First-Matsubara-frequency rule in a Fermi liquid. I. Fermionic self-energy

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We analyze in detail the fermionic self-energy $\Sigma(\omega,T)$ in a Fermi liquid (FL) at finite temperature T and frequency ω. We consider both canonical FLs, i.e., systems in spatial dimension $D > 2$, where the leading term in the fermionic self-energy is analytic [the retarded $\mathrm{Im}\Sigma^R(\omega,T) = C(\omega^2 + \pi^2 T^2)$], and noncanonical FLs in $1 < D < 2$, where the leading term in $\mathrm{Im}\Sigma^R(\omega,T)$ scales as T^D or ω^D. We relate the $\omega^2 + \pi^2 T^2$ form to a special property of the self-energy, "the first-Matsubara-frequency rule," which stipulates that $\Sigma^R(i\pi T,T)$ in a canonical FL contains an $\mathcal{O}(T)$ but no T^2 term. We show that in any $D > 1$, the next term after $\mathcal{O}(T)$ in $\Sigma^R(i\pi T,T)$ is of order T^D ($T^3 \ln T$ in $D = 3$). This T^D term comes from only forward and backward scattering, and is expressed in terms of fully renormalized amplitudes for these processes. The overall prefactor of the T^D term vanishes in the "local approximation," when the interaction can be approximated by its value for the initial and final fermionic states right on the Fermi surface. The local approximation is justified near a Pomeranchuk instability, even if the vertex corrections are non-negligible. We show that the strength of the first-Matsubara-frequency rule is amplified in the local approximation, where it states that not only the T^D term vanishes, but also that $\Sigma^R(i\pi T,T)$ does not contain any terms beyond $\mathcal{O}(T)$. This rule imposes two constraints on the scaling form of the self-energy: upon replacing ω by $i\pi T$, $\mathrm{Im}\Sigma^R(\omega,T)$ must vanish and $\mathrm{Re}\Sigma^R(\omega,T)$ must reduce to $\mathcal{O}(T)$. These two constraints should be taken into consideration in extracting scaling forms of $\Sigma^R(\omega,T)$ from experimental and numerical data.

I. INTRODUCTION

Properties of single-particle and collective excitations in strongly interacting electron systems continue to attract substantial interest of the condensed-matter community. This interest is stimulated by the avalanche of discoveries of new materials, many of which fall into a category of strongly correlated electron systems, and by advances in experimental techniques, which allow one to extract, with good accuracy, the single-particle self-energy from angle-resolved photoemission spectroscopy (ARPES) data and the two-particle or "optical" self-energy from the real and imaginary parts of the optical conductivity.

One of the most actively explored directions in the study of strongly correlated electron systems is a search for non-Fermi liquids (non-FLs), i.e., systems in which electrons interact so strongly that they completely lose coherence. Many newly discovered systems were classified as non-FLs because their electron spectral functions, extracted from ARPES, are quite broad. However, a broad spectral function is an indication, but not the proof, that the system in question is a non-FL, as the Landau criterion for the FL only requires that the spectral function must be sharp for fermions in the immediate vicinity of the Fermi surface (FS). A mathematical formulation of this requirement is that the imaginary part of the retarded self-energy $\mathrm{Im}\Sigma^R(\omega)$ must be much smaller than $\omega + \mathrm{Re}\Sigma^R(\omega)$ at the smallest ω. This does not preclude that at higher frequencies $\mathrm{Im}\Sigma^R(\omega)$ can become comparable to $\omega + \mathrm{Re}\Sigma^R(\omega)$ or even exceed it.

To satisfy the Landau criterion, $\mathrm{Im}\Sigma^R(\omega)$ has to scale as ω^{1+a} with $a > 0$. The original argument by Landau, based on the Pauli principle and the assumption of analyticity, yields $\mathrm{Im}\Sigma^R(\omega) \propto \omega^2$, i.e., $a = 1$. Microscopic calculations show that $\mathrm{Im}\Sigma^R(\omega)$ does indeed scale as ω^2 in a three-dimensional

(3D) FL. The same holds for all "fractional" dimensions $D > 2$. For $D \leqslant 2$, the analyticity is, however, broken: $\mathrm{Im}\Sigma^R(\omega)$ scales as $\omega^2 \ln|\omega|$ in $D = 2$ and as $|\omega|^D$ in $D < 2$. Still, by Landau criterion, these systems are FLs, as long as $D > 1$. Hereafter, we refer to systems in which $\mathrm{Im}\Sigma^R(\omega) \propto \omega^2$ as "canonical FLs," and to systems in which $\mathrm{Im}\Sigma^R(\omega) \propto \omega^{1+a}$ with $0 < a < 1$ as "noncanonical FLs."

The goal of this paper is to analyze the form of the self-energy in both conventional and nonconventional FLs at finite frequency ω and temperature T. We will be particularly interested in how general is a certain property of the self-energy, which we will be referring to as the "first-Matsubara-frequency rule" or, for brevity, as the "first-Matsubara rule." This rule states that the self-energy $\Sigma(\omega_m,T)$, evaluated at discrete Matsubara points $\omega_m = \pi T(2m + 1)$, exhibits a special behavior at the first fermionic Matsubara frequency $\omega_0 = \pi T$, namely, $\Sigma(\pi T,T)$ does not contain terms higher than T. (The same happens at $\omega_{-1} = -\pi T$.) This rule was proven in the past for particular cases of the electron-phonon[1,2] and screened Coulomb[3,4] interactions. In the former case, this rule is sometimes being referred to as a "Fowler-Prange theorem."[5]

Although the first-Matsubara rule operates on the imaginary frequency axis, it is relevant to properties of physical fermions with real frequencies: it requires that the retarded self-energy $\Sigma^R(\omega,T)$, with ω replaced by $i\pi T$, should not contain terms beyond $\mathcal{O}(T)$, and thus imposes a constraint on the interplay between the ω and T terms in $\Sigma^R(\omega,T)$.

A 3D FL provides a simple example of how the first-Matsubara rule works. To order ω^2 and T^2, we have in this case $\mathrm{Re}\Sigma^R(\omega,T) = \lambda\omega$, with no ωT term, and $\mathrm{Im}\Sigma^R(\omega,T) = C(\omega^2 + \pi^2 T^2)$, with a factor of exactly π^2 in front of T^2. At $\omega = i\pi T$, $\mathrm{Im}\Sigma^R$ vanishes and $\mathrm{Re}\Sigma^R$ becomes of order T,

hence the total $\Sigma^R(i\pi T, T)$ contains only an $\mathcal{O}(T)$ term but no T^2 term.

In this paper, we analyze the validity of the first-Matsubara rule beyond the conventional FL paradigm. The proof of this rule in prior work[1–4] was based on demonstrating the nullification of the leading term in the imaginary part of the self-energy at $\omega = i\pi T$. We show here that the first-Matsubara rule does not hold beyond the leading order for conventional FLs, and does not hold at all for unconventional FLs. Our primary finding is that $\Sigma^R(i\pi T, T)$ scales as T^D in all D (with an extra $\ln T$ factor in $D = 3$); however, the consequences of this finding are different for conventional and unconventional FLs. For conventional FLs, i.e., for $2 < D \leqslant 3$, the T^D term is still subleading to T^2, and thus the first-Matsubara rule holds to order T^2. For unconventional FLs, i.e., for $1 < D < 2$, the T^D term is of the same order as the leading terms in $\text{Im}\Sigma^R(\omega, T)$, and thus the first-Matsubara rule is violated. In $D = 2$, which is a marginal case between conventional and unconventional FLs, $\text{Im}\Sigma^R(\omega, T) \propto (\omega^2 + \pi^2 T^2) \ln|\omega| + \mathcal{O}(\omega^2, T^2)$. While the logarithmic term vanishes at $\omega = i\pi T$, the T^2 term does not. As a result, the first-Matsubara rule is satisfied to logarithmic accuracy but not beyond.

We find that for $1 < D < 3$, the T^D term in $\Sigma^R(i\pi T, T)$ is universal, i.e., independent of the upper cutoff of the theory. Furthermore, its prefactor is expressed via exact spin and charge components of the forward-scattering and backscattering amplitudes.

At the same time, we find that the first-Matsubara rule holds to *all* orders in T in both conventional and nonconventional FLs, if the effective interaction between fermions, which includes dynamic screening by particle-hole bubbles, is assumed to connect only the states right on the Fermi surface. Hereafter, we refer to this approximation as the "local approximation," as it is generally valid when bosons which mediate interaction between fermions are slow compared to fermions.[6–8]

We show that, within the local approximation, the first-Matsubara rule relies only on the analytic properties of the local susceptibility. For the electron-phonon interaction, this approximation is a key ingredient of the Eliashberg theory,[9] and the small parameter which controls this approximation is the ratio of the Debye frequency to Fermi energy. We consider here the case of an electron-electron interaction. In certain limits, it can be approximated by an effective interaction mediated by collective modes of fermions in the spin or charge channel. The collective modes are generally not slow compared to fermions themselves (their velocity is of order of the Fermi velocity), but they do become slow near a Pomeranchuk instability, when the correlation length for critical collective modes diverges. As a result of this divergence, the system generates a low-energy scale, below which near-critical collective modes become overdamped and slow down.[7,10] The local approximation for collective modes is a necessary but not sufficient condition for the Eliashberg theory, as the latter also requires vertex corrections to be small. In the case of collective modes, vertex corrections are not controlled by the same parameter which makes the local approximation valid,[10–15] and are not necessarily small.[13,14,16–18] We show that the smallness of vertex corrections is not required for the first-Matsubara rule to work: the local approximation is sufficient.

We analyze the local approximation in more detail and show that the first-Matsubara rule imposes two conditions: (1) $\text{Im}\Sigma^R(i\pi T, T)$ vanishes to all orders in T, and (2) $\text{Re}\Sigma^R(i\pi T, T)$ contains an $\mathcal{O}(T)$ term, but all higher-order terms in T vanish. These two conditions are nontrivial because, beyond the conventional FL paradigm, $\Sigma^R(\omega, T)$ *can not* be obtained from the $T = 0$ result by a simple replacement $\omega \to \sqrt{\omega^2 + \pi^2 T^2}$. This is true for the subleading ω^3, T^3 terms in a 3D FL, and also for the leading ω^D, T^D terms in nonconventional FLs. In particular, $\text{Im}\Sigma^R(\omega, T)$ in nonconventional FLs has a complex form which is very different from $(\omega^2 + \pi^2 T^2)^{D/2}$, and $\text{Re}\Sigma^R(\omega, T)$ also contains a complex dependence on ω and T at order ω^D, in addition to the $\lambda\omega$ term. Nevertheless, as long as the local approximation is applicable, $\text{Im}\Sigma^R(\omega, T)$ vanishes at $\omega = i\pi T$, and $\text{Re}\Sigma^R(i\pi T, T)$ reduces to $i\pi\lambda T$.

Finally, we show that the first-Matsubara rule holds within the local approximation even for a non-FL, e.g., for a system in $D \leqslant 3$ right at a Pomeranchuk instability, except that in this case the coefficient λ in $\Sigma^R(i\pi T, T) = i\pi T\lambda$ diverges as $T \to 0$. In particular, the first-Matsubara rule holds for a marginal FL and for an itinerant two-dimensional (2D) system at a nematic quantum critical point (QCP).

The rest of the paper is organized as follows. In Sec. II, we review the derivation of the single-particle self-energy to order T^2 and ω^2 in a conventional FL, and show where the relation between the ω^2 and T^2 terms comes from. In Sec. III, we discuss the self-energy outside of the conventional FL paradigm. We show that, in general, the self-energy contains terms of order T^D, which do not vanish when ω is replaced by $i\pi T$. The case of $D = 2$ is marginal, and we consider it separately. In Sec. IV, we discuss the self-energy within the local approximation. We show that, at order T^D, $\Sigma^R(\omega, T)$ has quite a complex dependence on the ratio ω/T, yet the prefactor of the T^D term vanishes at $\omega = i\pi T$. We consider in detail 2D and 3D FLs, a 2D system at a nematic QCP, and also a marginal FL. We discuss under what conditions the local approximation is valid in all these cases. We also discuss in this section how one should properly construct the self-energy along real frequency axis to make sure that a replacement of ω by $i\pi T$ agrees with the analytical continuation of the self-energy into the upper half-plane. We present our conclusions in Sec V.

In the subsequent paper,[19] we discuss the constraints imposed by the first-Matsubara rule on the Ω/T scaling, the optical conductivity $\sigma(\Omega, T)$ of a FL, and the consequences of these constraints for the experiment.

Throughout the paper, we denote the retarded self-energy along the real frequency axis as $\Sigma_{\mathbf{k}}^R(\omega, T)$ and the self-energy along the Matsubara axis as $\Sigma_{\mathbf{k}}(\omega_m, T)$, where $\mathbf{k}$ is the electron (quasi)momentum. We set the overall sign of the retarded self-energy via

$$G_k^R(\omega, T) = \frac{1}{\omega + \Sigma_{\mathbf{k}}^R(\omega, T) - \varepsilon_{\mathbf{k}}}, \tag{1.1}$$

where $\varepsilon_{\mathbf{k}}$ is the electron dispersion, and define the Matsubara self-energy in such a way that it is real on the Fermi surface, i.e.,

$$G_{\mathbf{k}}(\omega_m, T) = \frac{1}{i[\omega_m + \Sigma_{\mathbf{k}}(\omega_m, T)] - \varepsilon_{\mathbf{k}}}. \tag{1.2}$$

II. SINGLE-PARTICLE SELF-ENERGY: CANONICAL FERMI LIQUID

In this section, we briefly review the derivation of the scaling forms for the self-energy in a conventional FL to order ω^2 and T^2: $\mathrm{Im}\Sigma^R(\omega,T) \propto \omega^2 + \pi^2 T^2$, $\mathrm{Re}\Sigma^R(\omega,T) = \lambda\omega$ with no $\mathcal{O}(\omega T)$ terms in either of these quantities. We first show how these forms are obtained in the perturbation theory, then use the Eliashberg's argument[9] to generalize the derivation to an arbitrary order in the interaction, and finally relate these forms to the first-Matsubara rule, a special property of the self-energy at the first fermionic Matsubara frequency $\omega_{m=0,-1} = \pm\pi T$ (Sec. II C).

A. Perturbation theory

We consider a system of fermions on a lattice with single-particle dispersion $\varepsilon_{\mathbf{k}}$. We assume that the Fermi surface does not have nested parts and is away from the van Hove singularities but otherwise arbitrary. Near the FS, $\varepsilon_{\mathbf{k}}$ can be approximated as $\varepsilon_{\mathbf{k}} = \mathbf{v}_{\mathbf{k}_F} \cdot (\mathbf{k} - \mathbf{k}_F)$, where $\mathbf{k}_F$ is a vector pointing in the direction of $\mathbf{k}$ and residing on the FS, and $\mathbf{v}_{\mathbf{k}} = \nabla_{\mathbf{k}}\varepsilon_{\mathbf{k}}$. We will see that ω^2 and T^2 terms in $\mathrm{Im}\Sigma_{\mathbf{k}}^R(\omega,T)$ come from low energies where the linear approximation is valid. Having this in mind, we follow a conventional reasoning of a FL theory, set the upper cutoff of the theory with the linearized dispersion at some energy Λ (generally comparable to the bandwidth W), and absorb all renormalizations from energies between Λ and W into nonsingular renormalizations of the effective mass and quasiparticle residue Z. The bare Green's function of low-energy fermions is then given by

$$G_{\mathbf{k}}^R(\omega) = Z_{\mathbf{k}_F}/(\omega - \varepsilon_{\mathbf{k}} + i\delta), \qquad (2.1)$$

where $\delta > 0$ is infinitesimally small and $Z_{\mathbf{k}_F}$, in general, varies along the FS. We further assume that fermion-fermion interaction $U_{\mathbf{q}}$ is static and nonsingular for all $\mathbf{q}$ connecting points on the FS, including $q = 0$. This is the case for, e.g., a screened Coulomb interaction.

The lowest-order diagrams which contribute to the imaginary part of the fermionic self-energy are shown in Fig. 1. The imaginary part of the fermionic self-energy arises from the convolutions of two Green's functions marked by slanted
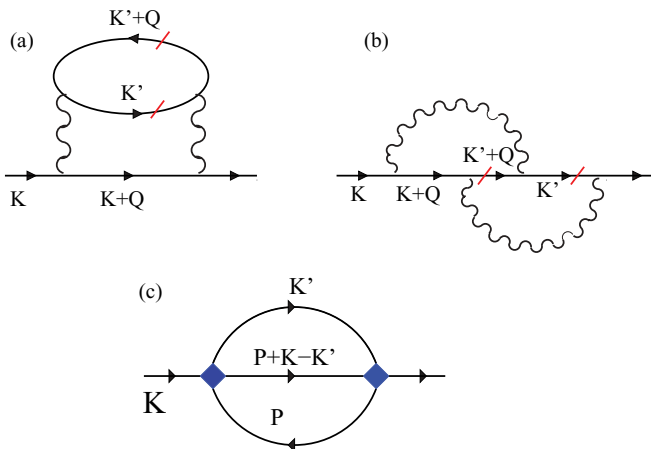

FIG. 1. (Color online) Diagrams for the self-energy. $K = (\omega_m, \mathbf{k})$, $K' = (\omega_{m'}, \mathbf{k}')$, $Q = (\Omega_n, \mathbf{q})$.

dashes in Fig. 1. In Fig. 1(a), such a convolution is just a particle-hole bubble, the imaginary part of which scales linearly with the bosonic frequency Ω. In Fig. 1(b), this convolution involves the momentum-dependent interaction, but the result still scales linearly with Ω.

To see this in more detail, we write down a Matsubara form of the self-energy from diagram Fig. 1(a) and obtain $\Sigma^R\mathbf{k}(\omega,T)$ by analytic continuation. With our definition for the self-energy (1.2), we have

$$\Sigma_{\mathbf{k}}^a(\omega_m,T) = -iT\sum_{\Omega_n}\int_{\mathbf{q}} U(\mathbf{q})^2 G_{\mathbf{k}+\mathbf{q}}(\omega_m + \Omega_n)\Pi_{\mathbf{q}}(\Omega_n),$$
$$(2.2)$$

where

$$\Pi_{\mathbf{q}}(\Omega_n) = 2T\sum_{\omega_{m'}}\int_{\mathbf{k}'} G_{\mathbf{k}'}(\omega_{m'})G_{\mathbf{k}'+\mathbf{q}}(\omega_{m'} + \Omega_n) \qquad (2.3)$$

with $\Omega_n = 2\pi nT$ and $\int_{\mathbf{l}} \equiv \int d^D l/(2\pi)^D$. Performing analytic continuation in both (2.2) and (2.3), we obtain the retarded self-energy along the real frequency axis

$$\Sigma_{\mathbf{k}}^{R,a}(\omega,T)$$
$$= \int_{\mathbf{q}} U_{\mathbf{q}}^2 \int \frac{d\Omega}{2\pi}\left[\coth\frac{\Omega}{2T} G_{\mathbf{k}+\mathbf{q}}^R(\omega + \Omega)\,\mathrm{Im}\,\Pi_{\mathbf{q}}^R(\Omega) \right.$$
$$\left. + \tanh\frac{\Omega + \omega}{2T}\,\mathrm{Im}\,G_{\mathbf{k}+\mathbf{q}}^R(\omega + \Omega)\Pi_{\mathbf{q}}^A(\Omega)\right], \qquad (2.4a)$$

$$\Pi_{\mathbf{q}}^R(\Omega)$$
$$= 2\int_{\mathbf{k}'} \int \frac{d\omega'}{2\pi}\left[\tanh\frac{\omega'}{2T}\,\mathrm{Im}\,G_{\mathbf{k}'}^R(\omega')G_{\mathbf{k}'+\mathbf{q}}^R(\omega' + \Omega) \right.$$
$$\left. + \tanh\frac{\omega' + \Omega}{2T} G_{\mathbf{k}'}^A(\omega')\,\mathrm{Im}\,G_{\mathbf{k}'+\mathbf{q}}^R(\omega' + \Omega)\right]. \qquad (2.4b)$$

Extracting the imaginary parts of Eqs. (2.4a) and (2.4b) and using the relations $\tanh(x/2) = 1 - 2n_F(x)$ and $\coth(x/2) = 2n_B(x) + 1$, where $n_F(x)$ and $n_B(x)$ are the Fermi and Bose functions, correspondingly, we obtain

$$\mathrm{Im}\Sigma_{\mathbf{k}}^{R,a}(\omega,T) = \int_{\mathbf{q}} U_{\mathbf{q}}^2 \int \frac{d\Omega}{\pi}\,[n_B(\Omega) + n_F(\omega + \Omega)]$$
$$\times \mathrm{Im}G_{\mathbf{k}+\mathbf{q}}^R(\omega + \Omega)\mathrm{Im}\Pi_{\mathbf{q}}^R(\Omega), \qquad (2.5a)$$

$$\mathrm{Im}\Pi_{\mathbf{q}}^R(\Omega) = 2\int_{\mathbf{k}'}\int \frac{d\omega'}{\pi}\,[n_F(\omega' + \Omega) - n_F(\omega')]$$
$$\times \mathrm{Im}G_{\mathbf{k}'}^R(\omega')\mathrm{Im}G_{\mathbf{k}'+\mathbf{q}}^R(\omega' + \Omega). \qquad (2.5b)$$

Equation (2.5b) can be rewritten as

$$\mathrm{Im}\Pi_{\mathbf{q}}^R(\Omega) = \int_{\mathbf{k}'}\int d\omega'[n_F(\omega' + \Omega) - n_F(\omega')]$$
$$\times \oint \frac{dA_{\mathbf{k}_F'}}{v_{\mathbf{k}_F'}(2\pi)^2} Z_{\mathbf{k}_F'}Z_{\mathbf{k}_F'+\mathbf{q}}\int d\varepsilon_{\mathbf{k}'}\delta(\omega' - \varepsilon_{\mathbf{k}'})$$
$$\times \delta(\omega' + \Omega - \epsilon_{\mathbf{k}'+\mathbf{q}}), \qquad (2.6)$$

where $dA_{\mathbf{k}_F'}$ is the element of the $(D - 1)$-dimensional FS. The integral over $\varepsilon_{\mathbf{k}'}$ gives $\delta(\omega' + \Omega - \epsilon_{\mathbf{k}'+\mathbf{q}})|_{\varepsilon_{\mathbf{k}'}=\omega'}$. The role of this δ function is to impose a constraint on the angle between $\mathbf{k}'$ and $\mathbf{q}$. Since this angle is not, in general, small, it suffices to resolve this constraint at $\omega' = \Omega = 0$ because,

as subsequent integration will show, $\omega' \sim \Omega \sim \max\{\omega, T\}$. The δ function thus reduces to $\delta(\epsilon_{\mathbf{k}'_F+\mathbf{q}})|_{\varepsilon_{\mathbf{k}'}=0}$, which means that both the initial and final states are on the FS. (Notice that this approximation corresponds to expanding the δ functions in $\max\{\omega, T\}/E_F$ rather than in $\omega/\varepsilon_{\mathbf{k}}$.) The integral over ω' now gives $\int_{-\infty}^{\infty} d\omega'[n_F(\omega') - n_F(\omega' + \Omega)] = \Omega$, and $\mathrm{Im}\Pi_{\mathbf{q}}^R(\Omega)$ reduces to Ω multiplied by a function of $\mathbf{q}$, averaged over the FS:

$$\mathrm{Im}\Pi_{\mathbf{q}}^R(\Omega) = -\frac{\Omega}{(2\pi)^2} \oint \frac{dA_{\mathbf{k}'_F}}{v_{\mathbf{k}'_F}} Z_{\mathbf{k}'_F} Z_{\mathbf{k}'_F+\mathbf{q}} \delta(\epsilon_{\mathbf{k}'_F+\mathbf{q}})|_{\varepsilon_{\mathbf{k}'}=0}. \tag{2.7}$$

For small q, the prefactor of Ω behaves as $1/q$. Substituting $\mathrm{Im}\Pi_{\mathbf{q}}^R(\Omega) \propto \Omega$ into (2.5a), and applying the same procedure as above to integrate over the momentum, we obtain, for $\mathbf{k} = \mathbf{k}_F$,

$$\mathrm{Im}\Sigma_{\mathbf{k}_F}^{R,a}(\omega, T) = 2C_a \int_{-\infty}^{\infty} d\Omega\, \Omega[n_B(\Omega) + n_F(\omega + \Omega)] \tag{2.8}$$

with

$$C_a = \frac{\pi}{2(2\pi)^{D-1}} \int_{\mathbf{q}} \oint \frac{dA_{\mathbf{k}'_F}}{v_{\mathbf{k}'_F}} Z_{\mathbf{k}_F+\mathbf{q}} Z_{\mathbf{k}'_F} Z_{\mathbf{k}'_F+\mathbf{q}}$$
$$\times \delta(\varepsilon_{\mathbf{k}_F+\mathbf{q}}) \delta(\epsilon_{\mathbf{k}'_F+\mathbf{q}}) U_{\mathbf{q}}^2. \tag{2.9}$$

[A factor of 2 in Eq. (2.8) is introduced for future convenience.] The frequency integral in Eq. (2.8) is readily evaluated as

$$\int_{-\infty}^{\infty} d\Omega\, \Omega[n_B(\Omega) + n_F(\omega + \Omega)] = \frac{1}{2}(\omega^2 + \pi^2 T^2), \tag{2.10}$$

hence

$$\mathrm{Im}\Sigma_{\mathbf{k}}^{R,a}(\omega, T) = C_a(\omega^2 + \pi^2 T^2). \tag{2.11}$$

We can now specify what actually makes the analysis above applicable only to conventional FLs rather than to all FLs: it is an assumption that the integral in Eq. (2.9) is convergent in the infrared. Power counting shows that the integrand behaves as $1/q^2$ for $q \to 0$; the integral over $d^{D-1}q$ then converges for $D > 2$ and diverges for $D \leqslant 2$. Infrared divergence for $D \leqslant 2$ will modify the ω and T dependences of $\Sigma_{\mathbf{k}_F}^R(\omega, T)$ compared to the canonical form valid for $D > 2$.

Figure 1(b) is analyzed in a similar way with the only difference that the quantity $U_q \mathrm{Im}\Pi_{\mathbf{q}}^R(\Omega)$ in Eq. (2.5a) is replaced by

$$\mathrm{Im}\mathcal{P}_{\mathbf{q},\mathbf{k}}^R(\Omega) = \int_{\mathbf{k}'} \int \frac{d\omega'}{\pi}[n_F(\omega' + \Omega) - n_F(\epsilon)]U_{\mathbf{k}-\mathbf{k}'}$$
$$\times \mathrm{Im}G_{\mathbf{k}'}^R(\epsilon)\mathrm{Im}G_{\mathbf{k}'+\mathbf{q}}^R(\epsilon + \Omega). \tag{2.12}$$

Still, $\mathrm{Im}\mathcal{P}_{\mathbf{q},\mathbf{k}}^R(\Omega)$ scales as Ω for $\Omega \to 0$. Evaluating the integrals in the same way as above, we find

$$\mathrm{Im}\Sigma_{\mathbf{k}_F}^{R,b}(\omega, T) = C_b(\omega^2 + \pi^2 T^2), \tag{2.13}$$

where

$$C_b = -\frac{\pi}{4(2\pi)^{D-1}} \int_{\mathbf{q}} \oint \frac{dA_{\mathbf{k}'_F}}{v_{\mathbf{k}'_F}} Z_{\mathbf{k}_F+\mathbf{q}} Z_{\mathbf{k}'_F} Z_{\mathbf{k}'_F+\mathbf{q}}$$
$$\times \delta(\varepsilon_{\mathbf{k}_F+\mathbf{q}}) \delta(\epsilon_{\mathbf{k}'_F+\mathbf{q}}) U_{\mathbf{q}} U_{\mathbf{k}'_F-\mathbf{k}_F}. \tag{2.14}$$

As before, the integral in Eq. (2.14) is convergent for $D > 2$. Comparing $\mathrm{Im}\Sigma_{\mathbf{k}_F}^{R,a}(\omega, T)$ and $\mathrm{Im}\Sigma_{\mathbf{k}_F}^{R,b}(\omega, T)$, we see that they

both have the same scaling form $\omega^2 + \pi^2 T^2$ and differ only in prefactors which, in the general case, depend on $\mathbf{k}_F$, i.e., on position along the FS.

The real part of the self-energy can be obtained either directly, e.g., from Eq. (2.4a) for Fig. 1(a), or via a Kramers-Kronig (KK) transformation of $\mathrm{Im}\Sigma_{\mathbf{k}}^R(\omega, T)$:

$$\mathrm{Re}\Sigma_{\mathbf{k}_F}^R(\omega, T) = \frac{2\omega}{\pi}\mathcal{P}\int_0^{\infty} d\omega' \frac{\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(\omega', T)}{\omega'^2 - \omega^2}, \tag{2.15}$$

where $\mathcal{P}$ stands for the principal part. The integral is ultraviolet divergent if Eq. (2.11) or (2.13) is used for $\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$, which implies that, to get the correct form of $\mathrm{Re}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ from the KK transformation, one has to use the full form of $\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ rather than its low-energy approximation. Nevertheless, one can easily make sure that, to quadratic order, $\mathrm{Re}\Sigma_{\mathbf{k}_F}(\omega, T) = \lambda_{\mathbf{k}_F}\omega$ (where $\lambda_{\mathbf{k}_F}$ varies, in general, along the FS) with no ωT term.

A comment is in order here. By applying (2.15) to (2.11) or (2.13), we can only show that there is no "universal" cutoff-independent ωT term in $\mathrm{Re}\Sigma_{\mathbf{k}}^R(\omega, T)$, and hence no T^2 term in $\mathrm{Re}\Sigma_{\mathbf{k}}^R(i\pi T, T)$. There is still a possibility that a T^2 term in $\mathrm{Re}\Sigma_{\mathbf{k}}^R(i\pi T, T)$ may come from internal frequencies in Eq. (2.15) comparable to the upper cutoff of the low-energy theory. We show later, in Sec. II C, that this is not the case, and that only a T^3 term emerges from high energies.

B. Arbitrary order in the interaction

We now follow the argument by Eliashberg[9] who showed that the $\omega^2 + \pi^2 T^2$ form of the self-energy at finite T holds to all orders in the interaction [a similar reasoning was also employed by Luttinger[20] to show that $\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T = 0) \propto \omega^2$].

The argument is as follows. In the second-order diagrams, the $\omega^2 + \pi^2 T^2$ form comes from the region where all three intermediate fermions are located within the window of width of order ω or T near the FS. Accordingly, the interactions $U_{\mathbf{q}}$ can be approximated by their values evaluated for the case when when the initial and final states are on the FS, i.e., $\mathbf{q} = \mathbf{l}_F - \mathbf{l}'_F$.

In a self-energy diagram of any order, one can select a cross section with three low-energy fermions, and sum over all other fermions without assuming that they are near the FS. The diagrams of this kind can be cast in the form of Fig. 1(c). The three selected fermions are near the FS and the shaded squares are the full vertex functions. Because integration over the fermionic lines already gives a function quadratic in ω or T, one can set $T = 0$ in the remainder of the diagram and project all momenta onto the FS. As long as the full vertex functions do not diverge, they do not affect integration over dispersions and frequencies of intermediate fermions. Self-energy corrections to fermionic lines are also irrelevant because the dressed Green's function still has the form of Eq. (2.1) at the lowest energies; adding one-loop self-energy to Eq. (2.1) simply replaces $i\delta$ by $iC(\omega^2 + \pi^2 T^2)$, which has an extra power of energy compared to ω and hence does not affect the ω^2 and T^2 terms in the full self-energy. As a result, the $\omega^2 + \pi^2 T^2$ form survives to an arbitrary order in the interaction: self-energy and vertex renormalizations only affect the overall factor in $\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$. We then have for a

conventional FL and to order ω^2, T^2

$$\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T) = C(\omega^2 + \pi^2 T^2). \qquad (2.16)$$

The prefactor C depends on model parameters, including the cutoff Λ, and is thus nonuniversal. Substituting this form into the KK formula (2.15), and using the same arguments as in the previous section, we find

$$\text{Re}\Sigma^R_{\mathbf{k}_F}(\omega, T) = \lambda\omega(1 + 0 \times T) \qquad (2.17)$$

[we spelled out the $0 \times T$ combination to emphasize that there $\text{Re}\Sigma^R(\omega, T)$ does not contain an ωT term]. The prefactor λ is again nonuniversal.

C. Self-energy along the imaginary axis: The first-Matsubara-frequency rule

We now show that the scaling form of $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ in Eq. (2.16) and the absence of the ωT term in $\text{Re}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ are related to a particular behavior of the self-energy at the first fermionic Matsubara frequency $\omega_{m=0} = \pi T$ (the same behavior holds at $\omega_{m=-1} = -\pi T$).

1. Analytic continuation

Let us first analytically continue $\text{Im}\Sigma^R_{\mathbf{k}}(\omega, T)$ and $\text{Re}\Sigma^R_{\mathbf{k}}(\omega, T)$ in Eqs. (2.16) and (2.17) into the upper-half plane of the complex variable $\omega \to z = z' + iz''$. Because $\text{Im}\Sigma^R_{\mathbf{k}}(\omega, T)$ and $\text{Re}\Sigma^R_{\mathbf{k}}(\omega, T)$ are analytic, their analytic continuation reduces to just a replacement of ω by z. The complex function $\text{Im}\Sigma^R(z, T) \propto z^2 + \pi^2 T^2$ then vanishes at $z = i\pi T$ and $\text{Re}\Sigma^R(z, T)$ becomes $i\pi\lambda T$, so that the full self-energy reduces to $i\pi T\lambda$ and does not contain a T^2 term. This is only true, however, if z is replaced by the first Matsubara frequency. For any other $\omega_m \neq \pi T$, the function $\Sigma^R(i\omega_m, T)$ contains a T^2 term. As we said in the Introduction, we will refer to this property as the first-Matsubara-frequency rule (first-Matsubara rule).

It is worth stressing that the rule formulated above applies only to the *first* Matsubara frequency. This may not seem to be the case if we replace ω by $i\pi(2m+1)T$ with arbitrary m in $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ given by Eq. (2.5a), *before* integrating over the bosonic frequency Ω. Doing so, and using the identity $n_F(\Omega + i\omega_m) = -n_B(\Omega)$, we seemingly find that $\text{Im}\Sigma^R(i\omega_m, T)$ vanishes not only at $\omega = i\pi T$, but also at *any* Matsubara frequency $i\pi T(2m+1)$.

This result is, however, false because the complex function, obtained by analytic continuation of, e.g., Eq. (2.5a), into the complex plane *before* the integral over Ω is performed, contains a sequence of branch cuts that run parallel to the real axis and intersect the imaginary axis at the Matsubara frequencies (see Fig. 2). As a result, the imaginary part of the function

$$F(z) = \int d\Omega \, \Omega[n_B(\Omega) + n_F(\Omega + z)] \qquad (2.18)$$

changes discontinuously at $z = z' + i\pi(2m+1)T$. For example, a discontinuity of $\text{Im}F(z)$ at $z = z' + i\pi T$ is

$$\{\text{Im}F[z' + iT(\pi + \delta/2)] - \text{Im}F[z' + iT(\pi - \delta/2)]\}|_{\delta \to 0}$$

$$= -\delta \int d\Omega \frac{\Omega}{\sinh^2\frac{\Omega + z'}{2T} + \delta^2} \approx 2\pi T z'. \qquad (2.19)$$
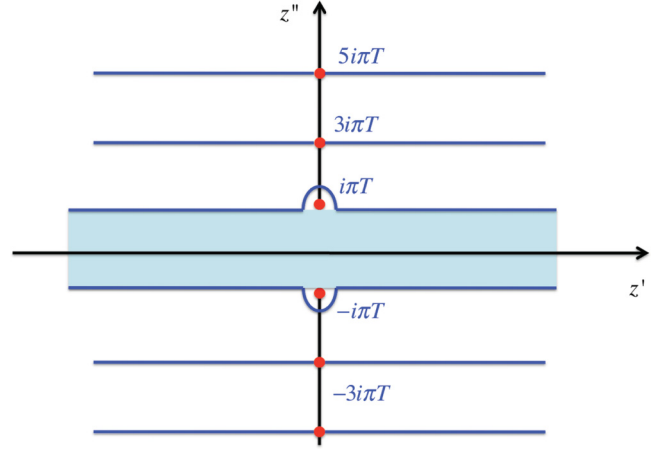

FIG. 2. (Color online) Analytic structure of the function $\text{Im}\Sigma^R_{\mathbf{k}_F}(z, T)$ in the complex z plane. Analytic continuation from the real axis is possible to any point within the shaded region, including the points $\pm i\pi T$, but not beyond this region.

This implies that the substitution $\omega \to z = z' + iz''$ into the integral form of $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ [Eq. (2.8)] gives the same result for $\text{Im}\Sigma^R_{\mathbf{k}_F}(z, T)$ as the actual analytical continuation only in the region bounded by two branch cuts at $z = z' + i\pi T$ and $z' - i\pi T$, but not outside this region. In other words, the substitution $\omega = i\omega_m$ into (2.8) gives the correct result for only for the first, but not for all Matsubara frequencies.

2. Direct proof of the first-Matsubara-frequency rule

The first-Matsubara rule can be also proven directly by computing the self-energy for a conventional FL in Matsubara frequencies. For the electron-electron interaction, this was done in Refs. 3 and 4; however, the proofs presented in these two papers are valid under two additional assumptions, namely, of small-angle scattering and of a quadratic dispersion $\varepsilon_\mathbf{k} = (k^2 - k_F^2)/2m^*$, where m^* is the renormalized effective mass. In fact, neither of these two assumptions is necessary. In what follows, we first consider the case of arbitrary-angle scattering but still keep an assumption of a quadratic dispersion, and then generalize the argument for an arbitrary dispersion.

Quadratic dispersion. To be specific, we consider the 3D case; other dimensions can be considered in a similar way. The clamshell self-energy diagram [Fig. 1(c)] reads as

$$\Sigma_{\mathbf{k}_F}(\omega_m, T) = -iZ^3 T \sum_{\Omega_n} T \sum_{\omega_{m'}} \int \frac{d^3k'}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3}$$

$$\times G_{\mathbf{k}'}(\omega_m + \Omega_n) G_\mathbf{p}(\omega_{m'}) G_{\mathbf{p}+\mathbf{k}-\mathbf{k}'}(\omega_{m'} + \Omega_n)$$

$$\times \Gamma_{\mathbf{k}_F, \mathbf{p}_F; \mathbf{k}'_F, \mathbf{p}_F + \mathbf{k}_F - \mathbf{k}'_F} \Gamma_{\mathbf{k}'_F, \mathbf{p}_F + \mathbf{k}_F - \mathbf{k}'_F; \mathbf{k}_F, \mathbf{p}_F}, \qquad (2.20)$$

where $\Gamma_{\mathbf{k}, \mathbf{p}; \mathbf{k}', \mathbf{p}'}$ is the renormalized vertex [a filled diamond in Fig. 1(c)]. Since we have already assumed that the dispersion is isotropic, the Z factor is assumed to be isotropic as well. The momentum transfers can be arbitrary, but all three intermediate momenta are assumed to be near the FS; this assumption has already been used in Eq. (2.20).

To evaluate the momentum integrals, the dispersion $\epsilon_{\mathbf{p}+\mathbf{k}-\mathbf{k}'}$ needs to be expanded in $\varepsilon_\mathbf{p}$, $\varepsilon_\mathbf{k}$, and $\varepsilon_{\mathbf{k}'}$. For a quadratic

dispersion, we obtain after some algebra

$$
\begin{aligned}
\epsilon_{\mathbf{p}+\mathbf{k}-\mathbf{k}'} = {} & \frac{2k_F^2}{m^*} \sin\frac{\theta_{\mathbf{k},\mathbf{k}'}}{2} \left(\sin\frac{\theta_{\mathbf{k},\mathbf{k}'}}{2} + \cos\theta_{\mathbf{p},\mathbf{k}-\mathbf{k}'} \right) \\
& + \varepsilon_{\mathbf{p}} \left(1 + 2\sin\frac{\theta_{\mathbf{k},\mathbf{k}'}}{2}\cos\theta_{\mathbf{p},\mathbf{k}-\mathbf{k}'} \right) + (\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}'}) \\
& \times \left(2\sin^2\frac{\theta_{\mathbf{k},\mathbf{k}'}}{2} + \sin\frac{\theta_{\mathbf{k},\mathbf{k}'}}{2}\cos\theta_{\mathbf{p},\mathbf{k}-\mathbf{k}'} \right), \quad (2.21)
\end{aligned}
$$

where $\theta_{\mathbf{l},\mathbf{m}}$ is the angle between momenta $\mathbf{l}$ and $\mathbf{m}$. A similar analysis can be carried out for any isotropic but not necessarily quadratic dispersion. For small-angle scattering $\theta_{\mathbf{k},\mathbf{k}'} \ll 1$, Eq. (2.21) reduces to a familiar form $\epsilon_{\mathbf{p}+\mathbf{q}} = \epsilon_{\mathbf{p}} + v_F q \cos\theta_{\mathbf{p},\mathbf{q}}$ with $q = 2k_F \sin(\theta_{\mathbf{k},\mathbf{k}'}/2)$. For the momentum $\mathbf{p} + \mathbf{k} - \mathbf{k}'$ to be

on the FS, the first term in Eq. (2.21) must be small; for generic values of $\theta_{\mathbf{k},\mathbf{k}'}$, this condition amounts to a geometric constraint

$$
\cos\theta_{\mathbf{p},\mathbf{k}-\mathbf{k}'} = -\sin\frac{\theta_{\mathbf{k},\mathbf{k}'}}{2} \quad (2.22)
$$

or $\theta_{\mathbf{p},\mathbf{k}-\mathbf{k}'} = \pm(\pi + \theta_{\mathbf{k},\mathbf{k}'})/2$. We expand the first term in Eq. (2.21) around this value as $\theta_{\mathbf{p},\mathbf{k}-\mathbf{k}'} = \pm(\pi + \theta_{\mathbf{k},\mathbf{k}'})/2 - \alpha$ with $\alpha \ll 1$, and set $\alpha = 0$ in the remaining two terms. This gives

$$
\begin{aligned}
\epsilon_{\mathbf{p}+\mathbf{k}-\mathbf{k}'} = {} & v_F^* k_F \alpha \sin\theta_{\mathbf{k},\mathbf{k}'} + \varepsilon_{\mathbf{p}} \cos^2\theta_{\mathbf{k},\mathbf{k}'} \\
& + (\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}'}) \sin^2\frac{\theta_{\mathbf{k},\mathbf{k}'}}{2}, \quad (2.23)
\end{aligned}
$$

where $v_F^* = k_F/m^*$. Substituting the last result into (2.21), we obtain

$$
\begin{aligned}
\Sigma_{\mathbf{k}_F}(\omega_m, T) = {} & -2iZ^3 T \sum_{\Omega_n} T \sum_{\omega_{m'}} \left(\frac{k_F^2}{v_F} \right)^2 \int \frac{d\varepsilon_{\mathbf{k}'}}{(2\pi)^2} \int d\theta_{\mathbf{k},\mathbf{k}'} \sin\theta_{\mathbf{k},\mathbf{k}'} \cos\frac{\theta_{\mathbf{k},\mathbf{k}'}}{2} \int \frac{d\varepsilon_{\mathbf{p}}}{(2\pi)^2} \int d\alpha \frac{1}{i(\Omega_n + \omega_m) - \varepsilon_{\mathbf{k}'}} \frac{1}{i\omega_{m'} - \varepsilon_{\mathbf{p}}} \\
& \times \frac{1}{i(\omega_{m'} + \Omega_n) - v_F k_F \alpha \sin\theta_{\mathbf{k},\mathbf{k}'} - \varepsilon_{\mathbf{p}} \cos^2\theta_{\mathbf{k},\mathbf{k}'} - (\varepsilon_{\mathbf{k}} + \varepsilon_{\mathbf{k}'})\sin^2\frac{\theta_{\mathbf{k},\mathbf{k}'}}{2}} \Gamma_{\mathbf{k}_F,\mathbf{p}_F;\mathbf{k}_F',\mathbf{p}_F+\mathbf{k}_F-\mathbf{k}_F'} \Gamma_{\mathbf{k}_F',\mathbf{p}_F+\mathbf{k}_F-\mathbf{k}_F';\mathbf{k}_F,\mathbf{p}_F}.
\end{aligned}
$$
$$(2.24)$$

Constraint (2.22) is assumed to be imposed on the momenta entering both vertices in the last equation. The integral over α gives

$$
-i\frac{\pi}{v_F^* k_F \sin\theta_{\mathbf{k},\mathbf{k}'}} \text{sgn}(\omega_{m'} + \Omega_n), \quad (2.25)
$$

while the integral over $\varepsilon_{\mathbf{p}}$ gives $-i\pi \, \text{sgn}\,\omega_{m'}$. Summing the product of the two sign functions over $\omega_{m'}$, we obtain a "local," i.e., integrated over the momentum, polarization bubble as a sum of two terms: $-|\Omega_n|/\pi$ and a constant, proportional to the ultraviolet cutoff of the theory. The constant contributes only to the $\mathcal{O}(T)$ term in $\Sigma_{\mathbf{k}_F}(\pi T, T)$, and we consider it separately later. The $|\Omega_n|$ term is the one relevant to our purposes as we need to verify that $\Sigma_{\mathbf{k}_F}(\pi T, T)$ does not contain a T^2 contribution. The prefactor of the $|\Omega_n|$ term is given by

$$
\begin{aligned}
2C = {} & \left(\frac{m^* Z}{2\pi} \right)^3 \int d\theta_{\mathbf{k},\mathbf{k}'} \cos\frac{\theta_{\mathbf{k},\mathbf{k}'}}{2} \Gamma_{\mathbf{k}_F,\mathbf{p}_F;\mathbf{k}_F',\mathbf{p}_F+\mathbf{k}_F-\mathbf{k}_F'} \\
& \times \Gamma_{\mathbf{k}_F',\mathbf{p}_F+\mathbf{k}_F-\mathbf{k}_F';\mathbf{k}_F,\mathbf{p}_F}. \quad (2.26)
\end{aligned}
$$

The remaining integral over $\varepsilon_{\mathbf{k}'}$ gives $\text{sgn}(\omega_m + \Omega_n)$, and the self-energy becomes

$$
\Sigma_{\mathbf{k}_F}(\omega_m, T) = -2C\pi T \sum_{\Omega_n} \text{sgn}(\omega_m + \Omega_n)|\Omega_n| + \cdots, \quad (2.27)
$$

where the ellipsis stands for $\mathcal{O}(T)$ terms. Summation over Ω_n is straightforward, and we obtain

$$
\Sigma_{\mathbf{k}_F}(\omega_m, T) = C(\pi^2 T^2 - \omega_m^2) + \cdots. \quad (2.28)
$$

The T^2 term in the Matsubara self-energy obviously vanishes for $\omega_m = \pm\pi T$.

Arbitrary dispersion. Equation (2.28) is also valid for an arbitrary fermionic dispersion, with the only difference that the prefactor C now depends on the position on the FS. To see how this works, we expand $\epsilon_{\mathbf{p}+\mathbf{k}-\mathbf{k}'}$ near a FS. Knowing that $\varepsilon_{\mathbf{k}}$ and $\varepsilon_{\mathbf{k}'}$ drop out anyway, we set them to zero and expand $\epsilon_{\mathbf{p}+\mathbf{k}-\mathbf{k}'}$ around $\mathbf{p}_F$ as

$$
\varepsilon_{\mathbf{p}+\mathbf{q}_F} = \varepsilon_{\mathbf{p}-\mathbf{p}_F+\mathbf{p}_F+\mathbf{q}_F} = \epsilon_{\mathbf{p}_F+\mathbf{q}_F} + \varepsilon_{\mathbf{p}} \frac{v_{\mathbf{p}_F+\mathbf{q}_F}^{||}}{v_{\mathbf{p}_F}^{||}}, \quad (2.29)
$$

where $\mathbf{q}_F \equiv \mathbf{k}_F - \mathbf{k}_F'$, $v_{\mathbf{l}}^{||} \equiv \mathbf{v}_{\mathbf{l}} \cdot \hat{\mathbf{p}}_F$, $\hat{\mathbf{p}}_F \equiv \mathbf{p}_F/p_F$, and we suppressed the * symbol in v_F for brevity.

Substituting this expansion into (2.20) and replacing integrals over 3D momenta by integrals over the FS and over the electron energy, we obtain

$$
\begin{aligned}
\Sigma_{\mathbf{k}_F}(\omega_m, T) = {} & -iT \sum_{\Omega_n} T \sum_{\omega_n'} \oint \frac{dA_{\mathbf{k}_F'}}{(2\pi)^3 v_{\mathbf{k}_F'}} \oint \frac{dA_{\mathbf{p}_F}}{(2\pi)^3 v_{\mathbf{p}_F}} \tilde{Z}^3 \\
& \times \int d\varepsilon_{\mathbf{k}'} \int d\varepsilon_{\mathbf{p}} \frac{1}{i(\Omega_n + \omega_m) - \varepsilon_{\mathbf{k}'}} \frac{1}{i\omega_n' - \varepsilon_{\mathbf{p}}} \\
& \times \frac{1}{i(\omega_n' + \Omega_n) - \varepsilon_{\mathbf{p}_F+\mathbf{q}_F} - \varepsilon_{\mathbf{p}} \frac{v_{\mathbf{p}_F+\mathbf{q}_F}^{||}}{v_{\mathbf{p}_F}^{||}}} \\
& \times \Gamma_{\mathbf{k}_F,\mathbf{p}_F;\mathbf{k}_F',\mathbf{p}_F+\mathbf{q}_F} \Gamma_{\mathbf{k}_F',\mathbf{p}_F+\mathbf{q}_F;\mathbf{k}_F,\mathbf{p}_F}, \quad (2.30)
\end{aligned}
$$

where $\tilde{Z}^3 \equiv Z_{\mathbf{k}_F'} Z_{\mathbf{p}_F'} Z_{\mathbf{p}_F+\mathbf{k}_F-\mathbf{k}_F'}$. The condition that all three internal fermions are located near the FS implies that, at fixed $\mathbf{q}_F$, the angle between $\mathbf{p}_F$ and $\mathbf{q}_F$ must be such that the first term in $\varepsilon_{\mathbf{p}_F+\mathbf{q}_F}$ is small. Suppose that, at fixed $\mathbf{q}_F$, the constraint $\varepsilon_{\mathbf{p}_F^{0,i}+\mathbf{q}_F} = 0$ is satisfied for a set of symmetry-related points on the FS, $\mathbf{p}_F^{0,i}$. The vector $\mathbf{p}_F$

spans a narrow solid angle around each of $\mathbf{p}_F^{0,i}$; therefore, we can expand the dispersion as $\varepsilon_{\mathbf{p}_F+\mathbf{q}_F} = (\mathbf{p}_F - \mathbf{p}_F^{0,i}) \cdot \tilde{\mathbf{v}}$, where $\tilde{\mathbf{v}} \equiv \mathbf{v}_{\mathbf{p}^{0,i}+\mathbf{q}_F}$. Since $\mathbf{q}_F$ is still fixed, it can be chosen as the polar axis of a local spherical system, in which a point on the FS is described by an equation $p_F = r(\theta, \phi)$. Vectors $\mathbf{p}_F^{0,i}$ are parametrized as $p_F = r(\theta^{0,i}, \phi^{0,i})$; correspondingly, $p_F = r(\theta^{0,i} - \alpha, \phi^{0,i} - \beta) \approx p_F^{0,i} - \alpha r_\theta - \beta r_\phi$ where r_θ and r_ϕ are the partial derivatives of r with respect to θ and ϕ, respectively, evaluated at the point $(\theta^{0,i}, \phi^{0,i})$. Suppose that $\tilde{\mathbf{v}}$ makes angle γ with the polar axis and, without a loss of generality, assume that the x axis belongs to the plane formed by vectors $\tilde{\mathbf{v}}$ and $\mathbf{q}_F$. Then, $\cos \theta_{\mathbf{p}^{0,i},\tilde{\mathbf{v}}} = \cos \theta^{0,i} \cos \gamma$ and $\cos \theta_{\mathbf{p}^{0,i}_F,\tilde{\mathbf{v}}} \approx (\cos \theta^{0,i} + \alpha \sin \theta^{0,i}) \cos \gamma$ to linear order in α. Substituting all of the results above into Eq. (2.29), we obtain

$$\varepsilon_{\mathbf{p}+\mathbf{q}_F} = \alpha \left(p_F^{0,i} \sin \theta^{0,i} - r_\theta \cos \theta^{0,i} \right) \cos \gamma + \varepsilon_{\mathbf{p}} \frac{v^{||}_{\mathbf{p}^{0,i}_F+\mathbf{q}_F}}{v^{||}_{\mathbf{p}^{0,i}_F}}, \quad (2.31)$$

which generalizes Eq. (2.23) for the arbitrary dispersion case. The measure of integration over the area $dA_{\mathbf{p}_F}$ reduces to

$$\frac{dA_{\mathbf{p}_F}}{v_{\mathbf{p}_F}} \approx \frac{\left(p_F^{0,i} \right)^2}{v^{||}_{\mathbf{p}^{0,i}_F}} \sin \theta^{0,i} d\alpha \, d\beta. \quad (2.32)$$

The rest of the calculations proceed in the same way as for the quadratic-dispersion case, namely, integrating first over α, then $\varepsilon_{\mathbf{p}}$, and, finally, over $\varepsilon_{\mathbf{k}}$, we reproduce the same product of the three sign factors as before. The final expression for the self-energy reduces to that in Eq. (2.27) with a different prefactor, which varies over the FS.

Interestingly, we found that the seemingly obvious result that frequency summation in Eq. (2.27) yields (2.28) can be reproduced only with a considerable effort if one uses the Euler-Maclaurin formula to sum over Ω_n. Namely, one has to keep not only the "conventional" terms with the integral over n and derivatives of the summand at $n = 0$, but also the "remainder" term, which is often neglected when the Euler-Maclaurin formula is applied in practice. We discuss this issue in Appendix A.

3. Linear-in-T term in the Matsubara self-energy

Finally, we consider in more detail the $\mathcal{O}(T)$ contribution to $\Sigma_{\mathbf{k}_F}(\pi T, T)$. For definiteness, we focus on the 3D case and restrict to quadratic dispersion. If we integrate in Eq. (2.20) over $\varepsilon_{\mathbf{p}}$ and $\varepsilon_{\mathbf{k}'}$ in infinite limits, as we did earlier in this section, and retain a constant term [denoted as $\Pi(0)$] instead of the $|\Omega|$ term in the local polarization bubble, we obtain

$$\Sigma_{\mathbf{k}_F}(\pi T, T) = \lambda T \sum_{\Omega_n} \text{sgn}(\pi T + \Omega_n), \quad (2.33)$$

where $\lambda \propto \Pi(0)$. Because only the $n = 0$ term contributes to the sum, $\Sigma_{\mathbf{k}_F}(\pi T, T) = \lambda T$.

This result holds only if we integrate over $\varepsilon_{\mathbf{p}}$ and $\varepsilon_{\mathbf{k}'}$ in infinite limits. Since, however, we have set the cutoff of our low-energy theory to Λ, integrations over $\varepsilon_{\mathbf{p}}$ and $\varepsilon_{\mathbf{k}'}$ should, strictly speaking, be performed between $-\Lambda$ and Λ. The

magnitude of λ then depends on the ratio Λ/E_F and reduces to the previous result only for $\Lambda \gg E_F$. In the opposite limit of $\Lambda \ll E_F$, which is more appropriate for systems in which E_F is of the same order as the bandwidth, λ is much smaller, namely, $\lambda \sim (\Lambda/E_F) \ln(E_F/\Lambda)$. We show this in Appendix B. We also checked if there is a T^2 contribution to $\Sigma_{\mathbf{k}_F}(\pi T, T)$ at finite Λ, but found no such term. The next term after the λT is of order T^3/Λ^2. This one is irrelevant to our purposes, as later in the text we show, that in a generic 3D FL, there are universal terms of order $(T^3/E_F^2) \ln E_F/T$, which are parametrically larger than a nonuniversal T^3 term.

III. SINGLE-PARTICLE SELF-ENERGY: NONCANONICAL FERMI LIQUIDS AND HIGHER-ORDER TERMS IN CANONICAL FERMI LIQUIDS

We remind the reader that the analysis in the previous section relied on the assumption that the momentum integrals, incorporated into the prefactor C in Eq. (2.16) for $\text{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$, are free from singularities. These integrals include quasiparticle renormalization factors, the effective interaction between the quasiparticles, and the prefactor of the Ω term in the imaginary part of the polarization operator [see Eqs. (2.9) and (2.14)]. The quasiparticle renormalization factors and the effective interaction are nonsingular at small q, but the prefactor of the Ω term scales as $1/q$ and may give rise to infrared divergencies. The momentum integral in the expression for C is over the $D - 1$ components of $\mathbf{q}$ lying in a plane tangential to the D-dimensional FS. This integral converges for $D > 2$, i.e., in a conventional FL, but diverges for $D \leqslant 2$, i.e., in a nonconventional FL.

The issue we discuss in this section is whether the first-Matsubara rule holds in a nonconventional FL and in a conventional FL beyond the T^2 order. We will show that the next after the $\mathcal{O}(T)$ term in $\Sigma_{\mathbf{k}_F}(\pi T, T)$ scales as T^D for any D, i.e., $\Sigma_{\mathbf{k}_F}(\pi T, T) = \lambda T + d T^D$.

The T^D term is subleading to the T^2 one in a conventional FL ($D > 2$), and thus the first-Matsubara rule holds to order T^2 in this case. However, the leading terms in a nonconventional FL ($1 < D \leqslant 2$) are also of the T^D order, and thus the first-Matsubara rule does not hold in this case. In the next section, we show that the first-Matsubara rule holds to all orders in T for any D near a QCP, when the local approximation becomes valid.

We consider first the marginal case of $D = 2$, and then discuss the cases of $2 < D < 3$, $D = 3$, and $1 < D < 2$.

A. $D = 2$

In $D = 2$, the self-energy is nonanalytic: $\text{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T) \propto \omega^2 \ln |\omega|$ at $T = 0$ and $T^2 \ln T$ at $\omega = 0$,[21–23] while the first subleading term in $\text{Re}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ scales as $\omega|\omega|$ at $T = 0$ and as $T^2 \text{sgn}\omega$ for $\omega \ll T$.[12] To logarithmic accuracy, the scaling form of $\text{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ is given by[3,4,24–30]

$$\text{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T) = C_2(\omega^2 + \pi^2 T^2) \ln \frac{\Lambda}{|\omega|}, \quad (3.1)$$

where C_2 is a constant. By the KK relation,

$$\text{Re}\Sigma_{\mathbf{k}_F}^R(\omega, T) = \lambda \omega - \frac{\pi C_2}{2} \text{sgn}\omega (\omega^2 + \pi^2 T^2). \quad (3.2)$$

At this level, the first-Matsubara rule is obviously satisfied. Beyond logarithmic accuracy, however, the situation is different, as we will now see.

Let us first calculate the self-energy in Matsubara frequencies. Consider diagram (a) in Fig. 1. The corresponding formula for the self-energy is given by Eq. (2.2). We explore an earlier observation[12,29–31] that the nonanalytic contributions to the fermionic self-energy come from forward scattering and backscattering rather than from scattering by an arbitrary angle. The internal structures of diagrams with forward scattering and backscattering are the same, i.e., it is sufficient to analyze only one of these two contributions. We consider forward scattering, i.e., focus on small momentum transfers q, and also assume that the FS is isotropic (a circle). Consequently, the self-energy does not depend on the position on the FS, but we will still keep the subscript $\mathbf{k}_F$ which indicates that the self-energy is evaluated on the FS, as opposed to the self-energy evaluated away from the FS also considered in this section.

At small q, the polarization bubble behaves as

$$\Pi_{\mathbf{q}}(\Omega_n) = -\frac{m}{\pi}\left(1 - \frac{|\Omega_n|}{\sqrt{\Omega_n^2 + (v_F q)^2}}\right). \qquad (3.3)$$

The constant term in Π gives rise to the $\mathcal{O}(T)$ term in $\Sigma_{\mathbf{k}_F}(i\pi T, T)$. We neglect it for now but will reinstate it in the final result for $\Sigma_{\mathbf{k}_F}(\pi T, T)$. Keeping the dynamic part in Eq. (3.3) and introducing polar coordinates for momentum integration, we obtain for the forward-scattering contribution

of Fig. 1(a) to the self-energy at arbitrary momentum $\mathbf{k}$

$$\Sigma_{\mathbf{k}}(\omega_m, T)$$
$$= -i A_2 T \sum_{\Omega_n} \int \frac{q\, dq\, d\phi}{(2\pi)^2} \frac{1}{i(\omega_m + \Omega_n) - \varepsilon_{\mathbf{k}} - v_F q \cos\phi}$$
$$\times \frac{|\Omega_n|}{\sqrt{\Omega_n^2 + (v_F q)^2}}, \qquad (3.4)$$

where $A_2 = 4\pi u^2(0)/m$ and $u(0) \equiv m U_{\mathbf{q}=0}/2\pi$ is the dimensionless coupling constant for forward scattering. Integrating over θ, we obtain

$$\Sigma_{\mathbf{k}}(\omega_m, T)$$
$$= -A_2 T \sum_{\Omega_n} \int \frac{dq\, q}{2\pi} \frac{\text{sgn}(\omega_m + \Omega_n)}{\sqrt{(\omega_m + \Omega_n + i\varepsilon_{\mathbf{k}})^2 + (v_F q)^2}}$$
$$\times \frac{|\Omega_n|}{\sqrt{\Omega_n^2 + (v_F q)^2}}. \qquad (3.5)$$

First, we discuss the self-energy on the FS. Substituting $\varepsilon_{\mathbf{k}} = 0$ into Eq. (3.5) and integrating over q up to Λ/v_F, we obtain

$$\Sigma_{\mathbf{k}_F}(\omega_m, T) = -\frac{T A_2}{2\pi v_F^2} \sum_{\Omega_n} |\Omega_n| \text{sgn}(\omega_m + \Omega_n)$$
$$\times \ln\left[\frac{\sqrt{\Lambda^2 + \Omega_m^2} + \sqrt{\Lambda^2 + (\omega_m + \Omega_n)^2}}{|\Omega_n| + |\omega_m + \Omega_n|}\right]. \qquad (3.6)$$

For $\omega_m = \pi T$, the last result reduces to

$$\Sigma_{\mathbf{k}_F}(\pi T, T) = -\frac{T^2 A_2}{v_F^2} \sum_{n=1}^{\infty} m \ln\left[\frac{2n - 1/2}{2n + 1/2} \frac{(\bar{\Lambda}^2 + n^2)^{1/2} + [\bar{\Lambda}^2 + (n + 1/2)^2]^{1/2}}{(\bar{\Lambda}^2 + n^2)^{1/2} + [\bar{\Lambda}^2 + (n - 1/2)^2]^{1/2}}\right], \qquad (3.7)$$

where $\bar{\Lambda} = \Lambda/(2\pi T) \gg 1$. To evaluate the frequency sum, we notice that the second fraction under the logarithm is close to unity in both regions of m that are relevant for the sum, namely, for $n \ll \bar{\Lambda}$ and for $m \sim \bar{\Lambda}$, when $n \ll n^2$. In either case,

$$\ln\frac{(\bar{\Lambda}^2 + n^2)^{1/2} + [\bar{\Lambda}^2 + (n + 1/2)^2]^{1/2}}{(\bar{\Lambda}^2 + n^2)^{1/2} + [\bar{\Lambda}^2 + (n - 1/2)^2]^{1/2}} \approx \frac{n}{2(\bar{\Lambda}^2 + n^2)}. \qquad (3.8)$$

With this simplification, the sum over n can be evaluated exactly. Performing summation, and adding the $\mathcal{O}(T)$ contribution from the static part of the polarization bubble, we obtain

$$\Sigma_{\mathbf{k}_F}(\pi T, T) = \pi T \lambda - \frac{A_2 T^2}{2\pi v_F^2}\left(K + \frac{\pi \ln 2}{4}\right), \qquad (3.9)$$

where $\lambda \sim (A_2 \Lambda)/v_F^2$ is a nonuniversal constant and $K = 0.9160$ is the Catalan's constant ($K + \pi \ln 2/4 = 1.460$). We see that $\Sigma(\pi T, T)$ does contain a universal, i.e., cutoff-independent T^2 term. We recall that there is no such term in $D > 2$, when the self-energy is analytic to order T^2.

The presence of such a term in $D = 2$ implies that the first-Matsubara rule breaks down once the self-energy becomes nonanalytic.

For completeness, we also reproduced Eq. (3.9) by evaluating first $\text{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ and then evaluating $\Sigma_{\mathbf{k}_F}(\pi T, T)$ using the general KK relation between the Matsubara self-energy and $\text{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$:

$$\Sigma_{\mathbf{k}_F}(\omega_m, T) = \frac{2\omega_m}{\pi} \int_0^{\infty} d\omega \frac{\text{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)}{\omega^2 + \omega_m^2}. \qquad (3.10)$$

Applying spectral representation to Eq. (3.4) and integrating over the momentum, we obtain for $\omega > 0$

$$\text{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T) = \frac{A_2}{4\pi^2 v_F^2} \int_{-\Lambda}^{\Lambda - \omega} d\Omega\, \Omega[n_B(\Omega) + n_F(\Omega + \omega)]$$
$$\times \ln\frac{[\sqrt{\Lambda^2 - \Omega^2} + \sqrt{\Lambda^2 - (\Omega + \omega)^2}]^2}{\omega|\omega + 2\Omega|}, \qquad (3.11)$$

if $\omega < 2\Lambda$ and $\text{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T) = 0$ otherwise. To logarithmic accuracy, this expression reduces to Eq. (3.1).

Substituting (3.11) into (3.10) and setting $\omega_m = \pi T$, we find that the main logarithmic term in $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ contributes only to the $\mathcal{O}(T)$ term in $\Sigma_{\mathbf{k}_F}(\pi T, T)$. The violation of the first-Matsubara rule comes from the subleading $\mathcal{O}(\omega^2)$ and $\mathcal{O}(T^2)$ terms. We obtained the first term in Eq. (3.9) analytically and reproduced the second term by integrating over ω in Eq. (3.10) numerically.

A complete expression for $\Sigma_{\mathbf{k}_F}(\pi T, T)$ to second order in the interaction contains contributions from diagrams in Figs. 1(a) and 1(b). Each of these diagrams contains contributions from the interaction with momentum transfers equal to zero and to $2k_F$ with amplitudes $U(0)$ and $U(2k_F)$, correspondingly. Collecting all these contributions, we obtain a complete result for $\Sigma_{\mathbf{k}_F}(\pi T, T)$ to second order in the interaction as[12,30]

$$\Sigma_{\mathbf{k}_F}(\pi T, T) = \pi T \lambda - \frac{T^2}{2E_F}[3u^2(0) + 2u^2(2k_F)$$
$$- 2u(0)u(2k_F)]\left(K + \frac{\pi \ln 2}{4}\right), \quad (3.12)$$

where $u(q) = mU_{\mathbf{q}}/(2\pi)$. The combination of the coupling constants in Eq. (3.12) can be expressed via the spin and charge components of the forward scattering (f) and backscattering (b) amplitudes Γ^f and Γ^b, defined by

$$\Gamma^{f,b}_{\alpha\gamma;\beta\delta} = \Gamma^{f,b}_c \delta_{\alpha\beta}\delta_{\gamma\delta} + \Gamma^{f,b}_s \boldsymbol{\sigma}_{\alpha\beta} \cdot \boldsymbol{\sigma}_{\gamma\delta}, \quad (3.13)$$

where subscripts c and s stand for "charge" and "spin," respectively. To first order in $U_{\mathbf{q}}$,

$$\Gamma^f_c = -\Gamma^f_s = u(0), \quad \Gamma^b_c = 2u(0) - u(2k_F), \quad \Gamma^b_s = -u(2k_F). \quad (3.14)$$

Using these relations, one can reexpress Eq. (3.12) as

$$\Sigma_{\mathbf{k}_F}(\pi T, T) = \pi T \lambda - \frac{T^2}{8E_F}\left(K + \frac{\pi \ln 2}{4}\right)$$
$$\times [2\{(\Gamma^b_c)^2 + 3(\Gamma^b_s)^2\} + (\Gamma^f_c)^2 + 3(\Gamma^f_s)^2]. \quad (3.15)$$

Equation (3.15) can be extended to a FL with an arbitrary interaction. One can show, using the same arguments as in Refs. 12 and 32, that the self-energy still contains the same combination of forward-scattering and backscattering amplitudes, except for in a general case $\Gamma^b_{c,s}$ and $\Gamma^f_{c,s}$ are expressed not via $u(0)$ and $u(2k_F)$, but rather via fully renormalized four-fermion vertices $\Gamma(\mathbf{k}, \mathbf{k}; \mathbf{k}, \mathbf{k})$, $\Gamma(\mathbf{k}, -\mathbf{k}; \mathbf{k}, -\mathbf{k})$, and $\Gamma(\mathbf{k}, -\mathbf{k}, -\mathbf{k}, \mathbf{k})$, which may depend on both transferred and total momenta. Explicitly, we have

$$\Gamma^f_c = \frac{Z^2 m^*}{2\pi}\Gamma(\mathbf{k}, \mathbf{k}; \mathbf{k}, \mathbf{k}); \quad \Gamma^f_s - \frac{Z^2 m^*}{2\pi}\Gamma(\mathbf{k}, \mathbf{k}; \mathbf{k}, \mathbf{k})$$
$$\Gamma^b_c = \frac{Z^2 m^*}{2\pi}[2\Gamma(\mathbf{k}, \mathbf{k}, -\mathbf{k}; \mathbf{k}, -\mathbf{k}) - \Gamma(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k})],$$
$$\Gamma^b_s = -\frac{Z^2 m^*}{2\pi}\Gamma(\mathbf{k}, -\mathbf{k}; -\mathbf{k}, \mathbf{k}). \quad (3.16)$$

A complete expression for the self-energy at the first Matsubara frequency is

$$\Sigma_{\mathbf{k}_F}(\pi T, T) = \pi T \lambda - \frac{T^2}{8E_F}\left(K + \frac{\pi \ln 2}{4}\right)\frac{m^*}{mZ}$$
$$\times [2\{(\Gamma^b_c)^2 + 3(\Gamma^b_s)^2\} + (\Gamma^f_c)^2 + 3(\Gamma^f_s)^2]. \quad (3.17)$$

There is one additional complication: The result in Eq. (3.17) is actually based on the expansion of the polarization bubble in frequency: for free fermions, this amounts to replacing (3.3) by $\Pi_{\mathbf{q}}(\Omega_n) = -(m/\pi)(1 - |\Omega_n|/v_F q)$. The static part of $\Pi_{\mathbf{q}}(\Omega_n)$ produces the T term in Eq. (3.17), while the (smaller) dynamic part produces the T^2 term. At weak coupling, one can safely set the lower limit of integration over q to zero because the contribution from the region $q \lesssim \Omega/v_F \sim T/v_F$ produces only higher than T^2 terms. In a generic FL, an expansion of the polarization bubble is possible for $|\Gamma\Omega_n|/v_F q \ll 1$, where Γ is the largest of the scattering amplitudes in Eq. (3.15). When $\Gamma \gg 1$, which happens either when the interaction is strong or when the system is near a Pomeranchuk instability,[15] the condition $v_F q \gg |\Gamma\Omega_n|$ sets a new lower cutoff for integration over q. We consider the case of large Γ in Sec. IV, where we show that the existence of this cutoff affects the prefactor for the T^2 term in Eq. (3.17), which gets smaller as Γ increases.

The consequences of the first-Matsubara rule for the de Haas–van Alphen (dHvA) oscillations in a 2D FL were analyzed in Refs. 3 and 4, where it was shown that the amplitude of these oscillations contains neither a $T^2 \ln T$ nor a T^2 term resulting from the self-energy of quasiparticles. This result seems to contradict Eq. (3.17), which shows that the self-energy evaluated at $\omega_m = \pi T$ does have a T^2 term. In fact, there is no contradiction because Eq. (3.17) refers to the self-energy evaluated on the FS, i.e., for $\varepsilon_{\mathbf{k}} = 0$, while the dHvA amplitude contains the self-energy evaluated at the "Matsubara mass shell," defined by a solution of the equation $G^{-1}_{\mathbf{k}}(\omega_m) = 0$. It turns out that these two self-energies do have different T dependences. The amplitude of dHvA oscillations in any thermodynamic quantity contains the following dimensionless combination:[5]

$$A_{\text{dHvA}} = \frac{iT}{2\pi\omega_c}\sum_{\omega_m > 0}\int d\varepsilon_{\mathbf{k}} G_{\mathbf{k}}(\omega_m) \exp\left(2\pi i \frac{\varepsilon_{\mathbf{k}}}{\omega_c}\right), \quad (3.18)$$

where ω_c is the cyclotron frequency. For simplicity, we omit $\mathcal{O}(\varepsilon_{\mathbf{k}})$ and $\mathcal{O}(\omega_m)$ terms in $\Sigma_{\mathbf{k}}$, which only renormalize the effective mass entering the cyclotron frequency, and focus on terms of order $T^2 \ln T$ and higher. We also focus on the weak-coupling regime, when the Matsubara mass shell can be determined perturbatively; to lowest order in the interaction, the mass shell simply coincides with the pole of the Matsubara Green's function $\varepsilon_{\mathbf{k}} = i\omega_m$. Substituting the self-energy (3.15) evaluated at $\varepsilon_{\mathbf{k}} = i\omega_m$ into Eq. (3.18) and integrating over $\varepsilon_{\mathbf{k}}$, we obtain

$$A_{\text{dHvA}} = \frac{T}{\omega_c}\sum_{\omega_m > 0}\exp\left(-2\pi\frac{\omega_m + \tilde{\Sigma}(\omega_m)}{\omega_c}\right), \quad (3.19)$$

where

$$\tilde{\Sigma}(\omega_m) = -\frac{A_2 T}{2\pi}\int_0^{\Lambda/v_F} dq\, q \sum_{\Omega_n}\frac{\text{sgn}(\omega_m + \Omega_n)|\Omega_n|}{(v_F q)^2 + \Omega_n^2}. \quad (3.20)$$

For high enough temperatures, i.e., for $T \gtrsim \omega_c$, one needs to keep only the $\omega_m = \pi T$ term in the sum of Eq. (3.19). This is where the first-Matsubara rule becomes useful because the Matsubara sum in Eq. (3.20) vanishes for $\omega_m = \pi T$, and A_{dHvA} reduces to the free-electron result (modulo renormalized effective mass) with no extra T-dependent terms.

A related point is the difference in the behavior of the self-energy at finite T and at $T = 0$. At $T = 0$, the perturbation theory in 2D for the self-energy diverges near the mass shell, and needs to be resummed to eliminate these divergences.[12,29] The mass-shell singularity shows up already in the second-order self-energy at $T = 0$, which is obtained by replacing the Matsubara sum in Eq. (3.5) by an integral over Ω_n. To logarithmic accuracy, this yields

$$\Sigma_{\mathbf{k}}(\omega_m, T = 0)$$
$$= -\frac{A_2}{8\pi^2 v_F^2}\left[\left(\omega_m^2 + \frac{1}{4}(\omega_m + i\varepsilon_{\mathbf{k}})^2\right)\ln\frac{\Lambda}{\omega_m + i\varepsilon_{\mathbf{k}}}\right.$$
$$\left.+ \left(\omega_m^2 - \frac{1}{4}(\omega_m + i\varepsilon_{\mathbf{k}})^2\right)\ln\frac{\Lambda}{\omega_m - i\varepsilon_{\mathbf{k}}}\right]. \quad (3.21)$$

The mass-shell singularity in this equation is manifested as a divergence of the first logarithmic term at $\varepsilon_{\mathbf{k}} = i\omega_m$. However, if we keep T in Eq. (3.5) finite, integrate over q at finite $\varepsilon_{\mathbf{k}}$, and then rearrange the resulting Matsubara sum, we obtain to logarithmic accuracy and for Matsubara frequencies with $m = \mathcal{O}(1)$

$$\Sigma_{\mathbf{k}}(\omega_m, T) = -\frac{A_2 T}{2\pi v_F^2}\sum_{2\pi T}^{\omega_m - \pi T}|\Omega_n|$$
$$\times \ln\frac{\Lambda^2}{(2\Omega_n - \omega_m - i\varepsilon_{\mathbf{k}})(\omega_m + i\varepsilon_{\mathbf{k}})}. \quad (3.22)$$

The limit of $T \to 0$ in this equation reproduces Eq. (3.21) with the same mass-shell singularity at $\varepsilon_{\mathbf{k}} = i\omega_m$. However, at $\omega_m = \pi T$, the sum in Eq. (3.22) contains no terms and thus the mass-shell singularity in $\Sigma_{\mathbf{k}}(\pi T, T)$ is absent. This is the reason why the mass-shell singularity does not show up in the dHvA amplitude.

B. Higher-order terms in canonical Fermi liquids ($2 < D < 3$)

In canonical FLs, the first-Matsubara rule holds to order T^2. Let us now verify whether it also holds to higher orders in T. To obtain $\Sigma_{\mathbf{k}}(\pi T, T)$ beyond the T^2 order, we need to go beyond the approximation we used in Sec. II, where we assumed that the interaction connects only the points right on the FS.

We verified that, as in 2D, the terms relevant to our analysis come both from small momentum transfers and momentum transfers near $2k_F$. Consider for definiteness a small momentum contribution to in Fig. 1(a). The corresponding formula for the self-energy is given by Eq. (2.2). To single out potential terms in $\Sigma_{\mathbf{k}_F}(\pi T, T)$ beyond the T^2 order, we subtract from the integrand in Eq. (2.2) its expression for the case when the effective interaction connects the points right on the FS. We parametrize the measure of the D-dimensional integral over $\mathbf{q}$ as $d^{D-1}q_\perp dq_\parallel$, where a $(D-1)$-dimensional vector $\mathbf{q}_\perp$ lies in the plane tangential to the FS and $q_\parallel$ is along the normal to the FS, and replace the integral over $q_\parallel$ by that over the fermionic dispersion in the final state $\epsilon_{\mathbf{k}_F + \mathbf{q}} \approx v_F q_\parallel \equiv \epsilon$. As before, we neglect the static part of $\Pi_{\mathbf{q}}(\Omega_n)$, which contributes only to the $\mathcal{O}(T)$ term in Σ, and approximate the dynamic part of $\Pi_{\mathbf{q}}(\Omega_n)$ by the $|\Omega_n|/q = |\Omega_n|/\sqrt{q_\perp^2 + q_\parallel^2}$ form. Using these simplifications, we express the part of the self-energy not captured in Sec. II as

$$\delta\Sigma(\pi T, T)$$
$$= -iA_D T\sum_{\Omega_n}|\Omega_n|\int\frac{q_\perp^{D-2}dq_\perp d\epsilon}{(2\pi)^D}\frac{1}{i(\pi T + \Omega_n) - \epsilon}$$
$$\times\left(\frac{1}{\sqrt{v_F^2 q_\perp^2 + \epsilon^2 + \Omega_n^2}} - \frac{1}{\sqrt{v_F^2 q_\perp^2 + \Omega_n^2}}\right), \quad (3.23)$$

where $A_D = 2\nu_D\pi^{\frac{D-1}{2}}/\Gamma[(D-1)/2]U_{\mathbf{q}=0}^2$, $\Gamma[x]$ is the gamma function, and ν_D is the density of states per spin projection in D dimensions. Because $\Sigma_{\mathbf{k}_F}(\pi T, T)$ obtained in Sec. II contains only linear-in-T terms and thus satisfies the first-Matsubara rule, potential deviations from this rule are due to $\delta\Sigma(\pi T, T)$. Integrating over $q_\perp$ and ϵ in (3.23), we find that $\delta\Sigma(\pi T, T)$ contains a contribution

$$\delta\Sigma(\pi T, T) = A_D\frac{T}{v_F^D}\sum_{\Omega_n}|\Omega_n|(\pi T + \Omega_n)|\pi T + \Omega_n|^{D-3}Q_D\left(\frac{\Omega_n}{\pi T + \Omega_n}\right)$$
$$= A_D\frac{T^D}{v_F^D}(2\pi)^{D-1}\sum_{n=1}^{\bar{\Lambda}}n\left[(n + 1/2)^{D-2}Q_D\left(\frac{n}{n + 1/2}\right) - (n - 1/2)^{D-2}Q_D\left(\frac{n}{n - 1/2}\right)\right], \quad (3.24)$$

where $\bar{\Lambda} = \Lambda/2\pi T$, and

$$Q_D(z)$$
$$= 2\int_0^{\bar{\Lambda}}\int_0^{\bar{\Lambda}}\frac{dx\, x^{D-2}dy}{(2\pi)^D}\frac{\sqrt{x^2 + y^2 + z^2} - \sqrt{x^2 + z^2}}{(y^2 + 1)\sqrt{x^2 + z^2}\sqrt{x^2 + y^2 + z^2}}. \quad (3.25)$$

The sum in Eq. (3.24) contains a contribution from the upper limit, which just adds an extra piece to the $\mathcal{O}(T)$ term, but it also contains a Λ-independent contribution from $n = \mathcal{O}(1)$, which yields $\delta\Sigma(i\pi T, T) \propto T^D$. We see therefore that the full $\Sigma_{\mathbf{k}_F}(\pi T, T) = \mathcal{O}(T) + \delta\Sigma(\pi T, T)$ contains a T^D term, i.e., the first-Matsubara rule breaks down at order T^D in conventional FLs. Still, $\Sigma_{\mathbf{k}_F}(\pi T, T)$ and $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ for $|\omega_m| \neq \pi T$ are qualitatively different: the next term after T in

$\Sigma_{\mathbf{k}_F}(\omega_m \neq \pi T, T)$ is T^2, while in $\Sigma_{\mathbf{k}_F}(\pi T, T)$ it is T^D, which for $D > 2$ is much smaller than T^2. We verified that in the limit $D \to 2$, the result matches the second term in (3.9). For arbitrary $2 < D < 3$, the sum has to be evaluated numerically.

The case $D = 3$ is special because Q_3 diverges logarithmically. In this case, we have, after integrating over x in Eq. (3.25) and neglecting nonlogarithmic terms,

$$\delta\Sigma(\pi T, T)$$
$$= A_3 \frac{4T^3}{\pi v_F^3} \sum_n |n|(n + 1/2) \int_0^{\bar{\Lambda}} dy \frac{\sqrt{y^2 + z^2}}{y^2 + 1}, \quad (3.26)$$

where $z = n/(n + 1/2)$. By power counting, $\delta\Sigma(\pi T, T)$ scales as T^3, but there is an additional logarithm, which can be captured by expanding the integral in Eq. (3.26) in $1/n$ to order $1/n^3$. The prefactor of the universal T^3 term is $1/|n|$, and the sum $T^3/|n|$ yields a $T^3 \ln \Lambda/T$ term in $\Sigma_{\mathbf{k}_F}(\pi T, T)$. Expanding the integrand of Eq. (3.26) in $1/n$, integrating over y, and collecting the prefactors for the $T^3 \ln \Lambda/T$ term, we obtain

$$\delta\Sigma(\pi T, T) = A_3 \frac{T^3}{15\pi v_F^3} \ln \frac{\Lambda}{T}. \quad (3.27)$$

The complete result in 3D again contains the contributions from Figs. 1(a) and 1(b) and includes terms coming from both forward scattering and backscattering.

Note that the signs of $\delta\Sigma(\pi T, T)$ are different in 2D and 3D [cf. Eqs. (3.9) and (3.27)], i.e., the prefactor of the T^D term vanishes at some D in-between 2 and 3.

C. Noncanonical Fermi liquids: $1 < D < 2$

The analysis for $1 < D < 2$ parallels that in the previous section. The extra term in the self-energy at $\omega_m = \pi T$, given by (3.24), is still of order T^D, and its prefactor is expressed via forward-scattering and backscattering amplitudes. The only difference between the $D < 2$ and $D > 2$ cases is that, for $D < 2$, the T^D term is larger than the T^2 one, and the first-Matsubara rule breaks down completely, i.e., the next term after T in $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ is of order T^D for all ω_m including $\omega_m = \pm \pi T$.

IV. THE FIRST-MATSUBARA-FREQUENCY RULE NEAR QUANTUM CRITICALITY

A. Local approximation

So far, we found that in a generic FL, either conventional or unconventional, the terms of order T^D in the self-energy do not distinguish between the first and other Matsubara frequencies, i.e., the prefactor of the T^D term in $\Sigma_{\mathbf{k}_F}(\omega_m)$ is nonzero for all m.

We now show that a different situation emerges when the system is tuned to the vicinity of a Pomeranchuk transition, at which a FL becomes unstable towards condensation of particle-hole excitations with zero-momentum transfer. A Pomeranchuk instability can occur in either the spin or charge channel. A magnetic (spin) instability is likely to trigger preemptive transitions[33,34] and, to keep the discussion focused on the first-Matsubara rule, we only consider here a Pomeranchuk instability in the charge channel. In the

bulk of this section, we focus on long-wavelength ($q = 0$) Pomeranchuk instability (a quantum phase transition with dynamical exponent $z = 3$). At the end of this section, we briefly discuss the first-Matsubara rule near an instability at finite q in a system on lattice (a quantum phase transition with dynamical exponent $z = 2$).

Near a Pomeranchuk instability, interactions generate a large length scale ξ (the correlation length) which diverges at the transition. In $D \leqslant 3$, a divergence in ξ brings the upper boundary of FL behavior down from $\mathcal{O}(E_F)$ to $\omega_{\text{FL}} \propto \xi^{-3}$.[15] At large enough ξ, ω_{FL} becomes smaller than Λ, and the low-energy theory with the upper cutoff Λ describes now both the FL and non-FL regimes. We first consider the case of $\Omega, T \ll \omega_{\text{FL}}$ and then discuss the first-Matsubara rule at energies above ω_{FL}.

The observation, which is most relevant to our analysis, concerns the low-energy cutoff in the integration over bosonic momentum q in the formula for the self-energy, once we cast it into the form of Eq. (3.4). As we mentioned in Sec. III A, the T^D term in $\Sigma_{\mathbf{k}_F}(\omega_m)$ with a prefactor that does not show any special features at $m = 0, -1$ is obtained by setting the lower momentum cutoff to zero. This approximation can be justified at $\xi = \mathcal{O}(1) \sim k_F^{-1}$, at least at weak coupling, but not at large ξ. To show this, we follow earlier work[35] and assume that, near a Pomeranchuk instability with some angular momentum ℓ, the fermionic self-energy given in Eq. (3.4) can be viewed as resulting from an exchange of low-energy and overdamped collective excitations. The propagator of these excitations at small q is given by

$$\chi_{\mathbf{q}}(\Omega_n) = \frac{\chi_0}{q^2 + \xi^{-2} + \gamma \Pi_{\mathbf{q}}(\Omega_n)}, \quad (4.1)$$

where γ depends on the original fermion-fermion interaction and fermionic dispersion and, in general, is different for different ℓ. As before, we keep only the dynamic part in $\Pi_{\mathbf{q}}(\Omega_n)$.

The one-loop self-energy is given by

$$\Sigma_{\mathbf{k}_F}(\omega_m, T) = iT \sum_{\Omega_n} \int \frac{d^D q}{(2\pi)^D} G_{\mathbf{k}_F + \mathbf{q}}(\omega_m + \Omega_n) \chi_{\mathbf{q}}(\Omega_n). \quad (4.2)$$

An order-of-magnitude estimate for $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ can be obtained by expanding $\chi_{\mathbf{q}}(\Omega_n)$ as

$$\chi_{\mathbf{q}}(\Omega_n) = \chi_{\mathbf{q}}(0) - \chi_0 \gamma \xi^4 \Pi_{\mathbf{q}}(\Omega_n). \quad (4.3)$$

Substituting this expansion into Eq. (4.2) and comparing the result to (3.4) in Sec. III A, and to its extension for an arbitrary interaction in Eq. (3.15), we see that $\chi_0 \gamma \xi^4$ plays the same role as the combination of the Γ^2 terms in Eq. (3.15), i.e., the overall prefactor of the dynamic part of $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ scales as ξ^4.

Let us now look more carefully at the limits of integration over q, which need to be imposed to ensure self-consistency expansion (4.3) for $\chi_{\mathbf{q}}$. Because $\Pi_q(\Omega_n)$ scales as $|\Omega_n|/q$ at the smallest Ω_n, the expansion in $\Pi_{\mathbf{q}}(\Omega_n)$ in Eq. (4.3) holds only for $q > \gamma |\Omega_n| \xi^2$, which sets the lower limit in the integral over q. The upper limit is set by ξ^{-1}. Now, we expand the dispersion as $\varepsilon_{\mathbf{k}_F + \mathbf{q}} = v_F q_\parallel + q_\perp^2/2m^*$ and express $q_\parallel$ as $q_\parallel = \varepsilon_{\mathbf{k}_F + \mathbf{q}}/v_F - q_\perp^2/(2v_F m^*)$. Consider momentarily a free-fermion propagator in Eq. (4.2). Typical $\varepsilon_{\mathbf{k}_F + \mathbf{q}}$ are then

of order $\omega_m + \Omega_n$, i.e., of order T for Matsubara indices $m, n \sim 1$. Since we expect the T^D term to come from the region when both typical $q_\parallel$ and $q_\perp$ are also proportional to T, the $q_\perp^2 / 2m^*$ term is of order T^2 and can be neglected compared to $\varepsilon_{\mathbf{k}_F + \mathbf{q}}$. Hence, typical $q_\parallel \sim \varepsilon_{\mathbf{k}_F + \mathbf{q}} / v_F$, and typical $q \sim \sqrt{q_\perp^2 + [(\omega_m + \Omega_n)/v_F]^2}$. For large ξ and $n, m \sim 1$, $(\omega_m + \Omega_n)/v_F \sim \Omega_n / v_F$ is then smaller than the lower limit for q, which is $\gamma |\Omega_n| \xi^2$. For $\gamma v_F \xi^2 \gg 1$, one can approximate q by $q_\perp$. This is equivalent to factorizing the momentum integral in (4.2) as $\int d\varepsilon_{\mathbf{k}_F + \mathbf{q}} G_{\mathbf{k}_F + \mathbf{q}}(\omega_m + \Omega_n) \int d^{D-1} q_\perp \chi_{\mathbf{q}_\perp}(\Omega_n)$. In this approximation, the dynamic part of the self-energy in Eq. (4.2) reduces to

$$\Sigma_{\mathbf{k}_F}(\omega_m, T) = \frac{T}{2 v_F} \sum_{\Omega_n} \mathrm{sgn}(\omega_m + \Omega_n) \chi_L(\Omega_n), \quad (4.4)$$

where

$$\chi_L(\Omega_n) = \int \frac{d^{D-1} q_\perp}{(2\pi)^{D-1}} \chi_{\mathbf{q}_\perp}(\Omega_n). \quad (4.5)$$

Because $\chi_L(\Omega_n)$ is an even function of Ω_n, the right-hand side of (4.4) vanishes at $|\omega_m| = \pi T$, i.e., the first-Matsubara rule holds. For all other frequencies, such that $|\omega_m| \neq \pi T$ but still $|\omega_m| \sim T$, $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ behaves as $T^D \xi^{2D}$ in nonconventional FLs, and as

$$\Sigma_{\mathbf{k}_F}(\omega_m, T) = T^2 \xi^{6-D} \{1 + \mathcal{O}((T\xi^3)^{D-2})\} \quad (4.6)$$

in conventional FLs.

We refer to an approximation, in which the momentum integral is factorized, as the local approximation. The name reflects the fact that the fermionic self-energy in this approximation is a convolution of the density of states (the Green's function integrated over fermionic dispersion) and the *local* susceptibility, obtained by integrating the nonlocal susceptibility over $D-1$ components of $\mathbf{q}_\perp$.

If we keep $\omega_m + \Omega_n$ in q and compute $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ without making any approximations, we find that the T^D term in $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ is present for all ω_m; however, its prefactor has different dependences on ξ for $|\omega_m| = \pi T$ and all other ω_m. For $|\omega_m| \neq \pi T$, the T^D term in $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ is present even in the local approximation, and the prefactor of this term scales as ξ^{2D}. For $|\omega_m| = \pi T$, the prefactor is zero in the local approximation, and scales as $\xi^{2(D-2)}$ if we compute $\Sigma_{\mathbf{k}_F}(\pi T, T)$ in (4.2) using a free-fermion propagator. Using a free-fermion propagator at large ξ is, however, not justified because the mass renormalization term $\lambda \omega_m$ in $\Sigma_{\mathbf{k}_F}$ is also proportional to ξ [this term involves a static susceptibility $\chi_L(0)$]. Including this term into the Green's function affects the estimate for a typical $\varepsilon_{\mathbf{k}_F + \mathbf{q}}$, which now becomes of order $(1 + \lambda)|\omega_m + \Omega_n|/v_F$. Accordingly, the prefactor of the T^D term scales as $\lambda^2 \xi^{2(D-2)}$ at $|\omega_m| = \pi T$. At one-loop order, $\lambda \propto \xi^{3-D}$ ($\propto \ln \xi$ in $D = 3$) and, hence, the self-energy at the first Matsubara frequency scales as $\Sigma_{\mathbf{k}_F}(\pi T, T) \propto T^D \xi^2$. Still, for all $D > 1$, this is parametrically smaller than the self-energy at larger Matsubara frequencies, which, we remind, scales as $T^2 \xi^{6-D}$ in conventional FLs, and as $T^D \xi^{2D}$ in nonconventional FLs.

The main outcome of this analysis is that, near a Pomeranchuk instability, the first-Matsubara rule approximately

holds, even if far from the instability this rule is broken, as it happens in nonconventional FLs. The distinction between the prefactors of T^D terms in $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ likely persists to higher orders in the loop expansion, even if higher-order corrections are not small. To verify this, we analyzed two- and three-loop contributions to the self-energy near a charge Pomeranchuk transition in $D = 2$. We recall that in $D = 2$, the self-energy at a generic ω_m scales as $T^2 \xi^4$. In 2D, a two-loop self-energy is small compared to the one-loop one [Eq. (4.2)], only if one extends the theory to N fermionic flavors and takes the $N \gg 1$ limit.[10,11] The three-loop self-energy is not small even in the large-N limit (Refs. 13, 14, and 17), and higher-order terms even bring in additional logarithmic singularities.[14,17,36] We computed two- and three-loop contributions to the self-energy along the Matsubara axis, and found that in both contributions the prefactor for the T^2 term still vanishes at $\omega_m = \pm \pi T$ if the local approximation is imposed, and scales as ξ^2 beyond this approximation. Higher-order corrections may, in principle, generate additional logarithms and eventually change the scaling of $\Sigma_{\mathbf{k}_F}(\pi, T)$ with ξ from ξ^2 to ξ^β with $\beta < 2$. However, because the one-loop results for $\Sigma_{\mathbf{k}_F}(\pi T, T)$ and $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ with $|\omega_m| \neq \pi T$ differ substantially (by a factor of ξ^2 in $D = 2$), it is likely that the difference between the prefactors of T^D terms in $\Sigma_{\mathbf{k}_F}(\pi T, T)$ and in $\Sigma_{\mathbf{k}_F}(|\omega_m| \neq \pi T, T)$ holds to infinite order in the loop expansion.

The difference between $\Sigma_{\mathbf{k}_F}(\pi T, T)$ and $\Sigma_{\mathbf{k}_F}(|\omega_m| \neq \pi T, T)$ becomes particularly pronounced right at the Pomeranchuk instability. Now $\omega_{FL} = 0$, and the self-energy exhibits a non-FL behavior at any finite ω or T. The self-energy for generic $\omega_m \neq \pm \pi T$ can be divided into two parts: dynamic Σ^{d} and static Σ^{s}. The dynamic part comes from processes with nonzero energy transfers, corresponding to $\Omega_n \neq 0$ in the Matsubara sum of Eq. (4.2). The critical form of the dynamic part is obtained by replacing ξ^{-3} in Eq. (4.6) by T, which gives $\Sigma_{\mathbf{k}_F}^{\mathrm{d}}(\omega_m, T) \sim T^{D/3}$. The static part comes from scattering of static critical fluctuations, corresponding to a single term with $\Omega_n = 0$ in Eq. (4.2). At finite ξ, this contribution behaves as $\Sigma_{\mathbf{k}_F}^{\mathrm{s}}(T) \propto T \xi^{3-D}$. At $\xi \to \infty$, the static contribution diverges for $D \leqslant 3$.[6,37] This divergence is usually regularized by introducing a temperature-dependent correlation length $\xi(T)$, which remains finite at $T > 0$ even right at criticality. On general grounds, one can postulate that $\xi(T) \propto T^{-\beta_T}$ with $\beta_T > 0$, and hence $\Sigma_{\mathbf{k}_F}^{\mathrm{s}}(T) \propto T^{1 - \beta_T(3-D)}$. At the one-loop level, $\beta_T = 1/2$ (modulo logarithms) for 2D quantum critical systems with dynamical exponents $Z = 2$ and 3,[37] but higher-order corrections may change the exponent. We will treat β_T as a phenomenological parameter of the theory. Comparing the exponents of the dynamic and static parts of the self-energy, we see that, for any $D < 3$, the leading T dependence of the self-energy is given by the dynamic part if $\beta_T < 1/3$ and by the static part if $\beta_T > 1/3$.

For the first Matsubara frequency, the static part of the self-energy is the same as for all other ω_m, but the dynamic part is different. To obtain $\Sigma_{\mathbf{k}_F}^{\mathrm{d}}(\pi T, T)$ at criticality, we reevaluate the self-energy diagram in (4.2) by replacing the frequency in the denominator of the Green's function by the self-energy at the same frequency. Now, typical $\varepsilon_{\mathbf{k}_F + \mathbf{q}} \sim \Sigma_{\mathbf{k}_F}(\omega_m + \Omega_n, T)|_{\omega_m + \Omega_n \sim T} \equiv \bar{\Sigma}(T)$. Expanding the bosonic propagator to leading (second) order in $\varepsilon_{\mathbf{k}_F + \mathbf{q}}$ and performing

power counting, we obtain

$$\Sigma^d_{\mathbf{k}_F}(\pi T, T) \propto T^{\frac{D-2}{3}} \bar{\Sigma}^2 \propto \begin{cases} T^{D-2/3}, & \text{if } \beta_T < 1/3, \\ T^{\frac{D+4}{3} - 2\beta_T(3-D)}, & \text{if } \beta_T > 1/3, \end{cases}$$
(4.7)

where we replaced $\bar{\Sigma}$ by Σ^d and Σ^s for $\beta_T < 1/3$ and $\beta_T > 1/3$, correspondingly. We see that $\Sigma^d_{\mathbf{k}_F}(\pi T, T)$ remains smaller than $\Sigma^d_{\mathbf{k}_F}(\omega_m \neq \pi T, T)$: the ratio of the two behaves as $\Sigma_{\mathbf{k}_F}(\pi T, T)/\Sigma^d \propto T^{2(D-1)/3}$ for $\beta_T < 1/3$, and $\Sigma_{\mathbf{k}_F}(\pi T, T)/\Sigma^d \propto T^{(D+1)/3 - \beta_T(3-D)}$ for $\beta_T > 1/3$. The exponent is positive for any $D > 1$ in the first expression and for $1/3 < \beta_T < (D+1)/3(3-D)$ in the second one. In both cases, the ratio of the self-energy at the first Matsubara frequency to that at a generic frequency scales to zero as T goes to zero. This smallness is a manifestation of the first-Matsubara rule at criticality.

B. Scaling form of the self-energy in the local approximation

A nontrivial aspect of the first-Matsubara rule near a Pomeranchuk instability shows up when we consider the self-energy along the real frequency axis. At order T^D, both the real and imaginary parts of $\Sigma^R_{\mathbf{k}_F}(\omega, T)$ are rather complicated functions of ω and T, and the extension of $\Sigma^R_{\mathbf{k}_F}(\omega, T = 0)$ to finite T by no means implies that ω is replaced by $\sqrt{\omega^2 + \pi^2 T^2}$. Still, within the local approximation, we obtain, analytically continuing (4.4) to real frequencies,

$$\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T) = \frac{1}{2\pi v_F} \int d\Omega \, \text{Im}\chi^R_L(\Omega) \left[n_B(\Omega) + n_F(\Omega + \omega) \right].$$
(4.8)

The right-hand side of Eq. (4.8) is an analytic function of complex variable $\omega \to z = z' + iz''$ within the stripe $|\text{Im}z| \leqslant \pi T$ (see Fig. 2). Within this stripe, one can then analytically continue $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ into the complex plane by just replacing $\omega \to z$. Aa a result, $\text{Im}\Sigma^R_{\mathbf{k}_F}(i\pi T, T)$ is still given by (4.8), but with $i\pi T$ instead of ω in the right-hand side of this equation. Because $n_B(\Omega) + n_F(\Omega + i\pi T) = 0$, $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega = i\pi T, T)$ vanishes. The full $\Sigma^R_{\mathbf{k}_F}(i\pi T, T)$ vanishes by the first-Matsubara rule, hence, $\text{Re}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ must also vanish [up to a $O(T)$ term], if we replace ω by $i\pi T$. These two requirements then set nontrivial constraints on the scaling functions of ω/T in $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T) \propto |\omega|^D f_{ID}(|\omega|/T)$ and $\text{Re}\Sigma^R_{\mathbf{k}_F}(\omega, T) \propto \omega|\omega|^{D-1} f_{RD}(|\omega|/T)$: both $f_{ID}(x)$ and $f_{RD}(x)$ must vanish at $x = i\pi$.

In the next two sections, we obtain explicit forms of $f_{ID}(|\omega|/T)$ and $f_{RD}(|\omega|/T)$ for near-critical FLs in $D = 2$ and 3 and show that they satisfy the constraint.

1. D = 2

We again use (4.1) for $\chi_q(\Omega_n)$. In $D = 2$, we have

$$\text{Im}\chi^R_L(\Omega) = \frac{\chi_0 \gamma \xi^4}{\pi v_F} \Omega \ln \frac{\omega_{\text{FL}}}{|\Omega|},$$
(4.9)

where $\omega_{\text{FL}} \sim 1/(\gamma \xi^3)$ is the upper boundary of the FL behavior. The imaginary part of the self-energy is given by

$$\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T) = B_0 \int d\Omega \, \Omega \ln \frac{\omega_{\text{FL}}}{|\Omega|} \left[n_B(\Omega) + n_F(\Omega + \omega) \right],$$
(4.10)

where $B_0 = \chi_0 \gamma \xi^4/(2\pi^2 v_F^2)$. The real part of the self-energy is obtained via the KK relation. We skip the details of calculations and show only the final results. It turns out that the real part of the self-energy (the one which does not contain logarithms) can be computed exactly, up to the term of order ω which we omit below. The real part of the self-energy is an odd function of the frequency at $\mathbf{k} = \mathbf{k}_F$. For $\omega > 0$, we find

$$\text{Re}\Sigma^R_{\mathbf{k}_F}(\omega, T) = -\frac{B_0}{4} \left[\pi \omega^2 + 4\pi T^2 \left(\frac{\pi^2}{12} + \text{Li}_2[-e^{-\omega/T}] \right) \right],$$
(4.11)

where

$$\text{Li}_s(y) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$
(4.12)

is a polylogarithmic function. This expression can be cast into the scaling form $\text{Re}\Sigma^R_{\mathbf{k}_F}(\omega, T) = \omega|\omega| f_{R2}(|\omega|/T)$. For $\omega = i\pi T$, $-e^{-\omega/T} = 1$ and $\text{Li}_2(1) = \pi^2/6$. Substituting these relations into (4.11), we find that $\text{Re}\Sigma^R_{\mathbf{k}_F}(i\pi T, T) = 0$, as expected.

The imaginary part of the self-energy is given by Eq. (4.10) in the form of a one-dimensional integral. The formula for $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ can be simplified if we extract from it the leading logarithmic term. Combining the remainder of $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ with $\text{Re}\Sigma^R_{\mathbf{k}_F}(\omega, T)$, we obtain

$$\begin{aligned}
\Sigma^R_{\mathbf{k}_F}(\omega, T) = {} & i\frac{B_0}{4}(\omega^2 + \pi^2 T^2) \ln \frac{e(\omega_{\text{FL}})^2}{\pi^2 T^2} \\
& + i\frac{B_0}{4}\left(\omega^2 + \frac{\pi^2 T^2}{3}\right) \ln \frac{\pi^2 T^2}{-\omega^2} \\
& + 2i B_0 T^2 \int_0^{\infty} x \, \text{Li}_2(-e^{-\pi x}) \\
& \times \left(\frac{1}{x^2 - (\omega/\pi T)^2} - \frac{1}{x^2 + 1} \right),
\end{aligned}$$
(4.13)

where $\ln(-\omega^2) = \ln \omega^2 - i\pi \, \text{sgn}\omega$. In Eq. (4.13), we singled out the leading, logarithmic term in $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T)$, and the rest has the form $\omega^2 f_{12}(|\omega|/T)$. The scaling function is rather nontrivial, yet we see from (4.13) that $\Sigma^R_{\mathbf{k}_F}(i\pi T, T)$ vanishes, as it should.

In Appendix C, we discuss several subtle issues related to analytic continuation of the self-energy to the complex ω plane in a situation when either $\text{Re}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ or $\text{Im}\Sigma^R_{\mathbf{k}_F}(\omega, T)$ can not be evaluated explicitly and has to be kept in an integral form, as in Eq. (4.10).

2. Subleading terms in D = 3

A very similar situation emerges in 3D systems if we go beyond the leading $\omega^2 + \pi^2 T^2$ term in the self-energy and consider the subleading terms of order T^3 and ω^3. At $T = 0$,

the real part of the self-energy scales as $\omega^3 \ln|\omega|$ and the imaginary part scales as $|\omega|^3$. At finite T, both parts contain scaling functions of $|\omega|/T$. The situation is somewhat similar to that in $D = 2$ in a sense that the behavior is marginal due to logarithms.

Using $\chi_\mathbf{q}(\Omega_n)$ from Eq. (4.1), we obtain

$$\mathrm{Im}\chi_L^R(\Omega) = \frac{\chi_0 \gamma \xi^3}{8 v_F}\Omega - \frac{\chi_0 \gamma^2 \xi^6}{4\pi v_F^2}\Omega|\Omega| + \cdots . \quad (4.14)$$

Substituting this form into Eq. (4.8), we obtain after some algebra an explicit expression for $\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ to order ω^3, T^3:

$$\begin{aligned}
\mathrm{Im}&\Sigma_{\mathbf{k}_F}^R(\omega, T) \\
&= C_0(\omega^2 + \pi^2 T^2) + D_0\bigg\{\frac{|\omega|}{3}(\omega^2 + \pi^2 T^2) \\
&\quad + 4T^3(\mathrm{Li}_3(-e^{|\omega|/T}) - \zeta(3))\bigg\}, \quad (4.15)
\end{aligned}$$

where $C_0 = \chi_0 \gamma \xi^3/(32\pi v_F^2)$, $D_0 = \chi_0 \gamma \xi^6/(24\pi^2 v_F^3)$, and $\zeta(x)$ is the zeta function. Using that $\mathrm{Li}_3(1) = \zeta(3)$, one can immediately verify that $\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(i\pi T, T) = 0$, as it should. This happens despite that the functional form $\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ is rather complicated at order ω^3, T^3; e.g., the prefactor of the ω^3 term is not the same as the prefactor of the T^3 term.

The real part of the self-energy contains logarithms and has to be left in an integral form. The calculation of $\mathrm{Re}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ using the KK formula, Eq. (2.15), requires some care as the integral is formally infrared divergent, if we use Eq. (4.15) for $\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$. The recipe is to (i) start with the general expression for $\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ in Eq. (4.8); (ii) substitute it into the KK formula and obtain $\mathrm{Re}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ in the form of a double integral; (iii) keep the full form of $\mathrm{Im}\chi_L^R(\Omega)$ (without expanding it) at intermediate stages of the calculation, and change the order of integrations when it is convenient; (iv) use the fact that $\mathrm{Im}\chi_L(\Omega)$ vanishes in the infrared and also that $\mathrm{Re}\chi_L^R(0) = (2/\pi)\int_0^\infty d\Omega\, \mathrm{Im}\chi_L^R(\Omega)/\Omega$. Evaluating $\mathrm{Re}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ this way, we obtain

$$\begin{aligned}
\mathrm{Re}\Sigma_{\mathbf{k}_F}^R(\omega, T) &= \omega \frac{\mathrm{Re}\chi_L^R(0)}{2\pi v_F} + \frac{1}{3\pi^2 v_F}\omega(\omega^2 + \pi^2 T^2)\int_0^\infty \mathrm{Im}\chi_L^R(x)\frac{x\,dx}{(x^2 - \omega^2)^2} + \frac{\omega}{\pi^2 v_F}\int_0^\infty \mathrm{Im}\chi_L^R(x)dx\bigg[\frac{1}{2\omega}\ln\frac{x+\omega}{x-\omega} - \frac{1}{x} \\
&\quad - \frac{x\omega^2}{3(x^2-\omega^2)^2}\bigg] + \frac{4\omega}{\pi^2 v_F}\int_0^\infty \mathrm{Im}\chi_L^R(x)x\,dx\int_0^\infty \frac{y\,dy}{e^{x/T}+1}\bigg[\frac{1}{(x^2+y^2-\omega^2)^2 - 4x^2 y^2} - \frac{1}{(x^2-\omega^2)^2}\bigg], \quad (4.16)
\end{aligned}$$

where all integrals are to be understood as principal values. The last two integrals are ultraviolet convergent for $\mathrm{Im}\chi_L^R(x)$ given by (4.14). The second term in Eq. (4.16) is singular, but only logarithmically, and accounts for the $\omega^3 \ln|\omega|$ term in $\mathrm{Re}\Sigma_{\mathbf{k}_F}^R$. Substituting $\chi_L^R(\omega)$ from Eq. (4.14) and combining $\mathrm{Im}\Sigma_{\mathbf{k}_F}^R(\omega, T)$ and $\mathrm{Re}\Sigma_{\mathbf{k}_F}(\omega, T)$, we obtain from (4.16)

$$\begin{aligned}
\Sigma_{\mathbf{k}_F}^R(\omega, T) &= \omega\frac{\mathrm{Re}\chi_L^R(0)}{2\pi v_F} + iC_0(\omega^2 + \pi^2 T^2) + \frac{D_0}{\pi}\omega(\omega^2 + \pi^2 T^2)\ln\left[\frac{\omega_{\mathrm{FL}}^2}{-e\omega^2}\right] + \frac{5D_0\omega^3}{3\pi} \\
&\quad - \frac{4D_0}{\pi}\omega\int_0^\infty \frac{dx}{x^2 + \omega^2}[x(x^2 + \pi^2 T^2) + 6T^3(\mathrm{Li}_3(-e^{x/T}) - \mathrm{Li}_3(1))]. \quad (4.17)
\end{aligned}$$

Equation (4.17) is a complete expression for the self-energy in a 3D FL within the local approximation.

One can easily make sure that $\Sigma_{\mathbf{k}_F}^R(\omega, T)$ in Eq. (4.17) is an analytic function of ω in the upper half-plane, hence, it can be straightforwardly continued from the real axis into the upper half-plane just by replacing ω by a complex z. At $z = i\pi T$, the second and third terms vanish, while the last two terms cancel each other, i.e., at the first Matsubara frequency, the self-energy contains a linear-in-T term but no terms of higher power of T, in agreement with the first-Matsubara rule.

C. Marginal FL

As another illustration, we consider the self-energy in a marginal FL (MFL).[38] The term marginal FL refers to a situation when the imaginary part of the self-energy is comparable to ω, hence by the Landau criterion, the system is at the boundary between FLs and non-FLs. By the KK relation, if $\mathrm{Im}\Sigma(\omega, T = 0) \propto |\omega|$, then $\mathrm{Re}\Sigma(\omega, T = 0) \propto \omega\ln(\Lambda/|\omega|)$.

Because in a generic nonconventional FL $\mathrm{Im}\Sigma^R(\omega, T) \propto \omega^D f_{ID}(|\omega|/T)$, the MFL behavior formally emerges when D approaches one. This limit is, however, special, and we follow earlier work[39] and assume that the MFL behavior is associated with some sort of quantum criticality rather than with $D = 1$. Specifically, the MFL behavior emerges if one assumes $\mathrm{Im}\chi_L^R(\Omega, T)$ to be a scaling function of Ω/T such that $\mathrm{Im}\chi_L^R(\Omega, T = 0) = \mathrm{const} \times \mathrm{sgn}(\Omega)$ and $\mathrm{Im}\chi_L^R(\Omega, T) \propto \Omega/T$ for $\Omega \ll T$.[39] A simple model form of $\mathrm{Im}\chi_L^R(\Omega, T)$ satisfying these conditions is

$$\mathrm{Im}\chi_L^R(\Omega, T) = \chi_{L0}\tanh\frac{\Omega}{T}. \quad (4.18)$$

This expression is valid for Ω smaller than some cutoff energy E^*. At larger Ω, $\mathrm{Im}\chi_L^R(\Omega, T)$ must decrease. To simplify calculations, we impose a hard cutoff, i.e., set $\mathrm{Im}\chi_L^R(\Omega, T)$ to be given by (4.18) for $|\Omega| < E^*$ and $\mathrm{Im}\chi_L^R(\Omega, T) = 0$ for $|\Omega| > E^*$.
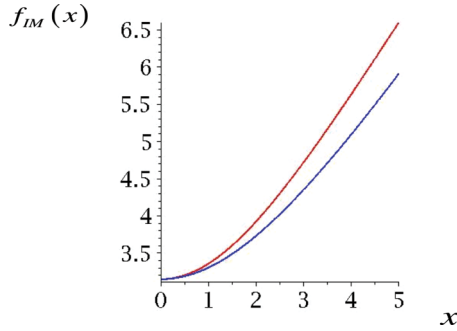
FIG. 3. (Color online) Red line: exact scaling function $f_{IM}(x)$ in $\mathrm{Im}\Sigma^R(\omega,T)$ for the marginal FL model [Eq. (4.20)]. Blue line: a scaling function obtained by replacing $\omega \to \sqrt{\omega^2 + \pi^2 T^2}$ in $\mathrm{Im}\Sigma^R(\omega,T=0)$. Square-root approximation (4.21) is practically indistinguishable from exact $f_{IM}(x)$ in the interval of x shown in the figure.

The first-Matsubara rule states that the self-energy at the first Matsubara frequency must be $\Sigma_{\mathbf{k}_F}(\pi T, T) = \pi T \chi_L(0,T)/(2\pi v_F)$. In all examples considered so far, we assumed that $D > 1$ and hence dropped this term, as it was of different order than the T^D term, which was our primary interest. Now, $\mathrm{Im}\Sigma^R_{\mathbf{k}_F}(\omega \sim T, T) = \mathcal{O}(T)$, and we should keep all $\mathcal{O}(T)$ terms.

Substituting Eq. (4.18) into Eq. (4.8), we obtain

$$\mathrm{Im}\Sigma^R_{\mathbf{k}_F}(\omega,T) = \frac{\chi_{L0}}{2\pi v_F} \int d\Omega \, \tanh \frac{\Omega}{T} \left[n_B(\Omega) + n_F(\Omega + \omega) \right].$$
(4.19)

Because the integral converges at large Ω, and we are interested in $\omega, T \ll E^*$, we can safely extend integration over Ω to the whole real axis. At $T = 0$, we have from (4.19) $\mathrm{Im}\Sigma^R_{\mathbf{k}_F}(\omega,0) = \chi_{L0}\omega/2\pi v_F$, and at $\omega = 0$, $\mathrm{Im}\Sigma^R_{\mathbf{k}_F}(0,T) = \chi_{L0}\pi T/2\pi v_F$. When ω and T are both finite, integration in Eq. (4.19) yields

$$\mathrm{Im}\Sigma^R_{\mathbf{k}_F}(\omega,T) = \frac{T\chi_{L0}}{2\pi v_F} f_{IM}\left(\frac{\omega}{T}\right),$$
$$f_{IM}(x) = \frac{(\pi/2)(e^x + 1)^2 + x(e^{2x} - 1)}{e^{2x} + 1}.$$
(4.20)

Function $f_{IM}(x)$ is plotted in Fig. 3.

Expanding (4.20) in ω/T and casting the result into the form of a square root, we obtain, approximately,

$$\mathrm{Im}\Sigma^R_{\mathbf{k}_F}(\omega,T) \approx \frac{\chi_{L0}}{2\pi v_F} \sqrt{\pi^2 T^2 + \omega^2 \frac{\pi(4 - \pi)}{2}}. \quad (4.21)$$

This form is obviously different from $\sqrt{\pi^2 T^2 + \omega^2}$ obtained by by replacing ω by $\sqrt{\pi^2 T^2 + \omega^2}$ in the $T = 0$ result. Nevertheless, by substituting $x = i\pi$ into (4.20), we find that $f_{IM}(i\pi)$ vanishes, as it should by the first-Matsubara rule.

The analysis of $\mathrm{Re}\Sigma^R_{\mathbf{k}_F}(\omega,T) = \omega f_{RM}(\omega/T)$ requires more effort as one has to take care of the upper cutoff of the theory. The calculation is similar to the one we did for $D = 3$ in the previous section. We use $\mathrm{Im}\Sigma^R_{\mathbf{k}_F}(\omega,T)$ in the form of Eq. (4.19), but keep the limits of the integration over Ω as $-E^*$ and E^* and set E^* to infinity only at the end of calculation. Without that, we would not reproduce the first-Matsubara rule for $\mathrm{Re}\Sigma^R_{\mathbf{k}_F}(\omega,T)$. Substituting $\mathrm{Im}\Sigma^R_{\mathbf{k}_F}(\omega,T)$ from Eq. (4.19)

into the KK formula, we obtain after some algebra

$$f_{RM}(x) = \frac{2}{\pi} \int_0^\infty \frac{dy}{y^2 - x^2} \left[f_{IM}(y) - y \right] + \frac{2}{\pi} \int_0^\infty \frac{dy}{y^2 - x^2}$$
$$\times (y - \ln[1 + e^{y - \bar{E}^*}]),$$
(4.22)

where $\bar{E}^* = E^*/T$. Both integrals are convergent and are easily evaluated numerically.

At $x = i\pi$, the first integral yields $(2/\pi) \times 0.963\,51$, while the second integral gives $(2/\pi)(\ln \bar{E}^* + 1 - \ln \pi) \approx (2/\pi)\left(\ln \bar{E}^* - 0.144\,73\right)$, up to terms exponentially small in E^*, which we neglect. Combining the two last expressions, we obtain

$$f_{RM}(i\pi) \approx \frac{2}{\pi}(\ln \bar{E}^* + 0.818\,78). \quad (4.23)$$

According to the first-Matsubara rule, the result in Eq. (4.23) should be exactly the same as $\mathrm{Re}\chi^R_L(0,T)$ [then $\mathrm{Re}\Sigma^R(i\pi T, T) = i(T/2v_F) f_{RM}(i\pi)$ becomes equal to $i(T/2v_F)\mathrm{Re}\chi^R_L(0,T)$]. The static local susceptibility is obtained by applying the KK transformation to $\mathrm{Im}\chi^R_L(\omega,T)$ in Eq. (4.18):

$$\mathrm{Re}\chi^R_L(0,T) = \frac{2}{\pi} \int_0^{\bar{E}^*} \frac{\tanh x}{x} dx$$
$$\approx \frac{2}{\pi}\left(\ln \bar{E}^* - \int_0^\infty \frac{dx \ln x}{\cosh^2 x}\right)$$
$$\approx \frac{2}{\pi}(\ln \bar{E}^* + 0.818\,78), \quad (4.24)$$

again, up to terms exponentially small in $\bar{E}^*$. Comparing Eqs. (4.23) and (4.24), we see that they are equal, as it should be, according to the first-Matsubara rule.

D. Finite-q instability

The discussion above is valid for a Pomeranchuk instability at $q = 0$, when the dynamical exponent z is equal to 3. In lattice systems, an instability may also occur at finite q, in which case $z = 2$, up to fluctuation corrections from multiloop diagrams.[6,16,40] Such an instability is often called either spin-density wave (SDW) or charge-density wave (CDW), depending on whether it occurs in the spin or charge channel. The $z = 2$ case is more involved because typical q along the FS now scale as $q_\perp \propto |\Omega_n|^{1/2}$, while $q_\parallel$ still scale as $\Sigma(\omega_m + \Omega_n, T)$. The one-loop self-energy for $z = 2$ problem scales as $\omega^{(D-1)/2}$, hence for $\Omega_n \sim \omega_m$, typical $q_\perp$ are of order $|\Omega_n|^{(D-1)/2}$. Local approximation is valid if typical $q_\parallel \ll q_\perp$, and is only justified for $D > 2$. At $D = 2$, $q_\perp$ and $q_\parallel$ are of the same $|\Omega_n|^{1/2}$ order. The local approximation in this case can be imposed by extending the system to a large number of fermionic flavors N, and the analysis up to two loops indeed shows that the local approximation, and the first-Matsubara rule associated with it, become exact at $N = \infty$. For a $z = 2$ transition, the first-Matsubara rule implies that $\Sigma(\omega_m, T)$ evaluated at a generic $\omega_m \neq \pm\pi T$ contains a $T^{1/2}$ term (or a $T^2\xi^3$ term in the FL regime), but the prefactor of this term vanishes at $\omega_m = \pm\pi T$. The vanishing is not exact, however, because some of the higher-order contributions to $\Sigma(\omega_m, T)$ can be viewed as coming from processes with small momentum transfers, mediated by small-q collective excitations of critical

$z = 2$ modes, and higher-order contributions to $\Sigma(\omega_m, T)$ from such processes do not vanish at $N = \infty$ (Refs. 13, 14, 16, and 17). Still, in $D = 2$, a local propagator of the collective mode made of two $z = 2$ excitations scales as $\chi_L(\Omega_n) \propto \int dq \, dq' d\Omega'_n \chi(q', \Omega'_n) \chi(q + q', \Omega_n + \Omega'_n) \propto \ln |\Omega_n|$ and is weaker than $\chi_L(\Omega_n) \propto 1/\sqrt{|\Omega_n|}$. As a result, the prefactor of the $T^{1/2}$ term in $\Sigma(\omega_m, T)$, although does not vanish exactly at $\omega_m = \pm\pi T$, is nevertheless reduced by a factor of $|\ln T|/T^{1/2}$. Contributions to this prefactor from even higher orders form series in $|\ln T|^n/T^{1/2}$ and may potentially give rise to an additional anomalous power $T^{\eta}/T^{1/2}$. The first-Matsubara rule then remains meaningful as long as $\eta < 1/2$.

V. SUMMARY

In this paper, we analyzed in detail the fermionic self-energy $\Sigma(\omega, T)$ in a FL at finite temperature T and frequency ω. Our main goal was to understand how general is a certain property of the self-energy, the first-Matsubara-frequency rule. This rule states that the self-energy $\Sigma(\omega_m, T)$, evaluated at discrete Matsubara points $\omega_m = \pi T(2m + 1)$, exhibits a special behavior at the first fermionic Matsubara frequency: namely, $\Sigma(\pi T, T)$ does not contain terms higher than $\mathcal{O}T$. As a particular manifestation of this rule, the imaginary part of the self-energy on the FS in a conventional 3D FL behaves as $\text{Im}\Sigma(\omega, T) \propto \omega^2 + \pi^2 T^2$, with exactly a π^2 factor in front of the T^2 term, and $\text{Re}\Sigma(\omega, T)$ contains an ω term but no ωT term. We found that the rule is not an exact one, i.e., $\Sigma(\pi T, T)$ in a generic FL does contain higher than linear terms in T. Still, the first term after $\mathcal{O}(T)$ in $\Sigma(\pi T, T)$ in any dimension $1 < D \leqslant 3$ is of order T^D ($T^3 \ln T$ in 3D). In $D > 2$, this term is parametrically smaller than the T^2 term which is present in $\Sigma(\omega_m, T)$ for $|\omega_m| \neq \pi T$. We found that the T^D term comes from only forward and backward scattering, and is expressed in terms of fully renormalized amplitudes for these processes. We further showed that the first-Matsubara-frequency rule becomes exact in the local approximation, when the interaction can be approximated by its value for the initial and final fermionic states right on the Fermi surface. In this approximation, which is justified, e.g., near a Pomeranchuk instability even if the vertex corrections are non-negligible, the T^D term and all higher-order terms in $\Sigma(\pi T, T)$ vanish, and only the $O(T)$ term survives. The first-Matsubara-frequency rule then imposes two constraints on the scaling form of the self-energy: upon replacing ω by $i\pi T$, $\text{Im}\Sigma^R(\omega, T)$ must vanish and $\text{Re}\Sigma^R(\omega, T)$ must reduce to an $\mathcal{O}(T)$ form. We considered several examples of the first-Matsubara rule, and argued that these two constraints should be taken into consideration in extracting scaling forms of $\Sigma^R(\omega, T)$ from experimental and numerical data.

ACKNOWLEDGMENTS

APPENDIX A: EVALUATION OF THE MATSUBARA SELF-ENERGY USING THE EULER-MACLAURIN SUMMATION FORMULA

In this Appendix, we show how to reproduce the first-Matsubara rule for the fermionic self-energy by using the Euler-Maclaurin (EM) formula for summation over the Matsubara frequencies. Unexpectedly, the calculations involving the EM formula turn out to be quite involved, and to reproduce the first-Matsubara rule, one has to keep not only the "conventional" terms in the EM formula, with the integral over a bosonic Matsubara frequency Ω_n and the sum over the derivatives of the summand at $n = 0$, but also the remainder term, which is often neglected when the EM formula is applied in practice.

To be specific, we consider Eq. (4.2) for the self-energy $\Sigma_{\mathbf{k}_F}(\omega_m, T)$ and set $\omega_m = \pi T$, which gives

$$\Sigma_{\mathbf{k}_F}(\pi T, T)$$
$$= iT \sum_n \int \frac{dq_\parallel d^{D-1}q_\perp}{(2\pi)^D} \frac{1}{i\pi T(2n+1) - v_F q_\parallel} \chi_{\mathbf{q}}(\Omega_n) \tag{A1}$$

with $\mathbf{q} = (q_\parallel, \mathbf{q}_\perp)$. We assume that the local approximation is valid, i.e., that typical $q_\parallel$ are small compared to typical $q_\perp$, and the dependence of the bosonic propagator on $q_\parallel$ can be neglected. Within this approximation, Eq. (A1) simplifies to

$$\Sigma_{\mathbf{k}_F}(\pi T, T) = \frac{T}{2v_F} \chi_L(0) + \frac{T}{2v_F} \sum_{n=1}^{\infty} \chi_L(2\pi T n)$$
$$\times [\text{sgn}(2n+1) - \text{sgn}(2n-1)], \tag{A2}$$

where

$$\chi_L(2\pi T n) = \int d^{D-1}q_\perp \chi_{\mathbf{q}_\perp}(2\pi n T)/(2\pi)^{D-1}. \tag{A3}$$

The first term in Eq. (A3) is proportional to T, and the second term vanishes identically because for any $n \geqslant 1$, $\text{sgn}(2n+1) = \text{sgn}(2n-1) = 1$. Hence, $\Sigma_{\mathbf{k}_F}(\pi T, T)$ does not contain terms beyond $\mathcal{O}(T)$, in accordance with the first-Matsubara rule.

An unexpected complication arises when one attempts to reproduce the vanishing of the second term in Eq. (A2) by applying the EM formula to the sum over n. Under the condition that $\bar{\chi}(x)$ and its derivatives vanish at $x \to \infty$, which we assume to hold in our case, the EM formula reads as[41]

$$\sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(x)dx - \frac{f(0)}{2} - \sum_{p=1}^{N} B_{2p} \frac{f^{(2p-1)}(0)}{(2p)!} - R_N, \tag{A4}$$

where B_k are the Bernoulli coefficients, $f^{(n)}$ is the nth derivative of f, and R_N is the Poisson remainder term

$$R_N = \int_0^\infty \frac{B_N(\{1-x\})}{N!} f^{(N)}(x) dx, \qquad (A5)$$

where $B_N(x)$ is Bernoulli polynomial, and $\{1-x\}$ denotes the fractional part of $1-x$.

In applications of this formula, it is often assumed that the remainder term R_N tends to zero in the limit $N \to \infty$ and is thus dropped. We show that in our case the remainder term can not be neglected and one should use the full EM formula [Eqs. (A4) and (A5)] instead of the truncated one.

Indeed, in our case,

$$f(x) = \frac{T}{2v_F} \chi_L(2\pi T x)[\text{sgn}(2x+1) - \text{sgn}(2x-1)]. \quad (A6)$$

The first term in the right-hand side of (A4) is the integral $\int_0^\infty f(x) dx$. By integrating $f(x)$ from (A6) over x, we obtain

$$\int_0^\infty f(x) dx = \frac{T}{v_F} \int_0^{1/2} dx \, \chi_L(2\pi T x)$$
$$= \frac{T}{2v_F} \chi_L(0) + \frac{\pi T^2}{4v_F} \chi_L'(0) + \cdots, \qquad (A7)$$

where the ellipsis stands for the terms of higher order in T. Combining (A7) with the boundary term $-f(0)/2 = -(T/2v_F)\chi_L(0)$, we see that the linear-in-T term cancels, but the quadratic term contributes

$$\Sigma_1 = T^2 \chi_L'(0) \frac{\pi}{4v_F} \qquad (A8)$$

to $\Sigma_{\mathbf{k}_F}(\pi T, T)$.

This T^2 term would violate the first-Matsubara rule and must be canceled by the terms with the derivatives $f^{(n)}(0)$. Because the derivatives of $\text{sgn}(2x+1)$ and of $\text{sgn}(2x-1)$ vanish at $x = 0$, one has to differentiate only $\chi_L(2\pi T x)$. A T^2 contribution to $\Sigma_{\mathbf{k}_F}(\pi T, T)$ comes from the first derivative of $f(x)$, i.e., from the $p = 1$ term in the sum over p in the right-hand side of (A4). Terms with $p > 1$ contribute higher powers of T. Using that $B_2 = 1/6$, we find the T^2 contribution from the infinite sum with the derivatives $f^{(n)}(0)$ as

$$\Sigma_2 = -T^2 \chi_L'(0) \frac{\pi}{6v_F}. \qquad (A9)$$

The sum $\Sigma_1 + \Sigma_2 = T^2 \chi_L'(0) \pi/(12v_F)$ is nonzero.

The sum over p in Eq. (A4) can be safely extended to infinity as only the $p = 1$ term contributes a T^2 to the self-energy. If we used the truncated EM formula without the remainder term, we would have then obtained an incorrect result that $\Sigma_{\mathbf{k}_F}(\pi T, T)$ does contain a T^2 term. In fact, the counterterm canceling the parasitic $\Sigma_1 + \Sigma_2$ contribution comes from the remainder term R_N in Eq. (A5), even if we take the $N = \infty$ limit. Indeed, let us focus on the T^2 term in the self-energy and replace $\chi_L(2\pi T x)$ by $2\pi T \chi_L'(0)x$. We then have

$$f(x) \to \pi \frac{T^2}{2v_F} x[1 - \text{sign}(2x-1)]. \qquad (A10)$$

One can easily make sure that $f(x)$ and its derivatives are nonzero only in the interval $0 < x \leqslant 1/2$, where the fractional

part of $1 - x$ in the argument of the Bernoulli polynomial in Eq. (A5) is equal to just $1 - x$. Furthermore, the derivatives $f^{(n)}(x)$ with $n \geqslant 2$ vanish at the boundaries of the integral in Eq. (A5), hence, one can integrate by parts $N - 2$ times and the boundary terms vanish. Using the property of Bernoulli polynomials $B_N'(x) = N B_{N-1}(x)$ and applying it $N - 2$ times, we rewrite R_N as

$$R_N = \frac{T^2}{2v_F} \chi_L'(0) \int_0^1 B_2(1-x) \frac{d^2}{dx^2} \{x[1 - \text{sign}(2x-1)]\}. \qquad (A11)$$

Using that $d/dx[1 - \text{sgn}(2x-1)] = -2\delta(x - 1/2)$ and also that $B_1(1/2) = 0$ and $B_2(1/2) = -1/12$, we obtain after integrating in Eq. (A11) by parts

$$\Sigma_3 = -R_N = \frac{T^2}{v_F} \chi_L'(0) B_2(1/2) = -T^2 \chi_L'(0) \frac{\pi}{12v_F}. \quad (A12)$$

Combining the three contributions, we see that $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$, as it should.

An alternative way to compute the sum over bosonic Matsubara frequencies using the EM formula would be to "smear" the discontinuity in $f(x)$ by integrating over $q_\parallel$ in Eq. (A1) in finite limits $-Q < q_\parallel < Q$ and take the limit $Q \to \infty$ only at the last stage. In this scheme, the remainder term R_∞ does not contribute, but terms with $p \sim \pi T/Q$ become relevant in the sum over p in Eq. (A4). This calculation is, however, more involved than the one we presented above, and we did not find a clear proof that the contribution from $p \sim \pi T/Q$ exactly cancels $\Sigma_1 + \Sigma_2$.

APPENDIX B: DEPENDENCE OF THE SELF-ENERGY ON THE UPPER CUTOFF OF LOW-ENERGY THEORY

In this Appendix, we show that the prefactor of the linear-in-T term in the fermionic self-energy at the first Matsubara frequency $\Sigma_{\mathbf{k}_F}(\pi T, T) = \lambda T$ depends on the ratio of the Fermi energy $E_F = v_F k_F/2$ to the upper cutoff of the low-energy theory denoted as Λ. The result shown in Eq. (2.33) with the "mass renormalization factor" $\lambda \propto \Pi(0)$ corresponds to the situation of $\Lambda \ll E_F$, when integration over intermediate energies in the expression for the self-energy [Eq. (2.30)] can be extended to infinity. In the opposite limit of $\Lambda \ll E_F$, λ is much smaller. To see this, we note that typical momentum transfers $q = |\mathbf{k} - \mathbf{k}'|$ are of order k_F, hence, typical internal energies in the self-energy diagram are of order E_F. The integration over $\varepsilon_{\mathbf{k}'}$ in Eq. (2.30) in finite limits changes the factor of $\text{sgn}(\omega_m + \Omega_n)$ to $(2/\pi) \arctan[\Lambda/(\omega_m + \Omega_n)]$, which becomes small when typical $\Omega_m \sim E_F$ is much larger than Λ. The polarization operator also changes, but the $\Pi(0)$ term remains the same because it comes from the smallest frequencies. To simplify the computations, we keep $\Pi_{\mathbf{q}}(\Omega_n)$ in the same form as before, but replace $v_F q$ by E_F, i.e., we set $\Pi_{\mathbf{q}}(\Omega_n) = -\Pi(0)(1 - |\Omega_n|/\sqrt{\Omega_n^2 + E_F^2})$. Substituting this expression along with the result of integration over $\varepsilon_{\mathbf{k}'}$ into the

self-energy, we obtain

$$\Sigma_{\mathbf{k}_F}(\pi T, T)$$

$$= \lambda T \frac{2}{\pi} \sum_{\Omega_n} \left(1 - \frac{|\Omega_n|}{\sqrt{\Omega_n^2 + E_F^2}} \right) \arctan \frac{\Lambda}{\pi T + \Omega_n}$$

$$= \lambda T \frac{2}{\pi} \left[\arctan 2\bar{\Lambda} + \sum_{n=1}^{\infty} \left(1 - \frac{n}{\sqrt{n^2 + \bar{E}_F}} \right) \right.$$

$$\times \left. \left(\arctan \frac{\bar{\Lambda}}{n + 1/2} - \arctan \frac{\bar{\Lambda}}{n - 1/2} \right) \right], \quad \text{(B1)}$$

where $\lambda \propto |\Pi_0|$ is the same as in Eq. (2.33), $\bar{\Lambda} = \Lambda/2\pi T$, and $\bar{E}_F = E_F/2\pi T$. For $\bar{\Lambda} \gg \bar{E}_F$, the term $(1 - n/\sqrt{n^2 + \bar{E}_F^2})$ decreases rapidly for $n \gtrsim \bar{E}_F$, when the difference between two arctangent functions is still small, of order $\bar{E}_F/\bar{\Lambda}$. Then, the first term in the last line in Eq. (B1) is the dominant one, and using that $\arctan 2\bar{\Lambda} \approx \pi/2$, one recovers $\Sigma_{\mathbf{k}_F}(\pi T, T) = \lambda T$ with $\lambda = \Pi(0)$. In the opposite limit of $\bar{E}_F \gg \bar{\Lambda}$, the first term can be approximated by unity for all n up to $n \sim \bar{E}_F \gg \bar{\Lambda}$. Because the difference of the two arctangents scales as $1/n^2$ for $n \gg \bar{\Lambda}$, and

$$\sum_{n=1}^{\infty} \left(\arctan \frac{\bar{\Lambda}}{n + 1/2} - \arctan \frac{\bar{\Lambda}}{n - 1/2} \right) = - \arctan 2\bar{\Lambda}, \quad \text{(B2)}$$

the contribution to the sum from positive n almost cancels that from $n = 0$. A straightforward analysis shows that in this limit, λ is small and scales as $\lambda \sim (\Lambda/E_F) \ln E_F/\Lambda$.

APPENDIX C: ANALYTIC CONTINUATION OF THE SELF-ENERGY IN THE LOCAL APPROXIMATION IN 2D

In this Appendix, we discuss some subtleties of analytic continuation of the self-energy into the complex ω plane in a situation when either $\text{Re}\Sigma^R(\omega, T)$ or $\text{Im}\Sigma^R(\omega, T)$ cannot be evaluated explicitly and have to be kept in an integral form, as in Eq. (4.10). One can use the fact that the integral converges in the ultraviolet and modify the integrand by shifting the variable. By doing so, one can obtain several different formulas for $\text{Im}\Sigma^R(\omega, T)$, which all nevertheless yield the same result along the real frequency axis. The danger of this trick is that, by shifting variables, one imposes the dependence on the external

ω onto $\text{Im}\chi_L^R$ which, in $D \leqslant 2$, is a nonanalytic function of its argument. As a consequence, if one now performs analytic continuation just by replacing $\omega \to z$, one obtains a branch cut which stretches down to $\text{Im}z \to 0$, and the self-energy will not obey the first-Matsubara rule at $z = i\pi T$. To make sure that this rule is satisfied, one has to use the Cauchy formula for analytical continuation which, in this case, is not equivalent to just replacing ω by a complex z. As an illustration, we consider Eq. (4.10) in the form it was presented in Ref. 29:

$$\text{Im}\Sigma^R(\omega, T)$$

$$= \frac{B_0}{4} \left\{ (\omega^2 + \pi^2 T^2) \ln \frac{e\Lambda^2}{\pi^2 T^2} + \omega^2 \ln \frac{\pi^2 T^2}{\omega^2} \right.$$

$$+ 1.1217 \pi^2 T^2 + 2 \int_0^{\infty} \frac{1}{e^{x/T} + 1} \left[\omega \ln \left(\frac{x - \omega}{x + \omega} \right)^2 \right.$$

$$+ \left. \left. x \ln \frac{x^4}{(x^2 - \omega^2)^2} \right] \right\}. \quad \text{(C1)}$$

Along the real frequency axis, this formula yields exactly the same result as Eq. (4.13). However, if we formally replace ω by $i\pi T + \delta$, with infinitesimally small $\delta > 0$, before integrating over x in the last term in Eq. (C1), we obtain $(B_0 \pi^2 T^2/4)(0.17 + i\pi)$, which obviously does not satisfy the first-Matsubara rule. The reason is that the ω dependence is under the logarithm in the last, integral term of (C1), and each of the two logarithms there has a branch cut. Let us set $\omega = \pi T e^{i\phi}$ and vary ϕ between zero (the real axis) and $\pi/2$ (the first Matsubara frequency along the imaginary axis). To understand what is going on, it is enough to move only a little off the real axis, i.e., to consider only small ϕ. The branch cuts in the first and second logarithms in the integral term of Eq. (C1) are at $x = 1$ and $\cos 2\phi$, correspondingly. Each of them gives rise to a discontinuity in the imaginary part ($\ln z = \ln |z| + i\pi \arg z$, and the argument of z changes discontinuously at the branch cut). At $\phi = 0$, the discontinuities coming from the two logarithms cancel each other, but at finite ϕ there is a range of x in-between $1 - \phi^2/4$ and 1, where the arguments add up to almost 2π. This additional contribution makes $\text{Im}\Sigma^R(T, \pi T e^{i\phi})$ to be different from the one obtained by analytical continuation of Eq. (C1). For small ϕ, the difference is $(\pi^2 T^2/8) \times (2\pi i)(\phi^2/4)/(e^\pi + 1)$. We verified numerically that this expression is exactly the difference between the analytical continuation of (C1) and the brute-force replacement $\omega \to \pi T \exp(i\phi)$ in Eq. (C1).

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