Quantum collapses and revivals of matter wave in dynamics of symmetry breaking

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Quantum processes can be described in terms of a quasiprobability distribution (the Wigner distribution) analogous to the phase-space probability distribution of the classical realm. In contrast to the incomplete glimpse of the wave function that is achievable in a single shot experiment, the Wigner distribution, accessible by quantum state tomography, reflects the full quantum state. We show that during the fundamental symmetry-breaking process of a generic quantum system, with a symmetry-breaking field driving the quantum system far from equilibrium, the Wigner distribution evolves continuously with the system undergoing a sequence of revivals into the symmetry-unbroken state, followed by collapses onto a quasiclassical state akin to the one realized in infinite-size systems. We show that generically this state is completely delocalized both in momentum and in real space.

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I. INTRODUCTION

Spontaneous symmetry breaking causes a macroscopic body under equilibrium condition to have less symmetry than its microscopic building blocks. Probably the phenomenon of superconductivity is the most spectacular example of the symmetry breaking which a macroscopic body spontaneously undergoes. Of course, this is not the only one. Antiferromagnets, liquid crystals, and other states of matter obey this rather general scheme of broken symmetries. The general idea is that when the number N of microscopic quantum constituents which, depending on the system, corresponds to the number of Cooper pairs, particles, or spins goes to infinity, the matter undergoes a "phase" transition to a state in which the microscopic symmetries are violated.[1–5] In the context of quantum magnetism,[6–8] the macroscopic classical state has been described as a combination of "thin spectrum" states emerging in the $N \to \infty$ limit because of the singular nature of the thermodynamic limit. The same description has been shown to apply also to the cases of quantum crystals, Bose-Einstein condensates, and superconductors.[9–13] The theory of spontaneous symmetry breaking explains the stability and rigidity of states which are not eigenstates of the underlying microscopic Hamiltonian, but it makes no assertion on whether or how a symmetry-broken ground state can evolve out of the symmetric state in a closed quantum system. To investigate this, one can perform a gedanken experiment in which a symmetry-breaking perturbation is slowly switched on in an arbitrary large but finite system initially prepared in a fully symmetric state. Using a particular antiferromagnetic model system (the Lieb-Mattis model[6]), it has been recently shown that the corresponding quantum dynamics is dominated by highly nonadiabatic processes triggering the appearance of a symmetric nonequilibrium state that recursively collapses at punctured times into a symmetry-broken state.[14]

Here, we shed light on this far-from-equilibrium symmetry-breaking process that is so ubiquitous in physics (in the formation of crystalline matter, atomic condensates, Josephson junction arrays, and local pairing superconductors to name but a few) by introducing quantum state tomography. In complete analogy with medical diagnostics where three-dimensional images of the inner part of a body can be extracted from NMR or x rays, two-dimensional images obtained at different directions, quantum state tomography determines a quasiprobability distribution in phase space from only position (Q) or momentum (Π) measurements.[15,16] This quasiprobability distribution has been introduced by Wigner in his phase-space formulation of quantum mechanics. For a pure quantum state, the Wigner distribution $W(Q,\Pi)$ is defined in terms of the position wave function $\Psi(Q)$ as $W(Q,\Pi) = \pi^{-1}\int_{-\infty}^{\infty}\Psi^{\star}(Q-S)\Psi(Q+S)e^{-2i\Pi S}dS$ (in $\hbar = 1$ units) and retains the marginal probability distributions[17]

$$\int W(Q,\Pi)d\Pi = |\Psi(Q)|^2,$$

$$\int W(Q,\Pi)dQ = |\Psi(\Pi)|^2.$$

By detecting the position of many objects prepared in the same quantum state yields the spatial distribution of the wave function $|\Psi(Q)|^2$ as a spacelike shadow of the Wigner distribution. This, in turns, allows for a tomographic reconstruction of $W(Q,\Pi)$ once various shadows at different directions in phase space have been observed.[18]

We unravel these snapshots in the dynamics of symmetry breaking by using the paradigmatic example of a harmonic crystal to show that the quantum dynamics is generically characterized by the appearance of revivals of the symmetric ground-state wave function followed by collapses towards a quasiclassical state akin to the symmetry-broken ground state of infinite-size system. In this quasiclassical state, however, the matter wave has maximum uncertainty both in total position (precisely as in the symmetric translational-invariant ground state) and in total momentum. The exceptions are punctured times which render a Dirac comb of symmetry-broken states.[14] Interestingly, we find this sequence of collapses/revivals of the ground-state wave function to occur on a characteristic time scale set by Zurek's equation of nonequilibrium quantum phase transition.[19]

II. SPONTANEOUS SYMMETRY BREAKING AND THE THIN SPECTRUM

The Nambu-Goldstone theorem[20,21] guarantees the existence of low-energy gapless collective excitations in systems with spontaneously broken continuous symmetries. The low-energy Hamiltonian for these normal modes, which depending on the particular system at hand correspond to phonons, spin waves, or Bogoliubov's excitations, can be always recast in the form

$$\mathcal{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \tag{1}$$

where $\omega_{\mathbf{k}}$ indicate the frequencies of the Goldstone mode excitations vanishing in the long-wavelength limit. If there is a spontaneously broken symmetry, the motion along the continuous symmetry axis characterizing the quantum mechanics of the macroscopic body as whole can not be of the form[12] $b^{\dagger}b$, but it will be either given by $\Pi^2/(2I)$ (as in crystals and Josephson junction arrays) or $Q^2/(2I)$ (e.g., antiferromagnets and local pairing superconductors)[10,13] where Q and Π are the coordinate and the conjugate momentum operators along the symmetry axis, whereas the parameter I depends on the total number of microscopic quantum constituents and diverges in the thermodynamic limit where $N \to \infty$. This is explicitly manifest by considering the specific example of a harmonic crystal with Hamiltonian

$$\mathcal{H} = \sum_j \frac{p_j^2}{2m} + \frac{\kappa}{2} \sum_j (x_j - x_{j+1})^2, \tag{2}$$

where the index j labels the N atoms in the lattice which have mass m, momentum p_j, and position x_j. The harmonic potential among neighboring atoms is parametrized by the constant κ. Let us now define the bosonic annihilation and creation operators $b_j = [C x_j + i p_j/C]/\sqrt{2}$ and $b_j^{\dagger} = [C x_j - i p_j/C]/\sqrt{2}$ where $C = (2 m \kappa)^{1/4}$. In momentum space, the Hamiltonian (2) reduces to

$$\mathcal{H} = \sqrt{\frac{\kappa}{2m}} \sum_k \left(A_k b_k^{\dagger} b_k + \frac{B_k}{2} (b_k^{\dagger} b_{-k}^{\dagger} + b_k b_{-k}) + 1 \right),$$

where $A_k = 2 - \cos(ka)$, $B_k = -\cos(ka)$, a is the lattice constant, and we have set $\hbar = 1$. This Hamiltonian is not diagonal since the terms $b_k^{\dagger} b_{-k}^{\dagger}$ and $b_k b_{-k}$ create and annihilate two bosons at the same time. One can get rid of them by performing a canonical Bogoliubov transformation. However, the parameters in the Bogoliubov transformation diverge as $k \to 0$ and thus the canonical transformation is not well defined.[10] This implies that one should investigate separately the $k = 0$ component

$$\mathcal{H}_{k=0} = \sqrt{\frac{\kappa}{2m}} \left[1 - \frac{1}{2} (b_0^{\dagger} - b_0)^2 \right].$$

This part of the Hamiltonian describes the fact that the quantum crystal carries a kinetic energy associated with the combined mass of all N atoms. Going back to real space, it reads as

$$\mathcal{H}_{k=0} = \frac{\Pi^2}{2 m N}, \tag{3}$$

where $\Pi = \sum_j p_j$ is the total momentum of the entire system and we left out a negligible constant. The ground state of this Hamiltonian has total zero momentum: it has no uncertainty in total momentum and maximum uncertainty in total position, thereby implying that the translational symmetry is unbroken. At finite N, the excitations over the ground state respecting the symmetry give rise to a tower of ultralow-energy states that form the so-called "thin spectrum."[10,11] It is "thin" because it contains states that are of such low energy that their contribution to thermodynamic quantities vanish in the thermodynamic limit. Nevertheless, when $N \to \infty$, the thin spectrum excitations collapse to form a degenerate continuum of states. Within this continuum, even a vanishingly small symmetry-breaking perturbation is enough to couple different thin spectrum states thereby stabilizing the symmetry-broken ground state. To show this, let us take into account the effect of a symmetry-breaking perturbation (a pinning potential for the individual atoms) rendering a symmetry-breaking Hamiltonian $\mathcal{H}_{\text{SB}} = -V \sum_j \cos(2\pi x_j/a)/(2\pi)^2$. For small deviations of the atoms from their mean positions, the zero-momentum term in lowest order is given[10] by $\mathcal{H}_{\text{SB}} = B x_{tot}^2/(2N)$ where $x_{tot} = \sum_j x_j$ and $B = V/a^2$. This, in turn, implies that the collective behavior of the harmonic crystal as a whole is governed by the harmonic-oscillator Hamiltonian

$$\mathcal{H}_c = \frac{\Pi^2}{2N} + \frac{B N Q^2}{2}, \tag{4}$$

where for simplicity we have set $m = 1$ and we introduced the center-of-mass coordinate $Q = x_{tot}/N$ satisfying the canonical commutation relation $[Q,\Pi] = i$. The quantum of energy of this Hamiltonian $\Delta E \propto \sqrt{B}$ and the excitations over the ground state realize a "dual" thin spectrum. The ground-state wave function corresponds to a Gaussian wave packet for the collective coordinate Q of the form $\Psi_0(Q) \propto e^{-Q^2/2L^2}$ with width $L \propto (N\sqrt{B})^{-1/2}$. For a vanishing symmetry-breaking field and a finite number of atoms, we find that the ground-state wave function obviously collapses onto the symmetric ground state of the microscopic Hamiltonian. However, by taking first the thermodynamic limit, the center-of-mass position becomes completely localized even if at the end the symmetry-breaking field is set to zero. Therefore, one finds that the system is in a stable state which is not an eigenstate of the underlying microscopic Hamiltonian: the system is inferred to spontaneously break the symmetry. Strictly speaking, only truly infinite-size systems are allowed to spontaneously break the symmetry. A large, but not infinitely large crystal requires a finite symmetry-breaking field to stabilize a symmetry-broken state over the symmetric ground state of the microscopic Hamiltonian.

III. QUANTUM DYNAMICS OF SYMMETRY BREAKING

Let us then consider such a large but finite system with a pinning potential whose strength is switched on linearly in time as $B(t) = \delta t$ with ramp rate δ. At initial time t_0, we consider a finite symmetry-breaking perturbation $B(t_0) \equiv B_0$ [cf. inset of Fig. 1(b)] and the wave function of the system in the corresponding ground state. We stress that the choice of a finite symmetry-breaking perturbation at initial time is essential in order to have a cutoff guaranteeing the continuity of the

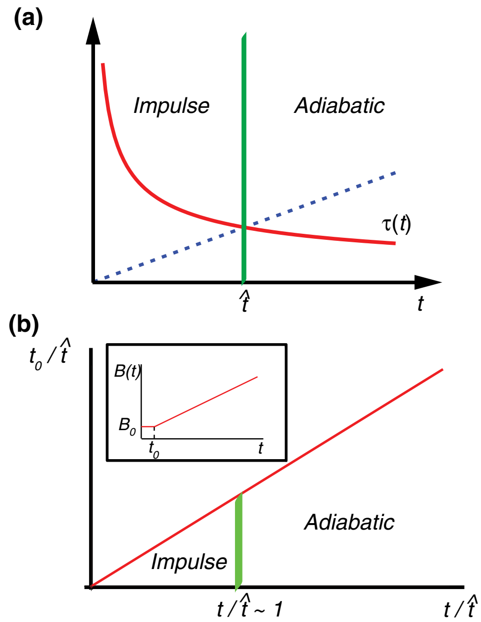

FIG. 1. (Color online) (a) Sketch of the behavior of the relaxation time (continuous line) as compared to the typical time on which the Hamiltonian is changed t determining the freeze-out time of Zurek's equation. (b) The corresponding regimes for the dynamics of symmetry breaking in the $t - t_0$ plane. Times have been measured in units of the freeze-out time $\hat{t}$. The inset shows the setup of the symmetry-breaking field.

wave-function basis. Later on we will consider the limit $B_0 \to 0$ corresponding for finite N to an initial completely symmetric ground state with gapless dual thin spectrum excitations. The inclusion of a cutoff B_0 renders two distinct regimes of the quantum dynamics. Whenever the characteristic relaxation time, which is set by the size of the gap among the ground state and the dual thin spectrum excitations, $\tau(t) = 1/\Delta E(t) = (\delta t)^{-1/2}$ is much smaller than the typical time scale t on which the Hamiltonian is changed, the system is able to react to the changing Hamiltonian thereby rendering an adiabatic passage. In the opposite regime $\tau(t) \gg t$, reflexes of the system are so deteriorated that the state can be considered effectively frozen and the dynamics is impulselike. Clearly, the crux of this story is the instant where the dynamics changes from impulse to adiabatic; it is determined by Zurek's equation[19] $\tau(\hat{t}) \equiv \hat{t}$ and defines the so-called freeze-out time $\hat{t} \propto \delta^{-1/3}$ [cf. Fig. 1(a)]. By considering an initial time $t_0 \gg \hat{t}$, the entire evolution will be thus nearly adiabatic. In this case, fluctuations of the center-of-mass coordinate decrease continuously in time as $\Delta Q^2 = \langle Q^2\rangle - \langle Q\rangle^2 \propto [N\sqrt{\delta t}]^{-1}$. However, for the dynamics of symmetry breaking to be adiabatic, the ramp rate δ is seen to be bounded by $\delta < B_0^{3/2}$. Henceforth, for a vanishing ramp rate δ at finite values of B_0, we recover a quasiadiabatic time evolution. But, taking the $B_0 \to 0$ limit at finite ramp rate δ, the dynamics will start in the impulse regime even if at the end the ramp rate is set to zero: the adiabatic limit can never be reached for a sufficiently small B_0. This is in agreement with the recent finding[22] that adiabatic processes in low-dimensional systems with broken continuous symmetries are absent.

In the same spirit of Ref. 14, to analyze the dynamics of symmetry breaking in the strongly nonadiabatic regime, we first make use of the adiabatic-impulse (AI) approximation which underlies the Kibble-Zurek theory[23–25] of nonequilibrium phase transition. In the AI scheme,[26] the initial state is considered effectively frozen in the impulselike regime $t_0 < t < \hat{t}$ changing only by a trivial overall phase factor. At freeze-out time, the system therefore reaches a state that is a superposition of dual thin spectrum excitations $|\Psi_0(Q,t_0)\rangle = \sum_n c_n |\Psi_n(Q,\hat{t})\rangle$ where the coefficients c_n are nonzero only for even values of the quantum number n. The evolution at $t > \hat{t}$ can be considered to be adiabatic and therefore the dynamics of the wave function is governed by $|\Psi(Q,t)\rangle = \sum_n c_n |\Psi_n(Q,t)\rangle\, e^{-i\Omega_n(t)}$ where we have defined the dynamical phase factor $\Omega_n(t) = \Delta\Omega(t)(n+1/2)$ and $\Delta\Omega(t) = 2/3 \times [(t/\hat{t})^{3/2} - 1]$. Within the AI approximation, we can obtain the time evolution of the symmetry-breaking order parameter (defined by the inverse of the fluctuations of the center-of-mass coordinate) taking explicitly into account quantum phase interference effects. This is unlike, for instance, the case of the Lieb-Mattis model where in the AI scheme the time evolution of the staggered magnetization can be computed only by neglecting interference effects.[14] We find

$$[\Delta Q^2(t)]^{-1} = 2\frac{N}{\hat{t}}\sqrt{\frac{t\,t_0}{\hat{t}^2}}\left[1 - \left(1 - \frac{t_0}{\hat{t}}\right)\sin^2 \Delta\Omega(t)\right]^{-1}, \tag{5}$$

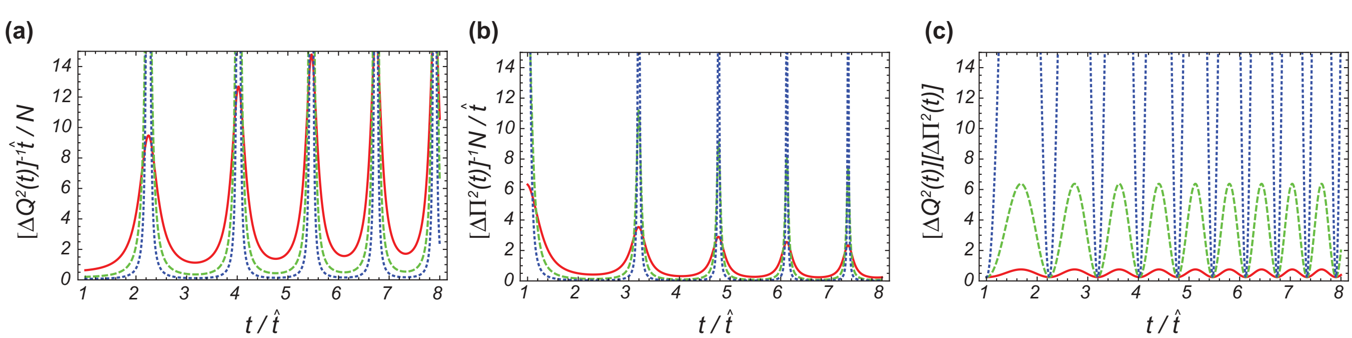

FIG. 2. (Color online) (a) Time evolution of $(\Delta Q^2)^{-1}$ for different values of the initial time t_0. The continuous thick line is for $t_0 = 10^{-1}$, the dashed line for $t_0 = 10^{-2}$, and the dotted line corresponds to $t_0 = 10^{-3}$. By decreasing the initial time t_0, a Dirac comb of symmetry-broken states is approached. (b) Same for the inverse of the fluctuations of the total momentum. Also in this case a Dirac comb is approached. (c) Same of (a) for the time evolution of the total uncertainty of the matter wave. Apart from punctured times, it is seen to diverge in the $t_0 \to 0$ limit.

the behavior of which is shown in Fig. 2(a). When decreasing the initial time t_0 we observe that the behavior of the order parameter calculated above corresponds precisely to a Dirac comb of symmetry-broken states in perfect agreement with the case of antiferromagnets.[14] Aside from this, we find that at the punctured times where the fluctuation of the center-of-mass coordinate vanishes, the dynamical phases of the excited dual thin spectrum excitations have π shifts, i.e., for $t = t_\kappa = [3\kappa\pi/2 + 3\pi/4 + 1]^{2/3}\hat{t}$ with κ integer. This shows that independent of the actual strength of the symmetry-breaking perturbation, destructive quantum phase interference leads to an instantaneous breaking of the translational symmetry. It is also manifested by the fact that a direct computation shows at these instants $\hat{Q}\,|\Psi(Q,t_\kappa)\rangle \equiv 0$ and therefore the harmonic crystal is completely localized in the center of the potential well.

To further show the nature of the nonequilibrium state realized in the remaining time evolution, we have determined the time evolution of the inverse of the fluctuations of the total momentum of the entire crystal and find

$$[\Delta\Pi^2(t)]^{-1} = 2\frac{\hat{t}}{N}\sqrt{\frac{\hat{t}^2}{t\,t_0}}\left[1 - \left(1 - \frac{\hat{t}}{t_0}\right)\sin^2\Delta\Omega(t)\right]^{-1}, \tag{6}$$

the behavior of which is shown in Fig. 2(b). A Dirac comb is also the result. The instants where the system is an eigenstate of the total momentum Π, which correspond to revivals of the initial completely delocalized symmetric state even in the presence of a sizable pinning potential, are realized for $t = t_\kappa = [3\kappa\pi/2 + 1]^{2/3}\hat{t}$ with κ integer in which case quantum phase interference effects are absent. This is again in line with the dynamics of the Lieb-Mattis model and henceforth guarantees the universality of the dynamical symmetry-breaking phenomenon independent of the specific microscopic model taken into account.

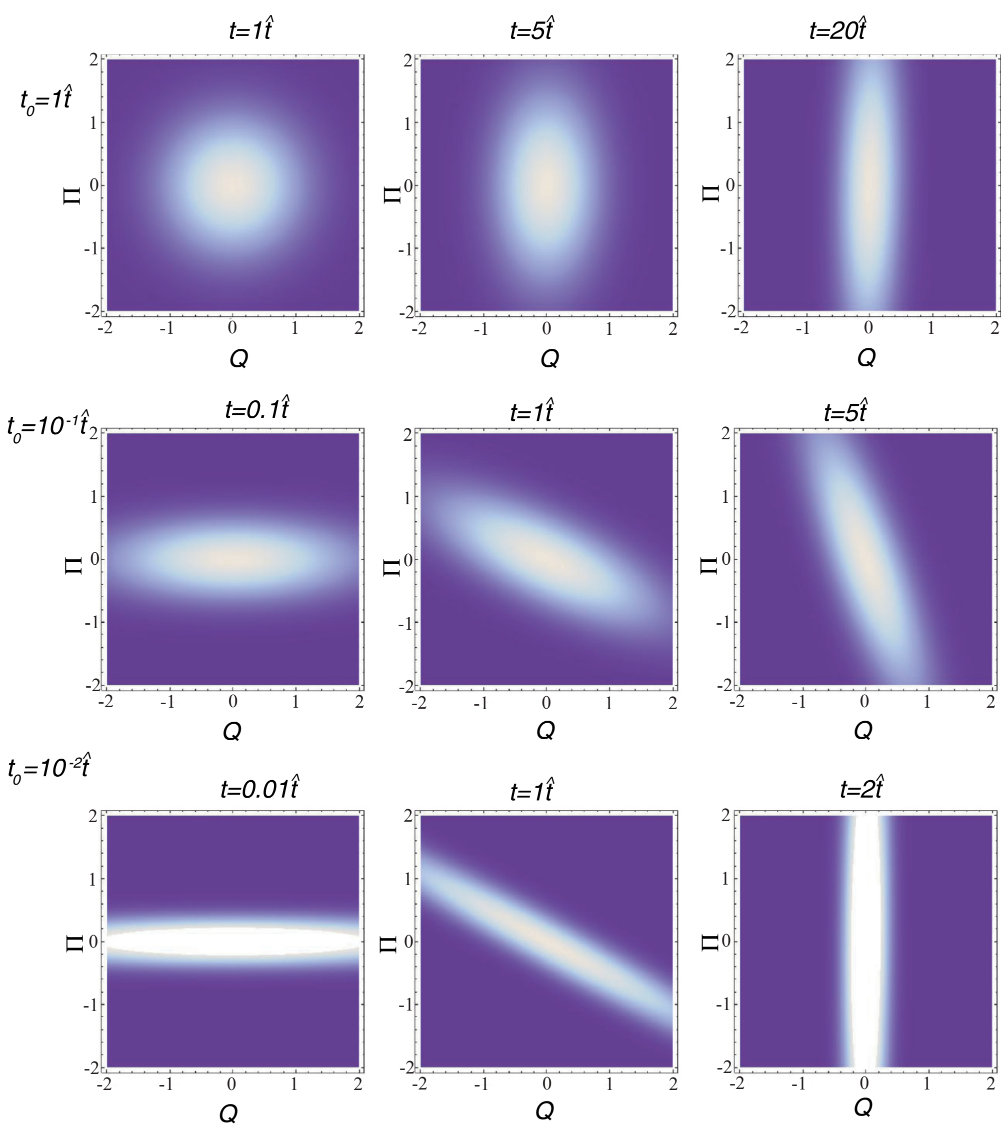

FIG. 3. (Color online) Density plots of the Wigner function in phase space (Q,Π). The center-of-mass coordinate Q and the total momentum Π have been rescaled, respectively, as $Q \to Q\sqrt{N}$ and $\Pi \to \Pi/\sqrt{N}$ in order to absorb the N dependence of the Wigner function. By decreasing the initial time t_0, the time evolution changes from an adiabatic shear motion to a strongly nonequilibrium rotative motion.

Apart from the punctured times where the matter wave has either no uncertainty in total position or no uncertainty in total momentum, a strongly nonequilibrium state is realized. This state is rather interesting as it retains a perfect delocalization of both the center-of-mass position and its corresponding momentum. It has an infinite uncertainty, i.e., $\Delta\Pi^2\Delta Q^2 \to \infty$ as it is shown in Fig. 2(c).

IV. QUANTUM DYNAMICS IN PHASE SPACE

To unravel the origin of this nonequilibrium state, we analyze the dynamics of symmetry breaking with quantum state tomography. The time-dependent Hamiltonian

$$\mathcal{H}(t) = \frac{\Pi^2}{2\,N} + \frac{N}{2}\,\delta\, t\, Q^2$$

represents a simple example of generalized time-dependent harmonic oscillator whose exact quantum theory has been extensively studied in the literature.[27–39] In particular, within the Feynman path-integral approach, it has been shown[38] that the spectral decomposition of the propagator $\mathcal{G}(Q_b,t_b|Q_a,t_a) = \sum_n \Psi_n^\star(Q_a,t_a)\,\Psi_n(Q_b,t_b)$ is defined by a complete set of wave functions of the form

$$\Psi_n(Q,t) = \sqrt{\frac{1}{2^n\,n!}}\left[\frac{\mathrm{Re}[\omega(t)]}{\pi}\right]^{1/4} H_n[\sqrt{\mathrm{Re}[\omega(t)]}\,Q] \times e^{-\frac{Q^2}{2}\omega(t)} \times e^{-\mathrm{i}(n+\frac{1}{2})\phi(t)}, \tag{7}$$

where H_n are the Hermite polynomials and $\mathrm{Re}[\omega(t)] > 0$ in order to guarantee square integrability. The quantal phase $\phi(t)$ and the complex parameter $\omega(t)$ are uniquely determined by solving the classical Euler-Lagrange equation of motion. It is worth mentioning that the complex parameter $\omega(t)$ differs from the dynamical phase factors $\Omega_n(t)$ of Sec. III. Different sets of wave functions of the form Eq. (7) correspond to take different pairs of linearly independent solutions to the classical equation of motion. This enables us to choose two particular solutions guaranteeing that at the initial time $\Psi_0(Q,t_0)$ corresponds to the initial static Gaussian wave packet and implies that the wave function at all times remains an $n = 0$ state of the form of Eq. (7).

The exact solution of the time-dependent Schrödinger equation allows us to determine the time evolution of the Wigner function (see Fig. 3) given by

$$W(Q,\Pi,t) = \frac{1}{\pi}e^{-\mathrm{Re}[\omega(t)]Q^2} \times e^{-(\Pi+\mathrm{Im}[\omega(t)]Q)^2/\mathrm{Re}[\omega(t)]}. \tag{8}$$

By considering a nearly adiabatic process (an initial time $t_0 \gg \hat{t}$), one finds that the Wigner function shears in time in agreement with the time evolution of a Gaussian wave packet reacting adiabatically to the time change of the harmonic-oscillator angular frequency. On the contrary, in the out-of-equilibrium regime, i.e., for $t_0 < \hat{t}$, the Wigner distribution shears and rotates in phase space as it follows from the fact that the initial width of the Gaussian wave packet acquires a non-negligible imaginary part. Finally, in the $t_0 \to 0$ limit instead, any shear is absent and the motion simply corresponds to a rigid rotation in phase space. For a completely symmetric initial state, the initial Wigner function corresponds to an infinite line in phase space $\delta(Q)$ and a rigid rotation is the only motion preserving this one-dimensional character. As a result, we find in the $t_0 \to 0$ limit the time evolution of the Wigner function as

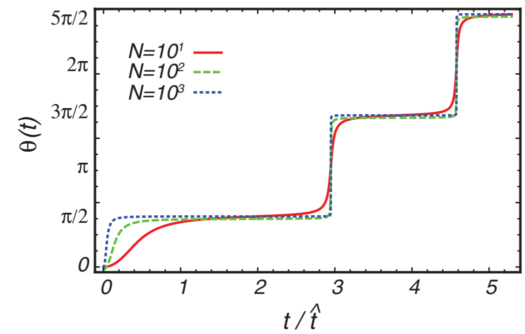

FIG. 4. (Color online) The time evolution of the angle θ characterizing the rotation in phase space of the Wigner function. By increasing the number of microscopic constituents, a steplike behavior is realized.

$$W(Q,\Pi,t) \propto \delta[\cos\theta(t)\,Q + \sin\theta(t)\,\Pi], \tag{9}$$

where the angle $\theta(t) \simeq \tan^{-1}\mathrm{Im}[\omega(t)]$. By increasing the number of microscopic quantum constituents N, the time dependence of the angle $\theta(t)$ approaches a steplike behavior as $\mathrm{Im}[\omega(t)] \propto N$ (cf. Fig. 4). This, in turns, implies that the quantum dynamics of symmetry breaking in a macroscopic body is characterized by revivals of the initial symmetric state and collapses of the initial quantum state towards a "quasiclassical" state, tomographically indistinguishable from the symmetry-broken state of infinite-size systems but, as we have shown above, completely delocalized both in momentum and in real space. The exceptions are the punctured times where the angle $\theta(t) \equiv 0,\pi/2$ in which case the fully symmetric ground state and the completely localized ground state, respectively, become fact. This is in line with the foregoing adiabatic-impulse approximation of Sec. III.

V. CONCLUSIONS

In conclusion, by considering the paradigmatic example of a harmonic oscillator, we have shown that in the dynamical realm, symmetry breaking is characterized by far-from-equilibrium processes. No matter how slowly a symmetry-breaking perturbation is driven, the adiabatic limit can never be reached in a macroscopic body. By means of quantum state tomography, we have shown that nevertheless the evolution of symmetry breaking corresponds to a continuous, rigid rotation of the Wigner distribution. This rotation yields at the same time a sequence of steplike revivals of the symmetric state followed by collapses onto a symmetry-broken ground state akin to the one realized in infinite-size systems but with maximum uncertainty both in total position and in total momentum.

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[1]P. W. Anderson, Science **177**, 393 (1972).
[2]P. W. Anderson, Phys. Rev. **86**, 694 (1952).
[3]P. W. Anderson, Phys. Rev. **112**, 1900 (1958).
[4]Y. Nambu, Phys. Rev. **117**, 648 (1960).
[5]J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).
[6]E. Lieb and D. Mattis, J. Math. Phys. **3**, 749 (1962).
[7]C. Kaiser and I. Peschel, J. Phys. A **22**, 4257 (1989).
[8]T. A. Kaplan, W. von der Linden, and P. Horsch, Phys. Rev. B **42**, 4663 (1990).
[9]J. van Wezel, J. van den Brink, and J. Zaanen, Phys. Rev. Lett. **94**, 230401 (2005).
[10]J. van Wezel, J. Zaanen, and J. van den Brink, Phys. Rev. B **74**, 094430 (2006).
[11]J. van Wezel and J. van den Brink, Am. J. Phys. **75**, 635 (2007).
[12]T. Birol, T. Dereli, O. E. Müstecaplıoğlu, and L. You, Phys. Rev. A **76**, 043616 (2007).
[13]J. van Wezel and J. van den Brink, Phys. Rev. B **77**, 064523 (2008).
[14]C. Ortix, J. Rijnbeek, and J. van den Brink, Phys. Rev. B **84**, 144423 (2011).
[15]D. Leibfried, D. M. Meekhof, B. E. King, C. Monroe, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **77**, 4281 (1996).
[16]E. Skovsen, H. Stapelfeldt, S. Juhl, and K. Molmer, Phys. Rev. Lett. **91**, 090406 (2003).
[17]D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, Rev. Mod. Phys. **75**, 281 (2003).
[18]D. Leibfried, T. Pfau, and M. Christopher, Phys. Today **51**(4), 22 (1998).
[19]W. H. Zurek, U. Dorner, and P. Zoller, Phys. Rev. Lett. **95**, 105701 (2005).
[20]J. Goldstone, Nuovo Cimento **19**, 154 (1961).
[21]Y. Nambu and G. Jona Lasinio, Phys. Rev. **122**, 345 (1961).
[22]A. Polkovnikov and V. Gritsev, Nat. Phys. **4**, 477 (2008).
[23]T. W. B. Kibble, J. Phys. A **9**, 1387 (1976).
[24]W. H. Zurek, Nature (London) **317**, 505 (1985).
[25]W. H. Zurek, Phys. Rep. **276**, 177 (1996).
[26]B. Damski and W. H. Zurek, Phys. Rev. A **73**, 063405 (2006).
[27]D. C. Khandekar and S. V. Lawande, Phys. Rep. **137**, 115 (1986).
[28]D. C. Khandekar and S. V. Lawande, J. Math. Phys. **20**, 1870 (1979).
[29]R. Colegrave and M. Abdalla, J. Phys. A **14**, 2269 (1981).
[30]R. Colegrave and M. Abdalla, J. Phys. A **15**, 1549 (1982).
[31]R. Colegrave and M. Abdalla, J. Phys. A **16**, 3805 (1983).
[32]M. S. Abdalla and R. K. Colegrave, Phys. Rev. A **32**, 1958 (1985).
[33]M. S. Abdalla, Phys. Rev. A **34**, 4598 (1986).
[34]M. S. Abdalla, Phys. Rev. A **33**, 2870 (1986).
[35]G. J. Papadopoulos, J. Phys. A: Math., Nucl. Gen. **7**, 209 (1974).
[36]K. H. Yeon, K. K. Lee, C. I. Um, T. F. George, and L. N. Pandey, Phys. Rev. A **48**, 2716 (1993).
[37]A. Lopes de Lima, A. Rosas, and I. A. Pedrosa, Ann. Phys. (NY) **323**, 2253 (2008).
[38]D.-Y. Song, Phys. Rev. A **59**, 2616 (1999).
[39]D.-Y. Song, Phys. Rev. Lett. **85**, 1141 (2000).