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Theory and classification of interacting integer topological phases in two dimensions: A Chern-Simons approach

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We study topological phases of interacting systems in two spatial dimensions in the absence of topological order (i.e., with a unique ground state on closed manifolds and no fractional excitations). These are the closest interacting analogs of integer quantum Hall states, topological insulators, and superconductors. We adapt the well-known Chern-Simons K-matrix description of quantum Hall states to classify such "integer" topological phases. Our main result is a general formalism that incorporates symmetries into the K-matrix description. Remarkably, this simple analysis yields the same list of topological phases as a recent group cohomology classification, and in addition provides field theories and explicit edge theories for all these phases. The bosonic topological phases, which only appear in the presence of interactions and which remain well defined in the presence of disorder, include (i) bosonic insulators with a Hall conductance quantized to even integers, (ii) a bosonic analog of quantum spin Hall insulators, and (iii) a bosonic analog of a chiral topological superconductor, whose K matrix is the Cartan matrix of Lie group E_8. We also discuss interacting fermion systems where symmetries are realized in a projective fashion, where we find the present formalism can handle a wider range of symmetries than a recent group super-cohomology classification. Lastly, we construct microscopic models of these phases from coupled one-dimensional systems.

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I. INTRODUCTION

The recent discovery of topological insulators[1–16] has lead to a renewed interest in phases of matter that are not described within the usual Landau paradigm of symmetry breaking and order parameters.[17] Topological insulators and topological superconductors, like integer quantum Hall states, are gapped in the bulk but differ from trivial phases in the topology of their electronic states. They are characterized by gapless edge excitations that reflect the bulk topology. A new aspect of $\mathbb{Z}_2$ spin-orbit topological insulators is the role of symmetry (time reversal in this case) in protecting the topological distinction. In the absence of symmetry, the topological properties, such as gapless edge states, are generally destroyed. As with the integer Hall effect, topological insulators and superconductors can be described in terms of noninteracting particles. A complete classification of such free fermion topological phases that are stable to disorder, in all spatial dimensions, has been achieved.[18,19] The remaining outstanding questions for fundamental theory have to do with interacting systems.

Interacting topological phases have been studied at two levels. First, the stability of the noninteracting classification to interactions has been examined.[20–23] In some cases, interactions *reduce* the number of topological phases,[20–25] i.e., two topologically distinct phases of free fermions be continuously connected, via intermediate states that involve interactions. The second possibility, of interactions leading to *new* phases, not possible within noninteracting particles has also been studied. Largely, these have attempted to extend the analogy between integer and fractional quantum Hall states, to topological insulators. Thus fractional topological insulators have been theoretically discussed,[26–32] along with lattice realizations of fractional quantum Hall insulators.[33–42] These phases are topologically ordered—in that they involve fractional excitations in the bulk and feature ground-state degeneracies that depend on the topology of the space on which they are defined. They are also characterized by a finite topological entanglement entropy (TEE) in the ground state. In contrast, integer quantum Hall and topological insulators (and superconductors), despite being topologically distinct, are not topologically ordered. Bulk excitations are essentially like electrons or groups of electrons, and the ground state is unique when defined on a manifold without boundaries. The TEE vanishes for these phases. Henceforth we shall refer to gapped phases without topological order as being *short-range entangled* (SRE) states. (This terminology differs slightly from that of Chen-Gu-Wen,[47] who require a state to also be nonchiral to be short range entangled.) It seems appropriate to define interacting "integer" topological phases, as new topological phases without topological order, but which only appear in the presence of interactions. Do such phases exist? And if so, how can they be studied?

In one dimension, topological order is absent, and all topological phases found are "integer" (or SRE) phases. They include examples like the Haldane (or AKLT) state of gapped spin-1 chains.[43] Using the matrix product representation of gapped states,[24,44–46] they are argued to be classified by projective representations of the symmetry group (G) or equivalently by the second group cohomology $\mathcal{H}^2(G,\mathcal{C})$ of symmetry group G. In higher dimensions, such rigorous results are not available. Nevertheless, new work indicates that these are also amenable to theoretical study. Recently, Chen, Gu, and Wen[47] have proposed that higher-dimensional group cohomology describes $d = 2,3$-dimensional interacting topological states without topological order. For example, bosonic systems were studied, where there are no topological phases in the absence of interactions. With interactions, topological phases were

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predicted in two (and three) dimensions, without topological order. While Chen *et al.*[47] restrict attention to the nonchiral subset of these states (i.e., ones that do not have a net imbalance of left and right movers at the edge of a two-dimensional system) protected by symmetry, Kitaev[48] has also considered chiral states. Explicit examples of such phases in special cases have been given.[49,50] However, predictions in the general case rely on writing Wess-Zumino-Witten terms for generalized sigma models. While this is a powerful approach, the physical meaning of the phases that are predicted are obscure. For example, the nature of edge excitations in these phases is not apparent. Moreover, a knowledge of group-cohomolgy machinery is required, which is mathematically sophisticated even by the standards of the field.

(a) K-matrix formulation. Here, we take a completely different and simpler approach to the problem, focusing on the case of two spatial dimensions. We rely on the K matrix formulation of quantum Hall states, a symmetric integer matrix that appears in the Chern-Simons action ($\hbar = 1$, and summation is implied over repeated indices $\mu,\nu,\lambda = 0,1,2$):

$$4\pi \mathcal{S}_{\rm CS} = \int d^2x dt \sum_{I,J} \epsilon_{\mu\nu\lambda} a^I_\mu [\mathbf{K}]_{I,J} \partial_\nu a^J_\lambda. \quad (1)$$

While this has been utilized to discuss quantum Hall states with Abelian topological order, here we show that it is also a powerful tool to discuss topological phases in the *absence* of topological order. The latter requires $|\det \mathbf{K}| = 1$ (i.e., $\mathbf{K}$ is a *unimodular* matrix). The bulk action also determines topological properties of the edge states. For example, the signature of the K matrix (number of positive minus negative eigenvalues) is the chirality of edge states—the imbalance between number of right and left moving edge modes. Maximally chiral states have all edge modes moving in the same direction. Physically the fluxes $\epsilon^{\mu\nu\lambda}\partial_\mu a_\nu$ are related to densities and currents of bosons of different flavors.

(b) Strategy and results. Let us briefly review our strategy and results. Although we mainly focus on nonchiral states, we begin by looking for maximally chiral states of bosons without topological order. These are bosonic analogs of the integer quantum Hall effect or chiral superconductors of fermions. It is readily shown that the smallest dimension of bosonic K matrix that yields a maximally chiral state is eight. This is consistent with the prediction of Kitaev, derived from topological field theory. Here, we explicitly construct a candidate K matrix for this state, corresponding to the Cartan matrix of the group E_8.

We then consider nonchiral states of bosons, with equal number of left and right moving edge modes. In the absence of symmetry, we argue that there are no nontrivial topological phases with $|\det \mathbf{K}| = 1$. However, the presence of a symmetry can lead to new topological phases. The main result of this work is a scheme to classify topological phases with $|\det \mathbf{K}| = 1$, that are protected by symmetry.

Given a particular symmetry (e.g., time reversal, charge conservation, etc.), we study distinct ways in which the symmetry can act on the elementary quasiparticles. The symmetries are realized by a set of symmetry transformations on elementary quasiparticles, which form a (faithful) representation of the symmetry group G. Distinct realizations are potentially different phases—analogous to the space group classification of crystals. However, an additional requirement to realize a nontrivial topological phase is the existence of symmetry protected edge states, i.e., either the edge is gapless, or if it is gapped, it must spontaneously break the symmetry. Note, an internal symmetry can never provide such protection to a purely one-dimensional system—hence the edge states enjoy these special properties by virtue of their connection to the bulk topological phase. We will call such phases symmetry protected topological (SPT) phases following the terminology of Ref. 47. To access these states, we supplement the Chern-Simons action with insertion/removal of "local" quasiparticles that are bosonic and have trivial mutual statistics with any other excitation. Symmetry imposes additional, and crucial, constraints on the possible terms. The set of these symmetry-allowed perturbations can be used to analyze if stable edge states exist or not. While we outline the general rules that apply to a K matrix of any dimension, for the most part we restrict our attention to 2×2 K matrices.

Remarkably, this simple analysis yields the same set of interacting topological phases as the group cohomology classification of Chen *et al.* for a large set of symmetry groups G (see Table I and Fig. 1). For example, bosonic phases with $G = U(1)$ are classified by an integer which is just the quantized Hall conductance in units of $2Q^2/h$ where Q is the unit of boson charge. When $U(1)$ is broken to a discrete subgroup $\mathbb{Z}_n$, the set of topological phases is also reduced to $\mathbb{Z}_n$. Similarly, both schemes find a $\mathbb{Z}_2$ classification of bosonic insulators with conserved charge and time reversal symmetry [$G = U(1) \rtimes Z_2^T$, the semidirect product ensures that this is

TABLE I. Topological classification of gapped $D = 2 + 1$ dimensional phases of bosons with short-range entanglement (no topological order).

Symmetry	Topological classification	Comments
No symmetry (chiral)	$\mathbb{Z}$	E_8 state and derivatives with chiral central charge[51] $c_- = 8n$.
Z_2^T	$\mathbb{Z}_1$	Time-reversal symmetry
$U(1)$	$\mathbb{Z}$	Charge conserved. Quantized Hall conductance $\sigma_{xy} = 2nq^2/h$ with $n \in \mathbb{Z}$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}_2$	Bosonic quantum Spin Hall with charge $U(1)$ and time reversal $\mathcal{T}^2 = +1$.
$U(1) \times Z_2^T$	$\mathbb{Z}_1$	$U(1)$ spin conservation and time reversal.
Z_n	$\mathbb{Z}_n$	$U(1)$ broken down to a discrete subgroup
$Z_n \rtimes Z_2^T$	$\mathbb{Z}^2_{(n,2)}$	$(a,b) \equiv$ greatest common divisor of a and b.
$Z_n \times Z_2^T$	$\mathbb{Z}^2_{(n,2)}$	
$U(1) \times Z_2$	$\mathbb{Z} \times \mathbb{Z}_2^2$	

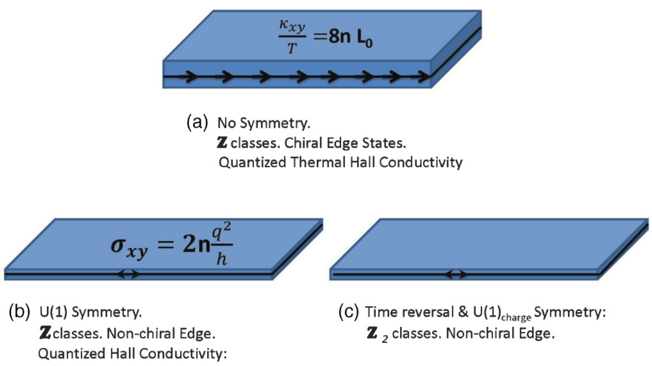

FIG. 1. (Color online) Summary of some simple integer bosonic topological phases. (a) A chiral phase of bosons (no symmetry required). An integer multiple of eight chiral bosons at the edge is needed to evade topological order, leading to a quantized thermal Hall conductance $\kappa_{xy}/T = 8nL_0$ in units of the the universal thermal conductance $L_0 = \frac{\pi^2 k_B^2}{3h}$. These are bosonic analogs of chiral superconductors. (b) A nonchiral phase of bosons protected by $U(1)$ symmetry (e.g., charge conservation). Distinct phases can be labeled by the quantized Hall conductance $\sigma_{xy} = 2n\sigma_0$, which are even integer multiples of the universal conductance $\sigma_0 = q^2/h$ for particles with charge q. These are bosonic analogs of the integer quantum Hall phases. (c) A nonchiral phase stabilized in the presence of time reversal and $U(1)$ charge conservation symmetries, the same symmetries used to define quantum spin Hall (topological) insulators. A Z_2 topological classification is obtained, although bosonic time reversal that squares to $+1$ is involved.

the usual relation between charge and time reversal], the analog of fermionic quantized spin Hall insulators, despite the fact that the time reversal operation is "bosonic" and squares to $+1$. An advantage of the present formulation is that the edge states of these phases are explicit—typically being nonchiral $c = 1$ conformal field theory (CFT) when gapless. Moreover, being cast in the familiar Abelian Chern-Simons form, it is amenable to further investigation using standard field theory methods. We focus on symmetries (such as time reversal) that are realized locally. Spatial symmetries such as translation invariance, inversion, etc., will be left for future work. Since we do not make any assumption about spatial uniformity, the topological phases we find are well defined in the presence of disorder.

A disadvantage of our method is that it is less suited to discuss non-Abelian Lie group symmetries, and we are currently restricted to two spatial dimensions, neither of which is a restriction for group cohomology theory.[47] Also, our method does not automatically produce a group structure for the set of topological states. On general grounds, one expects the set of topological phases protected by a particular symmetry to form an Abelian group, which is automatically satisfied in the group cohomology classification and in the classification of free fermion topological phases. We handle this by defining an Abelian group structure, addition and subtraction, on pairs of phases described within the K matrix formulation. With this refinement the group structure of the resulting sets of phases is readily determined. For phases with topological order ($|\det K| > 1$) and exotic bulk exceptions, it is less apparent whether such an Abelian group structure of topological phases will emerge. Nevertheless, a similar K matrix approach could be used to discuss topologically ordered phases in the presence of global symmetries, which is left for future work.

(c) Topological phases of interacting fermions. We extend our discussion to classifying topological phases of interacting fermion, in the absence of topological order. A key difference from the bosonic case is that since fermion insertion is a nonlocal operation, symmetries may be realized projectively on the fermion fields. We compare our results to a recent super-cohomology classification of interacting fermion phases.[25] In addition to the relative simplicity of our method, an advantage over supercohomology classification is that we are able to handle Kramers time reversal symmetry $\mathcal{T}^2 = (-1)^{\hat{N}_f}$ ($\hat{N}_f$ is the total fermion number operator). A disadvantage, shared by the super-cohomology classification, is that we are not able to capture chiral or nonchiral states with odd numbers of Majorana edge modes. As expected, we recover the $\mathbb{Z}_2$ classification of time reversal symmetric quantum spin Hall insulators, from this interacting formalism as well. We also compare our results with the recent work[21–23] on topological phases of interacting fermions with $\mathbb{Z}_2 \times \mathbb{Z}_2^T$ symmetry. We also discuss topological phases of the symmetry group $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$, corresponding to time reversal symmetric superconductors with four particle (charge $4e$) condensates. For the cases we considered, the topological phases of interacting fermions either descend from noninteracting phases, or correspond to a bosonic topological phases, where the bosons are bound states of fermions. Whether this is a general property of fermion topological phases is an open question. Our classifications of SPT phases of interacting fermions with various symmetries are summarized in Table II.

(d) Microscopic quasi-1D realization. Finally, to give a deeper insight into the obtained topological phases we utilize a quasi-one-dimensional (coupled wire[52,53]) approach to construct a candidate state consistent with the edge content that emerges from the classification. The K matrix approach naturally suggests such a construction. This sheds light on various paradoxical results such as the fact that there is a bosonic analog of the quantum spin Hall state although time reversal acts only on bosons with $T^2 = 1$.

Some aspects of this work are similar in spirit to a number of previous works that have discussed the role of symmetry and stability of edge states in various specific contexts.[28,50,54–60] For example, Refs. 28 and 60 discussed the stability of edge states in fractional topological insulators. However, the general machinery presented here to generate symmetry protected topological states has not previously been discussed.

II. K MATRIX FORMULATION OF 2 + 1-D TOPOLOGICAL PHASES

It is believed[61–63] that K matrix provides a complete classification of all Abelian fraction quantum Hall (FQH) states in $2 + 1$ dimensions. In this section, we briefly review the K matrix formulation of Abelian FQH states. We then discuss how it can be applied to study states without topological order. In particular, we point out that in the absence of symmetry, fields that have trivial (or bosonic) self and mutual statistics will be "Higgsed," and the stability of the edge is examined in

TABLE II. Topological classification of gapped $D = 2 + 1$-dimensional *nonchiral* phases of fermions with short range entanglement (no topological order). Here, $\boldsymbol{P}_f$ denotes fermion parity, which is always conserved. Note, states with an odd number of right(left)-moving Majorana edge modes (such as class DIII topological superconductor) are not captured in this formalism.

Symmetry	Minimal topological classification	Comments
Z_2^f (no symmetry)	$\mathbb{Z}_1$	No symmetry (fermion parity always conserved) nonchiral phase
$Z_2^T \times Z_2^f$	$\mathbb{Z}_1$	Time-reversal symmetric superconductor
$U(1) \rtimes Z_2^T, \boldsymbol{T}^2 = 1$	$\mathbb{Z}_2$	Bosonic quantum spin Hall insulator of Cooper pairs
$U(1) \rtimes Z_2^T, \boldsymbol{T}^2 = \boldsymbol{P}_f$	$\mathbb{Z}_2^2$	Including fermionic quantum spin Hall insulator
$U(1) \times Z_2^T \times Z_2^f$	$\mathbb{Z}_1$	$U(1)$ spin conservation and time reversal.
$Z_2 \times Z_2^f$	$\mathbb{Z}_4$	Superconductor with Ising-type symmetry
Z_4	$\mathbb{Z}_2$	Bosonic Z_2-symmetric SPT phase of Cooper pairs
$Z_2 \times Z_2^T \times Z_2^f$	$\mathbb{Z}_4 \times \mathbb{Z}_2$	Discussed in Ref. 21–23 with $\mathbb{Z}_8$ classification
$Z_4 \times Z_2^T$	$\mathbb{Z}_2^3$	Superconductor with Z_4 spin symmetry
$Z_4 \rtimes Z_2^T$	$\mathbb{Z}_2^2$	Time-reversal-symmetric charge-4e superconductor

the presence of these terms. This is then applied to study chiral topological phases of bosons that lack topological order.

A. A brief review of the K matrix formulation

The low-energy effective theory of an Abelian quantum Hall state is captured by ($\hbar = 1$)

$$\mathcal{L}_{\rm CS} = \frac{\epsilon^{\mu\nu\lambda}}{4\pi} a_\mu^I K_{I,J} \partial_\nu a_\lambda^J - a_\mu^I j_I^\mu + \cdots \qquad (2)$$

(summation over repeated indices is assumed). The a_μ^I, $I = 1,2,\ldots,N$ are internal gauge fields coupled to quasiparticles currents j_I^μ, and $\mathbf{K}$ is a symmetric matrix with integer entries. For states built entirely out of underlying bosons, the diagonal elements of $\mathbf{K}$ are all *even integers*, while for those built from underlying fermions (electrons), at least one diagonal entry is an odd integer.

The topological order is also characterized by the K matrix. The ground-state degeneracy (GSD) on a torus can be calculated by quantizing the Chern-Simons theory (2) and is given by[64]

$$\text{GSD on a torus} = |\det \mathbf{K}|. \qquad (3)$$

We will mainly be interested in states without topological order, i.e., with $|\det \mathbf{K}| = 1$.

Quasiparticles are characterized by integer vector $\mathbf{l}$ and couple minimally to the combination $\sum_I l_I a_\mu^I$. The self-(exchange) statistics θ of a quasiparticle is obtained by integrating out the gauge fields:

$$\theta = \pi \mathbf{l}^T \mathbf{K}^{-1} \mathbf{l}, \qquad (4)$$

while the mutual (braiding) statistics on taking quasiparticle $\mathbf{l_1}$ around quasiparticle $\mathbf{l_2}$ is

$$\theta_{12} = 2\pi \mathbf{l_1}^T \mathbf{K}^{-1} \mathbf{l_2}. \qquad (5)$$

The effective action describing the gapless edge excitations of a FQH state characterized by $\mathbf{K}$ can also be derived[65] from gauge invariance of Lagrangian (2) on a manifold with boundary:

$$\mathcal{S}_{\rm edge}^0 = \int \frac{dt dx}{4\pi} \sum_{I,J} (K_{I,J} \partial_t \phi_I \partial_x \phi_J - V_{I,J} \partial_x \phi_I \partial_x \phi_J). \qquad (6)$$

Here, $V_{I,J}$ is a positive definite constant matrix, that is nonuniversal. However, the commutation relations between fields is fixed by the first term that is simply the K matrix in the bulk. The number of right movers n_+ and left movers n_- are given by the signature of symmetric matrix $\mathbf{K}$, i.e., the matrix $\mathbf{K}$ has n_+ positive and n_- has negative eigenvalues.

One important question is are different FQH states characterized by different K matrices fundamentally different? In other words, can two different K matrices represent the same phase? This means that two FQH states have exactly the same set of quasiparticles but these quasiparticles are labeled in two different ways. It turns out[61,63,66] that a generic change of label (or change of basis) for the same set of quasiparticles is realized by the following $GL(N,\mathbb{Z})$ transformation:

$$a_\mu^I \to \sum_J W_{I,J} a_\mu^J, \quad \mathbf{W} \in GL(N,\mathbb{Z}). \qquad (7)$$

Here, $GL(N,\mathbb{Z})$ denotes all $N \times N$ integer matrix with determinant ± 1. After this relabeling of quasiparticles, the $\mathbf{K}$ matrix and currents j_μ^I transform as

$$\mathbf{K} \to \mathbf{W}^T \mathbf{K} \mathbf{W}, \quad j_\mu^I \to \sum_J W_{J,I} j_\mu^J. \qquad (8)$$

Any two K matrices related by such a $GL(N,\mathbb{Z})$ transformation represent the same state (in the absence of any global symmetry). It is straightforward to see that physical properties such as the determinant and the signature of a K matrix is invariant under such a $GL(N,\mathbb{Z})$ transformation.

When there is a $U(1)$ symmetry associated with charge conservation, one couples an external $U(1)$ gauge field A_μ to the conserved $U(1)$ current with charge q via an integer vector $\mathbf{t} \equiv (t_1,\ldots,t_N)^T$ called the *charge vector*.[63] This is incorporated by adding the following term to the Lagrangian (2) above: $2\pi \mathcal{L}_{\rm charge} = -q \epsilon_{\mu\nu\lambda} t_I A_\mu \partial_\nu a_\lambda^I$. By integrating out internal gauge fields $\{a_\mu^I\}$, one obtains the quantized Hall conductance

$$\sigma_{xy} = \frac{q^2}{2\pi} \mathbf{t}^T \mathbf{K}^{-1} \mathbf{t} \qquad (9)$$

and the $U(1)$ charge of a quasiparticle with integer vector $\mathbf{l}$ is given by $Q = q \mathbf{t}^T \mathbf{K}^{-1} \mathbf{l}$.

The many-body wave function of a multilayer FQH state described by effective theory (2) is given by[64,67]

$$\Psi_{\mathbf{K}} = \prod_{i<j,I,J} \left(z_i^{(I)} - z_j^{(J)}\right)^{\mathbf{K}_{I,J}} e^{-\sum_{i,I} |z_i^{(I)}|^2/4} \tag{10}$$

in a disk geometry. Here, $z_i^{(I)} \equiv x_i^{(I)} + \mathrm{i}\, y_i^{(I)}$ denotes the two-dimensional coordinates of the ith particle in the Ith layer. Multiparticle pseudopotentials can be constructed[68,69,71] as ideal Hamiltonians, whose zero-energy ground states are the above multilayer FQH states.

B. *K* matrix + Higgs formulation

The Lagrangian (2) seems to have $U(1)^N$ symmetry (or N conserved currents) due to the existence of N internal gauge fields $\{a_\mu^I\}$. When these correspond to bosonic excitations (featured by trivial self and mutual statistics with other quasiparticles), and in the absence of any symmetry, one generically does not expect them to be conserved. This can be implemented by adding terms to the action (2) that create and destroy these bosonic particles, which we (in the absence of a better phrase) call Higgs terms. To be precise, denote the annihilation operator for a quasiparticle of Ith type as b_I and the associated creation operator as $b_I^{-1} \equiv b_I^\dagger$. If an integer vector $\mathbf{l} = (l_1, \ldots, l_N)^T$ characterizes a boson, then we demand

$$\pi \mathbf{l}^T \mathbf{K}^{-1} \mathbf{l} = 0 \bmod 2\pi \tag{11}$$

and

$$2\pi \mathbf{l}^T \mathbf{K}^{-1} \mathbf{l}' = 0 \bmod 2\pi \tag{12}$$

for trivial mutual (braiding) statistics with all other quasiparticles $\mathbf{l}'$. Then, in the absence of symmetry, we can add a Higgs term

$$\delta\mathcal{L}_{\mathrm{CS}} = C_{\mathbf{l}} \prod_I b_I^{l_I} + \mathrm{H.c.}, \quad C_{\mathbf{l}} = \mathrm{const.}, \tag{13}$$

to the Lagrangian $\mathcal{L}_{\mathrm{CS}}$ that condenses the boson with vector $\mathbf{l}$. Note, since this particle has trivial statistics, we can dispense with the gauge field in this expression, whose only role here is to keep track of statistics. One can show that the topological properties of the corresponding state, such as the quasiparticle statistics (4) and ground-state degeneracy (3) are not influenced by these Higgs terms. Therefore the generic Lagrangian describing a 2 + 1-D gapped Abelian phase is the following:

$$\mathcal{L}_{2+1} = \mathcal{L}_{\mathrm{CS}} + \sum_{\{\mathbf{l}=\text{bosonic}\}} \left(C_{\mathbf{l}} \prod_I b_I^{l_I} + \mathrm{H.c.}\right). \tag{14}$$

Taken at face value, the Chern-Simons theory, which attaches flux to particles, would require monopole terms to account for a change in particle number. We have argued above why this may be unnecessary.[70] In any event, as we will see below, the only action of the Higgs terms we will need is their effect on the boundary, which does not suffer from these problems.

1. Stability of edge states

Now, the action of edge excitations corresponding to bulk Lagrangian (14) is

$$\begin{aligned} \mathcal{S}_{\mathrm{edge}} &= \mathcal{S}^0_{\mathrm{edge}} + \mathcal{S}^1_{\mathrm{edge}}, \\ \mathcal{S}^1_{\mathrm{edge}} &= \sum_{\{\mathbf{l}=\text{bosonic}\}} \tilde{C}_{\mathbf{l}} \int dt dx \cos\left(\sum_I l_I \phi_I + \alpha_{\mathbf{l}}\right), \end{aligned} \tag{15}$$

where $\mathcal{S}^0_{\mathrm{edge}}$ is given in Eq. (6). The bare action $\mathcal{S}^0_{\mathrm{edge}}$ indicates the following Kac-Moody algebra:

$$[\partial_x \phi_I(x), \partial_y \phi_J(y)] = 2\pi\,\mathrm{i}\,\mathbf{K}^{-1}_{I,J} \partial_x \delta(x-y). \tag{16}$$

Notice that each allowed Higgs term (13) in the bulk has a one-to-one correspondence with those on the edge in $\mathcal{S}^1_{\mathrm{edge}}$, i.e.,

$$\left[C_{\mathbf{l}} \prod_I b_I^{l_I} + \mathrm{H.c.}\right] \rightarrow \left[\tilde{C}_{\mathbf{l}} \cos\left(\sum_I l_I \phi_I + \alpha_{\mathbf{l}}\right)\right]. \tag{17}$$

While all these perturbations are present at the edge a more stringent requirement needs to be placed if they are to gap out the edge modes. For example, we expect a maximally chiral edge, where all modes move in the same direction, to be stable even in the absence of any symmetry. The required condition can be deduced by studying the commutation relation implied by the Kac-Moody algebra above for the field $\mathbf{l}^T \cdot \boldsymbol{\phi} = \sum_I l_I \phi_I$:

$$[\mathbf{l}^T \cdot \partial_x \phi(x), \mathbf{l}^T \cdot \partial_y \phi(y)] = 2\pi\,\mathrm{i}\,(\mathbf{l}^T \mathbf{K}^{-1} \mathbf{l}) \partial_x \delta(x-y), \tag{18}$$

thus, in order to be able to localize this field at a classical value, and gap out an edge mode, we require that the commutator vanishes i.e.,

$$\mathbf{l}^T \mathbf{K}^{-1} \mathbf{l} = 0. \tag{19}$$

For a maximally chiral state where $\mathbf{K}^{-1}$ is a positive definite matrix, no nonvanishing vector satisfies this condition. Hence the edge states cannot be gapped. Similarly, when there are an imbalanced number of right and left moving modes, $n_+ \neq n_-$ the system has a net number of chiral modes and we call it a 2 + 1-D chiral phase. In a 2 + 1-D chiral phase, even in the absence of any symmetry, there will be gapless edge excitations.[54]

To completely gap out an edge, one requires equal number of counterpropagating modes, i.e., a nonchiral edge. Then, dimension of $\mathbf{K}$ matrix N is even and $N/2 = n_+ = n_-$. Let us call, i.e., $\cos(\mathbf{l_1}^T \phi + \alpha_{\mathbf{l_1}})$ and $\cos(\mathbf{l_2}^T \phi + \alpha_{\mathbf{l_2}})$ *independent Higgs terms* if and only if:

$$\mathbf{l_1}^T \mathbf{K}^{-1} \mathbf{l_1} = \mathbf{l_2}^T \mathbf{K}^{-1} \mathbf{l_2} = \mathbf{l_1}^T \mathbf{K}^{-1} \mathbf{l_2} = 0. \tag{20}$$

In this case, they form a pair of commuting variables according to the Kac-Moody algebra. According to Heisenberg's uncertainty principle, these mutually commuting fields $\{\mathbf{l_n}^T \phi | n = 1,2,\ldots\}$ can be pinned at certain classical values simultaneously, and consequently, their associated edge excitations will be gapped. Then, to completely gap out the edge, one needs a set of $N/2$ independent Higgs terms that are pairwise commuting. In the absence of any symmetry, this is typically possible. However, in the next section, we will see that symmetry can forbid some Higgs terms leading to SPT phases with nontrivial edge structure. Now let us first consider a chiral state of bosons without topological order.

C. A chiral bosonic phase without topological order: The E_8 state

A phase without topological order is characterized by a symmetric K matrix with $|\det \mathbf{K}| = 1$. A chiral state in 2 + 1-D requires the signature (n_+,n_-) of its K matrix to satisfy that $n_+ \neq n_-$. Such a state has gapless edge excitations and a nonzero quantized thermal Hall conductance.[72] There are many such examples for a fermionic system, e.g., an integer quantum Hall state whose K matrix is the unit matrix of size N. On the other hand, in a bosonic system without topological order, the existence of such states is less obvious.

We therefore seek a K matrix with the following properties (i) $|\det \mathbf{K}| = 1$ (ii) the diagonal elements $K_{I,I}$ are all even integers and (iii) a maximally chiral phases, where all the edge states propagate in a single direction. Then, all eigenvalues of K must have the same sign (say positive), so $\mathbf{K}$ is a positive definite symmetric unimodular matrix.

It is helpful to map the problem of finding such a $\mathbf{K}$ to the following crystallographic problem. Diagonalizing $\mathbf{K}$ and multiplying each normalized eigenvector by the square root of its eigenvalue one obtains a set of primitive lattice vectors $\mathbf{e}_I$ such that $K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$. The inner product of a pair of vectors $l_I\mathbf{e}_I$ and $l'_I\mathbf{e}_I$ are given by $l'_I K_{IJ} l_J$, while the volume of the unit cell is given by $[\text{Det } K]^{1/2}$. The latter can be seen by writing the components of the vectors as a square matrix: $[\mathbf{k}]_{aI} = [e_I]_a$. Then $\text{Det}\, k$ is the volume of the unit cell. However, $K_{IJ} = \sum_a k_{aI}k_{aJ} = (\mathbf{k}^{\mathrm{T}}\mathbf{k})_{IJ}$. Thus $\text{Det}\,\mathbf{K} = [\text{Det}\,\mathbf{k}]^2$.

Thus, for a phase without topological order, we require the volume of the lattice unit cell to be unity $[\text{Det}\, k] = 1$ (unimodular lattice). Furthermore, for a bosonic state, we need that all lattice vectors have even length $l_I K_{IJ} l_J =$ eveninteger, since the K matrix has even diagonal entries (even lattice). It is known that the minimum dimension this can occur in is eight.[73] In fact, the root lattice of the exceptional Lie group E_8 is the smallest dimensional unimodular, even lattice.[74] Such lattices only occur in dimensions that are a multiple of eight.

A specific form of the K matrix is

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{21}$$

This matrix has unit determinant and all eigenvalues are positive. It defines a topological phase of bosons without topological order, with eight chiral bosons at the edge. Note K^{-1} can be related to K by a $GL(8,\mathbb{Z})$ transformation $S^{\mathrm{T}}K^{-1}S$ if we take $S = K$, and so they are physically identical. Thus, even without computing the inverse, all particles are seen to have trivial statistics $(\pi l'^T K^{-1} l = 2\pi m)$. This $\mathbf{K}$ is the Cartan matrix for E_8, hence the name of the state.

This result was previously pointed out by Kitaev,[51,75] utilizing the fact that the central charge of the edge states $(c_- = c - \bar{c})$ of a chiral topological phase are determined by the statistics of emergent excitations only modulo 8. Thus phases with trivial statistics are allowed whenever $c_- = 0$ mod 8. Combining these phases leads to an integer classification of chiral topological states, which are characterized by a quantized thermal hall conductivity,[72] which are integer multiples of the universal thermal conductivity: $\lim_{T\to 0} \frac{\kappa_{xy}}{T} = 8\frac{\pi^2 k_B^2}{3h}$ [see Fig. 1(a)].

III. INCORPORATING SYMMETRIES IN K MATRIX FORMULATION

In this section, we will be interested in incorporating global symmetries into the K matrix + Higgs formulation. This will lead to new symmetry protected topological phases. We only consider internal symmetries, spatial symmetries like inversion, translation, etc., will not be discussed.

Now let us restrict ourselves to 2 + 1-D *nonchiral* phases with equal numbers of counterpropagating modes $n_+ = n_-$ for signature (n_+,n_-) of matrix $\mathbf{K}$. In the absence of symmetry, any edge Higgs term that satisfies (13) can be added. In such a phase, there will be no gapless edge excitations in the absence of any symmetry, i.e., all edge modes will be gapped out by the Higgs terms $\mathcal{S}^1_{\text{edge}}$.

However, this is not true any more when there are symmetries in the system. In the presence of symmetry, only those bosonic quasiparticles, which transform trivially under the symmetry operation, can condense. This means certain Higgs terms, which transform nontrivially under the symmetry operation, are not allowed and cannot be added to effective theory (14) or (15).

How do quasiparticles transform under a symmetry operation? In general, the Lagrangians (2) and (14) should be invariant under the symmetry transformation on quasiparticle currents $\{j^\mu_I\}$. Notice that when a K matrix is acted on by a $GL(N,\mathbb{Z})$ transformation (8), it describes the same physical state. Only the labels of different quasiparticles are changed. Given a state described by a certain $\mathbf{K}$ matrix, the allowed $GL(N,\mathbb{Z})$ transformations $\mathbf{W}$ which transform the quasiparticles under symmetry must leave the K matrix invariant, i.e., $\mathbf{K} = \mathbf{W}^T\mathbf{K}\mathbf{W}$. Besides any global $U(1)$ phase transformation on the quasiparticle annihilation operator $b_I \to e^{i\delta\phi_I}b_I$ also keeps the Lagrangian (2) invariant. Notice that such a phase shift $\delta\phi_I$ is defined modulo 2π due to the quantization of quasiparticle number in a compact theory.

Therefore a generic realization of unitary symmetry g on the $\{\phi_I\}$ fields at the edge (we will not require the transformation law in the bulk in the following) is given by

$$\phi_I \to \sum_J W^g_{I,J}\phi_I + \delta\phi^g_I, \tag{22}$$

where $\delta\phi^g_I \in [0,2\pi)$ are constants and matrix $W^g \in GL(N,\mathbb{Z})$ satisfies

$$\mathbf{K} = (W^g)^T\mathbf{K}W^g. \tag{23}$$

For an antiunitary symmetry h (such as time reversal symmetry Z_2^T), in general, it is realized in the following way:

$$\phi_I \to -\sum_J W^h_{I,J}\phi_J + \delta\phi^h_I, \tag{24}$$

where $\delta\phi_I^h \in [0,2\pi)$ are constants and matrix $W^h \in GL(N,\mathbb{Z})$ satisfies

$$\mathbf{K} = -(W^h)^T \mathbf{K} W^h. \tag{25}$$

The antiunitary symmetry operation h is realized by the above transformations followed by complex conjugation $\mathcal{C}$. Notice that K matrix changes sign under the above $GL(N,\mathbb{Z})$ transformation since a Chern-Simons term $\epsilon^{\mu\nu\lambda}a_\mu^I\partial_\nu a_\lambda^J$ always changes sign under time reversal.

1. Group compatibility conditions

It may appear we have wide latitude in determining how the generators of a symmetry group act on the quasiparticles in our theory. However, there is an important constraint. For an arbitrary symmetry group G, the multiplication rule of its group elements is completely determined by certain algebraic relations of the group generators $\{g_1,g_2,\ldots\}$:

$$\mathcal{A}_{\{n_a\}} \equiv \prod_a g_a^{n_a} = \boldsymbol{e}, \tag{26}$$

where $\boldsymbol{e}$ is the identity element of group G and $\{n_a\}$ are all integers. A bosonic quasiparticle [which satisfied Eq. (11)] is a physical excitation and must transform trivially under the identity element. Thus all boson insertion operators satisfying (11) and (12) should be invariant under the symmetry operation $\mathcal{A}_{\{n_a\}}$:

$$\begin{aligned}&\mathcal{A}_{\{n_a\}}: \sum_I l_I\phi_I \to \sum_I l_I\phi_I \bmod 2\pi,\\ &\forall \mathbf{l} \quad \text{satisfying} \quad \mathbf{l}^T\mathbf{K}^{-1}\mathbf{l} = 0 \bmod 2.\end{aligned} \tag{27}$$

These algebraic requirements serve as constraints to the possible $GL(N,\mathbb{Z})$ transformations W^{g_a} and $U(1)$ phase rotations $\{\delta\phi_I^{g_a}\}$ and we shall call them *group compatibility conditions* for the edge states described by effective theory (15) and their associated bulk topological phases. In the case of bosonic phases without topological order, $\mathbf{K}$ is a symmetric unimodular matrix whose diagonal elements are all even integers. Then, any integer vector $\mathbf{l}$ satisfies the conditions (11) and (12), i.e., all quasiparticles of a bosonic SRE phase are bosons. Therefore the group compatibility conditions (27) for symmetry transformations are simplified as

$$\begin{aligned}\text{Under} \quad &\mathcal{A}_{\{n_a\}} = \boldsymbol{e}: \phi_I \to \phi_I \bmod 2\pi,\\ &I = 1,2,\ldots,N.\end{aligned} \tag{28}$$

By solving these algebraic equations we can find out all sets of inequivalent symmetry transformations $\{W^{g_a},\delta\phi_I^{g_a}\}$ for generators $\{g_a\}$ of group G.

As an aside we note that for phases with topological order, or in the presence of fermionic quasiparticles, symmetries are realized projectively.[76] Then, even the identity elements (26) can induce a nontrivial transformation on quasiparticles.

2. Gauge equivalence

A question naturally arises: do different symmetry transformations represent different SPT phases? We answer this question in two parts. First, we comment on the equivalency or inequivalency between two sets of symmetry transformations $\{W^{g_a},\delta\phi_I^{g_a}\}$. Notice that one can always change the label of quasiparticles by a $GL(N,\mathbb{Z})$ transformation X (as long as $X^T\mathbf{K}X = \mathbf{K}$), or perform a global $U(1)$ gauge transformation $\phi_I \to \phi_I + \Delta\phi_I$ that keeps Lagrangian (2) invariant. Under such a "gauge" transformation the symmetry operations $\{W^{g_a},\delta\phi_I^{g_a}\}$ will transform as

$$\begin{aligned}&W^g \to X^{-1}W^gX,\\ &\delta\phi_I^g \to X^{-1}\left[\delta\phi_I^g - \Delta\phi_I + \eta\sum_J W_{I,J}^g\Delta\phi_J\right],\\ &\text{if} \quad X \in GL(N,\mathbb{Z}), \quad X^T\mathbf{K}X = \mathbf{K},\end{aligned} \tag{29}$$

where $\eta = \pm 1$ if g is a unitary (antiunitary) symmetry. If two sets of symmetry operations $\{W^{g_a},\delta\phi_I^{g_a}\}$ associated with group G generated by $\{g_a\}$ are related by the above gauge transformation (29), then these two sets of symmetry operations are essentially identical.

3. Edge stability and criteria for SPT phases

With a set of symmetry transformations $\{W^{g_a},\delta\phi_I^{g_a}\}$ one can determine what Higgs terms in Eqs. (14) and (15) that are allowed in the presence of the symmetry group $G = \{g\}$. In general, only those Higgs terms that transform trivially under the symmetry transformations $\{W^{g_a},\delta\phi_I^{g_a}\}$ are allowed, which induce certain allowed set of edge perturbations. To determine the fate of possible gapless modes at the edge, one considers terms that commute with each other and with themselves, i.e., terms involving variables $\sum_I l_I\phi_I$ that satisfy conditions (19) and (20). Then one can simultaneously minimize these terms like classical variables. If no commuting operator remains, then the field minima exhaust the independent degrees of freedom, which implies a gapped edge. Note, scaling dimensions of these edge terms are immaterial to this discussion.

Criterion for Trivial Phase: If there is a set of independent Higgs terms $\{C_a\cos(\mathbf{l}_a^T\phi + \alpha_a)\}$ allowed by symmetry, so that any other variables $\mathbf{l}^T\phi$ on the edge is either a linear combination of these bosonic variables $\{\mathbf{l}_a^T\phi\}$ or doesn't commute with every condensed bosonic quasparticles in $\{\mathbf{l}_a^T\phi\}$, then the edge of the system will be completely gapped in the presence of independent Higgs terms $\{C_a\cos(\mathbf{l}_a^T\phi + \alpha_a)\}$. When the independent Higgs terms $\{C_a\cos(\mathbf{l}_a^T\phi + \alpha_a)\}$ are simultaneously minimized on the edge, the *elementary bosonic variables* $\{\mathbf{v}_a^T\phi\}$

$$\mathbf{v}_a = (v_{a,1},\ldots,v_{a,N}) \equiv \frac{\mathbf{l}_a}{\gcd(l_{a,1},l_{a,2},\ldots,l_{a,N})}, \quad \forall\, a, \tag{30}$$

will all condense and be localized at various classical values $\langle\mathbf{v}_a^T\phi\rangle = B_a$ (gcd is short for greatest common divisor). Notice that if the set of independent elementary bosonic variables $\{\mathbf{v}_a^T\phi\}$ are invariant under any symmetry transformation, i.e.,

$$\text{Under} \quad \forall g \in G: \left\{\sum_I v_{a,I}\phi_I\right\} \to \left\{\sum_I v_{a,I}\phi_I\right\}, \tag{31}$$

then the edge states can be all gapped out without breaking the symmetry G at all. We call such a nonchiral SRE phase a *trivial* phase since, in general, it doesn't support gapless edge states in the presence of symmetry G. These principles will be illustrated by examples in detail in the following section. In comparison to the aforementioned trivial phase, *a nontrivial SPT phase has a gapless edge structure which* *cannot be gapped without breaking the symmetry G*. The only two possible situations for the edge structure of such a phase are (i) *gapless:* the maximal set of independent Higgs terms $\{C_a \cos(\mathbf{l}_a^T\phi+\alpha_a)\}$ allowed by symmetry cannot gap out all the edge states. In other words, on the edge there exist at least one variable $\mathbf{l}\phi$ that commutes with all the condensed bosonic variables $\{\mathbf{l}_a^T\phi\}$. Hence this degree of freedom $\mathbf{l}\phi$ remains gapless even in the presence of all the symmetry-allowed independent Higgs terms $\{C_a\cos(\mathbf{l}_a^T\phi+\alpha_a)\}$. An example is the bosonic SPT phases protected by $U(1)$ symmetry in $2+1$-D, as will be discussed in detail later.

(ii) *Spontaneous symmetry breaking on the edge:* although all the edge states will be gapped in the presence of independent Higgs terms $\{C_a\cos(\mathbf{l}_a^T\phi+\alpha_a)\}$ allowed by symmetry, not all of the associated elementary bosonic variables $\{\mathbf{v}_a^T\phi\}$ in Eq. (30) are invariant under symmetry transformations $\{\eta_g W^g,\delta\phi^g\},\forall g\in G$, where G is the symmetry group. This means at least one elementary bosonic variable $\mathbf{v}_a^T\phi$ in Eq. (30) would transform nontrivially under symmetry group G. Therefore in order to gap out the edge by condensing all the independent elementary bosons, one has to spontaneously break the symmetry on the edge. An example is the bosonic SPT phase protected by $U(1)\rtimes Z_2^T$ symmetry (or by Z_2 symmetry) in $2+1$-D, as will be shown later.

4. Group structure of phases protected by symmetry group G

In general, the set of different *phases* that appear in a topological classification are expected to form an Abelian group, as proved for noninteracting fermions[19,77] and conjectured for interacting bosonic systems[47] (since group cohomology classification leads to an Abelian group). How does this group structure appear within our K matrix formulation? Let $\{\Psi_G[\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}]\}$ to denote the set of trivial and SPT phases in the presence of symmetry G. Here $\Psi_G[\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}]$ represents a phase with matrix $\mathbf{K}$ in action (15) and transformation rules $\{W^{g_a},\delta\phi^{g_a}\}$ for symmetry group $G=\{g_a\}$. We would like to attach to this set a group product.

A natural Abelian product rule $\oplus$ of two phases $\Psi_G[\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}]$ and $\Psi_G[\tilde{\mathbf{K}},\{\tilde W^{g_a},\tilde{\delta\phi}^{g_a}\}]$ is to take their matrix direct sum:

$$\Psi_G[\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}]\oplus\Psi_G[\tilde{\mathbf{K}},\{\tilde W^{g_a},\tilde{\delta\phi}^{g_a}\}] = \Psi_G[\mathbf{K}\oplus\tilde{\mathbf{K}},\{W^{g_a}\oplus\tilde W^{g_a},\delta\phi^{g_a}\oplus\tilde{\delta\phi}^{g_a}\}]. \quad (32)$$

This seems to suggest that one cannot obtain a full classification of all different SPT phases when restricted to a K matrix with a fixed dimension. However, notice that two phases Ψ_G^1 and Ψ_G^2 can be identified as the same one if $\Psi_G^1\oplus\boldsymbol{e}_G=\Psi_G^2$, i.e., adding a trivial phase (denoted by $\boldsymbol{e}_G$) to Ψ_G^1 yields the phase Ψ_G^2. We can use this fact to reduce the dimensions of K matrix by throwing away the "trivial" parts of the edge structure, which can be gapped without breaking any symmetry.

The identity element in the group $\{\Psi_G\}$ corresponds to the trivial phase $\boldsymbol{e}_G\equiv\Psi_G[\mathbf{K}_0,W^{g_a}\equiv I_{N\times N},\delta\phi^{g_a}\equiv 0]$, where $\mathbf{K}_0$ can be any $N\times N$ unimodular symmetric matrix corresponding to a nonchiral SRE phase in $2+1$-D, its edge states can be gapped out without breaking the symmetry of group G.

We can also define the "inverse" of a phase $\Psi_G[\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}]$ in the group to be

$$\Psi_G[\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}]^{-1}=\Psi_G[-\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}], \quad (33)$$

i.e., by changing the sign of its K matrix we obtain the inverse of a phase. This is simply because we can always gap out the edge of

$$\Psi_G[\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}]^{-1}\oplus\Psi_G[-\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}]=\boldsymbol{e}_G$$

without breaking the symmetry. Consider the variables $\{\phi_I\}$ of phase $\Psi_G^{-1}[\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}]$ and $\{\tilde\phi_I\}$ of phase $\Psi_G[-\mathbf{K},\{W^{g_a},\delta\phi^{g_a}\}]$ on the edge. Then edge perturbations such as $\{C_a\cos[\mathbf{l}_a^T(\phi-\tilde\phi)+\alpha_a]\}$ will not be affected by the phase factors $\delta\phi^{g_a}$. We can then condense a set of independent elementary bosonic variables $\{\mathbf{l}_a^T(\phi-\tilde\phi)\}$ satisfying Eqs. (19) and (20). This can be readily shown when either $W=I$ (arbitrary $\mathbf{K}$) or when $\mathbf{K}$ is a 2×2 matrix (arbitrary W). This is simply because $\{\phi_I\}$ and $\{\tilde\phi_I\}$ satisfy the Kac-Moody algebra with opposite K matrices. No symmetry will be broken by condensing these bosons, when a proper set of vectors $\{\mathbf{l}_a\}$ are chosen. To check if two putative SPT states are the same phase or are different phases, we use the above group multiplication rules to combine one state with the inverse of the other and check if a trivial phase results. If so, these two are the same SPT phase.

Now that the identity element, the inverse of an element and the multiplication rules are defined, we can identify the group structure. We will perform this analysis below to clarify the connection between phases generated by our formalism.

5. Miscellaneous considerations

We focus on the cases where $\{W^{g_a},\delta\phi_I^{g_a}\}$ form a faithful representation of symmetry group G, where G is the symmetry group of the system's Hamiltonian. The case when it forms an unfaithful representation, i.e., when more than one group element acts like the identity on all quasiparticles (this set of group elements form an invariant subgroup H), actually corresponds to the faithful representation of the quotient group (G/H). Thus studying faithful representations suffices. This is discussed in more detail in Appendix D. In some cases, we will find that a solution for symmetry transformation may not be realizable in a theory with local fields (for example, one that exchanges a pair of fields that are canonical conjugate of each other in a $c=1$ edge), which will then not be included in the minimal set of SPT phases. For phases that are reported in Table I, we have checked that they have symmetries that can be realized starting from microscopic degrees of freedom (such as from the coupled wire construction). In the following, we illustrate the above principles, first by classifying bosonic SPT phases in $2+1$-D, whose topology is protected by a certain symmetry group G.

IV. *K*-MATRIX CLASSIFICATION OF BOSONIC SPT PHASES

In this section, we will focus on bosonic nonchiral SRE states in the presence of certain symmetry, i.e., bosonic nonchiral SPT phases in $2+1$ dimensions. They are described by a symmetric, unimodular K matrix, whose diagonal elements are even integers and with the same number of positive eigenvalues and negative eigenvalues ($n_+=n_-$). Therefore the dimension of matrix $\mathbf{K}$ must be *even* and

$$\det\mathbf{K}=(-1)^{\dim(\mathbf{K})/2}\quad\text{for a nonchiral SRE phase.}$$

In this section, we restrict ourselves to a 2×2 K matrix. SPT phases that are necessarily described by K matrices of a larger size, may be missed by this restriction. However, our results are internally consistent and also capture at least all the topological states of the group cohomology classification of bosonic SPT phases in Ref. 47. It appears that a K matrix of dimension 2 is sufficient to represent and classify SPT phases in $2+1$ dimensions in many cases. The reason behind this unexpected success is the following: although we focus on SPT phases described by a K matrix of size 2×2, when analyzing the group structure formed by SPT phases with symmetry G we need to multiply two phases together by a direct sum of their K matrices. Consequently, we are in fact considering K matrices of size $2n\times 2n$ obtained from direct sums of original 2×2 K matrices. Therefore it is actually not surprising that many of the bosonic SPT phases in $2+1$ dimensions can be described and classified by a 2×2 K matrix and associated symmetry transformations.

As proved in Appendix B, a $2\times 2K$ matrix with determinant -1 for a bosonic system [see $n=1$ in theorem (B3)] is always equivalent to the standard form $\sigma_x = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ by certain $GL(N,\mathbb{Z})$ transformations ($\sigma_\alpha,\alpha=x,y,z$ are Pauli matrices). In the following, we always choose the 2×2 matrix $\mathbf{K}=\sigma_x$ to represent a generic bosonic nonchiral SRE state. In the following, we use general principles discussed earlier in Sec. III to study nonchiral bosonic SPT phases with different symmetries. Note that the $GL(2,\mathbb{Z})$ transformations $\mathbf{X}$ that keeps $\mathbf{K}=\sigma_x$ invariant under (8) are $\mathbf{X}=\pm I_{2\times 2},\pm\sigma_x$. For such a nonchiral bosonic SRE phase, the unperturbed edge theory is

$$\mathcal{S}^0_{\text{edge}} = \frac{1}{4\pi}\int dt dx\left(\partial_t\phi_1\partial_x\phi_2+\partial_t\phi_2\partial_x\phi_1 - \sum_{I,J}V_{I,J}\partial_x\phi_I\partial_x\phi_J\right), \tag{34}$$

where $V_{I,J}$ is a positive definite constant matrix, as discussed in Sec. II B. This implies the following commutation relations (Kac-Moody algebra) for the edge fields $\{\phi_1,\phi_2\}$:

$$\begin{aligned}&[\partial_x\phi_1(x),\partial_y\phi_1(y)] = [\partial_x\phi_2(x),\partial_y\phi_2(y)] = 0,\\ &[\partial_x\phi_1(x),\partial_y\phi_2(y)] = 2\pi\,\mathrm{i}\,\partial_x\delta(x-y).\end{aligned} \tag{35}$$

In the absence of any symmetry, the edge states in Eq. (34) can always be gapped, since a set of independent Higgs terms satisfying Eqs. (19) and (20) on the edge can be choose as either $\{C_l\cos(l\phi_1+\alpha_l),l\in\mathbb{Z}\}$ or $\{C_l\cos(l\phi_2+\alpha_l),l\in\mathbb{Z}\}$. These added terms destroy the edge states. All degrees of freedom on the edge are gapped when variable ϕ_1 (or ϕ_2) is localized at a classical value. Now, let us consider various symmetries.

A. Z_2^T symmetry: $\mathbb{Z}_1$ class

Z_2^T symmetry (time reversal) is generated by $\boldsymbol{T}$. The algebra (26), which defines Z_2^T symmetry group, is

$$\boldsymbol{T}^2 = \boldsymbol{e}, \tag{36}$$

where $\boldsymbol{e}$ is the identity operation. Time-reversal symmetry $\boldsymbol{T}$ is implemented [following rules (24)] by a matrix $W^{\boldsymbol{T}}\in GL(2,\mathbb{Z})$ and a vector of $U(1)$ phase changes $\delta\phi_{\boldsymbol{T}}$ (defined modulo 2π) satisfying the constraints (25) and (28) for a bosonic SRE system:

$$(W^{\boldsymbol{T}})^2 = I_{2\times 2},\quad (W^{\boldsymbol{T}})^T\mathbf{K}W^{\boldsymbol{T}} = -\mathbf{K}; \tag{37}$$

$$\delta\phi_I^{\boldsymbol{T}} - \sum_J W^{\boldsymbol{T}}_{I,J}\delta\phi_J^{\boldsymbol{T}} = 0 \bmod 2\pi,\quad I=1,2. \tag{38}$$

The only $GL(2,\mathbb{Z})$ matrix solutions to Eq. (37) are $W^{\boldsymbol{T}}=\pm\sigma_z$. Notice that $W^{\boldsymbol{T}}=\sigma_z$ and $W^{\boldsymbol{T}}=-\sigma_z$ are related a gauge transformation (29) $X=\sigma_x$. Therefore one can always choose

$$W^{\boldsymbol{T}} = +\sigma_z$$

by a proper gauge fixing. Then the constraint (38) becomes $(I_{2\times 2}-\sigma_z)\delta\phi^{\boldsymbol{T}}=0 \bmod 2\pi$ and it leads to

$$\delta\phi_2^{\boldsymbol{T}} = n_2\pi \bmod 2\pi,\quad n_2=0,1.$$

Under a gauge transformation $\Delta\phi_I$ in Eq. (29) the compact $U(1)$ phase shift $\delta\phi^{\boldsymbol{T}}$ transforms to $\delta\phi^{\boldsymbol{T}}-(I_{2\times 2}+\sigma_z)\Delta\phi$. As a result, we can always choose a gauge so that

$$\delta\phi_1^{\boldsymbol{T}} = 0 \bmod 2\pi.$$

So a generic bosonic nonchiral SPT phase in the presence of time reversal symmetry $\boldsymbol{T}$ has symmetry transformation

$$\{W^{\boldsymbol{T}},\delta\phi^{\boldsymbol{T}}\} = \left\{\sigma_z,\begin{pmatrix}0\\ n_2\pi\end{pmatrix}\right\}\quad n_2=0,1. \tag{39}$$

Since each bulk Higgs term has a one-to-one correspondence with that on the edge, hereafter, we will only write down those Higgs terms $\tilde{C}_{\mathbf{l}}\cos(\sum_I l_I\phi_I+\alpha_{\mathbf{l}})$ on the edge. In the case of Z_2^T symmetry, the allowed Higgs terms are

$$\begin{aligned}\mathcal{S}^1_{\text{edge}} = \sum_{l_1\geqslant 0,l_2} C_{\mathbf{l}}\int dxdt[&\cos(l_1\phi_1+l_2\phi_2+\alpha_{\mathbf{l}})\\ &+\cos(-l_1\phi_1+l_2\phi_2+n_2l_2\pi+\alpha_{\mathbf{l}})].\end{aligned}$$

For $n_2=0,1$, the allowed Higgs terms are different, e.g., $\cos(\phi_2)$ terms are allowed for $n_2=0$ but not allowed for $n_2=1$. Thus there is a distinction between these states. However, this is not a topological distinction as argued below. For both $n_2=0$ and $n_2=1$ cases, we can write the same set of symmetry-allowed *independent* Higgs terms formed by mutually commuting operators (35): i.e.,

$$\mathcal{S}^1_{\text{edge}} = \sum_{l_1} C_{l_1,0}\int dxdt\,\cos[l_1\phi_1(x,t)]. \tag{40}$$

Thus, in both cases, the edge can be gapped, therefore they belong to the same trivial phase. If the variable ϕ_1 is localized at expectation value, e.g., $\langle\phi_1\rangle=0$ by the Higgs terms, all excitations on the edge would be gapped but the time reversal symmetry ($\phi_1\to-\phi_1$) is not broken by this expectation value.

B. $U(1)$ symmetry: $\mathbb{Z}$ classes

The elements of $U(1)$ group can be labeled as U_θ, where $\theta\in[0,2\pi)$ and the identity element is U_0. The multiplication rule is given by

$$U_{\theta_1}U_{\theta_2} = U_{(\theta_1+\theta_2 \bmod 2\pi)} \tag{41}$$

and, therefore, A generic form of symmetry transformations $\{W^{g_a},\delta\phi_I^{g_a}\}$ satisfying constraint (28) for $U(1)$ group is

$$\left\{W^{U_\theta}=I_{2\times2},\delta\phi^{U_\theta}=\theta\mathbf{t}=\theta\begin{pmatrix}t_1\\t_2\end{pmatrix}\right\},\quad (t_1,t_2)=1,$$

where (t_1,t_2) denotes the *greatest common divisor* of integers t_1 and t_2. Notice that only when $(t_1,t_2)=1$ the above symmetry transformations $\{W^{U_\theta},\phi^{U_\theta}\}$ form a faithful representation of symmetry group $U(1)$, which is what we assume here.[78]

Here, $\mathbf{t}'\equiv\mathbf{K}\mathbf{t}=\binom{t_2}{t_1}$ is nothing but the charge vector defined in the context of K matrix formulation of a FQH state (see Sec. II A). As proved in Appendix C for a bosonic SRE phase with $\mathbf{K}=\sigma_x$, an arbitrary charge vector $\mathbf{t}$ with $(t_1,t_2)=1$ is equivalent to the standard form

$$\mathbf{t}=\begin{pmatrix}t_1\\t_2\end{pmatrix}\simeq\begin{pmatrix}q\\1\end{pmatrix}\simeq\begin{pmatrix}1\\q\end{pmatrix},\quad q\in\mathbb{Z},$$

by certain $GL(N,\mathbb{Z})$ gauge transformations. Therefore the inequivalent symmetry transformations under constraint (28) for a bosonic nonchiral SRE phase with $U(1)$ symmetry are

$$\left\{W^{U_\theta}=I_{2\times2},\delta\phi^{U_\theta}=\theta\begin{pmatrix}1\\q\end{pmatrix}\right\},\quad q\in\mathbb{Z}.$$

The associated symmetry-allowed Higgs terms are

$$\mathcal{S}^1_{\text{edge}}=\sum_{l\in\mathbb{Z}}C_l\int dxdt\cos[l(q\phi_1-\phi_2)].\tag{42}$$

Due to Kac-Moody algebra (35), they do not form a set of independent Higgs terms satisfying Eqs. (19) and (20) unless $q=0$! When $q=0$ the Higgs term can localize the variables ϕ_2 to expectation value $\langle\phi_2\rangle=$ const., which gaps the edge excitations without breaking $U(1)$ symmetry ($\phi_1\to\phi_1+\theta$).

Now we will determine the group structure formed by these phases. Let us label a phase by

$$[q]\equiv\Psi_{U(1)}\left[\mathbf{K}=\sigma_x,\left\{W^{U_\theta}=I_{2\times2},\delta\phi^{U_\theta}=\theta\begin{pmatrix}1\\q\end{pmatrix}\right\}\right],$$

where $\boldsymbol{e}_{U(1)}=[0]$ is the trivial phase. Consider two states $[q_1]$ with edge variables $\{\phi_1,\phi_2\}$ and $[q_2]$ with edge variables $\{\phi_1',\phi_2'\}$. When they are put together the following independent edge terms are symmetry allowed: $\sum_{l\in\mathbb{Z}}C_l\cos[l(\phi_1-\phi_1')+\alpha_l]$ and their associated elementary bosonic variable is $\phi_1-\phi_1'$. If $q_2=-q_1$, independent Higgs terms $\sum_{l\in\mathbb{Z}}C_l'\cos[l(\phi_2+\phi_2')+\alpha_l']$ are also allowed by $U(1)$ symmetry. Therefore the edge states will be fully gapped without breaking the $G=U(1)$ symmetry, by condensing elementary independent bosons $\{\phi_1-\phi_1',\phi_2+\phi_2'\}$. Therefore we have

$$[q]^{-1}=[-q],\quad\forall\,q\in\mathbb{Z}.\tag{43}$$

On the other hand, if $q_1\neq q_2$, the bosonic variable $\equiv\phi_2+\phi_2'\}$ cannot be gapped since $\cos[l(\phi_2+\phi_2')+\alpha_l']$ terms are not allowed by symmetry. Now the new variables describing the gapless edge structure can be chosen as $\{\tilde\phi_1=\phi_1,\tilde\phi_2\equiv\phi_2+\phi_2'\}$ satisfying Kac-Moody algebra (35). Notice that under $U(1)$ they transform as $\tilde W^{U_\theta}=I_{2\times2}$ and

$$\begin{pmatrix}\delta\tilde\phi_1^{U_\theta}\\\delta\tilde\phi_2^{U_\theta}\end{pmatrix}=\theta\begin{pmatrix}1\\q_1+q_2\end{pmatrix}.$$

Therefore we have the multiplication rule of the group formed by phases $[q]$ with $U(1)$ symmetry:

$$[q_1]\oplus[q_2]=[q_1+q_2].\tag{44}$$

This means phases $[q]$ labeled by different integer q's are different phases in the presence of $U(1)$ symmetry, and they form nothing but the integer group $\mathbb{Z}$! Any phase $[q]$ with $q\neq0$ corresponds to a nontrivial SPT phase, whose gapless edge states cannot be gapped without breaking $U(1)$ symmetry.

There is a simple physical reason underlying these observations [see Fig. 1(b) for an illustration]. The Hall conductance is a physical invariant that distinguishes these different phases:

$$\sigma_{xy}=(\mathbf{t}')^T\mathbf{K}^{-1}\mathbf{t}'=\begin{pmatrix}1\\q\end{pmatrix}^T\mathbf{K}\begin{pmatrix}1\\q\end{pmatrix}=2q.\tag{45}$$

C. $U(1)\rtimes Z_2^T$ symmetry: $\mathbb{Z}_2$ classes

In the presence of both "charge" $U(1)$ (group elements U_θ) and time reversal Z_2^T symmetry (generator $\boldsymbol{T}$), the extra algebraic relation in addition to Eqs. (36) and (41) is given by

$$U_{-\theta}\boldsymbol{T}=\boldsymbol{T}U_\theta$$

since the charge $U(1)$ symmetry doesn't commute with time reversal symmetry. The algebraic relations for $U(1)\rtimes Z_2^T$ are

$$\boldsymbol{T}^2=\boldsymbol{T}U_\theta\boldsymbol{T}U_\theta=\boldsymbol{e}\tag{46}$$

in addition to Eq. (41). The corresponding constraints (28) for symmetry transformations $\{W^{\boldsymbol{T}},\delta\phi^{\boldsymbol{T}}\}$ and $\{W^{U_\theta}=I_{2\times2},\delta\phi^{U_\theta}=\theta\mathbf{t}\}$ are

$$(I_{2\times2}-W^{\boldsymbol{T}})(\delta\phi^{\boldsymbol{T}}+\theta\mathbf{t})=\begin{pmatrix}0\\0\end{pmatrix}\mod 2\pi,\quad\forall\,\theta,\tag{47}$$

and Eqs. (37) and (38). Just like in the case of Z_2^T symmetry, again using gauge transformation $X=\sigma_x$ in Eq. (29) one can fix $W^{\boldsymbol{T}}=\sigma_z$ to satisfy Eq. (37). Solving Eqs. (38) and (47), we have $t_2=0$ and $\delta\phi_2^{\boldsymbol{T}}=n\pi, n=0,1$. Using gauge transformation $\Delta\phi$ in Eq. (29) one can fix $\delta\phi_1^{\boldsymbol{T}}=0$. Hence the inequivalent symmetry transformations for $U(1)\rtimes Z_2^T$ symmetry is

$$W^{U_\theta}=I_{2\times2},\quad\delta\phi^{U_\theta}=\theta\begin{pmatrix}1\\0\end{pmatrix},\tag{48}$$

$$W^{\boldsymbol{T}}=\sigma_z,\quad\delta\phi^{\boldsymbol{T}}=\begin{pmatrix}0\\n\pi\end{pmatrix},\quad n=0,1.\tag{49}$$

The symmetry-allowed independent Higgs terms are

$$\mathcal{S}^1_{\text{edge}}=\sum_{l\in\mathbb{Z}}C_l\int dxdt[\cos(l\phi_2)+\cos(l\phi_2+nl\pi)].\tag{50}$$

Apparently, they consist of only the ϕ_2 variable and hence are independent of each other. When $n=0$ all $\cos(l\phi_2), l\in\mathbb{Z}$ terms are allowed in Eq. (50) and it corresponds to the trivial phase. Variable ϕ_2 can be localized at value $\langle\phi_2\rangle=0$ and the edge excitations will be gapped without breaking any symmetry. On the other hand, when $n=1$, only $\cos(l\phi_2), l=$ even are allowed in Eq. (50) and it corresponds to the nontrivial SPT phase. If ϕ_2 is localized at any value, the edge becomes gapped. However, time reversal symmetry $\boldsymbol{T}$ will be broken, since under $\boldsymbol{T}$ we have $\phi_2\to\phi_2+\pi$.

Now let us analyze the group structure formed by these phases. We denote the two phases with $\mathbf{K} = \sigma_x$ and symmetry transformations (49) where $n = 0,1$ as [0] and [1]. $[0] = \boldsymbol{e}_{U(1)\rtimes Z_2^T}$ is the trivial phase. Consider two copies of nontrivial SPT phases [1] put together: they have edge variables $\{\phi_1,\phi_2\}$ and $\{\phi_1',\phi_2'\}$. Apparently one can always condense the pair of elementary independent bosons $\{\phi_1 - \phi_1',\phi_2 + \phi_2'\}$ on the edge, and the edge states will be fully gapped without breaking any symmetry. Therefore we have

$$[1] \oplus [1] = [0]. \tag{51}$$

Clearly, [0] and [1] form a $\mathbb{Z}_2$ group. As a result, $n = 0$ and $n = 1$ label the $\mathbb{Z}_2$ classes of bosonic nonchiral SPT phases for $U(1) \rtimes Z_2^T$ symmetry.

Note, the generator of charge $U(1)$ $U_\theta = e^{i\hat{n}\theta}$ does not commute with time reversal which involves complex conjugation which sends $i \to -i$ in the exponential. Thus $U_{-\theta}\boldsymbol{T} = \boldsymbol{T}U_\theta$, which implies time reversal and charge conjugation are combined via the semidirect product $U(1) \rtimes Z_2^T$ [see Fig. 1(c)]. However, if the $U(1)$ was associated with spin rotation about S_z, for example, of an integer spin system, the relation with time reversal would be that of a direct product $U(1) \times Z_2^T$, since now $U_\theta \boldsymbol{T} = \boldsymbol{T} U_\theta$. This completely changes the topological classification and leads to no nontrivial phases as shown in Appendix E1.

D. Z_N symmetry: $\mathbb{Z}_N$ classes

Denoting the generator of Z_N group as $\boldsymbol{g}$, the algebraic structure (26) of Z_N group is given by

$$\boldsymbol{g}^N = \boldsymbol{e}. \tag{52}$$

The corresponding constraints (28) for symmetry transformations $\{W^g,\delta\phi^g\}$ are

$$(W^g)^N = I_{2\times2}, \quad (W^g)^T \sigma_x W^g = \sigma_x, \tag{53}$$

$$\sum_{a=1}^{N} (W^g)^{a-1}\delta\phi^g = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \mod 2\pi. \tag{54}$$

In the following, we discuss the cases of N being an odd and even integers, respectively.

1. N = odd integer: $\mathbb{Z}_N$ classes

It is straightforward to check that the only solution to Eq. (53) is $W^g = I_{2\times2}$. So the solutions to Eq. (54) have the general form of $\delta\phi^g = \frac{2\pi k}{N}\mathbf{t}$, where $\mathbf{t} = \binom{t_1}{t_2}$ and $(t_1,t_2) = (k,N) = 1$ for $k,t_1,t_2 = 0,1,\ldots,N-1$. Here, we require $(t_1,t_2) = (k,N) = 1$ so that the transformations (55) form a faithful representation of symmetry group $G = Z_N$. Making use of theorem (C1), we can always reduce an arbitrary "charge vector" $\mathbf{t}$ with $(t_1,t_2) = 1$ to its standard form $\binom{1}{q}$ and hence the inequivalent symmetry transformations $\{W^g,\delta\phi^g\}$ for $Z_N, N = odd$ symmetry are

$$W^g = I_{2\times2}, \quad \delta\phi^g = \frac{2\pi k}{N}\begin{pmatrix} 1 \\ q \end{pmatrix}, \quad (k,N) = 1, \quad q = 0,1,\ldots,N-1. \tag{55}$$

It is easy to show that the Higgs terms allowed by symmetry do not depend on k and they are

$$\mathcal{S}^1_{\text{edge}} = \sum_{l_1+ql_2=0 \text{ mod } N} C_{\mathbf{l}} \int dx dt \cos(\mathbf{l}^T\phi + \alpha_{\mathbf{l}}). \tag{56}$$

Notice that a Higgs term labeled by vector $\mathbf{l}$ is allowed only if $l_1 + ql_2 = 0 \mod N$. Apparently, when $q = 0$, this is a trivial phase with a set of independent Higgs terms being $\sum_{l\in\mathbb{Z}} C_l \int dxdt \cos(l\phi_2 + \alpha_l)$, and the variable ϕ_2 can be localized at any value without breaking the Z_N symmetry. Since for different k values in transformations (55), the symmetry-allowed Higgs terms are exactly the same, we believe different k values correspond to the same phase and we will assume a representative $k = 1$ in Eq. (55) hereafter.

To analyze the group structure of these states, let us denote the various phases with symmetry transformations (55) under Z_N symmetry by

$$[q] \equiv \Psi_{Z_N}\left[\sigma_x, \left\{W^g = I_{2\times2}, \delta\phi^g = \frac{2\pi}{N}\begin{pmatrix} 1 \\ q \end{pmatrix}\right\}\right], \tag{57}$$

where $[0] = [N] = \boldsymbol{e}_{Z_N}$ is the trivial phase. Again, consider two states $[q_1]$ with edge variables $\{\phi_1,\phi_2\}$ and $[q_2]$ with edge variables $\{\phi_1',\phi_2'\}$. Completely in parallel with the discussions for $U(1)$ symmetry, it is straightforward to show that

$$[q]^{-1} = [N-q], \; [q_1] \oplus [q_2] = [q_1 + q_2 \text{ mod } N]. \tag{58}$$

Therefore different phases $[q]$ with $q = 0,1,\ldots,N-1$ form a $\mathbb{Z}_N$ group. There are $\mathbb{Z}_N$ classes of different phases labeled by $q = 0,1,\ldots,N-1$ in the presence of Z_N symmetry, when N = odd.

2. N = even integer: $\mathbb{Z}_N$ classes

Now the inequivalent solutions to Eq. (53) are $W^g = \pm I_{2\times2}, \pm\sigma_x$. (i) For $W^g = I_{2\times2}$, we have exactly the same solutions as Eq. (55) in N = odd case, and hence $\mathbb{Z}_N$ different classes of bosonic nonchiral SPT phases. All these $\mathbb{Z}_N$ phases can be realized by coupled wire construction, as will be discussed in Sec. VI. (ii) For $W^g = -I_{2\times2}$, one can always choose a gauge $\Delta\phi$ so that $\delta\phi^g = 0$. The symmetry-allowed Higgs terms are $\sum_{\forall\, \mathbf{l}} \cos(\mathbf{l}^T\phi)$ and this describes nothing but the trivial phase as $[q = 0]$ in Eq. (55). Its edge states can be gapped out without breaking any symmetry.

3. N = even integer: Other solutions

We discuss below additional representations of the symmetry group that appear for this particular case, which we believe are unphysical for a SRE phase with no ground state degeneracy on a torus.[79] These require interchanging the two edge fields ϕ_1,ϕ_2, which have very different character when they describe fundamental bosons (one is like the phase field, and is compact, while the other is related to the integrated density). Therefore we believe it is unphysical to exchange them. Also, unlike for the other symmetry transformations, a microscopic model with this realization of symmetries was not found. Finally, these additional phases are not naturally accommodated into a group structure. These points taken together lead us to drop them from the final list of topological phases with this symmetry.

(iii) For $W^g = \sigma_x$, the gauge inequivalent solutions to (54) are $\delta\phi^g = \frac{2\pi k}{N}\binom{1}{1}$ with $k = 0,1,\ldots,N-1$. We require $(k,N/2) = 1$ so that these transformations $\{W^g,\delta\phi^g\}$ form a faithful representation of symmetry group Z_N, $N =$ even. The symmetry-allowed Higgs terms are

$$\mathcal{S}^1_{\text{edge}} = \sum_{2k(l_1+l_2)=0 \mod N} C_{\mathbf{l}} \int dxdt \times \left[\cos(\mathbf{l}^T\phi + \alpha_{\mathbf{l}}) + \cos\left(l_1\phi_2 + l_2\phi_1 + \frac{2\pi k(l_1+l_2)}{N} + \alpha_{\mathbf{l}}\right)\right]. \quad (59)$$

One can verify that they all corresponds to nontrivial SPT phases, whose edge states cannot be gapped without breaking the symmetry. More precisely, variables ϕ_1 and ϕ_2 cannot be localized simultaneously since they do not commute, and if only one variable (say ϕ_1) is localized the symmetry $\boldsymbol{g}$ will be broken since $\phi_1 \leftrightarrow \phi_2 + \frac{2\pi k}{N}$. Their symmetry transformations are summarized as

$$W^g = \sigma_x, \quad \delta\phi^g = \frac{2\pi k}{N}\begin{pmatrix}1\\1\end{pmatrix}, \quad \left(k,\frac{N}{2}\right) = 1. \quad (60)$$

But it is not clear how to realize these phases in a microscopic model or what group structure they form. We label these phases as $[\sigma_x,k] \equiv \Psi_{Z_N}[\sigma_x,\{W^g = \sigma_x,\delta\phi^g = \frac{2\pi k}{N}\binom{1}{1}\}]$.

(iv) For $W^g = -\sigma_x$, the gauge inequivalent "faithful" solutions to (54) are $\delta\phi^g = \frac{2\pi k}{N}\binom{1}{-1}$ with $k = 0,1,\ldots,N-1$ and $(k,N/2) = 1$. The symmetry-allowed Higgs terms are

$$\mathcal{S}^1_{\text{edge}} = \sum_{2k(l_1-l_2)=0 \mod N} C_{\mathbf{l}} \int dxdt \times \left[\cos(\mathbf{l}^T\phi + \alpha_{\mathbf{l}}) + \cos\left(-l_1\phi_2 - l_2\phi_1 + \frac{2\pi k(l_1-l_2)}{N} + \alpha_{\mathbf{l}}\right)\right]. \quad (61)$$

If we label these phases as $[-\sigma_x,k] \equiv \Psi_{Z_N}[\sigma_x,\{W^g = -\sigma_x,\delta\phi^g = \frac{2\pi k}{N}\binom{1}{-1}\}]$, it is easy to see that

$$[\sigma_x,k]^{-1} = [-\sigma_x,k]. \quad (62)$$

This is because once a $[\sigma_x,k]$ state with edge variable $\{\phi_1,\phi_2\}$ and a $[-\sigma_x,k]$ state with edge variable $\{\phi_1',\phi_2'\}$ are put together, one can always condense bosons $\{\phi_1+\phi_1',\phi_2-\phi_2'\}$, and the edge will be gapped without breaking Z_N symmetry.

To summarize, no matter $N =$ odd or $N =$ even, there are $\mathbb{Z}_N$ different bosonic nonchiral phases in the presence of Z_N symmetry. They are characterized by different symmetry operations (55) associated with Z_N generator $\boldsymbol{g}$ and symmetry-allowed Higgs terms (56). All these $\mathbb{Z}_N$ phases can be realized in a coupled-wire construction as will be shown in Sec. VI. Besides, when $N =$ even, there are extra solutions (60) to constraint (54) for symmetry transformations associated with Z_N symmetry. However, the physical realization of these states and their group structure are not clear.

E. $Z_N \rtimes Z_2^T$ symmetry

The generators of $Z_N \rtimes Z_2^T$ symmetry group are $\boldsymbol{g}$ for Z_N and $\boldsymbol{T}$ for Z_2^T satisfying the following algebra:

$$\boldsymbol{g}^N = \boldsymbol{T}^2 = \boldsymbol{T}\boldsymbol{g}\boldsymbol{T}\boldsymbol{g} = \boldsymbol{e}. \quad (63)$$

The associated constraints on symmetry operations are

$$W^g W^T W^g W^T = I_{2\times2}, \qquad (I_{2\times2} - W^g W^T)(\delta\phi^g + W^g\delta\phi^T) = \begin{pmatrix}0\\0\end{pmatrix} \mod 2\pi, \quad (64)$$

in addition to Eqs. (37), (38), (53), and (54).

1. $N =$ odd integer: $\mathbb{Z}_1$ class

The gauge inequivalent solutions to these constraint equations are Eq. (39) and

$$W^g = I_{2\times2}, \frac{2\pi k}{N}\begin{pmatrix}1\\0\end{pmatrix}, \quad (k,N) = 1. \quad (65)$$

Let us label these phases by $[n_2,k]$, where n_2 is defined in Eq. (39). Notice that when $n_2 = 0$ in Eq. (39) one can always destroy the gapless edge excitations by the localizing variable ϕ_2 without breaking any symmetry (under $\boldsymbol{T}$ we have $\phi_2 \to \phi_2 + n_2\pi$). Similarly, when $k = 0$ in Eq. (65), the edge can be gapped out by the localizing bosonic variable ϕ_1. So $n_2 = 0$ or $k = 0$ both correspond to the trivial phase. On the other hand, when $n_2 = 1$, the symmetry-allowed Higgs terms are

$$\mathcal{S}^1_{\text{edge}} = \sum_{\mathbf{l}} C_{\mathbf{l}} \int dxdt[\cos(Nl_1\phi_1 + l_2\phi_2 + \alpha_{\mathbf{l}}) + \cos(-Nl_1\phi_1 + l_2\phi_2 + \alpha_{\mathbf{l}} + l_2 n_2\pi)]. \quad (66)$$

At first sight, it seems the edge states cannot be gapped without breaking the symmetry, i.e., neither ϕ_1 nor ϕ_2 can be localized due to symmetry. However, when a state $[n_2 = 1,k \neq 0]$ with edge variable $\{\phi_1,\phi_2\}$ is put together with a trivial state $[1,0] = \boldsymbol{e}_{Z_N\rtimes Z_2^T}$ with edge variable $\{\phi_1',\phi_2'\}$, the edge can be fully gapped by condensing bosons $\{N\phi_1+\phi_1',\phi_2-N\phi_2'\}$ without breaking the Z_N symmetry ($N =$ odd). Therefore $[1,k] \oplus [1,0] = \boldsymbol{e}_{Z_N\rtimes Z_2^T}$ and $[1,k]$ are all trivial phases, which, in general, does not have gapless edge states. As a result, there is no nontrivial SPT phases in the presence of symmetry $Z_N \rtimes Z_2^T, N =$ odd.

2. $N =$ even integer: Minimal set, $\mathbb{Z}_2^2$ classes

When $N =$ even, we always have $W^g = \pm I_{2\times2}$. (i) For $W^g = I_{2\times2}$, the gauge inequivalent faithful solutions to the constraint equations are Eq. (39) and

$$\delta\phi^g = \pi\begin{pmatrix}2k/N\\n\end{pmatrix}, \quad (k,N/2) = 1, \quad n = 0,1. \quad (67)$$

Let us label various phases with symmetry transformations (39) and (67) as $[k,n_2,n]$ where $n_2 = 0,1$ is defined in Eq. (39). When $k = 0$, variable ϕ_1 can be localized without breaking any symmetry and it is the trivial SPT phase. When $n_2 = n = 0$, the variable ϕ_2 can be localized and it is the trivial phase again. Therefore

$$\boldsymbol{e}_{Z_N\rtimes Z_2^T} = [0,n_2,n] = [k,0,0] \quad (68)$$

for $N=$ even. In the following, we analyze the group structure formed by these states.

Following discussions in Sec. III, we can obtain the inverse of a phase by merely changing the sign of its K matrix. Now let us put together a state $[k,n_2,n]$ with edge variable $\{\phi_1,\phi_2\}$ is put together with a state $[k',n_2',n']^{-1}$ with edge variable $\{\phi_1',\phi_2'\}$, we can condense the following independent bosonic variables $\{k'\phi_1-k\phi_1',k\phi_2-k'\phi_2'\}$ and destroy the gapless edge states if $(k,k')=1$. The associated Higgs terms will not break the $Z_N\rtimes Z_2^T$ symmetry if $k'n-kn'=0$ mod 2 and $kn_2-k'n_2'=0$ mod 2. As a result, $[k,n_2,n]\oplus[k',n_2',n']^{-1}=\boldsymbol{e}_{Z_N\rtimes Z_2^T}$

$$\text{for}\quad (k,k')=1:[k,n_2,n]=[k',n_2',n'],$$
$$\text{if}\quad k'n-kn'=0 \text{ mod } 2,\quad kn_2-k'n_2'=0\text{ mod } 2.$$

Therefore we have $[2k+1,n_2,n]=[1,n_2,n]$. For $k=$ even on the other hand, we know that $N/2$ must be odd since $(k,N/2)=1$ for a faithful representation. Then we can choose $k'=0$ and condense independent bosons $\{\frac{N}{2}\phi_1-\phi_1',\phi_2-\frac{N}{2}\phi_2'\}$ to destroy all edge state. No symmetry will be broken by doing so. Hence we showed $[2k,n_2,n]=[0,n_2,n]=\boldsymbol{e}_{Z_N\rtimes Z_2^T}$. Consequently, the only three nontrivial SPT phases are $[1,1,0]$, $[1,0,1]$ and $[1,1,1]$.

Similarly, by putting together a state $[1,n_2,n]$ with edge variable, $\{\phi_1,\phi_2\}$ is put together with a state $[1,n_2',n']$ with edge variable $\{\phi_1',\phi_2'\}$, we can always localize bosonic variable $\phi_1-\phi_1'$ and gap out part of the edge. What is left on the edge is described by variables $\{\tilde\phi_1=\phi_1,\tilde\phi_2=\phi_2+\phi_2'\}$. They obey Kac-Moody algebra (35) and transform as a $[1,n_2+n_2',n+n']$ state. Hence we have shown that

$$[1,n_2,n]\oplus[1,n_2',n']=[1,n_2+n_2',n+n']. \tag{69}$$

Since $n,n_2=0,1$ are both $\mathbb{Z}_2$ integers, so clearly all different 4 states $[1,n_2,n]$ form a $\mathbb{Z}_2^2$ group. Consequently, there are three nontrivial SPT phases labeled by $k=1$ and $[n_2,n]=[0,1],[1,0]$ or $[1,1]$ in Eqs. (39) and (67).

(ii) For $W^g=-I_{2\times2}$ we can always choose a gauge in Eq. (29) so that $\delta\phi^g=\binom{0}{0}$. From constraint equations, one can derive $W^{\boldsymbol{T}}=\sigma_z$ and $(I_{2\times2}\pm W^{\boldsymbol{T}})\delta\phi^{\boldsymbol{T}}=\binom{0}{0}$ and we have

$$\delta\phi^{\boldsymbol{T}}=\begin{pmatrix}n_1\\n_2\end{pmatrix}\pi,\quad n_1,n_2=0,1. \tag{70}$$

Apparently, if $n_1=0$, variable ϕ_1 can be localized without breaking any symmetry, and similarly, if $n_2=0$, variable ϕ_2 can be localized without breaking the symmetry. For the nontrivial SPT phase with $n_1=n_2=1$, the edge states cannot be destroyed without breaking any symmetry. If we label this SPT phase by $[n_1=1,n_2=1]$, one can show that the group structure formed by states with symmetry transformations $W^g=-I$ is the integer group $\mathbb{Z}$, i.e., $\{[1,1]^n,n\in\mathbb{Z}\}$. However, the above symmetry transformations $\{W^g=-I_{2\times2},\delta\phi^g=\binom{0}{0}\}$ do not correspond to a faithful representation of $Z_N\rtimes Z_2^T$ group for $N=$ even unless $N=2$. And it is not clear whether the states with symmetry transformations $W^g=-I$ can be realized in a physical bosonic system. Therefore we will not include the states with symmetry transformations $W^g=-I$ in the minimal set of topological phases with $Z_N\rtimes Z_2^T$ symmetry.

To summarize, there are $\mathbb{Z}_2^2$ classes of different nonchiral bosonic SRE phases (including one trivial phase and three SPT phases) with $Z_N\rtimes Z_2^T$ symmetry when $N=$ even. To compare, the classification and analysis of bosonic SPT phases with $Z_N\times Z_2^T$ symmetry (the *direct* product of Z_N and Z_2^T in contrast to the semidirect product discussed here) is shown in Appendix E2.

V. *K*-MATRIX CLASSIFICATION OF FERMIONIC SPT PHASES

According to theorem (B3) in Appendix B, a $2\times2K$ matrix with determinant -1 for a fermionic system is always equivalent to the standard form $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)\simeq\sigma_z$ by certain $GL(2,\mathbb{Z})$ transformations. In the following, we always choose the 2×2 matrix $\mathbf{K}=\sigma_z$ to represent a generic fermionic nonchiral SRE state. In the following, we use general principles discussed in Sec. III to study nonchiral fermionic SPT phases with different symmetries. Note that the only $GL(2,\mathbb{Z})$ transformations $\mathbf{X}$ that keeps $\mathbf{K}=\sigma_z$ invariant under Eq. (8) are $\mathbf{X}=\pm I_{2\times2},\ \pm\sigma_z$. For such a nonchiral fermionic SRE phase, its "bare" Chern-Simons effective theory with no Higgs terms added is

$$\mathcal{L}_{\mathbf{K}}=\frac{1}{4\pi}\epsilon^{\mu\nu\lambda}\left(a_\mu^1\partial_\nu a_\lambda^1-a_\mu^2\partial_\nu a_\lambda^2\right)-\sum_{I=1}^{2}a_\mu^I j_I^\mu \tag{71}$$

in the bulk and

$$\mathcal{S}_{\text{edge}}^0=\frac{1}{4\pi}\int dtdx\left(\partial_t\phi_1\partial_x\phi_1-\partial_t\phi_2\partial_x\phi_2-\sum_{I,J}V_{I,J}\partial_x\phi_I\partial_x\phi_J\right) \tag{72}$$

on the edge where $V_{I,J}$ is a positive definite constant matrix, as discussed in Sec. II B. The Kac-Moody algebra satisfied by fields $\{\phi_1,\phi_2\}$ on the edge is

$$[\partial_x\phi_1(x),\partial_y\phi_1(y)]=-[\partial_x\phi_2(x),\partial_y\phi_2(y)]=2\pi\mathrm{i}\partial_x\delta(x-y),$$
$$[\partial_x\phi_1(x),\partial_y\phi_2(y)]=0. \tag{73}$$

In the absence of any symmetry, a set of independent Higgs terms satisfying Eqs. (19) and (20) on the edge can be chosen as either $\{C_l\cos(l\phi_1+l\phi_2+\alpha_l),l\in\mathbb{Z}\}$ or $\{C_l\cos(l\phi_1-l\phi_2+\alpha_l),l\in\mathbb{Z}\}$. All degrees of freedom on the edge will be gapped once the "bosonic" variable $\phi_1+\phi_2$ (or $\phi_2-\phi_1$) is localized at a classical value by the Higgs terms. Again, the two bosonic variables $\phi_1+\phi_2$ and $\phi_1-\phi_2$ cannot be localized simultaneously, according to Heisenberg uncertainty relation implied by Kac-Moody algebra (73).

There is an intrinsic difference between fermions and bosons: i.e., only bosonic quasiparticles can "condense" in a bosonic/fermionic system described by a local Hamiltonian. This means in a bosonic system, all quasiparticles are bosons and should transform trivially under identity element $\boldsymbol{e}=\prod_a\boldsymbol{g}_a^{n_a}$ of group G as shown in Eq. (28). In a fermionic system, on the other hand, any bosonic quasiparticle consists of an even number of fermions and is always invariant if every fermion creation (annihilation) operator obtains a minus sign. Therefore under symmetry transformation $\{W^{\boldsymbol{e}},\delta\phi^{\boldsymbol{e}}\}$ corresponding to identity element $\boldsymbol{e}$ of the same symmetry group G in any local fermionic system, only those Higgs terms satisfying condition (11) and (12) should transform trivially. This can be generalized to a more universal situation where anyonic quasiparticles are present ($|\det \mathbf{K}| > 1$): under identity element of group G only local operators (i.e., Higgs terms $\cos(\mathbf{l}^T\phi + \alpha_\mathbf{l})$ satisfying (11) and (12) which condense bosonic quasiparticles) should transform trivially. This means with the same symmetry group $G = \{g_a\}$, the symmetry transformations $\{W_b^{g_a}, \delta\phi_b^{g_a}\}$ of a bosonic SRE state form a faithful representation of group G, while symmetry transformations $\{W_f^{g_a}, \delta\phi_f^{g_a}\}$ of a fermionic SRE state (or more generally a gapped Abelian phase containing fermionic and anyonic quasiparticles) form a *projective representation*[77] of group $G = \{g_a\}$. And for these systems the identity element $\boldsymbol{e}$ in group compatibility conditions (26) and (27) does not always correspond to a trivial transformation on the fermionic (anyonic) quasiparticles.

In a fermionic nonchiral SRE phase with $\mathbf{K} = \sigma_z$ here, it is easy to verify that such a bosonic quasiparticle is labeled by any vector $\mathbf{l} = \binom{l_1}{l_2}$ satisfying

$$l_1 = l_2 \bmod 2. \tag{74}$$

Locality requires that only Higgs terms $\cos(\mathbf{l}^T\phi + \alpha_\mathbf{l})$ satisfying Eq. (74) can be added to bare action (71) and (72), i.e., fermions are not allowed to condense. Hence in the absence of any symmetry, all these Higgs terms $\cos(\mathbf{l}^T\phi + \alpha_\mathbf{l})$ satisfying Eq. (74) are allowed and should be added to a fermionic nonchiral SRE state. As a result, the identity element $\boldsymbol{e}$ (no symmetry)in a fermionic system is implemented by the following generic form of symmetry transformations:

$$W^{\boldsymbol{e}} = I_{2\times 2}, \quad \delta\phi^{\boldsymbol{e}} = \eta_f \pi \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{75}$$

where $\eta_f = 0,1$. Notice that the above symmetry transformations are invariant under any gauge transformation (29). When $\eta_f = 1$, the fermionic operators $\sim\exp[\mathrm{i}\phi_\alpha]$, $\alpha = 1,2$ obtains a minus sign, corresponding to the *fermion number parity* operation $\boldsymbol{P}_f = (-1)^{\hat{N}_f}$. When $\eta_f = 0$, on the other hand, every fermion remains invariant and it corresponds to the actual identity element $\boldsymbol{e}_f$ of the symmetry group G_f (including $\boldsymbol{P}_f$) for the underlying fermions. If we incorporate the fermion number parity $\boldsymbol{P}_f$ into the symmetry group G_f, one easily notices that $Z_2^f = \{\boldsymbol{e}_f, \boldsymbol{P}_f\}$ is always a normal subgroup of fermion symmetry group G_f, which means $\boldsymbol{P}_f$ is involutory ($\boldsymbol{P}_f{}^2 = \boldsymbol{e}_f$) and central in G_f.[44]

If we always incorporate fermion number parity $\boldsymbol{P}_f$ into fermion symmetry group G_f, then a fermionic system with symmetry G_f is naturally related to a bosonic system with symmetry $G = G_f/Z_2^f$ (since Z_2^f is a normal subgroup of G_f the quotient group G_f/Z_2^f can be defined). Physically, this means if fermions with symmetry G_f pair up to form Cooper pairs (which are bosons), these bosonic Cooper pairs have symmetry $G = G_f/Z_2^f$. Different SPT phases of bosonic Cooper pairs (where fermions are confined) with symmetry $G = G_f/Z_2^f$ are necessarily different fermionic SRE phases with symmetry G_f. Hence the different classes of fermionic SRE phases with symmetry G_f must contain all different bosonic SRE phases with symmetry $G = G_f/Z_2^f$ as a subset.[25] To be more precise, the group $H_b(G_f/Z_2^f)$ formed by different bosonic (nonchiral) SRE phases with symmetry $G = G_f/Z_2^f$ is always a subgroup of $H_f(G_f)$, the group formed by different fermionic (nonchiral) SRE phases with symmetry G_f.

Before discussing specific examples of nonchiral topological phases, we point out that SRE *chiral* phases of fermions are readily obtained (e.g., integer quantum Hall states) so we skip their discussion. It should also be noted that our formalism currently is restricted to topological phases in which the gapless edge, when present, has integer central charge (e.g., $c = 1$ in many cases). So chiral and nonchiral Majorana modes (e.g., of a $p_x + ip_y$ superconductor, or the $D = 2 + 1$ class DIII topological superconductor,[18,19] with a pair of counter-propagating Majorana modes) are not captured in our formulation.

A. $G_f/Z_2^f = \{\boldsymbol{e}\} \Rightarrow G_f = Z_2^f$ symmetry: $\mathbb{Z}_1$ class

If we choose $G = G_f/Z_2^f = \{\boldsymbol{e}\}$ to be the trivial group, then the fermion symmetry group is $G_f = Z_2^f$. The generator of Z_2^f, i.e., fermion number parity operator, is

$$\boldsymbol{P}_f \equiv (-1)^{\hat{N}_f}, \tag{76}$$

where $\hat{N}_f$ denotes the total fermion number. The existence of Z_2^f symmetry is a basic requirement for any fermionic system described by a *local* Hamiltonian. Simply speaking, $\boldsymbol{P}_f$ guarantees that one single fermion cannot condense like the bosons: only a bosonic conglomerate containing an even number of fermions can condense and obtain a nonvanishing expectation value. A general form for such a bosonic quasiparticle in a fermionic system is labeled by a integer vector $\mathbf{l}$ satisfying condition (11).

This means Z_2^f is more like a constraint for a fermionic system due to locality, rather than a true "symmetry." As discussed earlier, it is implemented by nothing but the nontrivial ($n_f = 1$) realization for $\boldsymbol{e}$ in $G = G_f/Z_2^f$:

$$W^{\boldsymbol{P}_f} = I_{2\times 2}, \quad \delta\phi^{\boldsymbol{P}_f} = \pi \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{77}$$

It guarantees that in the absence of any symmetry, all Higgs terms $\cos(\mathbf{l}^T\phi + \alpha_\mathbf{l})$ satisfying Eq. (74) can be added to a fermionic nonchiral SRE state and these are the only terms that can be added. Notice that Eq. (77) is invariant under any gauge transformations (29) for fermions with $\mathbf{K} = \sigma_z$. Apparently, we have $\boldsymbol{P}_f{}^2 = \boldsymbol{e}_f$, i.e., the fermion number parity acting twice would yield the identity operation for fermions.

The Z_2^f symmetry-allowed Higgs terms are all terms associated with bosonic quasiparticles (74):

$$\mathcal{S}^1_{\text{edge}} = \sum_{\{l_1 = l_2 \bmod 2\}} C_\mathbf{l} \int dx\,dt \cos(\mathbf{l}^T\phi + \alpha_\mathbf{l}). \tag{78}$$

Apparently the bosonic variable $\phi_1 + \phi_2$ (or $\phi_1 - \phi_2$) can be localized at a classical value and the edge will be gapped. Physically, $\phi_1 + \phi_2$ corresponds to the pairing between right mover and left mover, while $\phi_1 - \phi_2$ is backscattering between right and left movers. They are both allowed in the absence of any symmetry. This describes the (trivial) $\mathbb{Z}_1$ class of nonchiral fermionic phase with Z_2^f symmetry.

Since fermion number parity $\boldsymbol{P}_f$ is always realized by Eq. (75), in the following, we will not specifically mention this symmetry but only requires Z_2^f to be a normal subgroup of the full symmetry group G_f of fermions. And we use $\boldsymbol{e}$ to denote the identity element in the "bosonic" symmetry group $G = G_f/Z_2^f$. Therefore in the fermion system $\boldsymbol{e}$ can be either $\boldsymbol{e}_f$ (all fermion operators keep invariant) or $\boldsymbol{P}_f$ (all fermion operators change sign).

B. $G_f/Z_2^f = Z_2^T \Rightarrow G_f = Z_2^T \times Z_2^f$ symmetry: $\mathbb{Z}_1$ class

In the presence of time-reversal Z_2^T symmetry with generator $\boldsymbol{T}$, the algebraic structure of full symmetry group $G_f/Z_2^T = Z_2^T$ is given by

$$\boldsymbol{T}^2 = \boldsymbol{e}. \tag{79}$$

Here, in our notation, $\boldsymbol{e}$ can be either $\boldsymbol{e}_f$, the identity element for fermions or $\boldsymbol{P}_f$, the fermion number parity operation. From Eq. (27), this leads to the following constraint on symmetry transformations $\{W^{\boldsymbol{T}},\delta\phi^{\boldsymbol{T}}\}$:

$$(W^{\boldsymbol{T}})^2 = I_{2\times2}, \quad (W^{\boldsymbol{T}})^T \mathbf{K} W^{\boldsymbol{T}} = -\mathbf{K}; \tag{80}$$

$$(I_{2\times2} - W^{\boldsymbol{T}})\delta\phi^{\boldsymbol{T}} = \eta_{\boldsymbol{T}}\pi\begin{pmatrix}1\\1\end{pmatrix} \bmod 2\pi, \tag{81}$$

$$(I_{2\times2} - W^{\boldsymbol{T}})(\delta\phi^{\boldsymbol{T}} + \delta\phi^{\boldsymbol{P}_f}) = \eta_{\boldsymbol{T}\boldsymbol{P}_f}\pi\begin{pmatrix}1\\1\end{pmatrix} \bmod 2\pi, \tag{82}$$

where $\eta_{\boldsymbol{T}},\eta_{\boldsymbol{T}\boldsymbol{P}_f} = 0,1$. Notice that fermion number parity symmetry $\boldsymbol{P}_f$ is always implemented by Eq. (77), independent of the gauge choice. Here, with $\mathbf{K} = \sigma_z$ the gauge inequivalent solutions to Eq. (80) is $W^{\boldsymbol{T}} = \sigma_x$. Then solving Eqs. (81) and (82), we get $\eta_{\boldsymbol{T}} = \eta_{\boldsymbol{T}\boldsymbol{P}_f}$ and $\delta\phi^{\boldsymbol{T}} = (\begin{smallmatrix}0\\ \eta_{\boldsymbol{T}}\pi\end{smallmatrix})$. Therefore the inequivalent symmetry transformations for $Z_2^T \times Z_2^f$ group is Eq. (77):

$$W^{\boldsymbol{T}} = \sigma_x, \quad \delta\phi^{\boldsymbol{T}} = \eta_{\boldsymbol{T}}\begin{pmatrix}0\\ \pi\end{pmatrix}, \quad \eta_{\boldsymbol{T}} = 0,1. \tag{83}$$

And the symmetry-allowed Higgs terms are

$$\mathcal{S}^1_{\text{edge}} = \sum_{\{l_1 = l_2 \bmod 2\}} C_{\mathbf{l}} \int dx dt [\cos(\mathbf{l}^T\phi + \alpha_{\mathbf{l}}) + \cos(-\mathbf{l}^T\sigma_x\phi + l_2\eta_{\boldsymbol{T}}\pi + \alpha_{\mathbf{l}})], \tag{84}$$

where $\eta_{\boldsymbol{T}} = 0,1$. It turns out the above Higgs term always describes the same trivial SPT phase no matter $\eta_{\boldsymbol{T}} = 0$ or 1, e.g., variable $\phi_1 + \phi_2$ can be always localized at an expectation value $\langle\phi_1 + \phi_2\rangle = \eta_2\pi/2$ by the Higgs terms, and the gapless edge states will be destroyed without breaking any symmetry. Therefore Eq. (72) together with Eq. (84) describes the (trivial) $\mathbb{Z}_1$ class of fermionic nonchiral SPT with $Z_2^T \times Z_2^f$ symmetry, no matter $\boldsymbol{T}^2 = \boldsymbol{e}_f$ ($\eta_{\boldsymbol{T}} = \eta_{\boldsymbol{T}\boldsymbol{P}_f} = 0$) or $\boldsymbol{T}^2 = \boldsymbol{P}_f$ ($\eta_{\boldsymbol{T}} = \eta_{\boldsymbol{T}\boldsymbol{P}_f} = 1$). Note, the Z_2 classification of free fermions in class DIII with these symmetries is missed by this classification. The reason of course is that we are not able to describe Majorana modes (the class DIII topological superconductor has a counter-propagating pair of Majorana modes with central charge $c = 1/2$) within the K-matrix formulation.

C. $G_f/Z_2^f = U(1) \rtimes Z_2^T$ symmetry

By labeling the $U(1)$ group elements as U_θ, the algebraic structure of $G = U(1) \rtimes Z_2^T$ group is given by

$$\boldsymbol{T}^2 = \boldsymbol{T}U_\theta\boldsymbol{T}U_\theta = U_{(\theta=0 \bmod 2\pi)} = \boldsymbol{e} \tag{85}$$

in addition to Eq. (41). Again, here in our notation, $\boldsymbol{e}$ can be either identity element $\boldsymbol{e}_f$ for fermions or fermion number parity $\boldsymbol{P}_f$. Again, a general form of symmetry transformation for U_θ is given by

$$W^{U_\theta} = I_{2\times2}, \quad \delta\phi^{U_\theta} = \theta\mathbf{t}, \tag{86}$$

and the we have the following constraints for the symmetry transformations:

$$2\pi\mathbf{t} = \eta_{U(1)}\pi\begin{pmatrix}1\\1\end{pmatrix} \bmod 2\pi, \tag{87}$$

$$(I_{2\times2} - W^{\boldsymbol{T}})(\theta\mathbf{t} + \delta\phi^{\boldsymbol{T}}) = \eta\pi\begin{pmatrix}1\\1\end{pmatrix} \bmod 2\pi, \tag{88}$$

in addition to Eqs. (80) and (81). The last line in Eq. (85) is automatically satisfied. The gauge inequivalent solutions to these constraint equations are

$$\mathbf{t} = \left[t + \frac{\eta_{U(1)}}{2}\right]\begin{pmatrix}1\\1\end{pmatrix}, \quad t \in \mathbb{Z}; \tag{89}$$

$$W^{\boldsymbol{T}} = \sigma_x, \quad \delta\phi^{\boldsymbol{T}} = \begin{pmatrix}0\\ \eta\pi\end{pmatrix} \bmod 2\pi, \tag{90}$$

where $\eta_{U(1)},\eta = 0,1$. Notice that the fermion parity generator $\boldsymbol{P}_f$ is always a subgroup of $U(1)$ since $t + \eta_{U(1)}/2 \neq 0$. Hence Z_2^f is always a subgroup of $U(1)$ group associated with fermion number conservation. If $\eta = 1$, we have $\boldsymbol{T}^2 = \boldsymbol{P}_f$, while $\eta = 0$ corresponds to $\boldsymbol{T}^2 = 1$.

1. $G_f = U(1) \rtimes Z_2^T$ with $\boldsymbol{T}^2 = 1$: $\mathbb{Z}_2$ classes

When $\eta = 0$, the algebra of symmetry group $G_f = U(1) \rtimes Z_2^T$ is

$$\boldsymbol{T}^2 = \boldsymbol{T}U_\theta\boldsymbol{T}U_\theta = U_{(\theta=0 \bmod 2\pi)} = \boldsymbol{e}_f, \tag{91}$$

where $\boldsymbol{e}_f$ is the identity element of symmetry group G_f for fermions. The symmetry-allowed Higgs terms associated with symmetry transformations (86) and (89) are

$$\mathcal{S}^1_{\text{edge}} = \sum_{l\in\mathbb{Z}} C_l \int dx dt \cos[l(\phi_1 - \phi_2) + \alpha_l]$$

for $\eta = 0$. Hence in $\eta = 0$ case, there is only one trivial phase, since the independent bosonic variable $\phi_1 - \phi_2$ can be localized at a classical value by the Higgs terms and the edge will be gapped without breaking any symmetry (under $\boldsymbol{T}$ we have $\phi_1 - \phi_2 \to \phi_1 - \phi_2 - \eta\pi$). Meanwhile, notice that for a bosonic system with $G = U(1) \rtimes Z_2^T$ symmetry ($\boldsymbol{T}^2 = 1$), there are $\mathbb{Z}_2$ classes of different phases. Hence the nontrivial bosonic SPT phase of Cooper pairs (fermions are confined) protected by $U(1) \rtimes Z_2^T$ symmetry form a nontrivial SPT phase of fermions with $G_f = U(1) \rtimes Z_2^T$ symmetry. As a result, there are $\mathbb{Z}_2$ classes of different fermionic (nonchiral) SRE phases in the presence of $U(1) \rtimes Z_2^T$ symmetry with

$T^2 = 1$. The $\mathbb{Z}_2$ classification comes purely from the bosonic SPT phases (bosonic QSH insulator) of Cooper pairs in the molecule limit where fermions are confined.

2. $G_f = U(1) \rtimes Z_2^T$ with $T^2 = P_f$: $\mathbb{Z}_2^2$ classes

When $\eta = 1$, the algebra of symmetry group $G_f = U(1) \rtimes Z_2^T$ is

$$T^2 = TU_\theta TU_\theta = U_{(\theta=0 \bmod 2\pi)} = P_f. \tag{92}$$

And the symmetry-allowed Higgs terms on the edge are

$$\mathcal{S}^1_{\text{edge}} = \sum_{l\in\mathbb{Z}} C_l \int dxdt \cos[2l(\phi_1 - \phi_2) + \alpha_l]$$

for $\eta = 1$. This corresponds to the nontrivial SPT phase, whose edge cannot be gapped without breaking the symmetry. We use $[\eta]$ with $\eta = 1$ and $\eta = 1$ to label these two phases. Now let us examine the group structure $\{\Psi_{U(1)\rtimes Z_2^T\times Z_2^f}\}$ formed by these states.

The trivial state is labeled by identity element $e_{U(1)\rtimes Z_2^T}$. If we put two $[\eta = 1]$ states together, we can gap out the edge states without breaking the symmetry so $[1] \oplus [1] = e_{U(1)\rtimes Z_2^T}$. They form a $\mathbb{Z}_2$ group. As a result, $[\eta = 1]$ and $[1] \oplus [1]$ label the $\mathbb{Z}_2$ classes of fermionic nonchiral SRE phases in the presence of $U(1) \rtimes Z_2^T$ symmetry with $T^2 = P_f$. When $U(1)$ symmetry corresponds to fermion charge conservation, these two different phases are nothing but the trivial band insulator and Z_2 topological band insulator (quantum spin Hall insulator) of fermions in 2 + 1 dimensions.[55,80,81] Again, the bosonic SPT phases of Cooper pairs (where fermions are confined) with $G = U(1) \rtimes Z_2^T$ symmetry gives rise to another $\mathbb{Z}_2$ classification. Hence, in total, there are $\mathbb{Z}_2^2$ classes of different SPT phases with $U(1) \rtimes Z_2^T$ symmetry ($T^2 = P_f$).

D. $G_f/Z_2^f = U(1) \times Z_2^T$ symmetry: $\mathbb{Z}_1$ class

The algebraic structure of $G = G_f/Z_2^f = U(1) \times Z_2^T$ group is given by

$$T^2 = TU_{-\theta}TU_\theta = U_{(\theta=0 \bmod 2\pi)} = e \tag{93}$$

in addition to Eq. (41). The associated constraints (27) for symmetry transformations (86) and $\{W^T,\delta\phi^T\}$ are Eqs. (80), (81), (87), and

$$(I_{2\times2} - W^T)\delta\phi^T + \theta(I_{2\times2} + W^T)\mathbf{t} = \eta\pi\begin{pmatrix}1\\1\end{pmatrix} \bmod 2\pi.$$

Again, the last line of Eq. (93) is automatically satisfied. The gauge inequivalent solutions to these constraint equations are

$$\mathbf{t} = \left[t + \frac{\eta_{U(1)}}{2}\right]\begin{pmatrix}1\\-1\end{pmatrix}, \quad t \in \mathbb{Z}; \tag{94}$$

$$W^T = \sigma_x, \quad \delta\phi^T = \begin{pmatrix}0\\\eta\pi\end{pmatrix} \bmod 2\pi. \tag{95}$$

The associated symmetry-allowed Higgs terms are

$$\mathcal{S}^1_{\text{edge}} = \sum_{l\in\mathbb{Z}} C_l \int dxdt \cos[l(\phi_1 + \phi_2) + \alpha_l]$$

for $\eta = 0$ and

$$\mathcal{S}^1_{\text{edge}} = \sum_{l\in\mathbb{Z}} \int dxdt\{C_l \sin[(2l+1)(\phi_1 + \phi_2)] + D_l \cos[2l(\phi_1 + \phi_2)]\}$$

for $\eta = 1$. In both cases, variable $\phi_1 + \phi_2$ can be localized at a classical value by the Higgs term, so the gapless edge states will be destroyed without breaking any symmetry (under T we have $\phi_1 + \phi_2 \to \eta\pi - \phi_1 - \phi_2$). So there is a (trivial) $\mathbb{Z}_1$ class of fermionic nonchiral SRE phase in the presence of $U(1) \times Z_2^T \times Z_2^f$ symmetry. Note that $T^2 = e_f$ if $\eta = 0$ and $T^2 = P_f$ if $\eta = 1$. Since there is no bosonic SPT phases with $U(1) \rtimes Z_2^T = G_f/Z_2^T$ symmetry, there are no new fermionic SPT phases coming from bosonic SPT phases of Cooper pairs.

E. $G_f/Z_2^f = Z_2$ symmetry

The generator g of Z_2 symmetry satisfies the following algebra:

$$g^2 = e. \tag{96}$$

Here, e stands for either identity element e_f for fermions or fermion number parity P_f. This algebraic constraints (27) for symmetry transformations $\{W^g,\delta\phi^g\}$ are

$$(W^g)^2 = I_{2\times2}, \quad (W^g)^T\mathbf{K}W^g = \mathbf{K}, \tag{97}$$

$$(I_{2\times2} + W^g)\delta\phi^g = \eta\pi\begin{pmatrix}1\\1\end{pmatrix} \bmod 2\pi, \tag{98}$$

where $\eta = 0,1$. The gauge inequivalent solutions of Eq. (97) are $W^g = \pm I_{2\times2}, \pm\sigma_z$. In the following, we analyze those cases with $W^g = \pm I_{2\times2}$ and the discussions about cases with $W^g = \pm\sigma_z$ is put in Appendix F. It is not clear to us now whether the transformation laws with $W^g = \pm\sigma_z$ can be realized in a microscopic model, therefore, we did not include these cases in the minimal set of different fermion SRE phases with G_f symmetry.

(i) For $W^g = -I_{2\times2}$, the gauge inequivalent solution to Eq. (98) is $\delta\phi^g = \binom{0}{0}$ and $\eta = 0$. A set of independent symmetry-allowed Higgs terms can be either Eq. (101) or (102) with $\alpha_l \equiv 0, \forall\, l \in \mathbb{Z}$. Hence it corresponds to the trivial phase, whose edge can be gapped without breaking any symmetry.

(ii) For $W^g = I_{2\times2}$, the inequivalent solutions to Eq. (98) are

$$\delta\phi^g = \pi\begin{pmatrix}t_1\\t_2\end{pmatrix} + \frac{\pi}{2}\eta\begin{pmatrix}1\\1\end{pmatrix}, \quad t_1,t_2,\eta = 0,1. \tag{99}$$

The symmetry allowed Higgs terms are those $\cos(\mathbf{l}^T\phi + \alpha_\mathbf{l})$ terms satisfying

$$l_1t_1 + l_2t_2 + \frac{l_1 + l_2}{2}\eta = 0 \bmod 2 \tag{100}$$

and the condition (74) for local operators. It is straightforward to verify that when $t_1 + t_2 + \eta = 0 \bmod 2$ a set of independent symmetry-allowed Higgs terms satisfying Eq. (19) is

$$\mathcal{S}^1_{\text{edge}} = \sum_{l\in\mathbb{Z}} C_l \int dxdt \cos[l(\phi_1 + \phi_2) + \alpha_l], \tag{101}$$

and the edge states can be gapped without breaking the Z_2 symmetry [under $\boldsymbol{g}$ we have $\phi_1+\phi_2 \to \phi_1+\phi_2+(t_1+t_2+\eta)\pi$], if variable $\phi_1+\phi_2$ is localized at any classical value. Similarly, when $t_1-t_2=0$ mod 2, a set of independent symmetry-allowed Higgs terms satisfying Eq. (19) is

$$\mathcal{S}^1_{\text{edge}} = \sum_{l\in\mathbb{Z}} D_l \int dx dt \cos[l(\phi_1-\phi_2)+\beta_l], \quad (102)$$

and the edge states will be gapped without breaking the Z_2 symmetry [under $\boldsymbol{g}$ we have $\phi_1-\phi_2 \to \phi_1-\phi_2+(t_1-t_2)\pi$], if variable $\phi_1-\phi_2$ is localized at any value. They all correspond to the trivial phase. Notice that when $\eta=0$ we have $\boldsymbol{g}^2=\boldsymbol{e}_f$ while $\boldsymbol{g}^2=\boldsymbol{P}_f$ if $\eta=1$. This corresponds to the following two different symmetry groups.

1. $G_f = Z_2 \times Z_2^f$ symmetry: $\mathbb{Z}_4$ classes

(a) Intrinsic fermion phases. This means $\eta=0$ and $t_1-t_2=1$ mod 2. The algebra of symmetry group G_f is

$$\boldsymbol{g}^2=\boldsymbol{e}_f. \quad (103)$$

When $[\eta,t_1,t_2]=[0,0,1]$ or $[0,1,0]$, a set of independent symmetry-allowed Higgs terms satisfying (19) can be chosen to be either

$$\mathcal{S}^1_{\text{edge}} = \sum_{l\in\mathbb{Z}} C_l \int dx dt \cos[2l(\phi_1+\phi_2)+\alpha_l] \quad (104)$$

or

$$\mathcal{S}^1_{\text{edge}} = \sum_{l\in\mathbb{Z}} D_l \int dx dt \cos[2l(\phi_1-\phi_2)+\beta_l]. \quad (105)$$

They correspond to two nontrivial SPT phases, where the edge cannot be gapped without spontaneously breaking the Z_2 symmetry. Let us label these two states as $[\eta,t_1,t_2]=[0,0,1]$ and $[0,1,0]$. Notice that when we put a $[0,0,1]$ edge with variables $\{\phi_1,\phi_2\}$ together with a $[0,1,0]$ edge with variables $\{\phi_1',\phi_2'\}$, the edge can always be gapped by condensing, e.g., independent bosonic variables $\{\phi_1+\phi_2',\phi_1'-\phi_2\}$ and no symmetry will be broken. Hence $[0,0,1]\oplus[0,1,0]=\boldsymbol{e}_{Z_2\times Z_2^f}$ and we have $[0,0,1]=[0,1,0]^{-1}$. On the other hand, if we put four $[0,1,0]$ states with edge variables $\{\phi_R^a,\phi_L^a,a=1,2,3,4\}$ together, the edge can be gapped without breaking the symmetry, by localizing the following independent bosonic variables:

$$\phi_R^1+\phi_R^2+\phi_L^3+\phi_L^4, \quad \phi_R^3+\phi_R^4+\phi_L^1+\phi_L^2,$$
$$\phi_R^1+\phi_R^3+\phi_L^1+\phi_L^4, \quad \phi_R^1+\phi_R^4+\phi_L^1+\phi_L^3.$$

As a results, we have $[0,1,0]^4=\boldsymbol{e}_{Z_2\times Z_2^f}$ and hence $[0,1,0]^3=[0,0,1]$. Therefore all different fermionic phases form a $\mathbb{Z}_4$ group.

To summarize, with Z_2 symmetry transformation $W^g=\pm I_{2\times 2}$, there are $\mathbb{Z}_4$ classes of different fermionic nonchiral SRE phases in the presence of $Z_2\times Z_2^f$ symmetry. The nontrivial SPT phases such as $[0,1,0]$ can be realized by noninteracting fermions, as shown by the coupled wire construction in Sec. VI.

(b) Interacting fermionic SPT phases from the bosonic SPT phase with Z_2 symmetry. In the previous discussion of bosonic SPT phases, a $\mathbb{Z}_2$ classification was found for bosons with Z_2 symmetry. Here, in a fermionic system with $Z_2\times Z_2^f$ symmetry, we can always let the fermions combine to form bosonic Cooper pairs that can serve as the fundamental bosons, which then form the nontrivial bosonic SPT phase discussed in Sec. IV. Notice that the fermion parity Z_2^f symmetry can never be broken and have no effect on the Cooper pairs at all. Do these interacting bosonic SPT phases lead to an extra $\mathbb{Z}_2$ classification for fermions with $Z_2\times Z_2^T$ symmetry in the presence of deconfined fermions in the low-energy sector? If so, these nontrivial SPT phases cannot be obtained from perturbing noninteracting fermions. However, we show now that the bosonic SPT phase with Z_2 symmetry is contained within the fermion classification discussed previously when there are gapless fermions on the edge. And it is a Z_2 subgroup of the Z_4 classes that were found. Thus they can be obtained from adding perturbation to a noninteracting fermion Hamiltonian.

Consider one bosonic Z_2-symmetric SPT state with edge variables $\{\phi_1,\phi_2\}$, whose symmetry transformations are $\phi_a\to\phi_a+\pi, a=1,2$ under Z_2 generator $\boldsymbol{g}$. When this state is put together with two fermion $Z_2\times Z_2^f$-symmetric SPT states $[0,1,0]\oplus[0,1,0]$ with edge variables $\{\phi_R,\phi_L\}$ and $\{\phi_R',\phi_L'\}$, its edge can be gapped out by simultaneously localizing the following bosonic fields on the edge:

$$\phi_R+\phi_L+\phi_1, \quad -\phi_R'+\phi_L+\phi_2, \quad \phi_R-\phi_L'-\phi_2. \quad (106)$$

Notice that under Z_2 generator $\boldsymbol{g}$ the edge variables $\{\phi_R,\phi_L\}$ transform as $\phi_R\to\phi_R+\pi$, $\phi_L\to\phi_L$ and the same for $\{\phi_R',\phi_L'\}$. Notices that the inverse of the above bosonic SPT phase is itself, hence we have shown that $[0,1,0]^2\equiv[0,1,0]\oplus[0,1,0]$, i.e., the state obtained by putting two fermion SPT phases $[0,1,0]$ together is nothing but the bosonic Z_2-symmetric SPT phase. Therefore we conclude that bosonic SPT phase with Z_2 symmetry is contained within the fermionic classification Z_4.

2. $G_f = Z_4$ symmetry: $\mathbb{Z}_2$ classes

Also note that when $\eta=1$, we have the algebra

$$\boldsymbol{g}^2=\boldsymbol{P}_f \quad (107)$$

for symmetry group $G_f=Z_4$. Note that Z_2^f is a normal subgroup of Z_4. Since all phases with $\eta=1$ are trivial, they do not give rise to nontrivial (intrinsic) fermionic SPT phases with Z_4 symmetry. However, as discussed before, the bosonic SPT phase of Cooper pairs with symmetry $G_f/Z_2^f=Z_2$ (when fermions are confined) always automatically lead to one interacting fermionic SPT phase protected by symmetry $G_f=Z_4$. Hence all different SRE fermionic phases with symmetry group $G_f=Z_4$ have at least a $\mathbb{Z}_2$ classification. These are physically related to charge-4e superconductors in two dimensions protected by electron charge conservation modulo 4. The nontrivial fermionic SPT phase protected by $G_f=Z_4$ symmetry cannot be obtained by perturbing noninteracting fermions, and therefore they are intrinsic interaction-driven fermion SPT phases (charge-4e superconductors in this case) with Z_4 symmetry.

In summary, there are at least $\mathbb{Z}_4$ classes of different fermionic nonchiral SRE phases in the presence of $Z_2\times Z_2^f$

symmetry. Bosonic SPT phases with symmetry Z_2 do not add any new phases. On the other hand, in a fermion system with Z_4 symmetry ($\boldsymbol{g}^2=\boldsymbol{P}_f$ or $\eta=1$) there are $\mathbb{Z}_2$ classes of different fermionic SRE phases. This corresponds to two different classes of charge-4e superconductors in $2+1$ dimensions.

(a) Discussion of results. The fermionic topological phases protected by $Z_2\times Z_2^f$ symmetry form a $\mathbb{Z}_4$ group. In comparison, supercohomology theory[25] obtains the same number of phases but with group structure $\mathbb{Z}_2^2$. An advantage of the present formalism is that we can see how these phases connect to the bosonic SPT phases with the same symmetry and verify they do not add any new phases.

F. $G_f/Z_2^f=Z_2\times Z_2^T$ symmetry

The algebraic structure of $Z_2\times Z_2^T\times Z_2^f$ group is

$$\boldsymbol{g}^2=\boldsymbol{T}^2=\boldsymbol{gTgT}=\boldsymbol{e}, \tag{108}$$

where $\boldsymbol{g}$ is the Z_2 generator and time reversal operation $\boldsymbol{T}$ is the Z_2^T generator. In our notation $\boldsymbol{e}$ can be either identity element $\boldsymbol{e}_f$ for fermions or fermion number parity $\boldsymbol{P}_f$. The associate constraints (27) are given by

$$\begin{aligned}(W^g)^2=(W^T)^2=(W^gW^T)^2=I_{2\times2},\\ (W^g)^T\mathbf{K}W^g=-(W^T)^T\mathbf{K}W^T=\mathbf{K},\end{aligned} \tag{109}$$

$$\begin{aligned}(1+W^g)\delta\phi^g&=\eta_g\pi\begin{pmatrix}1\\1\end{pmatrix},\\ (1-W^T)\delta\phi^T&=\eta_T\pi\begin{pmatrix}1\\1\end{pmatrix},\\ (1-W^TW^g)(\delta\phi^T-W^T\delta\phi^g)&=\eta\pi\begin{pmatrix}1\\1\end{pmatrix},\end{aligned} \tag{110}$$

where $\eta,\eta_T,\eta_g=0,1$. We can always choose a gauge so that $W^T=\sigma_x$ and from Eq. (109) $W^g=\pm I_{2\times2}$. We have not found a microscopic realization of symmetry transformation $W^g=-I_{2\times2}$ so far, therefore we put the discussions of $W^g=-I_{2\times2}$ case to Appendix G. Here, we will focus on the symmetry transformation $W^g_{2\times2}$ case.

For $W^g=I_{2\times2}$, the inequivalent solutions to Eq. (110) are

$$\begin{aligned}\delta\phi^g&=\left(\alpha+\frac{\eta_g}{2}\right)\pi\begin{pmatrix}1\\1\end{pmatrix}+\pi\begin{pmatrix}\eta-\eta_T\\0\end{pmatrix},\\ \delta\phi^T&=\eta_T\pi\begin{pmatrix}0\\1\end{pmatrix},\quad \alpha,\eta,\eta_T,\eta_g=0,1.\end{aligned} \tag{111}$$

If $\eta_g+\eta-\eta_T=0$, the variable $\phi_1+\phi_2$ can be localized without breaking any symmetry. If $\eta=\eta_T=0$, the variable $\phi_1-\phi_2$ can be localized without breaking any symmetry.

Note that when $\eta_g=0$, we have $\boldsymbol{g}^2=\boldsymbol{e}_f$ and hence the symmetry group is $G_f=Z_2\times Z_2^T\times Z_2^f$ with $\boldsymbol{T}^2=\boldsymbol{e}_f$ if $\eta_T=0$ or $\boldsymbol{T}^2=\boldsymbol{P}_f$ if $\eta_T=1$. When $\eta_g=1$, on the other hand, we have $\boldsymbol{g}^2=\boldsymbol{P}_f$, and symmetry group for fermions is $G_f=Z_4\rtimes Z_2^T$ or $G_f=Z_4\times Z_2^T$.

1. $G_f=Z_2\times Z_2^T\times Z_2^f$ symmetry: $\mathbb{Z}_4\times\mathbb{Z}_2$ classes

There are four nontrivial SPT phases with $\eta_g=0$: they have $\eta-\eta_T=1 \bmod 2$ and $\alpha=0,1$. Let us label a state with symmetry transformations (111) as $[\eta_g,\eta_T,\eta,\alpha]$. We already showed $[1,0,0,\alpha]=[1,\eta+1,\eta,\alpha]=[0,\eta,\eta,\alpha]=\boldsymbol{e}_{Z_2\times Z_2^T\times Z_2^f}$. When a $[0,\eta+1,\eta,\alpha]$ state with edge variables $\{\phi_1,\phi_2\}$ is put together with a $[0,\eta'+1,\eta',\alpha]^{-1}$ state with edge variables $\{\phi_1',\phi_2'\}$, its edge cannot be gapped without breaking the symmetry by localizing independent bosonic fields $\{\phi_1+\phi_1',\phi_2+\phi_2'\}$. Therefore we know that $[0,1,0,\alpha]=[0,0,1,\alpha]$ is the same nontrivial SPT phase. In the case $\eta_g=0=\eta_T$ and $\eta=1$, the algebra of $Z_2\times Z_2^T\times Z_2^f$ group is

$$\boldsymbol{g}^2=\boldsymbol{T}^2=\boldsymbol{e}_f,\quad \boldsymbol{gTgT}=\boldsymbol{P}_f. \tag{112}$$

As discussed in Ref. 21, this is the same as $\eta_g=0=\eta$ and $\eta_T=1$, since one can always redefine the antiunitary time reversal as $\boldsymbol{T}'\equiv\boldsymbol{gT}$. Just as discussed earlier for fermionic SPT phases with $Z_2\times Z_2^T$ symmetry, it is easy to verify that $[0,1,0,0]=[0,1,0,1]^{-1}$ and when four $[0,1,0,0]$ states are put together, their edges can be gapped without breaking any symmetry, i.e., $[0,1,0,0]^4=\boldsymbol{e}_{Z_2\times Z_2^T\times Z_2^f}$. As a result $[0,1,0,0]^n,n=1,2,3$ are the only three nontrivial SPT phases, whose edge cannot be gapped without breaking the symmetry. Hence all different phases with $W^g=I_{2\times2}$ form a $\mathbb{Z}_4$ group for $Z_2\times Z_2^T\times Z_2^f$ symmetry ($\eta_g=0$).

For the same reason mentioned earlier for fermions with $Z_2\times Z_2^f$ symmetry, here in the presence of $Z_2\times Z_2^T\times Z_2^f$ symmetry, we can obtain interacting SPT phases from bosonic SPT phases of fermion pairs with $Z_2\times Z_2^T$ symmetry. Note that there are $\mathbb{Z}_2^2$ classes of bosonic nonchiral SRE phases with $Z_2\times Z_2^T$ symmetry. Do these lead to an extra $\mathbb{Z}_2^2$ group structure for fermions with this symmetry? As before, we now show that this is *not* the case. These phases are already accounted for within the fermionic classification when fermions are present on the edge. While one of these Z_2 classes is contained within the Z_4 classification, the other is connected to the trivial class, in the presence of fermions. Hence it brings in an extra $\mathbb{Z}_2$ structure due to the bosonic SPT phase of strongly bound Cooper pairs where fermions are absent (confined) on the edge. Such a $\mathbb{Z}_2$ index cannot be obtained from perturbing free fermions.

Again let us consider a bosonic $Z_2\times Z_2^T$-symmetric SPT state $[1,n_2,n]_b$ with edge variables $\{\phi_1,\phi_2\}$, which transforms as

$$\boldsymbol{g}:\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix}\rightarrow\begin{pmatrix}\phi_1+\pi\\\phi_2+n\pi\end{pmatrix},\quad \boldsymbol{T}:\begin{pmatrix}\phi_1\\\phi_2\end{pmatrix}\rightarrow\begin{pmatrix}-\phi_1\\\phi_2+n_2\pi\end{pmatrix}$$

under Z_2 generator $\boldsymbol{g}$ and time reversal $\boldsymbol{T}$. When this state is put together with two fermion $Z_2\times Z_2^f$-symmetric SPT states $[0,1,0,0]_f\oplus[0,1,0,0]_f$ with edge variables $\{\phi_R,\phi_L\}$ and $\{\phi_R',\phi_L'\}$, its edge can be gapped out by simultaneously localizing the bosonic fields (106) on the edge if $n=1$. This means

$$[1,n_2,1]_b\oplus[0,1,0,0]_f\oplus[0,1,0,0]_f=e_{Z_2\times Z_2^T\times Z_2^f}.$$

Similarly, one can show that

$$\begin{aligned}&[1,n_2,0]_b\oplus[0,1,0,0]_f\oplus[0,1,0,1]_f\\ &=e_{Z_2\times Z_2^T\times Z_2^f}=[0,1,0,0]_f\oplus[0,1,0,1]_f,\end{aligned}$$

again by localizing the same bosonic variables (106). Hence we have shown that $[1,n_2,1]_b=[0,1,0,0]_f\oplus$

$[0,1,0,0]_f$ and $[1,n_2,0]_b = [0,1,0,0]_f \oplus [0,1,0,1]_f$. So in the presence of fermions, all the bosonic SPT phases protected by $Z_2 \times Z_2^T \times Z_2^f$ can be obtained by perturbing noninteracting fermions (constructed by putting several noninteracting $[0,1,0,0]_f$ together and add interactions). Therefore we conclude that bosonic SPT phases with $Z_2 \times Z_2^T$ symmetry can all be obtained from perturbing noninteracting fermions in the presence of $Z_2 \times Z_2^T$ symmetry. However, when fermions are confined in the low-energy sector (on the edge), the bosonic SPT phase $[1,1,0]_b$ of strongly-bound Cooper pairs indeed brings in new fermionic SPT phases for fermions. Hence, in total, there are $\mathbb{Z}_4 \times \mathbb{Z}_2$ classes of different fermionic (nonchiral) SRE phases with $Z_2 \times Z_2^T \times Z_2^f$ symmetry: $\mathbb{Z}_4$ can be obtained from perturbing free fermions, and the other $\mathbb{Z}_2$ is from bosonic $[1,n_2,0]_b$ phases of molecule-like Cooper pairs protected by $Z_2 \times Z_2^T$ symmetry where fermions are confined in the low-energy limit. The last $\mathbb{Z}_2$ cannot be obtained from perturbing free fermions.

(a) Discussion of results. While this particular symmetry class cannot be discussed within group supercohomology theory,[25] recent work[21–23] has approached this problem from another angle, by starting with noninteracting fermions (which have a $\mathbb{Z}$ classification with this symmetry) and then turning on interactions. They find a $\mathbb{Z}_8$ classification that survives interactions. Odd integer members of this series have an odd number of pair of Majorana modes at the edge, that move in opposite directions. Although apparently quire different, these results are consistent with ours due to the following. Since we are unable to deal with unpaired Majorana modes, only the even members of the series are captured (hence $\mathbb{Z}_4$ in our $\mathbb{Z}_4 \times \mathbb{Z}_2$ classes here). An advantage though is that this classification of topological phases, which are stable to interaction, emerges directly from the formalism, without the need to begin from free fermions. And our formulation also includes a $\mathbb{Z}_2$ classes that cannot be obtained by perturbing free fermions.

2. $G_f = Z_4 \times Z_2^T$ symmetry: $\mathbb{Z}_2^3$ classes

On the other hand, when $\eta_g = 1$, we have $g^2 = P_f = (-1)^{\hat{N}_f}$ and hence the corresponding symmetry group is actually $Z_4 \times Z_2^T$, where Z_2^f is a subgroup of Z_4. In this case, the nontrivial SPT phases have $\eta_g = 1$ and $\eta = \eta_T = 1$, $\alpha = 0,1$. The algebraic structure of the symmetry group G_f is given by

$$g^2 = T^2 = gTgT = P_f. \tag{113}$$

It is easy to check that $Tg = g^{-1}T$ and hence the symmetry group is actually $Z_4 \rtimes Z_2^T$ with $T^2 = P_f$. We still label the phases with symmetry transformations (111) as $[\eta_g,\eta_T,\eta,\alpha]$. When a $[1,1,1,0]$ state with edge variables $\{\phi_1,\phi_2\}$ is put together with a $[1,1,1,1]^{-1}$ state with edge variables $\{\phi_1',\phi_2'\}$, its edge cannot be gapped without breaking the symmetry by localizing independent bosonic fields $\{\phi_1+\phi_1',\phi_2+\phi_2'\}$. Therefore $[1,1,1,0] = [1,1,1,1]$ is the same nontrivial SPT phase. Now let us put two $[1,1,1,0]$ states together with edge variables $\{\phi_L,\phi_R\}$ and $\{\phi_L',\phi_R'\}$, it is easy to see that the edge states will be all gapped out by simultaneously localizing the following bosonic variables: $\{\phi_R-\phi_L',\phi_R'-\phi_L\}$. Hence we have $[1,1,1,0]^2 = e_{Z_4\times Z_2^T}$ and these different intrinsic fermionic SRE phases with $Z_4 \times Z_2^T$ form a $\mathbb{Z}_2$ group. The nontrivial fermionic SPT phase can be obtained from free-fermion band structures, just like the $G_f = Z_2 \times Z_2^T \times Z_2^f$ case.

In the case of a fermion system with $Z_4 \times Z_2^T$ symmetry, the corresponding bosonic system of Cooper pairs have $Z_2 \times Z_2^T$ symmetry and has a $\mathbb{Z}_2^2$ topological classification. For fermions in the presence of $Z_4 \times Z_2^T$ symmetry, these bosonic SPT phases of strongly bound Cooper pairs (with $Z_2 \times Z_2^T$ symmetry and a $\mathbb{Z}_2^2$ classification) will bring in new phases, in addition to the the nontrivial fermionic SPT phase $[1,1,1,0]$ (with a $\mathbb{Z}_2$ group structure). In these new phases, the fermions can be either confined in the low-energy sector (associated with $[1,1,1,0]^2$ state) or deconfined (associated with $[1,1,1,0]$ state). Hence all different fermionic SRE phases with $Z_4 \times Z_2^T$ symmetry have a $\mathbb{Z}_2^3$ classification, where one $\mathbb{Z}_2^2$ structure comes from bosonic SPT phases $[1,n_2,n]_b$ of Cooper pairs, and the other $\mathbb{Z}_2$ associated with fermion state $[1,1,1,0]$ are intrinsic properties of fermionic systems.

3. $G_f = Z_4 \rtimes Z_2^T$ symmetry: $\mathbb{Z}_2^2$ classes

In this case, we have $\eta_g = 1$ and $\eta = 0$, and therefore

$$g^2 = P_f, \quad gTgT = e_f, \quad T^2 = e_f \text{ or } P_f. \tag{114}$$

In this case, there are no intrinsic fermionic SPT phases with $G_f = Z_4 \rtimes Z_2^T$ symmetry. However, bosonic SPT phases of Cooper pairs with $G_f/Z_2^f = Z_2 \times Z_2^T$ symmetry leads to $\mathbb{Z}_2^2$ classes of different fermion SRE phases. Hence there are at least $\mathbb{Z}_2^2$ classes of different fermion nonchiral SRE phases with $G_f = Z_4 \rtimes Z_2^T$ symmetry. These $\mathbb{Z}_2^2$ classes of phases physically correspond to different charge-$4e$ superconductors with time reversal symmetry $T^2 = e_f$ or $T^2 = P_f$. Recently, the possibility of realizing charge-$4e$ superconductivity (four fermion condensates), in imbalanced cold atomic gases[84] (which break time-reversal symmetry) and also in certain cuprate superconductors[85] (which preserve time reversal) has been discussed. While these phases were nontopological, the prospects for realizing topological phases with these symmetries remains to be seen.

VI. COUPLED-WIRE CONSTRUCTION OF BOSONIC AND FERMIONIC SPT PHASES

In the previous sections, we showed how to classify different SPT phases in the K matrix + Higgs formulation. The edge structure of SPT phases is explicit in this formulation, e.g., the edge of a bosonic SPT phase is characterized by bare action (34), Kac-Moody algebra (35) as well as symmetry transformation rules (22) for unitary symmetry g and (24) for antiunitary symmetry h on the bosonic variable $\{\phi_a, a = 1,2\}$. However, a more microscopic construction of these $2+1$-D SPT phases is still lacking in this formulation. In this section, we present a microscopic construction of these SPT phases in the anisotropic (quasi-1D) limit, from an array of coupled one-dimensional quantum wires. This approach has been used to construct Abelian[52] and non-Abelian FQH states.[53] We first give a short introduction to the coupled wire construction, and then use the this method to explicitly construct bosonic SPT phases in the presence of symmetry group $G = U(1)$

and $G = U(1) \rtimes Z_2^T$ as well as fermionic SPT phases with symmetry group $Z_2 \times Z_2^F$. Generalizations to other symmetry groups are straightforward.

A. Coupled wire construction in a nutshell

Consider an array of uncoupled identical one-dimensional quantum wires, each wire being described by a nonchiral Luttinger liquid. The bosonic fields associated with the Luttinger liquid in the lth wire ($1 \leqslant l \leqslant N_w$, N_w being the total number of quantum wires) are $\{\theta_l(x),\varphi_l(x)\}$ satisfying the following commutation relation:

$$[\theta_m(x),\varphi_l(y)] = i\frac{\pi}{2}\text{Sign}(x-y)\delta_{m,l}, \tag{115}$$

where $\varphi(x)$ is a bosonic phase field, while $\theta(x)$ describes the density fluctuations in the Luttinger liquid. The long-wavelength density fluctuations on lth wire is given by

$$\rho_l(x) - \bar{\rho}_l = \partial_x\theta_l(x)/\pi, \tag{116}$$

where $\bar{\rho}_l$ is the average particle (boson or fermion) density in the lth wire. In terms of these two variables, the Luttinger liquid Hamiltonian for decoupled quantum wires is given by

$$H_{\text{LL}} = \sum_l \frac{v_l}{2\pi}\int dx\left[\frac{1}{g_l}(\partial_x\theta_l)^2 + g_l(\partial\varphi_l)^2\right], \tag{117}$$

where $g_l > 0$ is the Luttinger parameter for the lth wire.

The idea of coupled wire construction is to add interwire and intrawire interactions between electrons as well as tunneling between wires. For example, the forward scattering terms between different wires is written as

$$H_{\text{FC}} = \sum_{k\neq l}\int dx(\partial\varphi_k,\partial\theta_k)\mathbf{M}_{k,l}\begin{pmatrix}\partial\varphi_l\\ \partial\theta_l\end{pmatrix}, \tag{118}$$

where $\mathbf{M}_{j,k}$ are all 2×2 matrices describing the forward scattering interactions between the jth and kth wires. Other interchannel scattering terms in general have the following form:

$$\mathcal{O}^{\{m_l,n_l\}}(x) \sim \cos\left\{\mathrm{i}\sum_l[m_l\theta_l(x) + n_l\varphi_l(x)] + \alpha_{\{m_l,n_l\}}\right\}, \tag{119}$$

where $\alpha_{\{m_l,n_l\}}$ are real constants and $\{m_l,n_l\}$ all take integer values. For a bosonic system in the absence of any symmetry (or associated conservation laws), the above interwire scattering term must satisfy the following condition:

$$m_l = 0 \mod 2, \quad \forall \quad 1 \leqslant l \leqslant N_w. \tag{120}$$

This is because the boson density fluctuations are mainly contributed by density waves at vector $q_n \sim 2\pi\bar{\rho}_l n$, $n \in \mathbb{Z}$ and the density fluctuations at q_n is given by

$$\rho_n^l(x) \propto e^{\mathrm{i}n[2\pi\bar{\rho}_l x + 2\theta_l(x)]}, \quad n \in \mathbb{Z}, \quad 1 \leqslant l \leqslant N_w,$$

for the lth quantum wire. For a fermionic system on the other hand, in the absence of any symmetry, the constraint on interwire scattering term (119) is

$$m_l = n_l \mod 2, \quad \forall \quad 1 \leqslant l \leqslant N_w. \tag{121}$$

This is because interchannel scattering terms must be composed of single-fermion operators, i.e., left mover $\psi_l^R \sim \exp[\mathrm{i}(\varphi_l - \theta_l - \pi\bar{\rho}_l x)]$ and right mover $\psi_l^R \sim \exp[\mathrm{i}(\varphi_l + \theta_l + \pi\bar{\rho}_l x)]$. The presence of symmetry group G will lead to further constraints on interwire coupling terms (119) and forward scattering terms (118): symmetry-allowed scattering terms must transform trivially under any symmetry operation. The bare Luttinger liquid Hamiltonian together with symmetry-allowed forward scattering (118) and interchannel scattering (119) forms the generic Hamiltonian for a coupled wire construction:

$$H = H_{\text{LL}} + H_{\text{FC}} + \sum_{\{m_l,n_l\}}\int dx C_{\{m_l,n_l\}}\mathcal{O}^{\{m_l,n_l\}}(x). \tag{122}$$

In the coupled-wire construction (122), one can properly choose a set of symmetry-allowed interwire scattering terms $\{\mathcal{O}^{\{m_l,n_l\}}(x)\}$ in Eq. (122), so that

$$\sum_l m_l n_l = \sum_l (m_l n_l' + m_l' n_l) = \sum_l m_l' n_l' = 0, \tag{123}$$

for any two terms $\mathcal{O}^{\{m_l,n_l\}}(x)$ and $\mathcal{O}^{\{m_l',n_l'\}}(x)$ in Hamiltonian (122). Therefore the set of bosonic variables $\{\sum_l(m_l\theta_l + n_l\varphi_l)\}$ can be simultaneously localized at classical values by minimizing the interwire scattering terms $\{\mathcal{O}^{\{m_l,n_l\}}(x)\}$. When chosen properly, all degree of freedom in the bulk will be gapped by these interwire scattering terms, and the only low-energy degrees of freedom left free are on the edge. To be specific, the variable $p_1\theta_1 + q_1\varphi_1$ on the left edge $l = 1$ will remain gapless if

$$p_1 n_1 + q_1 m_1 = 0, \quad \forall\,\{m_l,n_l\} \quad \text{in Eq. (123)}. \tag{124}$$

Note that one can always tune the forward scattering terms (118) so that interwire coupling terms $\{\mathcal{O}^{\{m_l,n_l\}}(x)\}$ are relevant in the renormalization group sense. Then one expects the coupled wire system will be driven into strong coupling phase of interwire scattering $\{\mathcal{O}^{\{m_l,n_l\}}(x)\}$, and hence all bosonic variables $\{\sum_l(m_l\theta_l + n_l\varphi_l)\}$ will be localized simultaneously at classical values.

One of the simplest examples is the Laughlin state[82] of spinless fermions, i.e., FQH state at filling fraction $\nu = 1/m$, m = odd integer. The interwire scattering terms whose strong coupling phase corresponding to the Laughlin state are[52]

$$H_{1/m} = \sum_{l=1}^{N_w-1}\int dx C_l\cos[\varphi_l - \varphi_{l+1} - m(\theta_l + \theta_{l+1})].$$

Its gapless variable on the left edge is $\phi_{1/m}^L = \varphi_1 + m\theta_1$, which satisfies the following Kac-Moody algebra:[83]

$$\left[\partial_x\phi_{1/m}^L(x),\partial_y\phi_{1/m}^L(y)\right] = 2\pi m\,\mathrm{i}\,\partial_x\delta(x-y).$$

It is easy to verify that $\phi_{1/m}^L$ satisfies condition (124) and is the only gapless degree of freedom on the edge.

B. Bosonic SPT phases with $U(1)$ symmetry

As discussed in Sec. IV, in the presence of $U(1)$ symmetry there are $\mathbb{Z}$ (integer group) different classes of bosonic nonchiral SRE states, which are labeled by their $U(1)$ charge vector $\mathbf{t} = (1,q)^T$. The bosonic variables $\{\phi_1,\phi_2\}$ on its edge satisfies Kac-Moody algebra (35). Under group element $U_{\Delta\phi}$ of symmetry group $U(1)$ they transform as

$$U_{\Delta\phi}:\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\to\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}+\Delta\phi\begin{pmatrix}1\\ q\end{pmatrix},\quad \Delta\phi\in[0,2\pi). \tag{125}$$

The nontrivial SPT phases correspond to different integers $q\neq 0$, whose edge cannot be gapped out without breaking the $U(1)$ symmetry. Here, we present an explicit construction of these SPT phases with $U(1)$ symmetry in the coupled wire approach.

We start from an array of quantum wires ($1\leqslant l\leqslant N_w$) where each wire is composed of two chains: a chain of charged bosons [each boson carries a unit of $U(1)$ electric charge] and a spin chain. Each chain forms a $c=1$ Luttinger liquid described by bosonic fields: $\{\varphi_l^s(x),\theta_l^s(x)\}$ for the spin chain and $\{\varphi_l^c(x),\theta_l^c(x)\}$ for the chain of charged bosons in the lth wire. These bosonic fields satisfy the commutation relation (115):

$$\left[\theta_m^\alpha(x),\varphi_l^\beta(y)\right]=\mathrm{i}\frac{\pi}{2}\mathrm{Sign}(x-y)\delta_{m,l}\delta_{\alpha,\beta}, \tag{126}$$

where $\alpha,\beta=c/s$ denotes charge/spin degree of freedom and $1\leqslant m,l\leqslant N_w$ are the wire index. φ_l^c are phase fields of charged bosons, while $2\partial_x\theta_l^c$ describes charged boson density fluctuations. For the spin chain, $\partial_x\theta_l^s(x)\sim S_l^z(x)$ and $\exp[\mathrm{i}\varphi_l^s(x)]\sim S_l^+(x)$. Without interwire scattering terms, the bare Hamiltonian density of the system takes the form (117) of Luttinger liquids:

$$\mathcal{H}_{\rm LL}=\sum_{l=1}^{N_w}\sum_{\alpha=c/s}\frac{v_l^\alpha}{2\pi}\left[\frac{1}{g_l^\alpha}\left(\partial_x\theta_l^\alpha\right)^2+g_l^\alpha\left(\partial\varphi_l^\alpha\right)^2\right]. \tag{127}$$

The $U(1)$ symmetry associated with $\{\varphi_l^c(x),\theta_l^c(x)\}$ boson charge conservation leads to the following symmetry transformations for the bosonic fields:

$$\begin{aligned}\varphi_l^c(x)&\to\hat U_{\Delta\phi}\varphi_l^c(x)\hat U_{\Delta\phi}^{-1}=\varphi_l^c(x)+\Delta\phi,\\ \hat U_{\Delta\phi}&\equiv e^{\mathrm{i}\Delta\phi\int dx\sum_l 2\partial_x\theta_l^c(x)},\quad 0\leqslant\Delta\phi<2\pi.\end{aligned} \tag{128}$$

The other fields $\theta_l^c,\varphi_l^s,\theta_l^s$ are invariant under the above $U(1)$ charge rotation $\hat U_{\Delta\phi}$.

In the presence of the above $U(1)$ symmetry associated with boson charge conservation, the different phases labeled by charge vector $\mathbf{t}=(1,q)^T$ are stabilized by the following interwire coupling terms:

$$\begin{aligned}\mathcal{H}^1_{(1,q)}=\sum_{l=1}^{N_w-1}&\big[C_l\cos\left(\varphi_l^c-\varphi_{l+1}^c-2\theta_l^s+\lambda_l\right)\\ &+D_l\cos\big(\varphi_l^s-\varphi_{l+1}^s+q\left(\varphi_l^c-\varphi_{l+1}^c\right)\\ &-2\left(\theta_{l+1}^c-q\theta_{l+1}^s\right)+\lambda_l'\big)\big],\end{aligned} \tag{129}$$

where $C_l,D_l,\lambda_l,\lambda_l'$ are real constants. A pictorial illustration of the above interwire scattering terms is given in Fig. 2. Clearly, the above interwire scattering terms all satisfy constraint (120) for bosonic systems, and they are also invariant under $U(1)$ rotation (128).

As argued in Refs. 52 and 53, one can always choose proper forward scattering terms (118) to make the above interwire coupling terms become relevant and drive the system into their strong-coupling phase. Notice that the arguments of the above

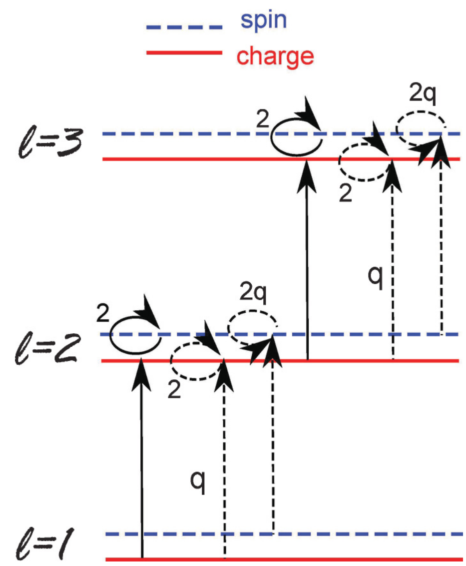

FIG. 2. (Color online) Schematic illustration of interwire coupling terms which stabilize the bosonic SPT phases protected by $U(1)$ symmetry, with Hall conductance $\sigma_{xy}=2q$. Solid horizontal lines stand for quantum wires of charged bosons [each carries unit $U(1)$ charge], while dashed horizontal lines represent quantum wires composed of neutral (say spin) degrees of freedom. Dashed and solid arrows illustrate the two interwire coupling terms in Eq. (130) that gap the bulk, but leave behind nontrivial edge states.

cosine terms commute with each other, so they can be localized at certain classical values simultaneously. It is straightforward to show that all bosonic fields in the bulk with $2\leqslant l\leqslant N_w-1$ are gapped, while the gapless edge states on the left edge $l=1$ are described by variables $\{\phi_1^1(x),\phi_1^2(x)\}$ defined as

$$\phi_l^1\equiv\varphi_l^c,\quad \phi_l^2\equiv\varphi_l^s+q\varphi_l^c+2\left(\theta_l^c-q\theta_l^s\right). \tag{130}$$

They transform exactly like $\{\phi_1,\phi_2\}$ in Eq. (125) under charge $U(1)$ symmetry (128). Besides, they also obtain the Kac-Moody algebra (35) for a bosonic nonchiral SRE system. As a result, the strong-coupling phase of interwire couplings (130) is nothing but the bosonic SPT phases labeled by charge vector $\mathbf{t}=(1,q)^T$ with charge $U(1)$ symmetry.

Now let us elaborate on why the interwire coupling (130) can gap out everything in the bulk and leave variables (130) on the edge. In addition to variables $\{\phi_l^1(x),\phi_l^2(x)\}$ in Eq. (130), one can define another pair of variables $\{\tilde\phi_l^1(x),\tilde\phi_l^2(x)\}$ as

$$\tilde\phi_l^1\equiv\varphi_l^c-2\theta_l^s,\quad \tilde\phi_l^2\equiv\varphi_l^s+q\varphi_l^c. \tag{131}$$

They also satisfy Kac-Moody algebra (35) except for an extra minus sign for all commutators. Notice that the two pairs of variables $\{\tilde\phi_l^1(x),\tilde\phi_l^2(x)\}$ and $\{\tilde\phi_l^1(x),\tilde\phi_l^2(x)\}$ commute with each other. They are just a linear combination of the original charge and spin variables $\{\varphi_l^c,\theta_l^c,\varphi_l^s,\theta_l^s\}$. The interwire scattering terms (130) can be written as

$$\begin{aligned}\mathcal{H}^1_{(1,q)}=\sum_{l=1}^{N_w-1}&\left[C_l\cos\left(\tilde\phi_l^1-\phi_{l+1}^1+\lambda_l\right)\right.\\ &\left.+D_l\cos\left(\tilde\phi_l^2-\phi_{l+1}^2+\lambda_l'\right)\right].\end{aligned}$$

Hence everything in the bulk, i.e., $\{\tilde\phi_l^{1,2}(x),\phi_l^{1,2}(x),2\leqslant l\leqslant N_w-1\}$, are all gapped since they don't commute with at least one of interwire scattering terms in Eq. (130). For the $l=1$ wire on the left edge, variables $\{\tilde\phi_1^1(x),\tilde\phi_1^2(x)\}$ are gapped for the same reason while variables $\{\phi_l^1(x),\phi_l^2(x)\}$ are left gapless. For the $l=N_w$ wire on the right edge, things happen in the opposite way: variables $\{\phi_{N_w}^1(x),\phi_{N_w}^2(x)\}$ are gapped, while variables $\{\tilde\phi_{N_w}^1(x),\tilde\phi_{N_w}^2(x)\}$ remain gapless.

C. Bosonic SPT phase with $U(1)\rtimes Z_2^T$ symmetry

As discussed in Sec. IV, in the presence of $U(1)\rtimes Z_2^T$ symmetry, there are $\mathbb{Z}_2$ different classes of nonchiral bosonic SRE phases in $2+1$ dimensions. Among them there is only one nontrivial SPT phase, whose edge cannot be gapped without breaking the $U(1)\rtimes Z_2^T$ symmetry. Its gapless edge is described by bosonic fields $\{\phi_1,\phi_2\}$, which satisfies the Kac-Moody algebra (35). Under charge $U(1)$ rotation, the two bosonic variables transform as Eq. (125), while under time reversal $\boldsymbol{T}$ they transform as

$$\boldsymbol{T}:\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\to\begin{pmatrix}-\phi_1\\ \phi_2+\pi\end{pmatrix} \tag{132}$$

for the nontrivial SPT phase.

Such a SPT phase is nothing but the strong coupling phase of interwire scattering terms (130) with $q=0$. Its gapless bosonic fields on the left edge $l=1$ are

$$\phi_l^1\equiv\varphi_1^c,\quad \phi_1^2\equiv\varphi_1^s+2\theta_l^c, \tag{133}$$

while on the right edge $l=N_w$ the gapless boson fields are

$$\tilde\phi_{N_w}^1\equiv\varphi_{N_w}^c-2\theta_{N_w}^s,\quad \tilde\phi_{N_w}^2\equiv\varphi_{N_w}^s. \tag{134}$$

Under time reversal $\boldsymbol{T}$, the original boson fields $\{\varphi_l^c,\theta_l^c,\varphi_l^s,\theta_l^s\}$ naturally transform as

$$\boldsymbol{T}\varphi_l^c\boldsymbol{T}^{-1}=-\varphi_l^c,\quad \boldsymbol{T}\theta_l^c\boldsymbol{T}^{-1}=\theta_l^c,$$
$$\boldsymbol{T}\varphi_l^s\boldsymbol{T}^{-1}=\varphi_l^s+\pi,\quad \boldsymbol{T}\theta_l^s\boldsymbol{T}^{-1}=-\theta_l^s,$$

since all components of the spin $S^z\sim\partial_x\theta^s,S^+\sim\exp(\mathrm{i}\varphi^s)$ change sign under time reversal. It's easy to verify that interwire scattering terms (130) with $q=0$ is invariant under time reversal as long as we choose $\lambda_l=0$. Hence these interwire couplings are allowed by symmetry. It is also straightforward to show that the pair of bosonic fields, i.e., both $\{\phi_l^1(x),\phi_l^2(x),l=1\}$ and $\{\tilde\phi_l^1(x),\tilde\phi_l^2(x),l=N_w\}$ transform in the same way as Eqs. (125) and (132) under $U(1)\rtimes Z_2^T$ symmetry. Hence the strong coupling phase of interwire couplings (130) with $q=0$, indeed, corresponds to the nontrivial bosonic SPT phase in the presence of $U(1)\rtimes Z_2^T$ symmetry.

D. Fermionic SPT phases with $Z_2\times Z_2^f$ symmetry

Here, we show that fermionic SPT phase $[\eta=0,t_1=1,t_2=0]$ with $W^g=I_{2\times2}$ discussed in Sec. V can be constructed in the coupled wire approach. Its coupled-wire construction also indicates this SPT phase responsible for the $\mathbb{Z}_4$ group structure can be obtained from noninteracting fermion band structures. The edge variables $\{\phi_1,\phi_2\}$ satisfy Kac-Moody algebra (73) and transform in the following way under Z_2 symmetry $\boldsymbol{g}$:

$$\boldsymbol{g}:\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\to\begin{pmatrix}\phi_1+\pi\\ \phi_2\end{pmatrix} \tag{135}$$

and under fermion parity $Z_2^f=\{\boldsymbol{e},\boldsymbol{P}_f\}$:

$$\boldsymbol{P}_f:\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\to\begin{pmatrix}\phi_1+\pi\\ \phi_2+\pi\end{pmatrix}. \tag{136}$$

Consider that right now each quantum wire contains electrons of both spins, i.e., two left movers $\psi_{\uparrow/\downarrow}^L$ and two right movers $\psi_{\uparrow/\downarrow}^R$:

$$\psi_{l,\uparrow/\downarrow}^R\sim\exp[\mathrm{i}(\varphi_{l,\uparrow/\downarrow}+\theta_{l,\uparrow/\downarrow}+k_{\uparrow/\downarrow}x)],$$
$$\psi_{l,\uparrow/\downarrow}^L\sim\exp[\mathrm{i}(\varphi_{l,\uparrow/\downarrow}-\theta_{l,\uparrow/\downarrow}-k_{\uparrow/\downarrow}x)],$$

where bosonic fields $\varphi_{l,\sigma}$ and $\theta_{l,\sigma}$ satisfy commutation relation (115). Let us assume here Z_2 symmetry $\boldsymbol{g}$ is the fermion number parity of spin-↑ fermions, which is naturally realized by

$$\boldsymbol{g}:\varphi_{l,\uparrow}(x)\to\varphi_{l,\uparrow}(x)+\pi, \tag{137}$$

while $\theta_{l,\uparrow},\varphi_{l,\downarrow},\theta_{l,\downarrow}$ remain invariant under $\boldsymbol{g}$. On the other hand, under total fermion parity $\boldsymbol{P}_f$, we have

$$\boldsymbol{P}_f:\varphi_{l,\uparrow/\downarrow}(x)\to\varphi_{l,\uparrow/\downarrow}(x)+\pi, \tag{138}$$

where $\theta_{l,\uparrow},\theta_{l,\downarrow}$ remain invariant under $\boldsymbol{P}_f$.

By defining the following variables:

$$\phi_l^R=\varphi_{l,\uparrow}+\theta_{l,\uparrow},\quad \phi_l^L=\varphi_{l,\downarrow}-\theta_{l,\downarrow}, \tag{139}$$

and

$$\tilde\phi_l^L=\varphi_{l,\uparrow}-\theta_{l,\uparrow},\quad \tilde\phi_l^R=\varphi_{l,\downarrow}+\theta_{l,\downarrow}, \tag{140}$$

it is easy to verify that both $\{\phi_l^R,\phi_l^L\}$ and $\{\tilde\phi_l^R,\tilde\phi_l^L\}$ satisfy Kac-Moody algebra (73) and the symmetry transformations (135) and (136). The two pairs of variables commute with each other. The two variables $\{\phi_l^R,\phi_l^L\}$ are nothing but the right mover for spin-↑ and left mover for spin-↓. Clearly, the following single-fermion tunneling terms

$$\begin{aligned}\mathcal{H}_{Z_2\times Z_2^f}^1&=\sum_{l=1}^{N_w-1}A_l\psi_{l,\uparrow}^{L\ \dagger}\psi_{l+1,\downarrow}^R+B_l\psi_{l,\downarrow}^{R\ \dagger}\psi_{l+1,\uparrow}^L+\text{H.c.}\\&=\sum_{l=1}^{N_w-1}C_l\cos\left(\tilde\phi_l^L-\phi_{l+1}^R+\lambda_l\right)\\&\quad+D_l\cos\left(\tilde\phi_l^R-\phi_{l+1}^L+\tilde\lambda_l\right)\end{aligned}$$

will gap out everything in the bulk in its strong coupling phase. The gapless variables are $\{\phi_1^R,\phi_1^L\}$ on the left edge $l=1$: they do transform as Eqs. (135) and (136). On the right edge $l=N_w$, the gapless variables are $\{\tilde\phi_{N_w}^L,\tilde\phi_{N_w}^R\}$. Since the interwire scattering term that stabilizes this SPT phase is just a single-electron tunneling term, we expect such a fermionic SPT phase should be realized in a noninteracting band structure with symmetry $Z_2\times Z_2^f$.

VII. CONCLUDING REMARKS

We have discussed an algebraic method to systematically classify interacting topological phases in two dimensions in the absence of topological order. The key development is a general formalism for incorporating symmetry transformations into the **K** matrix formalism. Various examples of interacting boson and fermion topological phases that are well defined in the presence of disorder were presented. The method provides both long-wavelength information of these phases (bulk effective field theory and edge theory) as well as suggests microscopic realizations in model systems (such as quasi 1D realizations presented here). It also opens the door to study various transitions out of these phases, e.g., topology or symmetry changing transitions driven by disorder and interactions. Future work will focus on extending these results to symmetry protected distinctions between topologically ordered states and extending this formalism to $d = 1$ and 3. It remains to be seen if a more general formalism that subsumes the present one can be devised, which, for example, can handle unpaired Majorana edge modes. A deeper understanding of the remarkable connection between this formalism and the Borel group cohomology/supercohomology is also needed. Perhaps the most important question is to determine how topological phases of the interacting variety can be obtained in an experimentally realistic setting. We leave these questions for future work.

While completing this manuscript, Ref. 60 appeared, which studies the specific case of $G = U(1) \rtimes Z_2^T$ (topological insulators) using a K matrix approach. Our results agree in the areas where they overlap.

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APPENDIX A: A SHORT NOTE ON $GL(N,\mathbb{Z})$

$GL(N,\mathbb{Z})$ is the group of all $N \times N$ unimodular matrices. All $GL(N,\mathbb{Z})$ matrices can be generated by the following basic transformations ($i \neq j$):

$$\begin{aligned} T^{(i,j)}_{a,b} &= \delta_{a,b} + \delta_{a,i}\delta_{b,j}, \\ S^{(i,j)}_{a,b} &= \delta_{a,b}(1-\delta_{a,i})(1-\delta_{a,j}) + \delta_{a,j}\delta_{b,i} - \delta_{a,i}\delta_{b,j}, \\ D_{a,b} &= \delta_{a,b} - 2\delta_{a,N}\delta_{b,N}. \end{aligned} \quad \text{(A1)}$$

$T^{(i,j)}\mathbf{K}$ will add the jth row of matrix $\mathbf{K}$ to the ith row of $\mathbf{K}$, while $S^{(i,j)}\mathbf{K}$ will exchange the ith and jth rows of $\mathbf{K}$ with a factor of -1 multiplied on the ith row. $D\mathbf{K}$ will just multiply the Nth row of $\mathbf{K}$ by a factor of -1. $\mathbf{K}T^{(i,j)}$, $\mathbf{K}S^{(i,j)}$, and $\mathbf{K}D$ correspond to similar operations to columns (instead of rows). A subgroup of $GL(N,\mathbb{Z})$ with determinant $+1$ is called $SL(N,\mathbb{Z})$ and it's generated by $\{T^{(i,j)},S^{(i,j)}\}$.

As a simple example when $N = 2$, group $GL(2,\mathbb{Z})$ is generated by the following basic transformations:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \text{(A2)}$$

The following results will be useful:

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad (-STS)^n = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}, \quad n \in \mathbb{Z}.$$

APPENDIX B: A THEOREM ON 2 × 2K MATRICES WITH DET K = $-n^2$

In general, a gauge transformation, which relabels the quasiparticles in the **K** matrix formulation (2), is implemented by a $GL(N,\mathbb{Z})$ matrix **W** as shown in Eqs. (7) and (8). Therefore in the absence of any symmetry, any two **K** matrices related by Eq. (8) are equivalent to each other, i.e.,

$$\mathbf{K} \simeq \mathbf{M}^T\mathbf{K}\mathbf{M}, \quad \forall\, \mathbf{M} \in GL(n,\mathbb{Z}). \quad \text{(B1)}$$

We use $\simeq$ to denote the equivalency. In the presence of $U(1)$ symmetry with a charge vector **t**, the equivalency requires

$$\{\mathbf{K},\mathbf{t}\} \simeq \{\mathbf{M}^T\mathbf{K}\mathbf{M},\mathbf{M}^T\mathbf{t}\}, \quad \forall\, \mathbf{M} \in GL(n,\mathbb{Z}). \quad \text{(B2)}$$

In the absence of any symmetry, here we prove the following theorem: *any* $2 \times 2\mathbf{K}$ *matrix with determinant* $\det \mathbf{K} = -n^2$ *can be transformed into the standard form*

$$\begin{pmatrix} 0 & n \\ n & a \end{pmatrix}, \quad 0 \leqslant a \leqslant 2n-1.$$

In the special case of SRE phases with $n = 1$,

$$\begin{aligned} &\begin{pmatrix} 0 & n \\ n & 2a \end{pmatrix}, \quad 0 \leqslant a \leqslant n-1 \text{ for bosons}, \\ &\begin{pmatrix} 0 & n \\ n & 2a+1 \end{pmatrix}, \quad 0 \leqslant a \leqslant n-1 \text{ for fermions}. \end{aligned} \quad \text{(B3)}$$

A generic $2 \times 2\mathbf{K}$ matrix with determinant $-n^2$ can be written as

$$\mathbf{K}_{2\times2} = \begin{pmatrix} a & n+k \\ n+k & b \end{pmatrix}, \quad ab = k(2n+k), \quad a,b,k \in \mathbb{Z}. \quad \text{(B4)}$$

Apparently, for a bosonic system, a and b are both even integers and k is also an even integer. For a fermionic system there are two possibilities: a,b,k are all odd integers, $k = 0$ or $-2n$ and one of a,b equals zero while the other is an odd integer.

Notice that under $GL(2,\mathbb{Z})$ transformations σ_x and $i\sigma_y$ ($\sigma_\alpha,\alpha = x,y,z$ are Pauli matrices), we have

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \simeq \begin{pmatrix} c & b \\ b & a \end{pmatrix} \simeq \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}. \quad \text{(B5)}$$

If $a = 0$ or $b = 0$, we have $k = 0$ or $k = -2n$, i.e., $n + k = \pm n$ in Eq. (B4). Using relations (B5) and generator T in Eq. (A2), one can show that

$$\mathbf{K} \simeq \begin{pmatrix} 0 & n \\ n & x \end{pmatrix} \simeq T\begin{pmatrix} 0 & n \\ n & x \end{pmatrix}T^T = \begin{pmatrix} 0 & n \\ n & x+2n \end{pmatrix},$$

and Eq. (B3) can be easily verified.

If $ab \neq 0$, without loss of generality, we can assume that $|a| \leqslant |b|$ and therefore $|a| \leqslant \max(|k|,|2n+k|) = \max(|\mathbf{K}_{1,2} - n|,|\mathbf{K}_{1,2} + n|)$. We use the following strategy: if $|k| \leqslant |2n+k|$, choose $\mathbf{M} = [\begin{smallmatrix} 1 & -\mathrm{sign}(a(2n+k)) \\ 0 & 1 \end{smallmatrix}]$ in Eq. (B1) so that $|\mathbf{K}_{1,2} + n| \to |\mathbf{K}_{1,2} + n| - |a|$; if $|k| > |2n+k|$, choose $\mathbf{M} = [\begin{smallmatrix} 1 & -\mathrm{sign}(ak) \\ 0 & 1 \end{smallmatrix}]$ in Eq. (B1) so that $|\mathbf{K}_{1,2} - n| \to |\mathbf{K}_{1,2} - n| - |a|$. The value of $\max(|\mathbf{K}_{1,2} - n|,|\mathbf{K}_{1,2} + n|)$ will decrease monotonically when this procedure is repeated and finally one will end up with a $2\times 2\mathbf{K}$ matrix whose off-diagonal elements are $\pm n$. This means $ab=0$ in Eq. (B4). Therefore theorem (B3) is proved.

APPENDIX C: A THEOREM ON BOSONIC 2 × 2K MATRICES WITH DET K = −1 AND CHARGE VECTOR T

In this section, we prove the following theorem [we use (a,b) to denote the greatest common divisor of two integers a and b]: *for a* $2+1$*-D bosonic system, any* $\mathbf{K}$ *with* $\det\mathbf{K}=-1$ *and a charge vector* $\mathbf{t}=\binom{t_1}{t_2}$ *with* $(t_1,t_2)=1$ *is equivalent to* $\mathbf{K}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ *and* $\mathbf{t}=\binom{1}{-l}, l\in\mathbb{Z}$ *by a* $GL(2,\mathbb{Z})$ *gauge transformation, i.e.,*

$$\left\{\mathbf{K},\mathbf{t}=\begin{pmatrix}t_1\\t_2\end{pmatrix}\right\}\simeq\left\{\begin{pmatrix}0&1\\1&2l\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}\right\}\simeq\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1\\-l\end{pmatrix}\right\},\quad \text{if}\quad (t_1,t_2)=1,\quad l\in\mathbb{Z}. \tag{C1}$$

First, we notice that according to the Euclidean division algorithm on integers $\mathbb{Z}$, for any pairs of integers, e.g., t_1 and t_2 here, there is always a list of arrays $[q_1,q_2,\ldots,q_{n+1}]$ and $[r_1,r_2,\ldots,r_n]$ such that (let's assume $|t_1|\geqslant|t_2|$ without loss of generality)

$$t_1=q_1t_2+r_1,\quad t_2=q_2r_1+r_2,$$
$$r_1=q_3r_2+r_3,\ldots,\quad r_{n-2}=q_nr_{n-1}+r_n,\quad r_{n-1}=q_{n+1}r_n,$$

where $r_n=(t_1,t_2)$ is the *greatest common divisor* of t_1 and t_2, and $1\leqslant|r_{m+1}|\leqslant|r_m|,\forall\, m$. Therefore one can always find two integers u_1 and u_2 such that

$$(t_1,t_2)=r_n=r_{n-2}-q_nr_{n-1}=\cdots=t_1u_2-t_2u_1. \tag{C2}$$

As a result for $(t_1,t_2)=1$, we have

$$\begin{pmatrix}t_1\\t_2\end{pmatrix}=\mathbf{M_0}^T\begin{pmatrix}1\\0\end{pmatrix},\quad \mathbf{M_0}^T\equiv\begin{pmatrix}t_1&u_1\\t_2&u_2\end{pmatrix}\in GL(2,\mathbb{Z}), \tag{C3}$$

and hence

$$\{\mathbf{K},\mathbf{t}\}\simeq\left\{\mathbf{K}'\equiv\left(\mathbf{M_0}^{-1}\right)^T\mathbf{K}\mathbf{M_0}^{-1},\begin{pmatrix}1\\0\end{pmatrix}\right\} \tag{C4}$$

as long as $(t_1,t_2)=1$. In the following, we prove that an arbitrary $2\times 2\mathbf{K}$ matrix with $\det\mathbf{K}=-1$ for a bosonic system is equivalent to the standard form $\begin{pmatrix}0&1\\1&2l\end{pmatrix}$ by a gauge transformation that keeps the charge vector $\binom{1}{0}$ invariant. To prove this, we need to enlarge the Hilbert space by introducing a 4×4 matrix $\tilde{\mathbf{K}}=\mathbf{K}_{2\times 2}\oplus\sigma_x$ and associated charge vector $\tilde{\mathbf{t}}\equiv(1,0,0,0)^T$. This describes a direct product of the original 2×2 bosonic $\mathbf{K}$ matrix with a $U(1)$ charge conservation and another trivial 2×2 bosonic $\mathbf{K}$ matrix without any symmetry. A generic form for $\mathbf{K}$ is $\begin{pmatrix}2a&2k+1\\2k+1&2b\end{pmatrix}$ satisfying $ab=k(k+1)$ ($\det\mathbf{K}=-1$). One can prove that

$$\left\{\tilde{\mathbf{K}}=\begin{pmatrix}2a&2k+1&0&0\\2k+1&2b&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}\simeq\left\{\begin{pmatrix}0&1&0&0\\1&2l&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\} \tag{C5}$$

by the following $GL(4,\mathbb{Z})$ transformations:

$$\mathbf{M_1}^T\begin{pmatrix}2a&2k+1&0&0\\2k+1&2b&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\mathbf{M_1}=\begin{pmatrix}0&1&0&0\\1&2b&2(ab-k)&-2b\\0&2(ab-k)&2a(ab-2k)&2(k-ab)+1\\0&-2b&2(k-ab)+1&2b\end{pmatrix}$$
$$\simeq\begin{pmatrix}0&1&0&0\\1&2b&a'&b'\\0&a'&0&1\\0&b'&1&0\end{pmatrix}=\left(\mathbf{M_2}^T\right)^{-1}\begin{pmatrix}0&1&0&0\\1&2(b-a'b')&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\mathbf{M_2}^{-1},\quad a,b,k,a',b'\in\mathbb{Z},$$

where we defined

$$\mathbf{M_1}\equiv\begin{pmatrix}1&0&0&0\\0&1&a&-1\\1&0&1&0\\-a&-2k&-2ka&2k+1\end{pmatrix},\quad \mathbf{M_2}\equiv\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-b'&1&0\\0&-a'&0&1\end{pmatrix}.$$

We have used the theorem (B3) for a $n=1$ bosonic system since one can easily verify that $\det\begin{bmatrix}2a(ab-2k)&2(k-ab)+1\\2(k-ab)+1&2b\end{bmatrix}=-1$. Notice that the charge vector $\mathbf{t}=(1,0,0,0)^T$ remains invariant under these $GL(4,\mathbb{Z})$ transformations. Combing relations (C4) and (C5), we have proved theorem (C1).

APPENDIX D: FAITHFUL VERSUS UNFAITHFUL REPRESENTATIONS OF THE SYMMETRY GROUP

Transformations $\{W^{g_a},\delta\phi_I^{g_a}\}$ *form a faithful or unfaithful representation of symmetry group* G. By solving the constraint equations (26) and choosing a proper gauge in Eq. (29), one can obtain a set of transformation rules $\{W^{g_a},\delta\phi_I^{g_a}\}$ as the solution. Apparently, the transformations $\{W^{g_a},\delta\phi_I^{g_a}\}$ always form a representation of the symmetry group G of the system (or the Hamiltonian of the system) in the sense that

$$\forall \quad g_1,g_2 \in G : W^{g_1\cdot g_2} = \eta_1\eta_2 W^{g_2}W^{g_1}, \\ \delta\phi^{g_1\cdot g_2} = \delta\phi^{g_2} + \eta_2 W^{g_2}\delta\phi^{g_1}, \tag{D1}$$

where $\eta_1 = \pm 1$ if g_1 is a unitary (antiunitary) symmetry and η_2 is associated with g_2. This representation of group G is *faithful* if and only if *the identity element* $\boldsymbol{e}$ *is the only symmetry element under which all bosonic quasiparticle fields* $\{\sum_I l_I\phi_I\}$ *on the edge (or* $\{\prod_I b_I^{l_I}\}$ *in the bulk) transform trivially*. In other words, under any element $g \neq \boldsymbol{e}$ of symmetry group G, at least one bosonic quasiparticles $\mathbf{l}^T\phi$ satisfying Eq. (11) will transform nontrivially.

In contrast to faithful representations, an *unfaithful* representation $\{W^{g_a},\delta\phi_I^{g_a}\}$ of symmetry group G means there exists a *nontrivial subgroup* G_ψ of G, so that *under any symmetry element* $g \in G_\psi$, all *bosonic quasiparticle fields* $\{\sum_I l_I\phi_I\}$ *on the edge (or* $\{\prod_I b_I^{l_I}\}$ *in the bulk) transform trivially*. This means for a phase described by $\mathbf{K}$ matrix and symmetry transformations $\{W^{g_a},\delta\phi_I^{g_a}\}$, its edge states can be gapped by condensing the bosonic quasiparticles without breaking the subgroup G_ψ of symmetry group G since under any symmetry $g \in G_\psi$ all bosonic quasiparticles are left invariant. As a result, when the edge is gapped, the symmetry group G of the Hamiltonian breaks down to its subgroup, the ground state symmetry group G_ψ. As a result, the symmetry breaking phases can be naturally incorporated in the $\mathbf{K}$ matrix + Higgs formulation.

APPENDIX E: OTHER BOSONIC SPT PHASES

1. $U(1) \times Z_2^T$ symmetry: $\mathbb{Z}_1$ class

In contrast to the $U(1) \rtimes Z_2^T$ symmetry discussed in the previous section, here, we study a direct product of $U(1)$ and time reversal Z_2^T symmetry. This can be realized by time reversal and $U(1)$ spin rotational symmetry in an integer spin system. The algebraic relations for the $U(1) \times Z_2^T$ group are

$$\boldsymbol{T}^2 = \boldsymbol{T}U_{-\theta}\boldsymbol{T}U_\theta = \boldsymbol{e}. \tag{E1}$$

The corresponding constraints (28) for symmetry transformations $\{W^{\boldsymbol{T}},\delta\phi^{\boldsymbol{T}}\}$ and $\{W^{U_\theta} = I_{2\times 2},\delta\phi^{U_\theta} = \theta\mathbf{t}\}$ are

$$(I_{2\times 2} - W^{\boldsymbol{T}})\delta\phi^{\boldsymbol{T}} + (I_{2\times 2} + W^{\boldsymbol{T}})\theta\mathbf{t} = \begin{pmatrix}0\\0\end{pmatrix} \bmod 2\pi,\ \forall\ \theta, \tag{E2}$$

and Eqs. (37) and (38). The gauge inequivalent solutions to these constraint equations lead to

$$W^{U_\theta} = I_{2\times 2}, \quad \delta\phi^{U_\theta} = \theta\begin{pmatrix}0\\1\end{pmatrix}, \tag{E3}$$

$$W^{\boldsymbol{T}} = \sigma_z, \quad \delta\phi^{\boldsymbol{T}} = \begin{pmatrix}0\\n\pi\end{pmatrix}, \quad n = 0,1. \tag{E4}$$

For both $n = 0,1$, the corresponding symmetry-allowed Higgs terms are

$$\mathcal{S}^1_{\text{edge}} = \sum_{l\in\mathbb{Z}} C_l \int dxdt\, \cos(l\phi_1). \tag{E5}$$

Apparently, there is only one ($\mathbb{Z}_1$ class) trivial phase $\boldsymbol{e}_{U(1)\times Z_2^T}$ with $U(1) \times Z_2^T$ symmetry, whose edge states can be gapped without breaking the symmetry.

2. $Z_N \times Z_2^T$ symmetry

The algebraic structure for $Z_N \times Z_2^T$ group is given by

$$\boldsymbol{g}^N = \boldsymbol{T}^2 = \boldsymbol{T}\boldsymbol{g}^{-1}\boldsymbol{T}\boldsymbol{g} = \boldsymbol{e}, \tag{E6}$$

where $\boldsymbol{g}$ is the Z_N symmetry generator and $\boldsymbol{T}$ is time reversal. The associated constraint equations for symmetry transformations are

$$W^{\boldsymbol{g}}W^{\boldsymbol{T}}(W^{\boldsymbol{g}})^{-1}W^{\boldsymbol{T}} = I_{2\times 2}, \tag{E7}$$

$$[I_{2\times 2} + W^{\boldsymbol{g}}W^{\boldsymbol{T}}(W^{\boldsymbol{g}})^{-1}]\delta\phi^{\boldsymbol{g}} + W^{\boldsymbol{g}}[1 - W^{\boldsymbol{T}}(W^{\boldsymbol{g}})^{-1}]\delta\phi^{\boldsymbol{T}} \\ = \begin{pmatrix}0\\0\end{pmatrix} \quad \bmod \quad 2\pi, \tag{E8}$$

in addition to Eqs. (37), (38), (53), and (54).

a. $N =$ odd integer: $\mathbb{Z}_1$ classes

The gauge inequivalent solutions to the above constraint equations are Eq. (39) and

$$W^{\boldsymbol{g}} = I_{2\times 2}, \quad \delta\phi^{\boldsymbol{g}} = \frac{2\pi k}{N}\begin{pmatrix}0\\1\end{pmatrix}, \quad k \in \mathbb{Z}. \tag{E9}$$

It is easy to verify that a set of independent Higgs terms are $\int dxdt \sum_l C_l \cos(l\phi_1)$. Hence the variable ϕ_1 can be localized at value $\langle\phi_1\rangle = 0$ without breaking any symmetry. So they all correspond to the trivial phase. There is only one trivial phase with $Z_N \times Z_2^T$ symmetry for $N =$ odd.

b. $N =$ even integer: Minimal set, $\mathbb{Z}_2^2$ classes

Solving Eqs. (53) and (E7), we have $W^{\boldsymbol{T}} = \sigma_z$ and $W^{\boldsymbol{g}} = \pm I_{2\times 2}$. (i) For $W^{\boldsymbol{g}} = I_{2\times 2}$, the gauge inequivalent solutions are

$$\delta\phi^{\boldsymbol{T}} = n_2\pi\begin{pmatrix}0\\1\end{pmatrix}, \quad \delta\phi^{\boldsymbol{g}} = \pi\begin{pmatrix}n_1\\2k/N\end{pmatrix}, \\ 0 \leqslant k \leqslant N-1, \quad n_1,n_2 = 0,1. \tag{E10}$$

If $n_1 = 0$, the variable ϕ_1 can be localized at $\langle\phi_1\rangle = 0$ by Higgs term $-\cos\phi_1$ without breaking the symmetry and it corresponds to the trivial phase. If $n_1 = 1$, $n_2 = k = 0$ corresponds to the trivial phase again since ϕ_2 can be localized. Notice that when $n_1 = 1$ we require $(k,N/2) = 1$ so that transformations (E10) form a faithful representation of symmetry group $G = Z_N \times Z_2^T, N =$ even. Let us label a state with the above transformations (E10) as $[k,n_1,n_2]$ and we have $[0,1,0] = [k,0,n_2] = \boldsymbol{e}_{Z_N\times Z_2^T}$. In the following, we analyze the group structure formed by states $[k,n_1,n_2]$.

Now let us put together a state $[k,n_1,n_2]$ with edge variable $\{\phi_1,\phi_2\}$ with a state $[k',n_1',n_2']^{-1}$ with edge variable $\{\phi_1',\phi_2'\}$, we can choose the following independent bosonic variables $\{k'\phi_2 - k\phi_2',k\phi_1 - k'\phi_1'\}$ and gap all the edge states if $(k,k') = 1$. The associated Higgs terms will preserve the $Z_N \times Z_2^T$ symmetry if $kn_1 - k'n_1' = 0 \bmod 2$ and $k'n_2 - kn_2' = 0 \bmod 2$. As a result, $[k,n_1,n_2] \oplus [k',n_1',n_2']^{-1} = e_{Z_N \rtimes Z_2^T}$ or, equivalently,

$$\text{for} \quad (k,k') = 1 : [k,n_1,n_2] = [k',n_1',n_2'],$$
$$\text{if} \quad kn_1 - k'n_1' = 0 \mod 2, \quad k'n_2 - kn_2' = 0 \mod 2.$$

Therefore we have $[2k+1,n_1,n_2] = [1,n_1,n_2]$ by choosing $k' = 1$. On the other hand, if $k = \text{even}$, we again have $N/2 = \text{odd}$ since $(k,N/2) = 1$ for a faithful representation of symmetry group $Z_N \times Z_2^T$. We can localize the bosonic variable $\{\frac{N}{2}\phi_2 - \phi_2', \frac{N}{2}\phi_1' - \phi_1\}$ without breaking any symmetry if we choose $k' = 0$. Hence we also have $[2k,n_1,n_2] = [0,n_1,n_2]$. These relations result in only three nontrivial SPT phases: [1,1,0], [1,1,1] and [0,1,1].

Similarly, by putting together a state $[k,1,n_2]$ with edge variable $\{\phi_1,\phi_2\}$ with a state $[k',1,n_2']$ with edge variable $\{\phi_1',\phi_2'\}$, we can always localize the bosonic variable $\phi_1 - \phi_1'$ and gap out part of the edge. What is left on the edge is described by variables $\{\tilde\phi_1 = \phi_1, \tilde\phi_2 = \phi_2 + \phi_2'\}$. They obey Kac-Moody algebra (35) and transform as a $[k+k',1,n_2+n_2']$ state. Hence, we have shown that

$$[k,1,n_2] \oplus [k',1,n_2'] = [k+k' \mod 2, 1, n_2 + n_2' \mod 2]. \tag{E11}$$

Since k,n_2 are both $\mathbb{Z}_2$ integers, so clearly all different four states $[k,1,n_2]$ form a $\mathbb{Z}_2^2$ group. Consequently, there are 3 nontrivial SPT phases labeled by $n_1 = 1$ and $[k,n_2] = [0,1],[1,0]$ or $[1,1]$ in Eqs. (39) and (E10).

(ii) For $W^g = -I_{2\times2}$, we can always choose a gauge so that inequivalent solutions to constraints are

$$\delta\phi^g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \delta\phi^T = \pi \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad n_1,n_2 = 0,1. \tag{E12}$$

However, the above symmetry transformations $\{W^g = -I_{2\times2}, \delta\phi^g = \binom{0}{0}\}$ do not correspond to a faithful representation of $Z_N \times Z_2^T$ group for $N = \text{even}$, unless $N = 2$. And it is not clear how the states with symmetry transformations $W^g = -I$ can be realized in a physical bosonic system. Therefore we will not include the states with symmetry transformations $W^g = -I$ in the minimal set of topological phases with $Z_N \times Z_2^T$ symmetry, as discussed earlier for $W^g = -I$ phases with $Z_N \rtimes Z_2^T$ symmetry.

In summary, there are $\mathbb{Z}_2^2$ classes of different bosonic nonchiral SRE phase in the presence of symmetry group $Z_N \times Z_2^T, N = \text{even}$. When $N = \text{odd}$, there are no nontrivial SPT phases.

3. $U(1) \times Z_2$ symmetry: $\mathbb{Z} \times \mathbb{Z}_2^2$ classes

Denoting the group elements of $U(1)$ by $U_\theta, 0 \leqslant \theta < 2\pi$ and generator of Z_2 by g, the group $U(1) \times Z_2$ has the following algebraic structure:

$$g^2 = U_\theta g U_{-\theta} g = e \tag{E13}$$

and Eq. (41). The associated constraints for symmetry transformations are

$$(W^g)^2 = I_{2\times2}, \quad (I_{2\times2} + W^g)\delta\phi^g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{E14}$$

$$(I_{2\times2} + W^g)\delta\phi^g + \theta(I_{2\times2} - W^g)\mathbf{t} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{E15}$$

where we have $W^{U_\theta} = I_{2\times2}$ $\delta\phi^{U_\theta} = \theta\mathbf{t}$ with $t_1,t_2 \in \mathbb{Z}$ and $(t_1,t_2) = 1$ for $U(1)$ symmetry. Solving Eq. (E14), we have $W^g = \pm I_{2\times2}$ or $\pm\sigma_x$.

a. $\mathbb{Z} \times \mathbb{Z}_2^2$ classes with $W^g = \pm I_{2\times2}$

(i) For $W^g = I_{2\times2}$, as guaranteed by theorem (C1), we can always transform the "charge vector" $\mathbf{t}$ into a standard form $\mathbf{t} = \binom{1}{q}, q \in \mathbb{Z}$. And the inequivalent "faithful" symmetry transformations satisfying constraints (E14) and (E15) are

$$\delta\phi^g = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}\pi, \quad \delta\phi^{U_\theta} = \theta\begin{pmatrix} 1 \\ q \end{pmatrix}, \quad n_1,n_2 = 0,1, \quad q \in \mathbb{Z}. \tag{E16}$$

Let us label a state with the above transformation rules as $[q,n_1,n_2]$. Similar as earlier discussions for other symmetries, by putting two states $[q,n_1,n_2]$ and $[q',n_1',n_2']$ together, one can show the following multiplication rule:

$$[q,n_1,n_2] \oplus [q',n_1',n_2'] = [q+q', n_1+n_1' \mod 2, n_2+n_2' \mod 2]. \tag{E17}$$

Hence there are $\mathbb{Z} \times (\mathbb{Z}_2)^2$ classes of different phases labeled by integer q and $\mathbb{Z}_2$ integers n_1,n_2. The trivial phase corresponds to $q = n_1 = n_2 = 0$.

(ii) For $W^g = -I_{2\times2}$, one can always choose a gauge so that $\delta\phi^g = \binom{0}{0}$ and by solving Eq. (E15), we get $\mathbf{t} = \binom{0}{0}$. It is easy to verify that this corresponds to the trivial phase.

b. Other solutions to Eqs. (E14) and (E15) with $W^g = \pm\sigma_x$

(iii) For $W^g = \sigma_x$, the inequivalent solutions to constraint equations are

$$\delta\phi^g = n\pi\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \delta\phi^{U_\theta} = \theta\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad n = 0,1. \tag{E18}$$

These two nontrivial SPT phases are labeled as $[\sigma_x,n]$ with $n = 0,1$. Their physical realization and group structure are not clear.

(iv) For $W^g = -\sigma_x$, the inequivalent solutions to constraint equations are

$$\delta\phi^g = n\pi\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \delta\phi^{U_\theta} = \theta\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad n = 0,1. \tag{E19}$$

These two nontrivial SPT phases are labeled as $[-\sigma_x,n]$ with $n = 0,1$. It is easy to show that

$$[\sigma_x,n]^{-1} = [-\sigma_x,n], \quad n = 0,1. \tag{E20}$$

Their physical realization and group structure are not clear either as with the discussion in Sec. IV D3 and therefore we do not include these phases.

To summarize, there are $\mathbb{Z} \times (\mathbb{Z}_2)^2$ classes of different phases with $W^T = \pm I_{2\times2}$ for symmetry group $U(1) \times Z_2$. Besides, there are four extra possible nontrivial SPT phases with $W^g = \pm\sigma_x$ for $U(1) \times Z_2$ symmetry in a bosonic nonchiral SRE system.

APPENDIX F: OTHER SOLUTIONS TO EQ. (98) FOR FERMION SPT PHASES WITH $G_f/Z_2^f = Z_2$ SYMMETRY

a. $W^g = \pm\sigma_z$: $\mathbb{Z}^2$ classes

(i) For $W^g = \sigma_z$, the inequivalent solutions to Eq. (98) are $\eta = 0$ and $\delta\phi^g = \binom{n\pi}{0}$, where $n = 0,1$. Since under symmetry $\boldsymbol{g}$ we have $\phi_1 \pm \phi_2 \to \phi_1 \mp \phi_2 + n\pi$, the variables cannot be localized without breaking the Z_2 symmetry. These two different nontrivial SPT phases are labeled as $[\sigma_z,n]$, where $n = 0,1$. In the following, we identify their group structure.

First, notice that when a $[\sigma_z,0]$ state with edge variables $\{\phi_1,\phi_2\}$ is put together with a $[\sigma_z,1]^{-1}$ state with edge variables $\{\phi_1',\phi_2'\}$, its edge cannot be gapped without breaking the symmetry, suggesting $[\sigma_z,0] \neq [\sigma_z,1]$. Then let us consider N copies of $[\sigma_z,n]$ states put together and their edge variables are $\{\phi_1^a,\phi_2^a,1 \leqslant a \leqslant N\}$. A generic bosonic variable that can be localized on the edge is written as $\sum_{a=1}^N (A_a\phi_1^a + B_a\phi_2^a), A_a, B_a \in \mathbb{Z}$ satisfying $\sum_a A_a^2 - B_a^2 = 0$ due to condition Eq. (19). Under Z_2 symmetry generator $\boldsymbol{g}$, this bosonic variable becomes $\sum_{a=1}^N (A_a\phi_1^a - B_a\phi_2^a)$. In order for the two bosonic variables to be localized simultaneously (i.e., they are independent bosons), they have to satisfy Eq. (20) and hence $\sum_a A_a^2 + B_a^2 = 0$. This leads to $A_a = B_a = 0$ and hence no bosonic variable on the edge can be localized without breaking the symmetry. Hence, whenever we add an extra $[\sigma_z,0]$ state into the system, there is one more $c = 1$ gapless state on the edge. Hence all the different states $\{[\sigma_z,0]^M \oplus [\sigma,1]^N, M,N \in \mathbb{Z}\}$ form the $\mathbb{Z}^2$ group.

(ii) For $W^g = -\sigma_z$, the inequivalent solutions to (98) are $\eta = 0$ and $\delta\phi^g = \binom{0}{n\pi}$ where $n = 0,1$. We label these states by $[-\sigma_z,n]$ and it is straightforward to show that $[\sigma_z,n]^{-1} = [-\sigma_z,n]$.

To summarize, with Z_2 symmetry transformation $W^g = \pm\sigma_z$, there are $\mathbb{Z}^2$ classes of different fermionic nonchiral SRE phases in the presence of $Z_2 \times Z_2^f$ symmetry. It is presently unclear to us if these transformation laws can be realized in a physical system of fermions. We have not found a microscopic realization, hence we do not include it in the minimal set of topological phases with this symmetry.

APPENDIX G: OTHER SOLUTIONS TO (110) FOR FERMION SPT PHASES WITH $G_f/Z_2^f = Z_2 \times Z_2^T$ SYMMETRY

a. $W^g = -I_{2\times 2}$: $\mathbb{Z}_2$ classes

(i) If $W^g = -I_{2\times 2}$, the gauge inequivalent solutions to Eq. (110) are $\eta_g = 0$ and

$$\delta\phi^g = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \delta\phi^T = \left(\frac{\eta}{2} + n\right)\pi\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\eta_T\pi}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad n,\eta,\eta_T = 0,1. \tag{G1}$$

If $\eta_T = 0$, we can destroy the gapless edge states without breaking the symmetry, by localizing the bosonic variable $\phi_1 - \phi_2$ at a classical value. If $\eta = 0$, we can destroy the gapless edge states without breaking the symmetry, by localizing a diffent bosonic variable $\phi_1 + \phi_2$ at a classical value. Hence only $\eta = \eta_T = 1$ and $n = 0,1$ correspond to nontrivial SPT phases, with $\delta\phi^T = (\pi,0)^T$ or $(0,\pi)^T$ ($n = 0$ or 1). Let's label the states with symmetry transformations (G1) by $[\eta,\eta_T,n]$. It is easy to verify that $[1,1,0] \oplus [1,1,0] = [1,1,0] \oplus [1,1,1] = \boldsymbol{e}_{Z_2\times Z_2^T\times Z_2^f}$, since the edge from two phases put together can be gapped by condensing independent bosons $\{\phi_1 + \phi_2', \phi_2 + \phi_1'\}$. Since $[0,0,n] = [0,1,n] = [1,0,n] = \boldsymbol{e}_{Z_2\times Z_2^T\times Z_2^f}$ we see that different phases form a $\mathbb{Z}_2$ group. The only nontrivial SPT phase is $[1,1,0] = [1,1,1]$. The microscopic realization of this particular SPT phase is not clear, (we have not found a realization in the coupled wire approach) and as discussed previously, we omit it from the minimal set of topological phases.

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