# Fluctuations of the vortex line density in turbulent flows of quantum fluids

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We present an analytical study of fluctuations of the vortex line density (VLD) $\langle \delta\mathcal{L}(\omega)\delta\mathcal{L}(-\omega)\rangle$ in turbulent flows of quantum fluids. Two cases are considered. The first is the counterflowing (Vinen) turbulence, where the vortex lines are disordered, and the evolution of quantity $\mathcal{L}(t)$ obeys the Vinen equation. The second case is the fluctuations of the VLD in a single vortex bundle, which develops inside the domain of the concentrated normal-fluid vorticity. The dynamics of the vortex bundle is described by the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) equations. The latter case is of special interest, because the set of the quantum vortex bundles is believed to mimic classical hydrodynamic turbulence. In steady states the VLD is related to the normal velocity as $\mathcal{L} = (\rho\gamma/\rho_s)^2 v_n^2$ for the Vinen case. In the vortex bundle case, which appears inside the domain of a concentrated vorticity of normal fluid, the stationary quantity $\mathcal{L}$ can be found from the matching of velocities and is described by $\mathcal{L} = |\nabla \times \mathbf{v}_n|/\kappa$. In nonstationary situations, and particularly in the fluctuating turbulent flow, there is a retardation between the instantaneous value of the normal velocity and the quantity $\mathcal{L}$. This retardation tends to decrease in accordance with the inner dynamics, which has a relaxation character. In both cases, the relaxation dynamics of the VLD is related to fluctuations of the relative velocity. However, for the Vinen case the rate of temporal change for $\mathcal{L}(t)$ is directly dependent upon $\delta\mathbf{v}_{ns}$, whereas for HVBK dynamics it depends on $\nabla \times \delta\mathbf{v}_{ns}$. Therefore, for the disordered case the spectrum $\langle \delta\mathcal{L}(\omega)\delta\mathcal{L}(-\omega)\rangle$ coincides with the spectrum $\omega^{-5/3}$. In the case of the bundle arrangement, the spectrum of the VLD varies (at different temperatures) from $\omega^{1/3}$ to $\omega^{-5/3}$ dependencies. This conclusion may serve as a basis for the experimental determination of what kind of turbulence is implemented in different types of generation.

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## I. INTRODUCTION

The problem of modeling classical turbulence with a set of chaotic quantized vortices is the hottest topic in modern studies of vortex tangles in quantum fluids (see, e.g., recent reviews articles by Vinen,[1] Procaccia and Sreenivasan,[2] and Skrbek and Sreenivasan[3]). The most common view of quasiclassical turbulence is the model of vortex bundles, the collections of near-parallel quantized vortices. In vortex bundles, the coarse-grained vorticity field of $\boldsymbol{\omega} = \nabla \times \mathbf{v}_s$ and the vortex line density (VLD) $\mathcal{L}$ are related to each other via Feynman's rule:

$$|\boldsymbol{\omega}| = \kappa\mathcal{L}. \tag{1}$$

This formula reflects the fact that the VLD $\mathcal{L}$ in this case coincides with the areal density of lines in a plane perpendicular to the bundle. The main motivation for this conception is that the quantized vortices have a fixed core radius and thus they do not possess the important property of classical turbulence—the stretching of tubes—which is responsible for the turbulence energy cascade from the large scales to the small scales. However, the vortex bundles do possess this property and therefore the idea that quasiclassical turbulence in quantum fluids is realized via vortex bundles of different sizes and intensities (number of threads) seems to be quite natural.

Recently, several numerical evidences of the structure of vortex bundles were obtained. Thus, in the numerical work by Ref. 4, in $2048^3$ simulations of quantum turbulence within the Gross-Pitaevskii equation (GPE), the authors observed nonuniform structures. The authors claimed that "the visualization of vortices clearly shows the bundle-like structure, which has never been confirmed in GPE simulations on smaller grids." In other numerical works,[5,6] the authors studied the evolution of the vortex structures (at zero temperature) on the basis of the Biot-Savart law. They also observed structures reminiscent of the field of vorticity in classical turbulence (see, e.g., Ref. 7).

To date, strong evidence of the bundle structure has not been obtained by experimental confirmation. There are experimental results which would seem to refute the idea of bundles. Thus, in experiments by Roche *et al.*[8] and by Bradley *et al.*,[9] it was observed that the spectrum of the fluctuation of the VLD $\langle \delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ ($f$ is the frequency) is compatible with a $-5/3$ power law. Sometimes this fact is interpreted as contradictory to the idea of the bundle structure, because the spectrum of vorticity [and, correspondingly, of the VLD $\mathcal{L}$, via Eq. (1)] should scale as the $1/3$ power law. However, this conclusion is not entirely accurate, because the scalar (and positive) quantity $\mathcal{L}$ is not identical to the vector variable, vorticity $\boldsymbol{\omega}$, and therefore the spectra of their fluctuations may differ. However, the situation is not so obvious. Formally, the spectra are the Fourier transforms from the correlation functions and, at least inside the bundle, the vorticity $\boldsymbol{\omega}(\mathbf{r})$ and the VLD $\mathcal{L}(\mathbf{r})$ should have coordinated changes [via Eq. (1)]. Accordingly, the correlation functions of fluctuations of $\delta L$ and $\delta\omega$ should coincide (up to a coefficient). In the case of many bundles randomly oriented in space, the correlation functions $\langle \boldsymbol{\omega}(\mathbf{r}_1)\boldsymbol{\omega}(\mathbf{r}_2)\rangle$ and $\langle \mathcal{L}(\mathbf{r}_1)\mathcal{L}(\mathbf{r}_2)\rangle$ behave differently and the spectra should also differ. It is not clear which scenario is realized in experiment (if any). For instance, in the numerical paper by Salort *et al.*,[10] made on the basis of Hall-Vinen-Bekarevich-Khalatnikov (HVBK) theory, the spectra of absolute value of $|\boldsymbol{\omega}_s|$ has different slopes (depending on the temperature) from $-5/3$ up to the positive slope that may be 1/3. In numerical papers by Baggaley *et al.*,[5,11] made on the basis of the vortex filament method, the authors reported about a $-5/3$ power law for spectrum fluctuations of the VLD, and the 1/3 dependence for the spectrum of enstrophy $\langle \boldsymbol{\omega}(k)\boldsymbol{\omega}(-k)\rangle$.

A theoretical study of the VLD fluctuation in turbulent flows was offered by Roche and Barenghi.[12] The authors considered the VLD $\mathcal{L}$ to be decomposed into two components, where the smaller part consisting of the polarized component is responsible for the large-scale turbulent phenomena, and the other disordered part evolves as a "passive" scalar, thereby taking the $-5/3$ velocity spectrum.

We present an analytical evaluation of the spectrum of fluctuations VLD $\langle \delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ in turbulent flows. The principal proposition of this paper is that the vortex line density $L(t)$ is the secondary quantity, whose evolution depends (in general as a functional) on the applied normal velocity $\mathbf{v}_n$. The evolution of $\mathcal{L}(t)$ obeys dynamical equations, which in general are different depending upon the arrangement of the experiment. All these equations include the normal velocity $\mathbf{v}_n$ and therefore small deviations of $\mathcal{L}(t)$ depend on fluctuations of quantity $\mathbf{v}_n$. By analyzing these equations, one is in a position to obtain the spectrum $\langle \delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ as a functional of the energy spectrum $\langle \delta\mathbf{v}_n(k)\delta\mathbf{v}_n(-k)\rangle$. Two cases are considered. The first is the counterflowing (Vinen) turbulence, where vortex lines are disordered and the dynamics of quantity $\mathcal{L}(t)$ is governed by the Vinen equation. The second case is the fluctuations of VLD in the vortex bundle, which develops inside the domain of the concentrated normal-fluid vorticity. The dynamics of the vortex bundle is described by the HVBK equations.

In steady states the VLD is related to the normal velocity as $\mathcal{L} = (\rho\gamma/\rho_s)^2 v_n^2$ for the Vinen case. For the HVBK case, the stationary values of VLD should be found from the HVBK equations (see, e.g. Refs. 13–18). Due to mutual frictions, which enter the HVBK equations and are proportional $(v_n - v_s)$, a stationary solution satisfies (with great accuracy) the condition of matching velocities $\mathbf{v}_n - \mathbf{v}_s - > \mathbf{0}$ . Mismatch is usually small and therefore $\mathcal{L} = |\nabla \times \mathbf{v}_n|/\kappa$ can serve as a good approximation for stationary solutions under various realization, e.g., in rotation, or in the concentrated normal-fluid vorticity.

In the fluctuating flow in nonstationary situations in particular, there is a retardation between the instantaneous value of the normal velocity and the quantity $\mathcal{L}(t)$. This retardation tends to decrease, according to the inner dynamics, which has a relaxation character. In both cases, the relaxation of $\delta\mathcal{L}(t)$ is related to fluctuations of the normal velocity $\delta\mathbf{v}_{ns}$. However, for the Vinen disordered turbulence, the rate of temporal change for $\mathcal{L}(t)$ depends directly on $\delta\mathbf{v}_{ns}$, whereas for the HVBK dynamics it depends on the quantity $\nabla \times \delta\mathbf{v}_{ns}$. In addition, the relaxation mechanisms, and, consequently, the times of relaxation are different. The factors outlined above lead to different formulas for spectra $\langle \delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ and their dependence on temperature.

## II. VINEN EQUATION CASE

Let us study the reaction of the vortex line density in a fluctuating flow of normal velocity supposing that the dynamics of $\mathcal{L}(t)$ obeys the Vinen equation

$$\frac{\partial \mathcal{L}}{\partial t} = \alpha_V |\mathbf{v}_{ns}| \mathcal{L}^{3/2} - \beta_V \mathcal{L}^2. \tag{2}$$

Equation (2) was initially derived phenomenologically for pure counterflowing superfluid helium.[19–22] Attempts to derive it an analytic form[23–26] demonstrated that this equation is seemingly valid for any nonstructured turbulence. Under the term "nonstructured turbulence," we understand the vortex tangle, which consists of closed vortex loops of different sizes, which are uniformly distributed in space. It differs, for instance, from the turbulent fronts in rotating fluids, which deal with the lines terminating on lateral walls. It also differs from the mechanically excited turbulence, which is believed to consist of the so-called vortex bundles composed of very polarized vortex filaments. In addition to the counterflow turbulence, the "nonstructured turbulence" is generated by intensive sounds (both by the first and the second sounds). The case of disordered vortex tangles also appears after the quench due to the Kibble-Zurek mechanism, or by the proliferation of vortices when approaching the critical temperature.

Our goal now is to study the stochastic properties of $\mathcal{L}(t)$, when $\mathbf{v}_n$ fluctuates with a given spectral density $\langle \delta\mathbf{v}_n(f)\delta\mathbf{v}_n(-f)\rangle = f(f)$. Furthermore, for simplicity, we will study the pure counterflowing case in the sense that the net flow is absent: $\mathbf{j} = \rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s = \mathbf{0}$. Then, the average value $\mathcal{L}_{0V}$ of the vortex line density in the counterflow is related to a relative velocity $\mathbf{v}_{ns} = \mathbf{v}_n - \mathbf{v}_n$ by the usual relation

$$\mathcal{L}_{0V} = \frac{\alpha_V^2}{\beta_V^2}\mathbf{v}_{ns}^2 = \gamma^2\mathbf{v}_{ns}^2. \tag{3}$$

To take into account the fluctuations, we put

$$\mathcal{L} = \frac{\alpha_V^2}{\beta_V^2}\,\mathbf{v}_{ns}^2 + \delta\mathcal{L}, \quad \mathbf{v}_n = \mathbf{v}_{n0} + \delta\mathbf{v}_n. \tag{4}$$

From the net mass condition $\mathbf{j} = \mathbf{0}$, the following relations between the fluctuations of $v_s$ and $v_{ns}$ and the fluctuations of $v_n$ occur:

$$\mathbf{v}_s = \mathbf{v}_{s0} - \frac{\rho_n}{\rho_s}\delta\mathbf{v}_n, \quad \mathbf{v}_{ns} = \mathbf{v}_{ns0} + \frac{\rho}{\rho_s}\delta\mathbf{v}_n. \tag{5}$$

Substituting Eqs. (4) and (5) into the Vinen equation (2), we arrive at

$$\frac{\partial \delta\mathcal{L}}{\partial t} = \alpha_V\,\frac{\rho}{\rho_s}\mathcal{L}_{0V}^{3/2}\delta\mathbf{v}_n - \beta_V\frac{1}{2}\mathcal{L}_{0V}\delta\mathcal{L}. \tag{6}$$

Equation (6) shows that the evolution of the fluctuating part of the vortex line density $\delta\mathcal{L}$ bears the relaxation-type character with a characteristic time

$$\tau_V = 2/(\beta_V\mathcal{L}_0), \tag{7}$$

and with a "force" proportional to $\delta\mathbf{v}_n$. In the Fourier transform, Eq. (6) takes the form

$$i2\pi f\,\delta\mathcal{L}(f) = \alpha_V\frac{\rho}{\rho_s}\mathcal{L}_0^{3/2}\delta\mathbf{v}_n(f) - \beta_V\frac{1}{2}\mathcal{L}_0\delta\mathcal{L}(f), \tag{8}$$

and therefore the spectrum $\langle \delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ is

$$\langle \delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle = \frac{4(\alpha_V/\beta_V)^2\left(\frac{\rho}{\rho_s}\right)^2\mathcal{L}_{0V}\langle \delta\mathbf{v}_n(f)\delta\mathbf{v}_n(-f)\rangle}{1+(2\pi f\tau_V)^2}. \tag{9}$$

An equation of this type was previously obtained by Barenghi *et al.*[27] to test the Vinen equation with the use of the heater fluctuations. It allows to express the spectrum of the VLD $\langle \delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ via the spectrum of a normal velocity. We will use it to study the response of the the VLD $\mathcal{L}(t)$ field to the fluctuations of the normal component $\delta\mathbf{v}_n(f)$. To do this, we will further apply the relationship $k = 2\pi f/v_n$ between the wave number $k$ and the frequency $2\pi f$ ($v_n$ is the mean flow velocity). This is widely accepted in the theory of the turbulence assumption, the so-called Taylor hypothesis (see, e.g., the book by Frisch[28]).

Relation (9) shows that for small frequencies, $2\pi f < 1/\tau_V$, the spectrum of the VLD reproduces the spectrum of fluctuations of the normal component, and if the Kolmogorov-type turbulence is developed in the normal component, then the quantity $\langle \delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ scales as $f^{-5/3}$. For larger frequencies $f$, or for a larger relaxation time $\tau_V$ (which takes place at lower temperatures), the decrease will be faster, approaching an $f^{-11/3}$ dependence.

### III. HVBK CASE

We have already discussed that the most popular model of quasiclassical turbulence in quantum fluids is the conception of chaotic vortex bundles. It is believed that the quantum turbulent flow consists of many bundles containing a large number of threads within them. Even though the conception of bundles is frequently used, too little is known about the chaotic structure that is formed by them. It is not clear what these bundles look like, how they fill the volume of quantum fluids, and how they decay and appear. Recently, several numerical studies have appeared where this structure was observed,[4–6,29–33] however, it is problematic to use these results for quantitative studies. To date, the only purpose of these bundles is that they mimic the stretching of the vortex, a very important property of classical turbulence, which is responsible for the energy cascade from the large scale to the small scale. Therefore, to avoid all these difficulties, we consider the case of a single vortex bundle that develops inside the domain of concentrated normal-fluid vorticity. However, because the vortex line density is an additive quantity, the total effect from all the bundles, regardless of how they are distributed in space, might reflect the features of individual bundles.

The coarse-grained hydrodynamics of the vortex bundles have been studied by many authors (see, e.g., Refs. 15–17 and 34), but the basis of these studies was the hydrodynamics of rotating superfluids, or the HVBK model (see, e.g., Ref. 35). The essence of the HVBK theory is that it does not consider individual filaments, but deals with averaged (cross-grained) motion, which takes into account the contribution from all near-parallel quantized vortices composing vortex bundles. In this case, the vorticity field $\boldsymbol{\omega}(\mathbf{r},t)$ of this coarse-grained superfluid motion is not zero (unlike the vortex-free motion, or the Vinen counterflow case), but is expressed via a vortex line density field $\mathcal{L}(\mathbf{r},t)$ with the use of Eq. (1).

In terms of HVBK dynamics, the vorticity field $\boldsymbol{\omega}(\mathbf{r},t)$ obeys the following equation (see Ref. 35),

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times [\mathbf{v}_L \times \boldsymbol{\omega}], \tag{10}$$

where $\mathbf{v}_L$ is the velocity of lines. In a comoving frame of reference the quantity $\mathbf{v}_L$ is

$$\mathbf{v}_L = \alpha[\hat{\boldsymbol{\omega}}\times(\mathbf{v}_n - \mathbf{v}_s)]. \tag{11}$$

Equation (10) describes the motion of vortex lines in the transverse (with respect to the unit vector $\hat{\boldsymbol{\omega}}$ along the vorticity) direction, when the coarse-grained superfluid velocity $\mathbf{v}_s$ differs from the normal velocity $\mathbf{v}_n$. In the stationary case, the coarse-grained superfluid velocity coincides with the normal velocity in the bundle $\mathbf{v}_s = \mathbf{v}_n$, and the steady value $\mathcal{L}_{0b}$ of the vortex line density in the bundle is

$$\mathcal{L}_{0b} = |\nabla \times \mathbf{v}_n|/\kappa. \tag{12}$$

Again, this equation is derived without fluctuations. To take into account the latter, we use the linearization $\mathcal{L} = \mathcal{L}_{0b} + \delta\mathcal{L}$, $\mathbf{v}_n = \mathbf{v}_{n0} + \delta\mathbf{v}_n$ into Eq. (10) and have

$$\frac{\partial \hat{\boldsymbol{\omega}}(\mathcal{L}_{0b}+\delta\mathcal{L})}{\partial t} = \nabla \times [(\mathbf{v}_{ns}+\delta\mathbf{v}_{ns})(\mathcal{L}_{0b}+\delta\mathcal{L})]. \tag{13}$$

After a little algebra this equation is transformed to

$$\frac{\partial \hat{\boldsymbol{\omega}}(\delta\mathcal{L})}{\partial t} = \alpha\mathcal{L}_{0b}\nabla \times \delta\mathbf{v}_{ns} = \alpha\mathcal{L}_{0b}(\nabla \times \delta\mathbf{v}_n - \nabla \times \delta\mathbf{v}_s). \tag{14}$$

Consequently using HVBK theory, and assuming that Eq. (1) relating the superfluid (coarse-grained) vorticity and number of lines is also valid for disturbances $\delta\boldsymbol{\omega}$ (which is $\nabla \times \delta\mathbf{v}_s$) and $\delta\mathcal{L}$, we rewrite the $\hat{\boldsymbol{\omega}}$ component of Eq. (14) in the form

$$\frac{\partial(\delta\mathcal{L})}{\partial t} = \alpha\mathcal{L}_{0b}\nabla \times \delta\mathbf{v}_{ns} = \alpha\mathcal{L}_{0b}\nabla \times \delta\mathbf{v}_n - \frac{1}{\tau_b}\delta\mathcal{L}, \tag{15}$$

where the relaxation time $\tau_b$ of the vortex lines population inside the bundle is

$$\tau_b = \frac{1}{\alpha\mathcal{L}_{0b}\kappa}. \tag{16}$$

Before moving further, we would like to make one comment concerning the HVBK theory and relations (14)–(16). This theory describes the redistribution of preexisting vortex lines in the transverse direction [see Eq. (11)], while it does not include a mechanism of the appearance of new lines. In Refs. 30 and 36–38 it was shown that the bundle structure of quantized vortices develops inside the eddies of the normal component due to the proliferation of vortex filaments. The mechanism for this proliferation is quite involved. It is similar to the development of the vortex tangle in an applied counterflow with the growth of the number of lines due to reconnections, and with the growth of length due to the relative velocity with the subsequent polarization. This effect is not considered in the HVBK approach and it is not included in Eqs. (14)–(16). In the phenomenological HVBK approach this process can be taken into account just by the proper enlarging of the inverse relaxation time $1/\tau_b$. Indeed, analyzing the numerical results, Samuels[36] observed the exponential-like saturation of the vortex tube (of the normal component) with the superfluid vortex filaments. He offered an expression for typical $\tau_{b1}$ of this saturation, which included the circulation of the vortex tube (of normal component) and its size (for details, see Ref. 36). The important fact is that quantity $\tau_{b1}$ is proportional to $1/\sqrt{\alpha}$, and thus it decreases with an increase of temperature.

In Fourier component equation (14) takes a form

$$-i2\pi f\delta\mathcal{L}(\omega) = i\alpha\mathcal{L}_{0b}\frac{\rho}{\rho_s}\mathbf{k}\times\delta\mathbf{v}_{ns}(f) + \frac{1}{\tau_b}\delta\mathcal{L}(f),$$

which leads to the following spectral density:

$$\langle\delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle = \frac{\alpha^2\mathcal{L}_{0b}^2\left(\frac{\rho}{\rho_s}\right)^2\tau_b^2k^2\langle\delta\mathbf{v}_n(f)\delta\mathbf{v}_n(-f)\rangle}{1+(2\pi f\tau_b)^2}. \tag{17}$$

Again, as in the case of the Vinen turbulence, the shape of the spectrum depends on the value of $2\pi f\tau_b$.

For large $\tau_s$, which implies a small coupling between the normal and superfluid components and is realized for a low temperature, the spectrum $\langle\delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ is proportional to $\langle\delta\mathbf{v}_n(f)\delta\mathbf{v}_n(-f)\rangle$. Thus, the spectrum $\langle\delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle \propto f^{-5/3}$ for Kolmogorov turbulence in the normal component. In this case the result is similar to the previous (Vinen turbulence) case, considered above. However, for small $2\pi f\tau_s < 1$, which corresponds to a large temperature (strong coupling due to the larger mutual friction), the spectrum behaves as $\langle\delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ is proportional to $f^2\langle\delta\mathbf{v}_n(f)\delta\mathbf{v}_n(-f)\rangle$ ($f^{1/3}$ for Kolmogorov turbulence), and the intensity of this spectrum is much lower. This result is in good qualitative agreement with the numerical results.[10]

## IV. DISCUSSION AND CONCLUSION

We studied analytically the spectrum of fluctuations of the vortex line density, both in the counterflowing (Vinen) turbulence and based on HVBK theory. In both cases, these deviations of quantity $\delta\mathcal{L}(t)$ appear due to the strongly fluctuating field of the normal velocity. Deviations of $\delta\mathcal{L}(t)$ evolve in a relaxationlike manner, which is determined either from the Vinen equation or from the HVBK equation. It allows the determination of the Fourier transform and evaluation of the spectra $\langle\delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$. The crucial difference between these two cases is that the stirring force for $\mathcal{L}(t)$ in the Vinen case is proportional to perturbations of the normal velocity $\delta\mathbf{v}_n(t)$, whereas for HVBK this force is related to $\nabla\times\delta\mathbf{v}_n(t)$. This difference results in different spectra and their dependence on the temperature. Specifically, as the temperature increases, the slope of the spectrum of the VLD fluctuation $\langle\delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ runs from $1/3$ to $-5/3$ in the HVBK model. In the Vinen equation model the interval of the change of slope is from $-5/3$ to $-11/3$. The point of the bending of the curve is determined by the inverse relaxation time (either $\tau_V$ [Eq. (7)] or $\tau_b$ [Eq. (16)]) of the vortex structure. This conclusion may serve as a basis for the experimental determination of what kind of turbulence is realized in different types of generation of the vortex tangle. Of course the same result would be obtained by varying the frequency, but this way is restricted by the presence of the inertial interval.

Let us consider the experiment by Roche *et al.*[8] from the point of view of relations (9) and (17). We will restrict ourselves only by the scaling (power law) behavior of the spectra as functions of frequency. The absolute values are difficult to discuss because of uncertainties both in the experiment and in the theoretical models. The authors of Ref. 8 suggested that for the mean flow $v_n \approx 1$ m/s and at $T = 1.6$ K, the value of the vortex line density is about $6\times10^6$ 1/cm$^2$. The difficult question concerns the value of the coefficient $\beta_V$, the point being that only the quantity $(\alpha_V/\beta_V)$ has been studied and determined and, separately, the coefficients of the Vinen equation have been poorly studied. The values proposed by Vinen in his pioneering work[20] were revised later by Schwarz,[24] who concluded that both $\alpha_V$ and $\beta_V$ should be larger (by about a factor of 6). The latter result was also confirmed in experiment by Stamm *et al.*[39] Taking the Vinen value $\beta_V = 0.0034$ cm$^2$/s we obtain that $\tau_V$ is about 0.002 s (we took $\beta_V = 0.0034$ cm$^2$/s). Therefore, the change of slope of the curve $\langle\delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ as a function of frequency $f$ should happen at $f_c = 1/2\pi\tau_V \approx 160$ Hz. Adopting the Schwarz values, one can conclude that the bending of the curve $\langle\delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ (as a function of $f$) occurs at $f_c \approx 1000$ Hz. Therefore, for the condition of experiment,[8] the spectrum VLD should have the $-5/3$ dependence. Let us now explore the HVBK case. The inverse relaxation time $1/\tau_b$ [Eq. (16)] for the experiment[8] is about 600 s$^{-1}$. This implies for frequencies $f$ (in Hz) exceeding ≈100 Hz the slope of the curve $\langle\delta\mathcal{L}(f)\delta\mathcal{L}(-f)\rangle$ should be also $-5/3$.

The above speculations mean that experiment by Roche *et al.*[8] is not able to identify which scenario (chaotic vortex tangle, or the bundle structure) is realized in the turbulent flow. At the same time results obtained in the present work may serve as a basis for the experimental determination of what kind of turbulence is implemented in different types of generation.

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