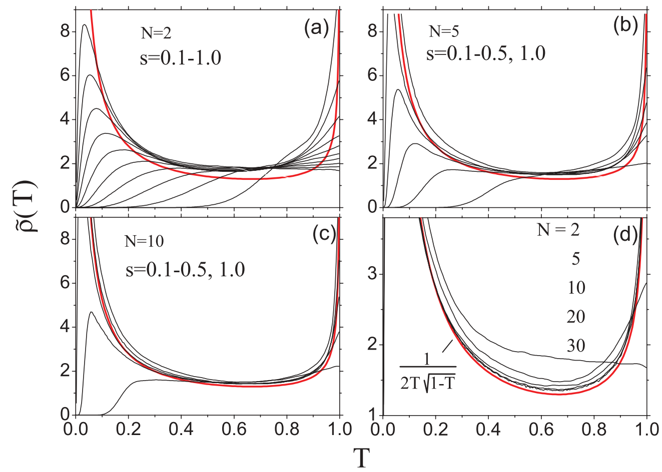

FIG. 3. (Color online) The normalized transmission channel density $\tilde{\rho}(T) = \frac{1}{\langle g\rangle}\left\langle \sum_n \delta(T - T_n)\right\rangle$. (a)–(c) Evolution of $\tilde{\rho}(T)$ in the interval $s \in [0:1]$ (the step size $\Delta s = 0.1$) for $N = 2$, 5, and 10. (d) The "closest-to-universal" density is approaching the bimodal distribution $\frac{1}{2T\sqrt{1-T}}$ at increasing N (see the text). The number of channels N is shown in the same order (from the top down) as the corresponding curves.

The shot-noise power is another important transport characteristics. In the absence of correlation among the carriers, the charge transfer can be considered to be a Poisson process with the shot-noise power $P_{\rm Poisson} = 2eI$ proportional to the time-averaged current. In a phase-coherent conductor, the quantum theory predicts the noise reduction $P/P_{\rm Poisson} = \frac{1}{\langle g\rangle}\langle \sum_n T_n(1 - T_n)\rangle$, which is sensitive to the carrier statistics.[36,37] P thus contains different information on the quantum transport which is not given by the conductance and can be directly measured in the experiments. The results of calculations for $N = 10$ and 50 are presented in Fig. 4. The ratio $P/P_{\rm Poisson}$ reaches the universal value $1/3$ at $s \approx s_{max}$ and then shows a nearly linear moderate growth which slows down in the localization regime. The inset in Fig. 4 shows in more detail the shot-noise suppression at various N in the crossover regime as a function of length s and average conductance $\langle g\rangle$. $P/P_{\rm Poisson}$ is seen to approach the large-N asymptote (thicker line), obtained from the leading terms in the perturbation expansion.[34]

In conclusion, we have shown that any statistics of the transmission eigenvalues in the DMPK equation can be calculated by solving a single Riccati equation. The random

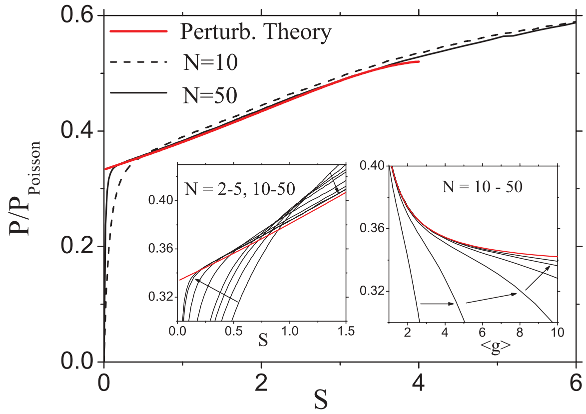

FIG. 4. (Color online) Suppression of the Poisson shot noise in a wire with 10 and 50 channels. Insets: N dependence of P/P_{Poisson} in the crossover regime. The arrows show the direction of increasing N. The thicker (red) lines correspond to the g^{-1} perturbation expansion (Ref. 34).

evolution approach makes it possible to obtain a full statistical description of quasi-one-dimensional conductors from the ballistic to localization regime. Examples of such calculations have been presented for a disordered wire sandwiched between ideal leads. We have obtained the conductance distribution, transmission channel density, and have calculated the conductance fluctuations and shot-noise reduction in a wide range of wire sizes. Our results reproduce all the previously reported data and predict a number of different general properties. We have also found that universal statistical behavior can be observed in thin quantum wires with a small number of propagated channels. Such a situation can be realized in one-dimensional nanomaterials which have recently attracted much attention for their interesting physical properties and wide range of potential applications. Apart from its theoretical interest, a detailed understanding of the statistical behavior of these systems may become important in addressing practical issues related to sample-to-sample variability.[38] In this paper, we have only considered a spinless system with time-reversal symmetry. A similar approach can be developed for other types of symmetries. We can show that the random evolution for the DMPK equation with $\beta = 2$ ($\beta = 4$) is defined by the Riccati equation with a general complex (self-dual complex quaternion) δ-correlated coefficient matrix. Further details are beyond the scope of this paper and will be given elsewhere.

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