Interplay of superconductivity and spin-density-wave order in doped graphene

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We study the interplay between superconductivity and spin-density-wave order in graphene doped to 3/8 or 5/8 filling (a van Hove doping). At this doping level, the system is known to exhibit weak-coupling instabilities to both chiral $d + id$ superconductivity and to a uniaxial spin density wave. Right at van Hove doping, the superconducting instability is strongest and emerges at the highest T_c, but slightly away from van Hove doping, a spin density wave likely emerges first. We investigate whether at some lower temperature superconductivity and spin density waves coexist. We derive the Landau-Ginzburg functional describing interplay of the two order parameters. Our calculations show that superconductivity and spin-density-wave order do not coexist and are separated by first-order transitions, either as a function of doping or as a function of T.

I. INTRODUCTION

Two-dimensional electron systems provide an ideal environment for exploration of many-body physics. Graphene, as a new two-dimensional electron system, may allow us to access new many-body phases that have not been hitherto observed. Unfortunately, undoped single-layer graphene seems to be well described by a noninteracting model,[1] with the vanishing density of states suppressing interaction effects. In order to access many-body physics in graphene, one must sidestep the vanishing density of states. One way to do this is by doping. When graphene is doped to the M point of the Brillouin zone, a doping level that corresponds to 3/8 (or 5/8) filling (undoped graphene corresponds to 1/2 filling), the Fermi surface undergoes a topological transition from a two-piece to a one-piece Fermi surface.[2] Associated with this topological transition is a divergent density of states, which gives rise to weak-coupling instabilities to unusual many-body states. Doped graphene thus provides a promising playground for exploration of new quantum many-body states.

The recent success of experimental efforts to dope graphene to the M point[3] has inspired a flurry of theoretical works studying many-body physics in doped graphene.[4–14] It has been established[4] that the principal weak-coupling instabilities are to chiral $d + id$ superconductivity and to a uniaxial spin density wave (SDW). The superconductivity arises from spin fluctuation exchange in a model that starts with weak, purely repulsive electronic interactions. This is in contrast to studies of graphene at half-filling, where superconductivity arises either due to phonons[15] or due to strong interelectron interactions.[16,17] It is known from a renormalization-group analysis[4] that the superconducting instability is leading at van Hove doping,[4] and the SDW is leading somewhat away from van Hove doping.[11] However, it is not known whether these two orders are mutually exclusive, or whether they coexist in some range of temperatures and dopings.

In this paper, we demonstrate that for graphene near the M point, superconductivity and SDW magnetism are mutually exclusive orders. We derive Landau-Ginzburg action for

two order parameters and show that the interplay between quartic terms is such that the minimum of the action is when only one order parameter is nonzero. This result stands in stark contrast to pnictide materials, where Landau-Ginzburg analysis shows that superconducting and SDW orders do coexist.[18,19] In doped graphene, one expects to observe pure chiral superconductivity at van Hove doping, with a first-order transition to pure spin-density-wave order upon doping away from the van Hove point. Our conclusions apply also to doped triangular lattice systems,[20] which have an identical low-energy description near 3/4 filling.

II. THE MODEL

Our point of departure is the tight-binding model[2] with the nearest-neighbor dispersion

$$\varepsilon_{\mathbf{k}} = \pm t \sqrt{1 + 4\cos\frac{k_y\sqrt{3}}{2}\cos\frac{3k_x}{2} + 4\cos^2\frac{k_y\sqrt{3}}{2}} - \mu, \quad (1)$$

where the overall sign is $+$ or $-$ depending on whether we are above or below half-filling. For definiteness, we take a plus sign. The van Hove doping then corresponds to $\mu = t$, at which point the Fermi surface has the form shown in Fig. 1. The Fermi velocity vanishes near the hexagon corners $\mathbf{M_1} = (2\pi/3, 0)$, $\mathbf{M_2} = (\pi/3, \pi/\sqrt{3})$, $\mathbf{M_3} = (-\pi/3, \pi/\sqrt{3})$, which are saddle points of the dispersion:

$$\varepsilon_{k\approx M_1} = \frac{3t}{4}(3k_x^2 - k_y^2), \quad \varepsilon_{k\approx M_2} = \frac{3t}{4}2k_y(k_y - \sqrt{3}k_x),$$
$$\varepsilon_{k\approx M_3} = \frac{3t}{4}2k_y(k_y + \sqrt{3}k_x). \quad (2)$$

Each time, k is a deviation from a saddle point. Saddle points give rise to a logarithmic singularity in the density of states (DOS) and control physics at weak coupling. There are three inequivalent nesting vectors Q_{ab} connecting inequivalent pairs of saddle points $\mathbf{M}_a$ and $\mathbf{M}_b$ (see Fig. 1):

$$Q_1 = Q_{23} = (\pi, \pi/\sqrt{3}), \quad Q_2 = Q_{31} = (\pi, -\pi/\sqrt{3}),$$
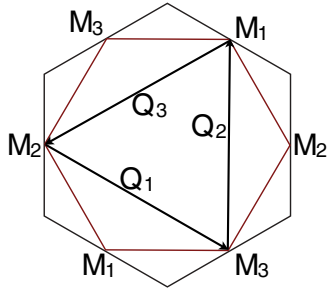$$Q_3 = Q_{12} = (0, 2\pi/\sqrt{3}). \quad (3)$$

FIG. 1. (Color online) The Fermi surface at van Hove doping is a perfect hexagon inscribed within the hexagonal Brillouin zone. The hexagon has three inequivalent corners, labeled $M_{1,2,3}$, which are saddle points of the dispersion and give rise to a divergent density of states. Each saddle point is perfectly nested with each other saddle point. The perfect nesting of the Fermi surface is broken only by third and higher neighbor hoppings, which are generally quite small. Meanwhile, the existence of saddle points is fully robust, being a consequence of a topological transition from a Fermi surface with two inequivalent pieces to a one-piece Fermi surface.

Each $\mathbf{Q}_i$ is physically the same as $-\mathbf{Q}_i$ because $\mathbf{Q}_i$ is half of a reciprocal lattice vector.

There are four different interactions between fermions near saddle points g_i, $i = 1 - 4$, with momentum transfer near zero and near Q_i.[4,7–12,14] For our purposes, relevant interactions are density-density interaction within one patch (g_4) and between patches (g_2) and the interaction which describes hopping of a pair of fermions from one patch to the other (g_3). The fourth interaction g_1 is the exchange interaction between patches. Interactions g_2 and g_3 renormalize particle-hole vertices and control the SDW instability, while interactions g_3 and g_4 renormalize particle-particle vertices and control the superconducting instability (note that g_3 contributes to both instabilities).

The partition function Z of the model can be written as a functional integral over Grassmann valued (fermionic) fields ψ. We have $Z = \int D[\bar{\psi}, \psi] \exp[-S(\bar{\psi}, \psi)]$, where $S = \int_0^{1/T} \mathcal{L}(\mathbf{k}, \tau)$ (T is the temperature) and

$$
\begin{aligned}
\mathcal{L} = \sum_{a\alpha} \Big[& \bar{\psi}_{a,\alpha}(\partial_\tau + \varepsilon_\mathbf{k} - \mu - g_4\bar{\psi}_{a,\bar{\alpha}}\psi_{a,\bar{\alpha}})\psi_{a,\alpha} \\
& - \sum_{b\neq a}\sum_\beta g_1\bar{\psi}_{a,\alpha}\bar{\psi}_{b,\beta}\psi_{a,\beta}\psi_{b,\alpha} + g_2\bar{\psi}_{a,\alpha}\bar{\psi}_{b,\beta}\psi_{b,\beta}\psi_{a,\alpha} \\
& + g_3\bar{\psi}_{a,\alpha}\bar{\psi}_{a,\beta}\psi_{b,\beta}\psi_{b,\alpha} \Big].
\end{aligned}
\tag{4}
$$

Here, $a,b = 1,2,3$ label which saddle point we are closest to, α and β are spin labels, and $\bar{\alpha}$ is the opposite spin state to α. We have retained only those states that are close to the saddle points; this "patch model" is exact in the limit of weak coupling.[4]

III. THE SUPERCONDUCTING AND SDW ORDERS

This action displays instabilities towards d-wave superconductivity and SDW. We therefore decouple the interactions in the d-wave superconducting and SDW channels simultane-

ously by means of two Hubbard-Stratonovich transformations. We introduce the Hubbard-Stratonovich superconducting fields $\Delta_a = (g_3 - g_4)\langle\psi_{a,\uparrow}\psi_{a,\downarrow}\rangle$. Since the superconductivity is known to be $d + id$[4], we set $(\Delta_1, \Delta_2, \Delta_3) = \Delta(1, e^{2i\pi/3}, e^{-2i\pi/3})$ and describe superconducting fields by a single complex order parameter Δ. We also introduce the three SDW order parameters $M_{ab} = (g_2 + g_3)\langle\bar{\psi}_{a,\uparrow}\psi_{b,\downarrow}\rangle$. Since the SDW order is known to be uniaxial,[11] we can replace the three vector order parameters M_{12}, M_{23}, M_{31} by a single scalar SDW order parameter M, which represents the magnetic order along the SDW axis.

Since the system has $O(3)$ spin rotation symmetry, the SDW axis can be chosen to coincide with the z axis without loss of generality. Finally, we introduce the Nambu spinor χ_a, a four-component spinor defined according to $\chi_a = (\psi_{a,\uparrow}, \psi_{a,\downarrow}, \bar{\psi}_{a,\downarrow}, -\bar{\psi}_{a,\uparrow})$. The action after Hubbard-Stratonovich transformation can be written in the Nambu spinor basis as

$$
\mathcal{L} = \frac{M^2}{g_2 + g_3} + \frac{|\Delta|^2}{g_3 - g_4} + \sum_{ab} \bar{\chi}_a\mathcal{G}^{-1}_{ab}\chi_b,
\tag{5}
$$

$$
\begin{aligned}
\mathcal{G}^{-1}_{aa} &= [\partial_\tau 1_2 + (\varepsilon_\mathbf{k} - \mu)\sigma_3 + \Delta\sigma_+ + \Delta^*\sigma_-] \otimes 1_2, \\
\mathcal{G}^{-1}_{a\neq b} &= M(1_2 \otimes \eta_3).
\end{aligned}
\tag{6}
$$

Here, the σ_i are Pauli matrices acting in the particle-hole space, the η_i are Pauli matrices acting in the spin space, 1_2 is a two-dimensional identity matrix, and $\sigma_\pm = \sigma_1 \pm \sigma_2$. The notation we have used is borrowed from Ref. 21. We emphasize that this expression for the Ginzburg-Landau functional in terms of superconducting (SC) and SDW order parameters is unique because the superconducting and SDW channels are orthogonal.

We can now integrate out the fermions exactly to obtain an action purely in terms of the superconducting and SDW order-parameter fields

$$
\mathcal{L} = \frac{M^2}{g_2 + g_3} + \frac{|\Delta|^2}{g_3 - g_4} - \text{Tr}\ln\mathcal{G}^{-1}(\Delta, M),
\tag{7}
$$

where the trace goes over Nambu spinor indices, and also over imaginary time and over momentum. We now define $\mathbf{G}$ to be the "bare" (matrix) Green's function evaluated at $\Delta = 0, M = 0$, and define matrix order parameters $\boldsymbol{\Delta}$ and $\mathbf{M}$, such that $\mathcal{G}^{-1} = \mathbf{G}^{-1} + \boldsymbol{\Delta} + \mathbf{M}$. We can then write

$$
\begin{aligned}
\text{Tr}\ln\mathcal{G}^{-1} &= \text{Tr}\ln(\mathbf{G}^{-1}[1 + \mathbf{G}(\boldsymbol{\Delta} + \mathbf{M})]) \\
&= \text{constant} + \text{Tr}\ln[1 + \mathbf{G}(\boldsymbol{\Delta} + \mathbf{M})].
\end{aligned}
\tag{8}
$$

It is convenient to explicitly write out the expressions for $\mathbf{G}, \boldsymbol{\Delta}$, and $\mathbf{M}$. We adopt the shorthand $F^\pm(\omega_n, \mathbf{k}) = 1/[i\omega_n \pm (\varepsilon_\mathbf{k} - \mu)]$, where $\omega_n = (2n + 1)\pi T$ are fermionic Matsubara frequencies. Using the shorthand,

we can define the various matrices as

$$
\mathbf{G} = \begin{pmatrix}
F^+(\omega_n,\mathbf{k}) & 0 & 0 & 0 & 0 & 0 \\
0 & F^-(\omega_n,\mathbf{k}) & 0 & 0 & 0 & 0 \\
0 & 0 & F^+(\omega_n,\mathbf{k}+\mathbf{Q_1}) & 0 & 0 & 0 \\
0 & 0 & 0 & F^-(\omega_n,\mathbf{k}+\mathbf{Q_1}) & 0 & 0 \\
0 & 0 & 0 & 0 & F^+(\omega_n,\mathbf{k}+\mathbf{Q_2}) & 0 \\
0 & 0 & 0 & 0 & 0 & F^-(\omega_n\mathbf{k}+\mathbf{Q_2})
\end{pmatrix} \otimes 1_2,
$$

(9)

$$
\mathbf{\Delta} = \begin{pmatrix}
0 & \Delta & 0 & 0 & 0 & 0 \\
\Delta^* & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Delta e^{2i\pi/3} & 0 & 0 \\
0 & 0 & \Delta^* e^{-2i\pi/3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta e^{-2i\pi/3} \\
0 & 0 & 0 & 0 & \Delta^* e^{2i\pi/3} & 0
\end{pmatrix} \otimes 1_2, \quad
\mathbf{M} = M \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix} \otimes \sigma_3.
$$

IV. LANDAU-GINZBURG ANALYSIS

Thus far, everything we have done has been exact. We now work close to T_c and perform a double expansion of (8) in small $|\Delta|$ and small M. We terminate the expansion at quartic order in both fields and drop all terms that are odd in powers of $\mathbf{M}$ or $\mathbf{\Delta}$ as they vanish upon taking the trace. We then obtain for the Tr ln term the expression

$$
\mathrm{Tr}\Big[-\tfrac{1}{2}(\mathbf{G\Delta G\Delta} + \mathbf{GMGM}) - \tfrac{1}{4}(\mathbf{G\Delta G\Delta G\Delta} + \mathbf{GMGMGMGM} + 4\mathbf{GMGM\Delta G\Delta} + 2\mathbf{GMG\Delta GMG\Delta})\Big].
$$

We have made use of the fact that the trace of a product of matrices is invariant under a cyclic permutation of the matrices. Evaluating the traces and substituting back into (7) leads to the expression

$$
\mathcal{L} = \alpha_1(T - T_c)|\Delta|^2 + \alpha_2(T - T_N)M^2 + K_1|\Delta|^4 + K_2 M^4 + 2K_3|\Delta|^2 M^2,
$$

(10)

where we have defined the expansion coefficients

$$
K_1 = 3T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2}[F^+(\omega_n,\mathbf{k})F^-(\omega_n,\mathbf{k})]^2,
$$

$$
K_2 = 3T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} F^+(\omega_n,\mathbf{k})^2 F^+(\omega_n,\mathbf{k}+\mathbf{Q_1})^2
$$
$$
+ 2F^+(\omega_n,\mathbf{k})^2 F^+(\omega_n,\mathbf{k}+\mathbf{Q_1})F^+(\omega_n,\mathbf{k}+\mathbf{Q_2})
$$
$$
+ (F^+ \to F^-),
$$

$$
K_3 = 6T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} F^+(\omega_n,\mathbf{k})F^-(\omega_n,\mathbf{k})F^+(\omega_n,\mathbf{k}+\mathbf{Q_1})
$$

$$
\times F^-(\omega_n,\mathbf{k}+\mathbf{Q_1})\cos\frac{2\pi}{3} + F^+(\omega_n,\mathbf{k})^2 F^-(\omega_n,\mathbf{k})
$$
$$
\times F^+(\omega_n,\mathbf{k}+\mathbf{Q_1}) + (F^+ \leftrightarrow F^-).
$$

(11)

Terminating the expansion at quartic order in both order parameters is justified if the quadratic terms for superconducting and SDW order change sign at about the same critical temperature $T_c \approx T_N$.[23] Renormalization-group analysis shows that the couplings which determine both superconducting and SDW instabilities diverge at the onset of the first instability upon lowering T.[4,7,9] The critical temperatures, however, are determined by superconducting and SDW susceptibilities, which generally have different exponents (different anomalous di-

mensions). It has been demonstrated on general grounds[22] that different orders emerge simultaneously when their anomalous dimensions $\eta_i > 1$. In our case, the anomalous exponents for superconducting and SDW susceptibilities have been calculated in Ref. 4. Using results from that work, we find that $\eta_{\mathrm{SC}} = 1.48 > 1$, while $\eta_{\mathrm{SDW}} = 0.97$ (the results are for perfect nesting). Because $\eta_{\mathrm{SDW}} < 1$, $T_N < T_c$. However, since η_{SDW} is very close to one, we expect that T_N in (10) is only slightly lower than T_c, in which case the expansion up to quartic order in Δ and M is justified. For dopings slightly away from the van Hove one, we expect SDW to be the leading instability.[7,9,11,12] In this case, $T_N \geqslant T_c$; again, we assume that the difference between the two critical temperatures is small.

It was shown in the context of pnictides[18,19] that a free energy of the form (10) leads to coexistence of the two order parameters if $K_1 K_2 - K_3^2 > 0$. We computed the coefficients K_1, K_2, and K_3 in our case by explicitly integrating over fermionic momenta and summing over fermionic frequencies in (12) (for details, see the Appendix). We obtain, with logarithmic accuracy

$$
K_1 = \frac{1.05}{\sqrt{3}\pi^4 T_c^2 t} \ln\frac{t}{T_c} + \text{subleading},
$$

$$
K_2 = \frac{1.05}{\sqrt{3}\pi^4 T_c^2 t} \ln\frac{t}{T_c} + \text{subleading} = K_1,
$$

$$
K_3 = \frac{1.05\big[\cos\big(\frac{2\pi}{3}\big) + 2\big]}{\sqrt{3}\pi^4 t T_c^2} \ln\frac{t}{T_c} + \text{subleading}
$$

$$
= K_1\Big[\cos\Big(\frac{2\pi}{3}\Big) + 2\Big].
$$

(12)

Note that there are two processes which contribute to the coefficient K_3. The processes are represented diagrammatically
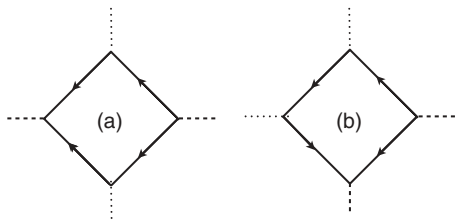
FIG. 2. Diagrammatic representation of the two processes that couple superconductivity and magnetism at quartic order in the Landau-Ginsburg expansion. Lines with arrows represent fermion propagators, dotted lines represent SDW order parameter M, dashed lines represent superconducting order parameter Δ.

in Fig. 2. The process shown in Fig. 2(a) is sensitive to the chirality of the superconducting order parameter because of the dependence on the phase difference between different saddle points, and gives rise to the $\cos(\frac{2\pi}{3})$ term. Because $\cos(2\pi/3) = -1/2$, this process gives rise to an effective attraction between superconductivity and spin density waves. This effective attraction is, however, outweighed by a larger (chirality-independent) repulsion between the two order parameters, coming from the processes shown in Fig. 2(b). The prefactors in our case are such that $K_3 = \frac{3}{2}K_1 > 0$. Comparing K_1K_2 and K_3^2, we see that in the case of doped graphene $K_1K_2 - K_3^2 < 0$, so that coexistence is disfavored. The system only allows one order parameter to exist, even when $T_c = T_N$. A direct second-order transition between superconducting and SDW orders is Landau forbidden since the symmetry group of one ordered phase is not a subgroup of the symmetry group of the other ordered phase, and as a result the transition separating the region when $\Delta \neq 0, M = 0$ from the region where $\Delta = 0, M \neq 0$ is expected to be first order (although we can not exclude a non-Landau continuous transition between the two ordered states).

The fact that $T_c \neq T_N$ makes coexistence even less likely. We therefore conclude that *there is no coexistence of superconducting and SDW order in doped graphene.*

In pnictides, the structure of K_1, K_2, K_3 is quite similar[19] (modulo that there is no $\ln t/T_c$ term), but the argument of cos in K_3 is the phase difference between the gaps on hole and electron FSs. For s^{+-} superconductivity, the argument is π, in which case $K_3 = K_1 = K_2$. Then, $K_1K_2 = K_3^2$, and one has to include subleading terms to verify whether the two orders can coexist. The subleading terms are those which break the nesting between hole and electron pockets, and the analysis shows[18,19] that s^{+-} superconducting and SDW orders coexist in some range of parameters. In graphene, the argument of cos is $2\pi/3$, and such coexistence does not occur.

V. CONCLUSIONS

To conclude, we have demonstrated that superconductivity and spin-density-wave order are mutually exclusive in graphene doped near the M point of the Brillouin zone (a van Hove doping). Sufficiently close to the van Hove point, we expect to see pure chiral superconductivity, and somewhat away from the van Hove point, we expect to see pure spin-density-wave order. The results stand in stark contrast to

pnictides, where there can be coexistence between spin density waves and superconductivity.

APPENDIX

1. Evaluating K_1

We start with the expression

$$K_1 = 3T_c \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} [F^+(\omega_n, \mathbf{k})F^-(\omega_n, \mathbf{k})]^2$$
$$= 3T_c \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} \frac{1}{\left[\omega_n^2 + \frac{9t^2}{16}\left(3k_x^2 - k_y\right)^2\right]^2}, \quad \text{(A1)}$$

where the sum goes over all Matsubara frequencies, and the integral goes over all wave vectors $\mathbf{k}$ up to a UV cutoff of $|\mathbf{k}| \sim O(1)$, at which point the dispersion relation changes. The UV cutoff must be retained in the integrals because the integrals are log divergent in the UV if we ignore the cutoff (the integrals are convergent in the infrared at nonzero temperature). We now scale out ω_n, and define the rescaled coordinates $x = \sqrt{3t/(4\omega_n)}k_x$, $y = \sqrt{3t/(4\omega_n)}k_y$. The expression for K_1 can then be recast as

$$K_1 = 3T_c \sum_{\omega_n} \frac{1}{3\pi^2 t |\omega_n|^3} \int_{-\sqrt{t/T_c}}^{\sqrt{t/T_c}} \frac{dx\, dy}{[1 + (3x^2 - y^2)^2]^2}. \quad \text{(A2)}$$

We now integrate first over $-\infty < y < \infty$, and then over $-\sqrt{t/T_c} < x < \sqrt{t/T_c}$, and expand the resulting expression to leading order in large t/T_c (the manipulations are all done on MATHEMATICA). We obtain the result

$$K_1 = T_c \sum_{\omega_n} \frac{1}{\pi^2 t |\omega_n|^3}\left(\frac{\pi}{2\sqrt{3}}\ln\frac{t}{T_c} + \text{subleading}\right). \quad \text{(A3)}$$

The same result is obtained, with logarithmic accuracy, if we first integrate over $-\infty < x < \infty$, and then over $-\sqrt{t/T_c} < y < \sqrt{t/T_c}$. We now recall that $\omega_n = (2n+1)\pi T_c$, and that $\sum_n \frac{1}{|n+1/2|^3} = 14\zeta(3) \approx 16.8$ (the sum may again be evaluated on MATHEMATICA). Thus, we obtain, with logarithmic accuracy, the result quoted in the main text, i.e.,

$$K_1 = \frac{14\zeta(3)}{16\sqrt{3}\pi^4 t T_c^2}\ln\frac{t}{T_c} \approx \frac{1.05}{\sqrt{3}\pi^4 t T_c^2}\ln\frac{t}{T_c}. \quad \text{(A4)}$$

2. Evaluating K_2

The coefficient K_2 was evaluated already in Ref. 11. We can write $K_2 = 6Z_1 + 12Z_2$, where Z_1 and Z_2 are coefficients that were defined in Ref. 11 and calculated in the supplement to Ref. 11. (Note that there is an overall factor of 2 relative to Ref. 11, which comes about because we have doubled the number of degrees of freedom in going to the Nambu spinor representation. However, this overall factor of 2 multiplies all terms in our free energy, and thus has no physical significance.)

In Ref. 11, it was shown that the term Z_1 was larger than Z_2 by a factor of $\ln t/T_c$, which is a large number at weak coupling. Thus, we can neglect Z_2 with logarithmic accuracy, and say $K_2 = 6Z_1$. The coefficient Z_1 was calculated in Ref. 11, however, the calculation there had a factor of 2 error, which was unimportant for the physics considered in Ref. 11 but is important here. Therefore, we redo the calculation of Z_1.

We wish to evaluate

$$Z_1 = T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} F^+(\mathbf{k}, \omega_n)^2 F^+(\mathbf{k} + \mathbf{Q}, \omega_n)^2. \quad (A5)$$

The integral over the Brillouin zone is dominated by those values of $\mathbf{k}$ where both Green's functions correspond to states near a saddle point. Expanding the energy about the saddle points, we rewrite the integral as

$$Z_1 \approx T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2}$$
$$\times \frac{1}{\left[i\omega_n - \frac{3t_1}{4}(3k_x^2 - k_y^2)\right]^2 \left[i\omega_n - \frac{3t_1}{4}2k_y(k_y - \sqrt{3}k_x)\right]^2}, \quad (A6)$$

where the integral is understood to have a UV cutoff for $\mathbf{k}$ of order 1. We now define $a = \sqrt{3t_1/4}(k_y - \sqrt{3}k_x)$ and $b = \sqrt{3t_1/4}(k_y + \sqrt{3}k_x)$, and rewrite the above integral as

$$Z_1 = T \sum_{\omega_n} \frac{2}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}}^{\sqrt{t_1}} \frac{da\, db}{(2\pi)^2}$$
$$\times \frac{1}{(i\omega_n + ab)^2[i\omega_n - a(a+b)]^2}. \quad (A7)$$

We now define $x = ab$ and rewrite the integral as

$$Z_1 = T \sum_{\omega_n} \frac{2}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}}^{\sqrt{t_1}} \frac{da}{2\pi} \frac{1}{|a|}$$
$$\times \int_{-\sqrt{t_1}a}^{\sqrt{t_1}a} \frac{dx}{2\pi} \frac{1}{(i\omega_n + x)^2(i\omega_n - a^2 - x)^2}. \quad (A8)$$

We now assume $T_N \ll t_1$ (which should certainly be the case for weak/moderate coupling). In this limit, we can perform the integral over x approximately, using the Cauchy integral formula, to get

$$Z_1 = T \sum_{\omega_n} \frac{2}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}}^{\sqrt{t_1}} \frac{da}{2\pi} \frac{1}{|a|} \frac{2i\, \text{sign}\omega_n}{(a^2 - 2i\omega_n)^3}$$
$$= T \sum_{\omega_n} \frac{4}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}}^{\sqrt{t_1}} \frac{da}{2\pi} \frac{1}{|a|} \frac{i\, \text{sign}\omega_n(a^2 + 2i\omega_n)^3}{(a^4 + 4\omega_n^2)^3}. \quad (A9)$$

The imaginary part of the above integral is odd in ω and hence vanishes upon performing the Matsubara sum to leave an integral that is purely real:

$$Z_1 = T \sum_{\omega_n} \frac{8|\omega_n|}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}}^{\sqrt{t_1}} \frac{da}{2\pi} \frac{1}{|a|} \frac{4\omega_n^2 - 3a^4}{(a^4 + 4\omega_n^2)^3}$$
$$\approx T \sum_{\omega_n} \frac{8|\omega_n|}{3\sqrt{3}t_1} \int_{-\sqrt{t_1}}^{\sqrt{t_1}} \frac{da}{2\pi} \frac{1}{|a|} \frac{4\omega_n^2}{(a^4 + 4\omega_n^2)^3} \quad (A10)$$

with logarithmic accuracy. Performing the integral over a (again with logarithmic accuracy) gives

$$Z_1 \approx T \sum_{\omega_n} \frac{1}{12\pi\sqrt{3}t_1} \frac{1}{|\omega_n|^3} \ln \frac{t_1}{\omega_n}$$
$$= \frac{1}{96\pi^4\sqrt{3}T_N^2 t_1} \left(16.8 \ln \frac{t_1}{2\pi T} + 10.5\right)$$
$$\approx \frac{16.8 \ln \frac{t_1}{T_N}}{96\pi^4\sqrt{3}T_N^2 t_1}, \quad (A11)$$

where we take $\omega_n = 2\pi(n + 1/2)T_N$, $T = T_N$, and perform the discrete sum on MATHEMATICA. The error in the supplement to Ref. 11 was in the last line of the calculation.

3. Evaluating K_3

There are two distinct contributions to K_3, and we evaluate both in turn. We can write $K_3 = K_3^a + K_3^b$, where

$$K_3^a = 6T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} F^+(\omega_n, \mathbf{k}) F^-(\omega_n, \mathbf{k}) F^+(\omega_n, \mathbf{k} + \mathbf{Q_1})$$
$$\times F^-(\omega_n, \mathbf{k} + \mathbf{Q_1}) \cos(\theta_\mathbf{k} - \theta_{\mathbf{k}+\mathbf{Q}}),$$
$$K_3^b = 6T \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} F^+(\omega_n, \mathbf{k})^2 F^-(\omega_n, \mathbf{k}) F^+(\omega_n, \mathbf{k} + \mathbf{Q_1})$$
$$+ (F^+ \leftrightarrow F^-). \quad (A12)$$

The first contribution K_3^a comes from processes of the form shown in Fig. 2(a), and is sensitive to the chirality of the superconducting order parameter (it depends on the difference in the phase of the superconducting order parameter at different points on the Fermi surface). This process leads to an attraction between chiral superconductivity and spin density waves. The second contribution K_3^b comes from processes of the form shown in Fig. 2(b), and is insensitive to the chirality of the superconducting order parameter. This process leads to a repulsion between any kind of superconductivity and spin density waves. The second process dominates (because of purely numerical prefactors), so superconductivity and spin density waves do repel, but the repulsion is too weak to prevent coexistence.

Let us first calculate K_3^a. For $d + id$ pairing, we have $\theta_\mathbf{k} - \theta_{\mathbf{k}+\mathbf{Q}} = 4\pi/3$. Thus, we have

$$K_3^a = 6T_c \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2} F^+(\omega_n, \mathbf{k}) F^-(\omega_n, \mathbf{k}) F^+(\omega_n, \mathbf{k} + \mathbf{Q_1})$$
$$\times F^-(\omega_n, \mathbf{k} + \mathbf{Q_1}) \cos \frac{4\pi}{3}$$
$$= 6T_c \cos \frac{4\pi}{3} \sum_{\omega_n} \int \frac{d^2k}{(2\pi)^2}$$
$$\times \frac{1}{\left[\omega_n^2 + \frac{9t^2}{16}(3k_x^2 - k_y^2)^2\right]\left[\omega_n^2 + \frac{9t^2}{16}4k_y^2(k_y - \sqrt{3}k_x)^2\right]}. \quad (A13)$$

We scale out ω_n and define rescaled variables $x = \sqrt{3t/(4\omega_n)}k_x$, $y = \sqrt{3t/(4\omega_n)}k_y$. The expression for K_3^a can

then be recast as

$$K_3^a = \cos\left(\frac{4\pi}{3}\right) \sum_{\omega_n} \frac{2T_c}{\pi^2 t |\omega_n|^3}$$
$$\times \int_{-\sqrt{t/T_c}}^{\sqrt{t/T_c}} \frac{dx\, dy}{[1+(3x^2-y^2)^2][1+4y^2(y-\sqrt{3}x)^2]}.$$

(A14)

We define the new coordinates $a = y - \sqrt{3}x$, $b = y + \sqrt{3}x$, and hence reexpress the above integral (with logarithmic accuracy) as

$$K_3^a = \cos\left(\frac{4\pi}{3}\right) \sum_{\omega_n} \frac{T_c}{\sqrt{3}\pi^2 t |\omega_n|^3}$$
$$\times \int_{-\sqrt{t/T_c}}^{\sqrt{t/T_c}} \frac{da\, db}{(1+a^2 b^2)[1+a^2(a+b)^2]}.$$

(A15)

We integrate over $-\sqrt{t/T_c} < b < \sqrt{t/T_c}$ (on MATHEMATICA), and expand the resulting expression to leading order in large

t/T_c. This leads to the expression

$$K_3^a = \cos\left(\frac{4\pi}{3}\right) \sum_{\omega_n} \frac{T_c}{\sqrt{3}\pi^2 t |\omega_n|^3} \left(\int da\, \frac{\pi}{2|a|(1+a^4/4)}\right).$$

(A16)

It should be remembered that the expansion in large t/T_c is valid only for $a^2 t/T_c \gg 1$, thus, the above integral implicitly carries an infrared cutoff on the scale $a \approx \sqrt{t/T_c}$. Performing the integral with this infrared cutoff, we obtain the expression

$$K_3^a = \cos\left(\frac{4\pi}{3}\right) \sum_{\omega_n} \frac{T_c}{2\sqrt{3}\pi t |\omega_n|^3} \ln\frac{t}{T_c}.$$

(A17)

Performing the summation over n on MATHEMATICA, as before, we obtain

$$K_3^a = \cos\left(\frac{4\pi}{3}\right) \frac{14\zeta 3}{16\sqrt{3}\pi^4 t T_c^2} \ln\frac{t}{T_c} \approx \frac{-1}{2} \frac{16.8}{16\sqrt{3}\pi^4 t T_c^2} \ln\frac{t}{T_c}$$
$$= \frac{-1.05}{2\sqrt{3}\pi^4 t T_c^2} \ln\frac{t}{T_c} = -\frac{1}{2}K_1.$$

(A18)

Note the crucial minus sign that comes from the chirality sensitive cos factor: this particular term represents an attraction between magnetism and chiral superconductivity.

We now turn our attention to the second term K_3^b. We have

$$K_3^b = 6T_c \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} F^+(\omega_n,\mathbf{k})^2 F^-(\omega_n,\mathbf{k}) F^+(\omega_n,\mathbf{k}+\mathbf{Q_1}) + (F^+ \leftrightarrow F^-)$$

$$= 6T_c \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \frac{-\left(\left[i\omega_n + \frac{3t}{4}\left(3k_x^2-k_y^2\right)\right]\left[i\omega_n + \frac{3t}{4}2k_y(k_y-\sqrt{3}k_x)\right] + (\omega_n \to -\omega_n)\right)}{\left[\omega_n^2 + \frac{9t^2}{16}\left(3k_x^2-k_y^2\right)^2\right]^2 \left[\omega_n^2 + \frac{9t^2}{16}4k_y^2(k_y-\sqrt{3}k_x)^2\right]}$$

$$= 12T_c \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \frac{\omega_n^2 - \frac{9t^2}{16}\left(3k_x^2-k_y^2\right)2k_y(k_y-\sqrt{3}k_x)}{\left[\omega_n^2 + \frac{9t^2}{16}\left(3k_x^2-k_y^2\right)^2\right]^2 \left[\omega_n^2 + \frac{9t^2}{16}4k_y^2(k_y-\sqrt{3}k_x)^2\right]}.$$

(A19)

Again, we scale out ω_n and define the rescaled variables $x = \sqrt{3t/(4\omega_n)}k_x$, $y = \sqrt{3t/(4\omega_n)}k_y$. The expression for K_3^b can then be recast as

$$K_3^b = \sum_{\omega_n} \frac{4T_c}{\pi^2 t |\omega_n|^3} \int_{-\sqrt{t/T_c}}^{\sqrt{t/T_c}} dx\, dy$$
$$\times \frac{1-(3x^2-y^2)2y(y-\sqrt{3}x)}{[1+(3x^2-y^2)^2]^2[1+4y^2(y-\sqrt{3}x)^2]}.$$

(A20)

We define the new coordinates $a = y - \sqrt{3}x$, $b = y + \sqrt{3}x$, and hence reexpress the above integral (with logarithmic accuracy) as

$$K_3^b = \sum_{\omega_n} \frac{2T_c}{\sqrt{3}\pi^2 t |\omega_n|^3} \int_{-\sqrt{t/T_c}}^{\sqrt{t/T_c}} da\, db$$
$$\times \frac{1+a^2 b(a+b)}{(1+a^2 b^2)^2[1+a^2(a+b)^2]}.$$

(A21)

We integrate over $-\sqrt{t/T_c} < b < \sqrt{t/T_c}$ (on MATHEMATICA), and expand the resulting expression to leading order in large

t/T_c. This leads to the expression

$$K_3^b = \sum_{\omega_n} \frac{2T_c}{\sqrt{3}\pi^2 t |\omega_n|^3} \left(\int da\, \frac{\pi}{2|a|(1+a^4/4)^2}\right).$$

(A22)

It should be remembered that the expansion in large t/T_c is valid only for $a^2 t/T_c \gg 1$, thus, the above integral implicitly carries an infrared cutoff on the scale $a \approx \sqrt{t/T_c}$. Performing the integral with this infrared cutoff, we obtain the expression

$$K_3^b = \sum_{\omega_n} \frac{T_c}{\sqrt{3}\pi t |\omega_n|^3} \ln\frac{t}{T_c} = \frac{2.01}{\sqrt{3}\pi^4 t T_c^2} \ln\frac{t}{T_c} = 2K_1.$$

(A23)

Putting things together, we have

$$K_3 = K_3^a + K_3^b = K_1\left(\cos\frac{2\pi}{3}+2\right)$$
$$= K_1\left(-\frac{1}{2}+2\right) = \frac{3}{2}K_1$$

(A24)

quoted in the main text.

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