Coexistence of antiferromagnetism and $d + id$ superconducting correlations in the graphene bilayer

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We discuss the t-J-U model on a honeycomb monolayer that has the same low-energy description of the kinetic term as the graphene bilayer, and in particular study coexistence of antiferromagnetism and superconducting correlations that originate from Cooper pairs without phase coherence. We show that the model is relevant for the description of the graphene bilayer and that the presence of the $d + id$ superconducting correlations with antiferromagnetism can lead to quadratic dependence in small magnetic fields of the gap of the effective monolayer consistent with the transport measurements of Velasco *et al.* on the graphene bilayer.

I. INTRODUCTION

The interaction effects are important for the physics of the graphene bilayer; recent experiments reveal gapped phase(s) in the undoped graphene bilayer which without interactions would represent a gapless system. In a recent experiment[1] with high quality samples, a completely insulating behavior was detected in transport measurements. Theoretical investigations, mean field, and renormalization group approaches[2–16] speak for a close competition of a few, mostly gapped, phases. One of the most prominent candidates for an explanation of the experiment in Ref. 1 is a layer antiferromagnetic (LAF) state. The main reason for the existence of this state would be an on-site Coulomb repulsion U; indeed as pointed out in Ref. 6, a Hubbard model on a honeycomb bilayer lattice would lead to the LAF state, both in weak and strong U limit. This may remind us of the behavior of the Hubbard model on the square lattice and the antiferromagnetic (AF) behavior due to nesting in the weak coupling limit. The Hubbard model on a square lattice is usually invoked as a model for cuprates in its strong coupling limit which forbids the double occupancy and leads to a "perfect" AF behavior at half-filling. On the other hand, the estimate for U is hard to know in the graphene bilayer and certainly depends on the computational scheme but it is expected to be stronger than both (inter and intralayer) hoppings. Due to the smallness of the gap revealed in the experiment in Ref. 1, we will not consider the large U limit (exclusion of double occupancy) when modeling the graphene bilayer. But we will keep the on-site repulsion as a main cause of the insulating behavior detected in the experiment. As expected from previous approaches this will lead to AF insulating behavior but seems insufficient to describe all phenomena detected in the experiment. An additional order parameter, besides the one that describes the antiferromagnetism, is necessary for the complete explanation of the transport data of the experiment.[1,8]

In this work we will look for the additional order parameter that can coexist with antiferromagnetism in the graphene bilayer at half-filling. We will argue that this is $d + id$ (broken time reversal symmetry)—wave superconducting order parameter. This ($d + id$) order parameter and its coexistence with antiferromagnetism was already found at finite (nonzero) dopings in a numerical (Grassman tensor product) approach to the t-J (large U) model on the honeycomb monolayer

in Ref. 17. Due to the assumed moderate (not large) value of U in our model of the graphene bilayer the AF and $d + id$ superconducting order parameter can coexist even at half-filling. Our model of the graphene bilayer can be described as a t-J-U model on an effective honeycomb lattice and, in the following, we will argue why this model is relevant for the description of graphene bilayer.

II. MODEL AND ITS MOTIVATION

The kinetic part of the Hamiltonian that describes the graphene bilayer on two honeycomb lattices, which are Bernal stacked, is

$$H_0 = -t \sum_{\vec{n},\sigma} \sum_{\vec{\delta}} (a^\dagger_{1,\vec{n},\sigma} b_{1,\vec{n}+\vec{\delta},\sigma} + a^\dagger_{2,\vec{n},\sigma} b_{2,\vec{n}-\vec{\delta},\sigma} + \text{H.c.})$$
$$+ t_\perp \sum_{\vec{n},\sigma} (a^\dagger_{1,\vec{n},\sigma} a_{2,\vec{n},\sigma} + \text{H.c.}). \quad (1)$$

The index $i = 1,2$ denotes the layer index. In Fig. 1 the relative positions of two triangular sublattices $A1$ and $B1$ for lattice 1, and $A2$ and $B2$ for lattice 2 are illustrated. In Eq. (1) t is the hopping energy between nearest sites in each layer, and $t_\perp$ is the same energy for hopping between the layers. The on-site creation (annihilation) operators $a^\dagger_{i,\vec{n},\sigma} (a_{i,\vec{n},\sigma})$ are for the electrons in the sublattice Ai of the layer i with spin $\sigma = \uparrow,\downarrow$, and $b^\dagger_{i,\vec{n},\sigma} (b_{i,\vec{n},\sigma})$ for the electrons in the sublattice Bi. $\vec{\delta}$'s are defined as $\delta_1 = a(0,1/\sqrt{3})$, $\delta_2 = a/2(1, -1/\sqrt{3})$, and $\delta_3 = a/2(-1, -1/\sqrt{3})$, and $a = \sqrt{3} \, a_{cc}$, a_{cc} is the distance between sites and a is the next to nearest neighbor distance.

The screening of the long-range part of the Coulomb interaction in two dimensions (2D), in the graphene bilayer,[18] is insufficient so this part can be neglected. Nevertheless, in the experiment we discuss,[1] two metallic gates are present and this must lead to an additional screening of the long-range part of the Coulomb interaction that will reduce the influence of the interaction to its short-range part. We will build our phenomenological model on the understanding based on the renormalization group method, as described in Refs. 5–10, of the influence of the short-range part in the context of the graphene bilayer. This understanding was successful in the explanation of the experiment of Mayorov *et al.*[19] Namely, the three most prominent instabilities of these studies are
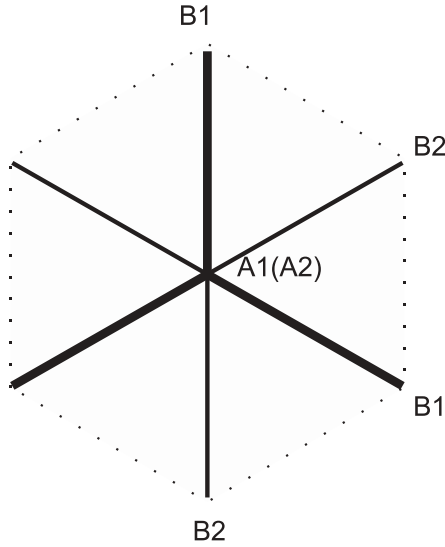
FIG. 1. A view of Bernal stacked honeycomb lattices 1 and 2 with corresponding sublattice sites $A1$ and $B1$, and $A2$ and $B2$.

antiferromagnetic (LAF), nematic, and anomalus Hall effect state. Also in mean field studies their close competition was observed and this indeed points out that the physics cannot be reduced to just (Hubbard) U, but must be extended to first and second near neighbors in an effective description. (This competition is not hard to understand in the graphene bilayer because the second near neighbors in the effective description belong to the same layer, while the first neighbors belong to opposite layers, see below.) But because the (edge) conductance of order e^2/h was not observed in the experiment,[1] we will neglect the influence of the second near neighbor interaction ($B1 - B1, B2 - B2$ in Fig. 1) that (according to the Haldane model[20,21]) would lead to the anomalous Hall effect.

Thus, according to the results of renormalization group approaches[5–10] and on phenomenological grounds (i.e., on the basis of the experiments[1,19]), the antiferromagnetic (LAF) state and a nematic state are the main instabilities that may arise due to interactions in the graphene bilayer. The nematic state is a result of an ordering in the particle-hole channel that can be described by a nonzero expectation value of the hopping (bond) operator

$$\sum_\sigma b^\dagger_{1,\vec{n}+\vec{\delta}_1,\sigma} b_{2,\vec{n}-\vec{\delta}_2,\sigma}, \tag{2}$$

where $\vec{n} + \vec{\delta}_1$ and $\vec{n} - \vec{\delta}_2$ denote sites that are near neighbors on the honeycomb lattice that make sublattices $B1$ and $B2$, as shown in Fig. 1. On the same lattice the antiferromagnetic ordering can occur which describes the LAF state.

The nematic, that is, bond ordering as described in Ref. 7, can be thought as a $d_{x^2-y^2}$ CDW (or a $d+id$ CDW as described in Ref. 5—see below Eq. (13) for the definition of the $d_{x^2-y^2}$ and $d+id$ ordering) in the language of Ref. 22. In the same reference this instability ("hidden order" on a square lattice) was proposed for cuprates. Just as in the case of the square lattice and cuprates, this ordering, in the graphene bilayer, can be thought as the result of short-range interactions and superexchange processes. These interactions and processes can also lead to the LAF state. The effective Heisenberg

interaction on the $B1$ and $B2$ honeycomb of neighboring sites i and j ($\vec{n}+\vec{\delta}$ and $\vec{n}-\vec{\delta}'$) can be rewritten in terms of the hopping operators [Eq. (2)] as

$$\vec{S}_i\vec{S}_j = -\frac{1}{2}\left(\sum_\sigma b^\dagger_{1i\sigma} b_{2j\sigma}\right)\left(\sum_\sigma b^\dagger_{2j\sigma} b_{1i\sigma}\right) - \frac{1}{4}, \tag{3}$$

and should be a part of an effective description of the graphene bilayer. According to Refs. 6 and 10, the only instability in the weak coupling limit of the Hubbard model on the graphene bilayer is antiferromagnetism. Thus, on phenomenological grounds, we will assume that the value of J is an independent parameter in the effective description, favorable for both instabilities, nematic and antiferromagnetic.

With respect to cuprates the on-site repulsion U is not that strong to preclude the double occupancy on the graphene bilayer. Therefore we will explicitly include this interaction in our model Hamiltonian, which can be described as

$$H = H_J + H_H, \tag{4}$$

where

$$H_J = J \sum_{\vec{n}} \sum_{\vec{\delta}_1,\vec{\delta}_2} \vec{S}_{b1}(\vec{n}+\vec{\delta}_1)\ \vec{S}_{b2}(\vec{n}-\vec{\delta}_2), \tag{5}$$

and the summation is over the near neighbors, and

$$H_H = -\sum_{k\in IBZ} t_{\text{eff}}\ \gamma^2_k b^\dagger_{1k}\ b_{2k} + \text{H.c.} + \sum_{i=1,2}\sum_{\vec{n}_i} U\hat{n}_{\uparrow\vec{n}_i}\hat{n}_{\downarrow\vec{n}_i}, \tag{6}$$

with $\gamma_k = \sum_{\vec{\delta}}\exp\{i\vec{k}\vec{\delta}\}$, $\vec{n}_1 \equiv \vec{n}+\vec{\delta}_1$, and $\vec{n}_2 \equiv \vec{n}-\vec{\delta}_2$. The kinetic term (written above in the momentum space) in real space, on the effective honeycomb monolayer of $B1$ and $B2$ sites, describes near-neighbor and two times weaker third-neighbor hopping. It can be recovered in the case of the noninteracting honeycomb bilayer by taking interlayer hopping to be large. In that case $t_{\text{eff}} = \frac{t^2}{t_\perp}$. In the small momentum limit the kinetic term in Eq. (6) becomes the one of the graphene bilayer, that is, $t_{\text{eff}}\ \gamma^2_k \to \frac{t^2}{t_\perp}(k_x \mp ik_y)^2$ near K points: $K_\pm = \pm\frac{2\pi}{a}(\frac{2}{3},0)$.[23,24] Because we look for the low energy properties we will keep the two-band extension with γ^2_k throughout the Brillouin zone.

This model is similar to the t-J-U model defined on the square lattice and known from previous investigations. The t-J-U model appeared also in the context of gossamer superconductivity,[25–29] the superconductivity that can exist even at half-filling.

III. MEAN FIELD APPROACH

In order to apply a mean field approach we can use the identity

$$\vec{S}_i\vec{S}_j = -\frac{1}{2}(b_{i\uparrow}b_{j\downarrow} - b_{i\downarrow}b_{j\uparrow})(b^\dagger_{j\downarrow}b^\dagger_{i\uparrow} - b^\dagger_{j\uparrow}b^\dagger_{i\downarrow}) + \frac{1}{4}. \tag{7}$$

We will define the superconducting order parameter,

$$\Delta_{\vec{\delta}} = \langle b_{1i\uparrow}b_{2i+\vec{\delta}\downarrow} - b_{2i+\vec{\delta}\uparrow}b_{1i\downarrow}\rangle, \tag{8}$$

where $\vec{\delta}$ can be any of the near-neighbor vectors on the honeycomb lattice, and

$$m = \langle n_{i\uparrow} - n_{i\downarrow}\rangle, \tag{9}$$

the antiferromagnetic order parameter. In the following we will use the following notation: $t \equiv t_{\text{eff}}$, and apply to the Hamiltonian in Eq. (4) the mean field ansatzes. We will generalize the derivation of Ref. 30 to the case with the antiferromagnetic order parameter.

If we use spinors,

$$\Psi_k = [b_{1k\uparrow} \quad b_{2k\uparrow} \quad b_{1-k\downarrow}^\dagger \quad b_{2-k\downarrow}^\dagger]^T, \qquad (10)$$

we can write the mean field Hamiltonian as

$$H_{\text{MF}} = \Psi_k^\dagger \mathcal{H} \Psi_k + \frac{1}{2} N J \sum_{\vec{\delta}} |\Delta_{\vec{\delta}}|^2 + U \frac{m^2}{2} N, \qquad (11)$$

where N is the number of unit cells,

$$\mathcal{H} = \begin{bmatrix} \frac{U}{2}m & -t\gamma_k^2 & 0 & -\frac{J}{2}\Delta(k) \\ -t\gamma_k^{*2} & -\frac{U}{2}m & -\frac{J}{2}\Delta(-k) & 0 \\ 0 & -\frac{J}{2}\Delta^*(-k) & \frac{U}{2}m & t\gamma_{-k}^{*2} \\ -\frac{J}{2}\Delta^*(k) & 0 & t\gamma_{-k}^2 & -\frac{U}{2}m \end{bmatrix}, \qquad (12)$$

and $\Delta(k) = \sum_{\vec{\delta}} \Delta_{\vec{\delta}} \exp\{i\vec{k}\vec{\delta}\}$. The symmetry analysis of the order parameter on a honeycomb lattice, first done in Ref. 31, concluded that there are three possibilities,

$$\Delta_{\vec{\delta}} : \begin{array}{l} \Delta\,(1, \quad 1, \quad 1) \\ \Delta\,(2, \; -1, \; -1) \\ \Delta\,(0, \quad 1, \; -1) \end{array} \qquad (13)$$

that span the space of order parameter. The last two possibilities belong to a two-dimensional subspace of irreducible representation of $\mathcal{S}_3$, permutation group.[32] The s-wave $\Delta_\delta = \Delta(1,1,1)$ has nodes at K points because $\Delta_k = \Delta\gamma_k$. For $d_{x^2-y^2}$ wave $\Delta_\delta = \Delta(2,-1,-1)$ near $K_\pm$ points we have $\Delta_1(K_\pm + k) = \Delta[3 \pm \frac{\sqrt{3}}{2}(k_x \pm ik_y)]$, and for d_{xy} wave $\Delta_\delta = \Delta(0,1,-1)$ the expansions are $\Delta_2(K_\pm + k) = \Delta i[\pm\sqrt{3} - \frac{1}{2}(k_x \pm ik_y)]$. Therefore if the $d + id$ combination $\Delta_1(k) \pm i\sqrt{3}\Delta_2(k)$ is taken near one of the K points the order parameter is a constant (6Δ) and at the other K point is linear in k_x and k_y. Therefore, instead of having the coefficients of the same absolute magnitude with $b_{1k\uparrow}^\dagger b_{2-k\downarrow}^\dagger$ and $b_{2k\uparrow}^\dagger b_{1-k\downarrow}^\dagger$ (and $b_{2-k\downarrow}b_{1k\uparrow}$ and $b_{1-k\downarrow}b_{2k\uparrow}$) for a fixed valley point $d + id$ singles out one spin projection (up or down) to be associated with sites on layer 1 and the opposite one to be associated with sites on layer 2. Thus it favors pairing (Cooper pairs) in which the layer index is associated with definite spin projection just as in an antiferromagnetically ordered state, that is, LAF state described above.

$d + id$ wave and s wave can coexist, as rotationally symmetric states, with the LAF state although only for certain values of J and U parameters. One can show that for the coexistence of the LAF state and s-wave $J \gg U$, which cannot be the case in the graphene bilayer. For the $d + id$ wave, on the other hand, one can find an interval for couplings J and U for which the LAF state and $d + id$-wave pairing can coexist. Expanding the mean field equations to fourth order in the ratio of superconducting and antiferromagnetic order parameter and in the weak coupling limit that is, $t > J, U$, we find $\frac{4}{3} - \frac{1}{4}Jw < \frac{J}{U} < \frac{4}{3} + \frac{1}{4}Jw$, with $w = \frac{1}{\sqrt{3}\pi t}$. We expect

that the interval will broaden when the short-range correlations (due to U) are properly taken into account that will renormalize (reduce) the effective value of t. This was worked out for the square lattice in Ref. 27 in a comprehensive (renormalized mean field) study of the t-J-U model, and a interval of couplings was identified for which antiferromagnetism and superconducting correlations can coexist at half-filling. Furthermore, in Ref. 28 a variational Monte Carlo method was applied to the same system, and a finite value of the pairing amplitude (Δ) was found in the antiferromagnetic region (with no superconducting phase coherence). We expect a similar situation in our case.

As in the case of honeycomb monolayer in Ref. 31, we can diagonalize the free part of the above Hamiltonian and come to the following expressions for order parameters:

$$C_{\vec{k}} = \sum_{\vec{\delta}} \Delta_{\vec{\delta}} \cos\{\vec{k}\vec{\delta} - 2\phi_{\vec{k}}\}$$

and

$$S_{\vec{k}} = \sum_{\vec{\delta}} \Delta_{\vec{\delta}} \sin\{\vec{k}\vec{\delta} - 2\phi_{\vec{k}}\},$$

where $\phi_{\vec{k}} = \arg[\gamma(k)]$. Due to the expansion of Δ_k around K points in the case of the $d + id$ wave we have

$$C_{K_\pm + k} \sim S_{K_\pm + k} \sim \frac{(k_x \pm ik_y)^2}{|k|^2}, \qquad (14)$$

where the last sign is independent of K points. Thus we recovered the basic signatures of $d + id$ pairing: (a) The order parameter is an eigenfunction of orbital angular momentum with eigenvalue equal to two, and (b) due to the same sign (chirality) at both K points this wave is a time reversal symmetry breaking wave on the bilayer honeycomb lattice. Therefore in analyzing this wave we can keep the leading behavior in $\Delta_1(k)$ and $\Delta_2(k)$ as this effectively captures the basic phenomenology of the $d + id$ wave. Thus we take

$$\mathcal{H}_{d-id} = \begin{bmatrix} \frac{U}{2}m & -t\gamma_k^2 & 0 & -J3\Delta \\ -t\gamma_k^{*2} & -\frac{U}{2}m & 0 & 0 \\ 0 & 0 & \frac{U}{2}m & t\gamma_{-k}^{*2} \\ -J3\Delta & 0 & t\gamma_{-k}^2 & -\frac{U}{2}m \end{bmatrix} \qquad (15)$$

in the case of $\Delta(k) = \Delta_1(k) - i\sqrt{3}\Delta_2(k)$ combination, or

$$\mathcal{H}_{d+id} = \begin{bmatrix} \frac{U}{2}m & -t\gamma_k^2 & 0 & 0 \\ -t\gamma_k^{*2} & -\frac{U}{2}m & -J3\Delta & 0 \\ 0 & -J3\Delta & \frac{U}{2}m & t\gamma_{-k}^{*2} \\ 0 & 0 & t\gamma_{-k}^2 & -\frac{U}{2}m \end{bmatrix} \qquad (16)$$

in the case when $\Delta(k) = \Delta_1(k) + i\sqrt{3}\Delta_2(k)$.

In the same low-momentum limit $\gamma_{K_\pm + k} \approx \mp a\frac{\sqrt{3}}{2}(k_x \mp ik_y)$. We will use redefinitions $\frac{U}{2}m \equiv m$, $J3\Delta \equiv \Delta$, and $ta^2\frac{3}{4} \equiv t$ in the following.

We take Δ to be purely real and without the phase [$U(1)$] degree of freedom, that is, phase coherence[33] that would lead to supercurrents proportional to the gradient of this phase that would screen a magnetic field that may be present. We assume that supercurrents cannot develop in the antiferromagnetic, insulating background.

The Bogoliubov spectrum is the same irrespective whether we ask for energy eigenvalues in the case defined by Eq. (15) or (16), and with the introduced redefinitions the eigenvalues are

$$E = \pm \sqrt{m^2 + \left[\frac{\Delta}{2} \pm \sqrt{t^2 k^4 + \left(\frac{\Delta}{2} \right)^2} \right]^2}. \quad (17)$$

Therefore the two different chirality states of the d wave are equally likely in the presence of the antiferromagnetic ordering.

IV. PRESENCE OF SMALL MAGNETIC FIELD

In the presence of magnetic field, due to the minimal prescription, we may introduce a pair of creation and annihilation operators and express the resulting Hamiltonian matrix around K_+ point as

$$\mathcal{H}^o_{+(d-id)B} = \begin{bmatrix} m & -\omega_c(a^\dagger)^2 & 0 & -\Delta \\ -\omega_c(a)^2 & -m & 0 & 0 \\ 0 & 0 & m & \omega_c(a^\dagger)^2 \\ -\Delta & 0 & \omega_c(a)^2 & -m \end{bmatrix}. \quad (18)$$

Here we introduced $\omega_c = \frac{eB}{mc}$, where B is the magnetic field and m is the effective mass of the graphene bilayer $\frac{1}{m} = 2t$. The eigenvectors can be expressed as 4-spinor coefficients of eigenvectors Ψ_n of $a^\dagger a$ operator $a^\dagger a \Psi_n = n \Psi_n$ classified by integer eigenvalues n: $0,1,2,\dots$. In the presence of small magnetic field we will look for the eigenstates in the form

$$\Psi_n = [c_1 \ c_2 \ c_3 \ c_4]^T |n\rangle, \quad n = 0,1,2,\dots. \quad (19)$$

The Nambu-Gorkov formalism with 4-spinors artificially doubles the degrees of freedom. This appears in spectra as doubling of energy levels ($\pm E$). Thus when solving the $\mathcal{H}_{+(d-id)B}$ we have to keep levels that are continuously related to energy levels with no superconducting instability ($\Delta \neq 0$) and are pertinent to the 2×2 upper, left block of the Hamiltonian matrix.

The Hamiltonian in Eq. (18) we will consider under the approximation of a small magnetic field and rewrite it as

$$\mathcal{H}^o_{+(d-id)B} = H_0 + V, \quad (20)$$

where

$$H_0 = \begin{bmatrix} m & 0 & 0 & -\Delta \\ 0 & -m & 0 & 0 \\ 0 & 0 & m & 0 \\ -\Delta & 0 & 0 & -m \end{bmatrix}, \quad (21)$$

and V denotes the perturbation

$$V = \begin{bmatrix} 0 & -\omega_c(a^\dagger)^2 & 0 & 0 \\ -\omega_c(a)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_c(a^\dagger)^2 \\ 0 & 0 & \omega_c(a)^2 & 0 \end{bmatrix}. \quad (22)$$

Taking as solutions only values that are connected continuously in the limit $\Delta \to 0$ to the upper 2×2 left part of H_0 we

get for the eigenvalues and eigenvectors of H_0:

$$E_1^n = -m, \quad \Psi_1 = [0, 1, 0, 0]^T |n\rangle,$$
$$E_2^n = \sqrt{m^2 + \Delta^2}, \quad \Psi_2 = c[m + E, 0, 0, -\Delta]^T |n\rangle,$$

where $E = \sqrt{m^2 + \Delta^2}$ and $c = \frac{1}{\sqrt{2E(E+m)}}$. Considering the small magnetic field to second order as perturbation we get $E_1^n = -m - \frac{(n+2)(n+1)}{2} \frac{\omega_c^2}{E}$ and $E_2^n = E + \frac{n(n-1)}{2} \frac{\omega_c^2}{E}$. Considering the same problem at $K' \equiv -K$ point we get $\tilde{E}_1^n = m + \frac{(n+2)(n+1)}{2} \frac{\omega_c^2}{E}$ and $\tilde{E}_2^n = -E - \frac{n(n-1)}{2} \frac{\omega_c^2}{E}$.

Thus, by analyzing the spectra of both K points together, we can conclude that with the inclusion of small magnetic fields the gap changes from $E_g = 2m$ value to $E_g = 2m + 2\frac{\omega_c^2}{E}$ in the presence of $d - id$ correlations. Without the correlations or with $d + id$ correlations the gap will not have the correction quadratic in small magnetic field, which direction is fixed in Eq. (18). $d - id$ correlations minimize the energy of the system by shifting also the energy levels closest to the Fermi point. In the Appendix we compare the energies of the states with $d + id$ and $d - id$ correlations, and show that $d - id$ are indeed of the lower energy.

The energy minimization, when the direction of perpendicular magnetic field is opposite, requires that the superconducting correlations are of $d + id$ kind. Thus the change in the direction of magnetic field is followed by the change in the chirality of the superconducting instability, that is, $B \to -B$ followed by $d - id \to d + id$, as one would expect from the superconducting instability that has orbital and therefore magnetic moment. This amounts to just switching of the previously found spectra between K and K' points. The gap is the same irrespective of the direction of the magnetic field although with the inclusion of superconducting correlations linear in k [on the diagonal in Eqs. (15) and (16)] leads to an asymmetry which may be related to the asymmetry detected in the transport measurements of Ref. 1 with respect to the change in the direction of the external field.

V. DISCUSSION AND CONCLUSION

In the literature we find several proposals[34–36] for the explanation of the data of Ref. 1. See also Ref. 37 for further experimental investigations on the same system and possible explanations based on anomalous quantum Hall physics. Reference 34 by Kharitonov introduces an additional order parameter to the Néel order parameter, but the resulting gap dependence does not have a minimum at B(magnetic field) = 0—compare Fig. 3 in Ref. 34, in contrast to what can be seen from the transport measurements—compare Fig. 3 in Ref. 1. Reference 35 by Zhu, Aji, and Varma with the interesting proposal of taking into account the full four band structure, still gives linear dependence on B of the gap—compare with Fig. 6 in Ref. 35, in contrast to the quadratic dependence on small B as seen in the experiment. Reference 36 by Roy does describe the quadratic dependence based on a mean field treatment of spin magnetism, where a phenomenological ferromagnetic interaction next to the Néel ordering among the spins of electrons is introduced which existence (with a precise magnitude) is necessary to obtain a correspondence to the experimental data.

Our approach is also mean field and phenomenological, though clearly motivated microscopically by the physics of the t-J-U model, as we introduce orbital magnetism of superconducting correlations. This leads to the quadratic dependence of the gap on small B as observed in the experiment. Thus we demonstrated a possibility that the quadratic dependence on small magnetic field observed in the experiment of Ref. 1 may be due to the time reversal symmetry breaking d-wave superconducting correlations that coexist with antiferromagnetism.

The $d + id$ wave superconductivity and antiferromagnetism at high dopings of the graphene monolayer were studied in Refs. 38 and 39. It was shown[39] that both instabilities are connected with the existence of the on-site repulsion U.

The last and important question we would like to discuss is how our proposal can explain the behavior of the system gap in strong (and moderate) magnetic fields. In other words, the question is how does the antiferromagnetic ground state with $d + id$ superconducting correlations evolve in the many-body state of half-filled zero-energy Landau level, which is eightfold degenerate due to flavor [spin and valley (layer)] and orbital ($n = 0,1$ Landau index) degrees of freedom. We expect a gradual formation of a QHFM (quantum Hall ferromagnet)[40] due to many-body correlations and the spontaneous ferromagnetic ordering of the spin degree of freedom. Thus we will fix the valley and orbital degree of freedom in the following and discuss how from two (spin up and spin down) Landau levels we can have effectively a single filled Landau level and ferromagnetic ordering. $d + id$ wave Cooper pairs, described in the long distance with the following Cooper pair wave function:[41]

$$f(\vec{r}_{1\uparrow} - \vec{r}_{2\downarrow}) \sim \frac{\bar{z}_{1\uparrow} - \bar{z}_{2\downarrow}}{z_{1\uparrow} - z_{2\downarrow}}, \tag{23}$$

in the presence of the magnetic flux will be modified by flux (vortex) attachment due to the particles of opposite spin as in

$$f(\vec{r}_{1\uparrow} - \vec{r}_{2\downarrow}) \sim \frac{\bar{z}_{1\uparrow} - \bar{z}_{2\downarrow}}{z_{1\uparrow} - z_{2\downarrow}} \prod_i (z_{1\uparrow} - z_{i\downarrow})^2 \prod_j (z_{2\downarrow} - z_{j\uparrow})^2. \tag{24}$$

This will lead to the following many-body state

$$\text{Det}\left(\frac{\bar{z}_{i\uparrow} - \bar{z}_{j\downarrow}}{z_{i\uparrow} - z_{j\downarrow}}\right) \prod_{i,j} (z_{i\uparrow} - z_{j\downarrow})^2 = \prod_{i,j} (z_{i\uparrow} - z_{j\downarrow}) \chi_2, \tag{25}$$

where Det denotes the determinant of the antisymmetrized product of Cooper pair wave functions, and χ_2 denotes the filled second Landau level wave function in the Jain notation. The identity used in Eq. (25) was proved in Ref. 42. The topological properties of the wave function in Eq. (25) (or the low energy properties of the system described with the wave function as discussed in Ref. 43) are equivalent to the Halperin (1,1,1) state or QHFM, that is, the following state:

$$\prod_{i<j} (z_{i\uparrow} - z_{j\uparrow}) \prod_{p<q} (z_{p\downarrow} - z_{q\downarrow}) \prod_{l,m} (z_{l\uparrow} - z_{m\downarrow}) \tag{26}$$

for fixed valley and orbital index, and thus lead to the QHFM state with the effective filling factor $\nu_{\text{eff}} = 4$. It was shown in Ref. 44 that this state would lead to the gap with linear dependence on the (strong) magnetic field as observed in Ref. 1. Thus we described a possible route from antiferromagnetic state with $d + id$ superconducting correlations to the spin QHFM state consistent with the experiment.

APPENDIX: ENERGY MINIMIZATION

To find whether $d - id$ or $d + id$ SC correlations coexist with antiferromagnetism in the presence of the magnetic field, which direction is defined as in Eq. (18), we should compare the two ground state energies,

$$E^{d-id} = \sum_{n=0}^{n'} [-m - (n+1)(n+2)\delta] + \sum_{n=0}^{n''} [-E - n(n-1)\delta] \tag{A1}$$

and

$$E^{d+id} = \sum_{n=0}^{\bar{n}'} [-E - (n+1)(n+2)\delta] + \sum_{n=0}^{\bar{n}''} [-m - n(n-1)\delta], \tag{A2}$$

where $\delta = \frac{\omega_c^2}{2E}$, and the bounds for the summations are determined by the lower cutoff $-E_c$, that is, we have in the $d - id$ case

$$(n'+2)(n'+1) = \frac{E_c - m}{\delta}, \tag{A3}$$

$$n''(n''-1) = \frac{E_c - E}{\delta}, \tag{A4}$$

$$(\bar{n}'+2)(\bar{n}'+1) = \frac{E_c - E}{\delta}, \tag{A5}$$

and

$$\bar{n}''(\bar{n}''-1) = \frac{E_c - m}{\delta} \tag{A6}$$

in the $d + id$ case. After a few steps of simple algebra we get

$$E^{d-id} - E^{d+id} = 2(m - E). \tag{A7}$$

Because $E > m$, the energy minimization favors $d - id$ SC correlations for the fixed direction of the magnetic field [Eq. (18)].

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