Classification and analysis of two-dimensional Abelian fractional topological insulators

Michael Levin[1] and Ady Stern[2]
[1]*Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland 20742, USA*
[2]*Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel*
(Received 10 May 2012; published 21 September 2012)

We present a general framework for analyzing fractionalized, time-reversal invariant electronic insulators in two dimensions. The framework applies to all insulators whose quasiparticles have Abelian braiding statistics. First, we construct the most general Chern-Simons theories that can describe these states. We then derive a criterion for when these systems have protected gapless edge modes, that is, edge modes that cannot be gapped out without breaking time-reversal or charge-conservation symmetry. The systems with protected edge modes can be regarded as fractionalized analogues of topological insulators. We show that previous examples of 2D fractional topological insulators are special cases of this general construction. As part of our derivation, we define the concept of "local Kramers degeneracy" and prove a local version of Kramers theorem.

DOI: 10.1103/PhysRevB.86.115131 PACS number(s): 71.10.Pm, 73.43.−f

I. INTRODUCTION

Recently, it was realized that there are two kinds of two-dimensional (2D) time-reversal invariant band insulators: topological insulators and trivial insulators. These two types of insulators are distinguished by the fact that topological insulators have robust gapless edge modes while trivial insulators do not.[1–4]

While much of the work on topological insulators has focused on noninteracting or weakly interacting systems, it is natural to wonder whether similar physics can occur in systems with strong interactions. Such strongly interacting insulators can be divided into two classes: systems that can be adiabatically deformed into (noninteracting) band insulators without closing the bulk gap, and systems that cannot. At some level the first case is understood: it is known[5] that the gapless edge modes of a topological insulator are stable to arbitrary local interactions as long as time-reversal symmetry and charge conservation are not broken (explicitly or spontaneously). Here we will consider the second case: strongly interacting, time-reversal invariant electron systems whose ground state cannot be adiabatically connected to a band insulator. These systems are typically fractionalized in the sense that they have quasiparticle excitations with fractional charge and fractional statistics.

Previous work has focused on particular classes of these fractionalized insulators. Reference 6 considered toy models[3] where spin-up and spin-down electrons each form decoupled Abelian fractional quantum Hall states with opposite chiralities.[7] The authors showed that some of these systems have protected edge modes, while some do not. More precisely, the authors found that these models have a protected edge mode if and only if σ_{xH}/e^* is odd, where e^* is the elementary charge (in units of e) and σ_{xH} is the spin-Hall conductivity (in units of $e/2\pi$). The two kinds of insulators were dubbed "fractional topological" and "fractional trivial" insulators by analogy to noninteracting topological and trivial insulators. Along the same lines, Refs. 8 and 9 considered generalizations of the above toy models where spin-up and spin-down electrons form *correlated* fractional quantum Hall states. Finally, Ref. 10 considered a somewhat different class of time reversal invariant fractionalized insulators that were realized by exactly soluble lattice models. In all these cases, it was found that some of the fractionalized insulators have protected edge modes, while some do not.

Taken together, these papers have analyzed a large class of time-reversal invariant fractionalized insulators. However, they have not exhausted all the possibilities. Here, we give a more complete analysis of these systems. We consider *general* time-reversal invariant 2D electronic insulators with Abelian quasiparticle statistics. We then derive a criterion for when these systems have protected gapless edge modes, that is, edge modes that cannot be gapped out without breaking time-reversal or charge-conservation symmetry, explicitly or spontaneously. We call the states with protected edge modes "fractional topological insulators," following the terminology of Ref. 6.

Our analysis proceeds in two steps. First, we find the most general Chern-Simons theory that can describe a time-reversal invariant, Abelian electronic insulator. Then, we investigate the stability of the edge modes of these states, using both a microscopic approach and a macroscopic flux insertion argument. Much of the edge stability analysis can be regarded as a direct generalization of the arguments of Ref. 6. However, there are a number of additional features coming from the more detailed exposition of this paper. In particular, we give a precise definition of "local Kramers degeneracy" and a proof of a local analog of Kramers theorem.

This paper is organized as follows. In Sec. II, we find a Chern-Simons description of general time-reversal invariant insulators with Abelian quasiparticle statistics. In Sec. III, we analyze the stability of the edges of these insulators using a microscopic approach. In Sec. IV, we analyze the edge stability using a flux insertion argument and in the process give a proof of a local analog of Kramers theorem. Appendixes contain some of the more technical calculations.

II. CHERN-SIMONS THEORIES FOR TIME-REVERSAL INVARIANT INSULATORS

A. The time-reversal breaking case

First, we review the Chern-Simons description for Abelian insulators without time-reversal symmetry.[11–13] In the next section, we discuss how this formalism needs to be modified to

describe the time-reversal symmetric case. To begin, recall that a general Abelian insulator can be described by a p-component $U(1)$ Chern-Simons theory of the form

$$L_B = \frac{\mathcal{K}_{IJ}}{4\pi}\epsilon^{\lambda\mu\nu}a_{I\lambda}\partial_\mu a_{J\nu} - \frac{1}{2\pi}\tau_I\epsilon^{\lambda\mu\nu}A_\lambda\partial_\mu a_{I\nu}, \quad (1)$$

where $\mathcal{K}$ is a $p \times p$ symmetric, nondegenerate integer matrix and τ is a p component integer vector. We will refer to $\mathcal{K}$ as the "K matrix" and τ as the "charge vector." The first term in L_B describes the different degrees of freedom in the insulator, while the second term describes their coupling to the electromagnetic vector potential A_λ. In general, L_B can also contain additional terms such as $(\partial_\mu a_{I\nu} - \partial_\nu a_{I\mu})^2$, but these terms do not affect the basic topological features of the insulator, our main interest here.

In this formalism, the quasiparticle excitations in the insulator are described by coupling L_B to bosonic particles that carry *integer* gauge charge l_I under each of the gauge fields a_I. Thus, each quasiparticle excitation corresponds to a p component integer vector l. The physical electric charge of each excitation is given by

$$q_l = \frac{1}{2\pi}l^T\mathcal{K}^{-1}\tau, \quad (2)$$

while the mutual statistics associated with braiding one particle around another is given by

$$\theta_{ll'} = 2\pi l^T\mathcal{K}^{-1}l'. \quad (3)$$

The statistical phase associated with exchanging two particles is $\theta_l = \theta_{ll}/2$. The quasiparticle excitations that are "local"—that is, composed out of the constituent electrons—correspond to vectors l of the form $l = \mathcal{K}\Lambda$, where Λ is an integer p component vector.

If we do not require time-reversal symmetry, the K matrix and charge vector are unconstrained except for two requirements. First, we must have

$$\tau_I \equiv \mathcal{K}_{II} \pmod 2. \quad (4)$$

To derive this constraint, consider the statistics and charges of the local excitations with $l = K\Lambda$. From Eqs. (2) and (3) we can see that the excitation corresponding to $l = \mathcal{K}\Lambda$ carries electric charge $\Lambda^T\tau$ and has exchange statistics $\theta = \pi\Lambda^T\mathcal{K}\Lambda$. At the same time, we know that local excitations with even charge must be bosons, while those with odd charge must be fermions, since they are composed out of electrons. Combining these two observations yields Eq. (4).

The second constraint on $\mathcal{K},\tau$ is that

$$\gcd(\tau_1,\ldots,\tau_p) = 1, \quad (5)$$

where the notation "gcd" denotes the greatest common divisor of these integers. This constraint follows from the requirement that at least one local excitation has the charge of an electron. [Readers familiar with the "strong pairing" fractional quantum Hall state may worry that the condition (5) is too strict, since this state is traditionally described using a Chern-Simons theory with $\mathcal{K} = 8, \tau = 2$, apparently violating Eq. (5). However, this state can be described equally well by the 3×3 K matrix with diagonal elements $8,1,-1$, and the charge vector $\tau = (2,1,1)$, thus satisfying Eq. (5). The same trick can be used to find a Chern-Simons representation satisfying Eq. (5) for any Abelian insulator built out of electrons].

B. Including time-reversal symmetry

We now show how to extend the above formalism to describe Abelian insulators with the additional symmetry of time reversal. The first step is to describe the action of time reversal on the field theory (1). We will assume that the time-reversal transformation $\mathcal{T}$ is of the form

$$a_I \to T_{IJ}a_J,$$

where T is an integer $p \times p$ matrix. More precisely, because time reversal acts differently on spatial and temporal coordinates, we will assume that

$$a_{I\mu} \to \pm T_{IJ}a_{J\mu}, \quad (6)$$

where the sign is $+$ for $\mu = 1,2$ and $-$ for $\mu = 0$. To see where this transformation law comes from, recall that, in our formalism, the quasiparticle excitations are described by sources carrying integer gauge charge. Therefore integer gauge charge must transform into integer gauge charge under time reversal. Equation (6) follows immediately.

In order for the action (1) to be invariant under the time-reversal transformation (6), we must have

$$T^T\mathcal{K}T = -\mathcal{K}, \quad (7)$$

$$T\tau = \tau. \quad (8)$$

Similarly, the requirement that $\mathcal{T}^2 = (-1)^{N_e}$, where N_e is the total number of electrons in the system, implies that

$$T^2 = \mathbf{1_P}, \quad (9)$$

where $\mathbf{1_P}$ is the $p \times p$ identity matrix.

We now come to an important point: Eq. (6) does not completely specify the action of time reversal. The problem is that Eq. (6) only tells us how time reversal acts on operators like $\epsilon^{\lambda\mu\nu}\partial_\mu a_{I\nu}$. These operators are all electrically neutral, that is, they preserve the total electric charge in the system. To complete our description of time reversal, we also have to specify the transformation properties of *charged* operators, like electron creation and annihilation operators.

Unfortunately, the Chern-Simons theory is not a convenient framework for describing these operators. The reason is that these charged operators correspond to magnetic monopole instantons in the gauge fields a_I. Thus, if we want to complete our description within the Chern-Simons theory, we need to introduce additional notation and formalism to describe these instantons and their transformation properties under $\mathcal{T}$.

We can avoid these (technical) complications by instead describing the action of time reversal at the *edge*. According to the usual bulk-edge correspondence for Abelian Chern-Simons theories, a valid edge theory for Eq. (1) is given by[11,12]

$$L = \frac{1}{4\pi}(\mathcal{K}_{IJ}\partial_x\Phi_I\partial_t\Phi_J - \mathcal{V}_{IJ}\partial_x\Phi_I\partial_x\Phi_J) + \frac{1}{2\pi}\epsilon^{\mu\nu}\tau_I\partial_\mu\Phi_I A_\nu, \quad (10)$$

where $\mathcal{V}$ is a $p \times p$ velocity matrix. Here, we are using a normalization convention where the quasiparticle excitations

are of the form $e^{il^T\Phi}$, where l is an integer p component vector. Local operators (i.e., operators composed out of products of electron creation and annihilation operators) are of the form $e^{i\Theta(\Lambda)}$ with $\Theta(\Lambda)\equiv\Lambda^T\mathcal{K}\Phi$ and Λ an integer vector.

Translating the transformation law (6) into the edge theory language gives

$$\mathcal{T}^{-1}(\partial_x\Phi_I)\mathcal{T}=T_{IJ}\partial_x\Phi_J, \tag{11}$$

along with a similar expression for the temporal derivative $\partial_t\Phi_I$. The missing component in our description is now clear: Eq. (11) does not determine the transformation properties for the local operators $e^{i\Theta(\Lambda)}$, i.e., the creation and annihilation operators for electrons.

We can see how to complete our description if we rewrite Eq. (11) as

$$\mathcal{T}^{-1}\Phi_I\mathcal{T}=T_{IJ}\Phi_J+\text{const.},$$

or equivalently

$$\mathcal{T}^{-1}\Phi_I\mathcal{T}=T_{IJ}\Phi_J+\pi\mathcal{K}^{-1}_{IJ}\chi_J, \tag{12}$$

where χ is some p-component real vector. Clearly, in order to complete our description of time reversal we need to specify χ. We will refer to χ as the "time-reversal vector."

Like T, the vector χ cannot be chosen arbitrarily: it is constrained by the requirement that electron creation and annihilation operators are odd under $\mathcal{T}^2$. This requirement implies that

$$\mathcal{T}^{-2}e^{i\Theta(\Lambda)}\mathcal{T}^2=e^{i\Theta(\Lambda)}(-1)^{\Lambda^T\tau}, \tag{13}$$

which is equivalent to the constraint that

$$(1-T^T)\chi\equiv\tau\ (\text{mod } 2). \tag{14}$$

We note that Klein factors are not relevant to this discussion of time-reversal properties and therefore we have dropped them for clarity. In fact, we will see that Klein factors do not affect our later analysis either and can be safely ignored throughout this paper.

In summary, we have shown that a general Abelian time-reversal invariant insulator is described by the data $(\mathcal{K},\tau,T,\chi)$. The K matrix $\mathcal{K}$ and charge vector τ specify the quasiparticle braiding statistics and charges, while the time-reversal matrix T and vector χ describe the action of time reversal. This data must satisfy the conditions (4) and (5), (7)–(9), and (14) in order to be self-consistent.

C. Explicit parametrization of time-reversal invariant Abelian Chern-Simons theories

An important point is that not all $(\mathcal{K},\tau,T,\chi)$ correspond to physically distinct Chern-Simons theories. For example, $(\mathcal{K},\tau,T,\chi)$ is physically equivalent to

$$\begin{aligned}\mathcal{K}'&=(U^{-1})^T\mathcal{K}U^{-1},\quad \tau'=(U^{-1})^T\tau,\\ T'&=UTU^{-1},\quad \chi'=(U^{-1})^T\chi,\end{aligned} \tag{15}$$

for any integer matrix U with determinant ±1. This equivalence can be derived by making the change of variables $a'=Ua$, $\Phi'=U\Phi$ in the bulk/edge theories (1) and (10). Another important equivalence is the one between χ and

$$\chi'=\chi+\frac{1}{\pi}\mathcal{K}(1-T)\xi \tag{16}$$

for any real vector ξ, which follows from the field redefinition $\Phi'=\Phi+\xi$. A final example is that χ is physically equivalent to

$$\chi'=\chi+2v \tag{17}$$

for any integer vector v. To see where this equivalence comes from, note that the only place where χ enters into the physical description of the edge is in the transformation law for $e^{i\Theta(\Lambda)}$:

$$\mathcal{T}^{-1}e^{i\Theta(\Lambda)}\mathcal{T}=e^{i\Theta(T\Lambda)-i\pi\Lambda^T\chi}. \tag{18}$$

We can see that χ appears in the combination $e^{i\pi\Lambda^T\chi}$, which only depends on the value of χ modulo 2.

We are now in a position to explicitly write down all time-reversal invariant Abelian Chern-Simons theories. Indeed, according to the above analysis, it suffices to find all $(\mathcal{K},\tau,T,\chi)$ satisfying conditions (4), (5), (7)–(9). This problem can be solved by straightforward linear algebra as we demonstrate in Appendix A. We find that the most general solution, up to the transformations (15)–(17), is of the form

$$\mathcal{K}=\begin{pmatrix}0&A&B&B\\A^T&0&C&-C\\B^T&C^T&K&W\\B^T&-C^T&W^T&-K\end{pmatrix},\quad \tau=\begin{pmatrix}0\\t'\\t\\t\end{pmatrix}, \tag{19}$$

$$T=\begin{pmatrix}-\mathbf{1_M}&0&0&0\\0&\mathbf{1_M}&0&0\\0&0&0&\mathbf{1_{N-M}}\\0&0&\mathbf{1_{N-M}}&0\end{pmatrix},\quad \chi=\begin{pmatrix}x\\0\\0\\t\end{pmatrix}. \tag{20}$$

Here, the matrix A is of dimension $M\times M$, while the matrices B,C are of dimension $M\times(N-M)$. The matrices K,W are both of dimension $(N-M)\times(N-M)$. Similarly, t' is of dimension M and t is of dimension $(N-M)$. Finally, the vector x is some $(N-M)$ dimensional vector of 1's and 0's. We note that that the total dimension of $\mathcal{K}$ is $2N\times2N$ so the above solution corresponds to $p=2N$ in our previous notation.

There are only a few constraints on (A,B,C,K,W,t,t',x). First, W must be antisymmetric: $W=-W^T$. This requirement follows from time-reversal invariance (7). Second, t' must be even valued. This constraint comes from the condition (4) that the insulator is composed out of electrons. For the same reason, the parity of t_I must match that of K_{II}, but can be either even or odd. Finally, the greatest common factor of $\{\tau_I\}$ must be 1 according to Eq. (5).

We would like to mention that while every time-reversal invariant Abelian Chern-Simons theory can be written in the form Eqs. (19) and (20), this representation is not unique in general. In other words, different (A,B,C,K,W,t,t',x) may correspond to physically equivalent Chern-Simons theories. For this reason, more work is necessary to turn (19) and (20) into a one-to-one classification of time-reversal invariant Abelian insulators.

D. Examples and physical realizations

In this section, we discuss special cases of the general time-reversal invariant Chern-Simons theories (19) and (20). A particularly simple case is when $M = 0$ and $W = 0$. In this case, Eqs. (19) and (20) reduce to

$$\mathcal{K} = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}, \quad T = \begin{pmatrix} 0 & \mathbf{1_N} \\ \mathbf{1_N} & 0 \end{pmatrix}, \tag{21}$$

and

$$\tau = \begin{pmatrix} t \\ t \end{pmatrix}, \quad \chi = \begin{pmatrix} 0 \\ t \end{pmatrix}. \tag{22}$$

This theory can be physically realized in a toy model in which spin-up and spin-down electrons each form completely decoupled quantum Hall states with opposite chiralities: a "fractional quantum spin Hall state".[3] This model, and in particular the stability of the edge modes, was analyzed in Ref. 6. It is also possible to construct models of the type (21) from bosons.[14] For example, we can imagine that bosons of spin $s_z = \pm 1$ form decoupled bosonic quantum Hall states of opposite chiralities. In that case, K would have only even numbers on the diagonal.

Similarly, one can consider the case where $W = -W^T \neq 0$, so that

$$\mathcal{K} = \begin{pmatrix} K & W \\ W^T & -K \end{pmatrix}. \tag{23}$$

This theory, which was analyzed in Refs. 8 and 9, can also be realized by a toy model in which spin-up and spin-down electrons form quantum Hall states with opposite chiralities. The main difference from the previous case is that here the ground state contains correlations between the two spin species. Obviously, the requirement $W = -W^T \neq 0$ necessitates at least a four-dimensional $\mathcal{K}$.

Next, it is instructive to consider cases where $M = N$, so that

$$\mathcal{K} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \quad T = \begin{pmatrix} -\mathbf{1_N} & 0 \\ 0 & \mathbf{1_N} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 \\ t' \end{pmatrix}. \tag{24}$$

In this case, $\mathcal{K}$ has only even numbers on the diagonal, so that the insulators are bosonic. The best known examples of this type have the matrix A being simply an integer number, m, and $t' = q_B$, where q_B is the boson charge. For $m = 1$, this theory describes a conventional bosonic Mott insulator. For $m \neq 1$, it describes a charge-conserving model of the "toric code"[10,15] type, with "charge" excitations of electric charge q_B/m, and "flux" excitations which are neutral. These examples all have the vector $x = 0$ in Eq. (20), so that $\chi = 0$.

We can also imagine the same class of bosonic insulators, but with $x = 1$. The simplest case, $m = 1$ is already interesting: this theory describes a new kind of bosonic Mott insulator. This insulator can be distinguished from a conventional Mott insulator by the fact that it has protected edge modes. To see this, consider the edge stability criterion derived in the next two sections. There, we show that the electronic insulators (19) have protected edge modes if and only if $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is odd, where e^* is the smallest charged excitation of the system in units of e. While our derivation focuses on electronic insulators, the same analysis applies to bosonic insulators, with the only difference being that we should measure the elementary charge e^* in units of the boson charge q_B instead of e. Applying this result to the case $m = 1, x = 1, t' = q_B$, we find $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau = 1$, implying the existence of a protected edge mode. Interestingly, this new kind of Mott insulator does not have excitations with fractional charge or statistics. In this sense, it is similar to a conventional topological insulator, except that it is built out of bosons instead of fermions. We expect that this "unfractionalized" bosonic topological insulator is equivalent to the one proposed in Ref. 16.

Finally, we discuss examples where all parts of the matrix (19) are at play. Recall that Ref. 10 constructed a 2D time-reversal invariant lattice model built out of electrons described by

$$\mathcal{K} = \begin{pmatrix} 0 & m & -k & -k \\ m & 0 & 0 & 0 \\ -k & 0 & 1 & 0 \\ -k & 0 & 0 & -1 \end{pmatrix}, \; T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{25}$$

and

$$\tau = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad \chi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{26}$$

In this model, the upper-left quadrant of the matrix describes bosons that form a charge-conserving toric code[15] state, with "charge" excitations that carry a charge $2/m$ and "flux" excitations that are neutral. The off-diagonal quadrant (the $-k$ terms) couple the charge excitations with electrons to form composite fermions whose electric charge is $1 + 2k/m$. The right-bottom quadrant then describes how these fermions form a quantum spin Hall state of $\nu = \pm 1$.

In this example, we can see that the role played by the matrix B in Eq. (19) is to couple the two types of charge excitations to one another. In order to be symmetric to time reversal, this coupling must be identical for the two spin directions of the electrons. In a similar fashion, the matrix C in Eq. (19) couples electrons to flux excitations of the bosonic system. In this case, in order for the coupling to preserve the symmetry to time reversal, the flux excitations couple oppositely to the two spin directions of the electrons.

III. STABILITY OF THE EDGE: MICROSCOPIC ANALYSIS

An important question is to determine which of the Abelian insulators described by Eqs. (19) and (20) have protected gapless edge modes, that is edge modes that cannot be gapped out without breaking time-reversal or charge-conservation symmetry, explicitly or spontaneously. In the next two sections we show that these systems have protected edge modes if and only if the quantity $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is odd. Here, e^* is the smallest charged excitation in the system (in units of e). Formally, e^* is defined by

$$e^* = \min_l (l^T\mathcal{K}^{-1}\tau), \tag{27}$$

where l ranges over all integer vectors.

This criterion was previously derived in Refs. 6, 8, and 10 for the three special cases (21), (23), and (25). In these cases,

the criterion can be rephrased in more physically transparent language, assuming that the system conserves the total electron spin s^z. To be specific, in these cases, we can identify the quantity $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ with $-\sigma_{xH}/e^*$, where σ_{xH} is the spin-Hall conductivity in units of $e/2\pi$. Thus the criterion reduces to the statement that these systems have protected edge modes if and only if σ_{xH}/e^* is odd.[6] More generally, this reformulation in terms of σ_{xH}/e^* is valid for any s^z conserving insulator for which $\chi \equiv \tau_\downarrow \pmod 2$, where $\tau_\downarrow$ is the charge vector corresponding to the spin-down electrons (see Appendix B). Physically, the condition $\chi \equiv \tau_\downarrow \pmod 2$ is equivalent to the requirement that the excitations of the insulator have a "local Kramers degeneracy" (see Sec. IV D) if and only if they carry half-integer spin. This connection between spin and time reversal is what allows us to reformulate the edge stability criterion in terms of σ_{xH}.

A. Basic setup

We derive the $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ criterion in two steps. First, we use a microscopic approach to explicitly show that when $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is even, the edge can be gapped out without breaking any symmetries (explicitly or spontaneously). We also show that there is an obstruction to gapping out the edge when $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is odd. Then, in Sec. IV, we complete the derivation by giving a general argument that the edge is protected when $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is odd.

Our analysis closely follows that of Ref. 6. Our starting point is the edge theory (10), which we reprint below for convenience:

$$L = \frac{1}{4\pi}(\mathcal{K}_{IJ}\partial_x\Phi_I\partial_t\Phi_J - \mathcal{V}_{IJ}\partial_x\Phi_I\partial_x\Phi_J) + \frac{1}{2\pi}\epsilon^{\mu\nu}\tau_I\partial_\mu\Phi_I A_\nu.$$

We recall that operators of the form $e^{i\Theta(\Lambda)}$, with $\Theta(\Lambda) \equiv \Lambda^T\mathcal{K}\Phi$, are local in the sense that they correspond to products of electron creation and annihilation operators.

This edge theory has $2N$ gapless edge modes, N for each chirality. Our goal is to find the conditions under which these modes can be gapped out by charge conserving, time reversal symmetric perturbations. We focus on a general class of scattering terms of the form

$$U(x)\cos[\Theta(\Lambda) - \alpha(x)], \tag{28}$$

where Λ is a $2N$-dimensional integer valued vector. Imposing charge conservation leads to the requirement $\Lambda^T\tau = 0$. As for time reversal, we note that Eq. (12) implies that Θ transforms under time reversal as

$$\mathcal{T}\Theta(\Lambda)\mathcal{T}^{-1} = \Theta(-T\Lambda) - Q(\Lambda)\pi, \tag{29}$$

where

$$Q(\Lambda) \equiv -\Lambda^T\chi. \tag{30}$$

Thus one can construct scattering terms that are even/odd under time reversal by defining

$$U_\pm(\Lambda) = U(x)\{\cos[\Theta(\Lambda) - \alpha(x)] \pm (-1)^{Q(\Lambda)}\cos[\Theta(-T\Lambda) - \alpha(x)]\}. \tag{31}$$

In general, we should include Klein factors in the definition of $U_\pm(\Lambda)$ to ensure that these terms obey the correct commutation relations, i.e., these terms should commute with each other when they act at spatially separated points. However, in the analysis below, we only consider sets of $\{\Lambda_i\}$ satisfying $\Lambda_i^T\mathcal{K}\Lambda_j = 0$. This condition guarantees that the $e^{i\Theta(\Lambda)}$ operators automatically commute with one another, without any need for Klein factors.[17] Thus the Klein factors can be safely ignored, and we will drop them for clarity.

We now examine whether time-reversal symmetric terms of the type U_+ (31) can gap the spectrum without spontaneously breaking time-reversal symmetry. Importantly, it will *not* matter to us whether these terms are relevant or irrelevant in the renormalization group sense. One reason is that we are interested in whether any term can gap out the edge, including those with large coefficients. Thus the perturbative stability of the edge is not our concern here. Another reason is that we can always make the $\{U_+(\Lambda)\}$ terms relevant by appropriately tuning $\mathcal{V}_{IJ}$—at least when $\{\Lambda_i\}$ satisfy $\Lambda_i^T\mathcal{K}\Lambda_j = 0$, as assumed below.

Before tackling the general case, we first warm up with two simple examples. Both examples were discussed (briefly) in Ref. 6.

B. Example 1: Laughlin quantum spin Hall state

In this section, we analyze the stability of the edge for the case

$$\mathcal{K} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix},\ T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tau = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{32}$$

Physically, this case can realized by a toy model in which the spin-up and spin-down electrons each form $\nu = 1/k$ Laughlin states with opposite chiralities.

We note that the elementary charge is

$$e^* = \min_l(l^T\mathcal{K}^{-1}\tau) = \frac{1}{k}, \tag{33}$$

while the quantity $\chi^T\mathcal{K}^{-1}\tau$ is given by

$$\chi^T\mathcal{K}^{-1}\tau = -\frac{1}{k}. \tag{34}$$

In particular, we have $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau = -1$ for all k. Thus, according to the general criterion, all of these states have protected edge modes.

To see how this stability manifests itself in a microscopic analysis, note that the only charge conserving vectors are of the form $\Lambda = (n, -n)$. The corresponding perturbation $U(x)\cos[\Theta(n,-n) - \alpha(x)]$ is even under time reversal for even n and odd for odd n. Thus time-reversal symmetry requires even n, say $n = 2$. Adding such a perturbation will indeed open a gap in the spectrum, at least for large U. However, hand in hand with gapping out the spectrum, this perturbation also spontaneously breaks time-reversal symmetry. To see this, note that when the perturbation gaps out the edge, it "freezes" the value of $\Theta(n,-n)$. As a result, it also freezes the value of $\Theta(1,-1)$. But then $\cos[\Theta(1,-1) - \alpha]$—an operator which is odd under time reversal—acquires a nonzero expectation value. It follows that time-reversal symmetry is broken spontaneously. In this way, we see that none of these

perturbations can gap the two edge modes without breaking time reversal symmetry, either explicitly or spontaneously.

C. Example 2: Two-component fractional quantum spin Hall state

In this section, we analyze the stability of the edge for the case

$$\mathcal{K} = \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix}, \quad T = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

$$\tau^T = (1,1,1,1), \quad \chi^T = (0,0,1,1), \tag{35}$$

where K is a 2×2 matrix. Physically, this case can be realized by a toy model in which the spin-up and spin-down electrons each form two component quantum Hall states with opposite chiralities.

In this case, there are two pairs of counter-propagating edge modes, so we need to fix the value of two different $\Theta(\Lambda)$'s to gap out the edge. The simplest way to do this is to add a perturbation of the form $U_+(\Lambda)$ (31) where $\Lambda, -T\Lambda$ are linearly independent and charge conserving. We write this term as

$$U(x)\{\cos[\Theta(\Lambda_1) - \alpha(x)] + (-1)^{Q(\Lambda_1)} \cos[\Theta(\Lambda_2) - \alpha(x)]\}, \tag{36}$$

where $\Lambda_1 \equiv \Lambda$ and $\Lambda_2 \equiv -T\Lambda$.

According to Haldane's null vector criterion,[18] such a perturbation can gap out the four edge modes if Λ_1 and Λ_2 satisfy

$$\Lambda_1^T \mathcal{K} \Lambda_1 = \Lambda_2^T \mathcal{K} \Lambda_2 = \Lambda_1^T \mathcal{K} \Lambda_2 = 0. \tag{37}$$

The origin of this criterion is that it guarantees that one can make a linear change of variables from Φ to Φ' such that the action for Φ' will be that of two decoupled nonchiral Luttinger liquids. The two terms in Eq. (31) will then gap the spectrum of these two liquids by freezing the values of $\Theta(\Lambda_1)$ and $\Theta(\Lambda_2)$. We review this result in Appendix C.

We now turn to search for charge conserving Λ_1 and Λ_2 such that $\Lambda_2 = -T\Lambda_1$, and such that Λ_1 and Λ_2 satisfy the condition (37). It is convenient to parameterize the matrix $\mathcal{K}$ as

$$\mathcal{K} = \begin{pmatrix} b+us & b & 0 & 0 \\ b & b+vs & 0 & 0 \\ 0 & 0 & -b-us & -b \\ 0 & 0 & -b & -b-vs \end{pmatrix} \tag{38}$$

with b,u,v,s integers and u and v having no common factor. In terms of these parameters, the elementary charge is

$$e^* = \min_l (l^T \mathcal{K}^{-1} \tau) = \frac{1}{(u+v)b + uvs}. \tag{39}$$

Also, the quantity $\chi^T \mathcal{K}^{-1} \tau$ is given by

$$\chi^T \mathcal{K}^{-1} \tau = -\frac{u+v}{(u+v)b + uvs}. \tag{40}$$

The ratio $\frac{1}{e^*} \chi^T \mathcal{K}^{-1} \tau$ is then $(u+v)$, so according to the general criterion, the parity of $u+v$ determines whether the spectrum can be gapped.

When $u+v$ is odd, it is indeed impossible to find Λ_1 and Λ_2 that do not spontaneously break time reversal symmetry. Imagine one had such a solution and define $\Lambda_\pm = \Lambda_1 \pm \Lambda_2$. Then,

$$T\Lambda_- = \Lambda_-, \quad Q(\Lambda_-) = 0, \tag{41}$$

so Λ_-^T must be an integer multiple of $(1,-1,1,-1)$. Also,

$$T\Lambda_+ = -\Lambda_+, \quad \Lambda_-^T \mathcal{K} \Lambda_+ = 0, \tag{42}$$

so Λ_+^T must be an integer multiple of $(v,u,-v,-u)$. But $\cos[\Theta(v,u,-v,-u) - \alpha]$ is odd under time reversal, according to Eq. (29). This means that the perturbation (36) will spontaneously break time-reversal symmetry: the scattering term (36) freezes the value of $\Theta(\Lambda_1)$, $\Theta(\Lambda_2)$ and therefore also freezes the value of $\Theta(\Lambda_1) + \Theta(\Lambda_2)$ and hence $\Theta(v,u,-v,-u)$. It follows that $\cos[\Theta(v,u,-v,-u) - \alpha]$ acquires an expectation value, spontaneously breaking time-reversal symmetry.

On the other hand, when $u+v$ is even (so that both u,v are odd), the above analysis suggests an obvious solution (Λ_1, Λ_2). We can take

$$\Lambda_-^T = (1,-1,1,-1), \tag{43}$$

$$\Lambda_+^T = (v,u,-v,-u), \tag{44}$$

so that

$$\begin{aligned} \Lambda_1^T &= \tfrac{1}{2}(1+v,-1+u,1-v,-1-u), \\ \Lambda_2^T &= \tfrac{1}{2}(-1+v,1+u,-1-v,1-u). \end{aligned} \tag{45}$$

To complete the analysis, we need to check that the perturbation (36) corresponding to Λ_1 and Λ_2 does not spontaneously break time-reversal symmetry. The only way that time-reversal symmetry (or any other symmetry) can be spontaneously broken is if, for some a_1 and a_2 with no common factors, the linear combination $a_1\Lambda_1 + a_2\Lambda_2$ is nonprimitive, that is, $a_1\Lambda_1 + a_2\Lambda_2 = k\Lambda$, where Λ is an integer vector, and k is an integer larger than 1. If such an a_1, a_2, and Λ exist, then when the perturbation freezes the value of $\Theta(\Lambda_i)$ it will also freeze the value of $\Theta(\Lambda)$, which may lead to spontaneous symmetry breaking, as in the discussion after Eq. (42). Conversely, if all such linear combinations are primitive, then the perturbation does not break any symmetries when it freezes $\Theta(\Lambda_i)$. Thus we can be assured that no symmetries are broken spontaneously, as long as we choose the Λ's so that all linear combinations $a_1\Lambda_1 + a_2\Lambda_2$ are primitive.

It is possible to show that such a nonprimitive $a_1\Lambda_1 + a_2\Lambda_2$ exists if and only if the six 2×2 minors generated from the 4×2 matrix with columns Λ_1 and Λ_2 have a common factor (see Appendix D). We now check this condition explicitly. Writing out the 4×2 matrix corresponding to Eq. (45) gives

$$\frac{1}{2} \cdot \begin{pmatrix} 1+v & -1+v \\ -1+u & 1+u \\ 1-v & -1-v \\ -1-u & 1-u \end{pmatrix}. \tag{46}$$

We can see that the minor corresponding to the first two rows is given by

$$\frac{1}{4}[(1+v)(1+u)-(-1+v)(-1+u)]=\frac{u+v}{2}, \quad (47)$$

while the minor corresponding to the second and third rows is

$$\frac{1}{4}[(-1+u)(-1-v)-(1+u)(1-v)]=\frac{v-u}{2}. \quad (48)$$

Clearly these two minors have no common factor since u,v have no common factor. We conclude that there are no nonprimitive linear combinations $a_1\Lambda_1+a_2\Lambda_2$ and therefore the perturbation does not break time-reversal symmetry (or any other symmetry) spontaneously.

D. Microscopic analysis in general case

In this section, we analyze the edge stability of the general Abelian insulators described by Eqs. (19) and (20). The simplest way to gap the edge in the general case is to add several perturbations of the form $U_+(\Lambda)$ (31) each with a different Λ. In general, these perturbations can be divided into two different classes: perturbations where $\Lambda,-T\Lambda$ are linearly independent and perturbations where $T\Lambda=\pm\Lambda$. Perturbations of the first type can gap out two pairs of edge modes while perturbations of the second type can gap out a single pair of edge modes. Therefore, in order to gap out all N pairs of edge modes, we need k perturbations of the first type and l perturbations of the second type, where $2k+l=N$.

We can describe the sum of all these perturbations by listing all the linearly independent Λ's that appear in the various cosine terms. All together, we will have $2k+l=N$ different Λ's since the first type of perturbation involves two Λ's (namely, $\Lambda,-T\Lambda$), while the second type of perturbation involves one Λ. We will label the Λ's by $\Lambda_1,\ldots,\Lambda_N$.

Just like the example discussed in the previous section, the perturbations corresponding to $\Lambda_1,\ldots,\Lambda_N$ can gap the edge if the $\{\Lambda_i\}$ satisfy Haldane's null vector condition[18]

$$\Lambda_i^T\mathcal{K}\Lambda_j=0. \quad (49)$$

This condition guarantees that we can make a change of variables from Φ to Φ' such that (a) the resulting action consists of N decoupled nonchiral Luttinger liquids, and (b) the cosine terms correspond to backscattering terms for each of these decoupled liquids. (For an example, see Appendix C.)

We now turn to search for charge conserving $\{\Lambda_i\}$ satisfying these conditions. It is convenient to define a vector

$$\Lambda_c=\frac{1}{e^*}\mathcal{K}^{-1}\tau. \quad (50)$$

We note that the definition of e^*, see Eq. (27), implies that Λ_c is an integer vector and that the greatest common divisor of $\{\Lambda_{ci}\}$ is 1. According to our criterion, the edge is protected if and only if the quantity $\chi^T\Lambda_c$ is odd. We now verify this claim.

When $\chi^T\Lambda_c$ is odd, it is indeed impossible to find $\{\Lambda_i\}$ that satisfy Eq. (49) and that do not spontaneously break time-reversal symmetry. The reason is that one can always find a linear combination of Λ_i, which is a multiple of Λ_c, as we prove in the next paragraph. But $\cos[\Theta(\Lambda_c)-\alpha]$ is odd under time reversal according to Eq. (29). It then follows that the perturbations corresponding to $\{\Lambda_i\}$ will spontaneously break time-reversal symmetry: these perturbations will freeze the values of $\Theta(\Lambda_i)$ and therefore also $\Theta(\Lambda_c)$, thus giving an expectation value to the time-reversal odd operator $\cos[\Theta(\Lambda_c)-\alpha]$.

To complete the argument, we explain why one can always find a linear combination of $\{\Lambda_i\}$ which is a multiple of Λ_c. First, we note that $\Lambda_i^T\mathcal{K}\Lambda_c=0$ for all i, by charge conservation. At the same time, we can see that the set of all vectors satisfying $\{\Lambda_i^T\mathcal{K}\Lambda=0\}$ has dimension N, since these relations describe N equations in $2N$ unknowns. Combining this observation with the fact that the $\{\Lambda_i\}$ themselves provide N linearly independent solutions to these equations, we deduce that Λ_c cannot be linearly independent from $\{\Lambda_i\}$. Hence, there must be a linear relation of the form $a_c\Lambda_c+\sum a_i\Lambda_i=0$ with all the a's being integers. Intuitively, the basic point of this argument is that one cannot gap the edge without gapping the "charge mode" $\Theta(\Lambda_c)$, and this mode is protected by time-reversal symmetry whenever $\chi^T\Lambda_c$ is odd.

On the other hand, if $\Lambda_c^T\chi$ is even, then the edge *can* be gapped out, as we now show. It is convenient to work in a basis where $t=(1,1,\ldots,1)$ and $t'=(2,2,\ldots,2)$. Using the fact that $T\Lambda_c=-\Lambda_c$, we can see that Λ_c is of the form

$$\Lambda_c^T=(w,0_M,u,-u), \quad (51)$$

where $w=(w_1,\ldots,w_M)$ is an M component integer vector, $u=(u_1,\ldots,u_{N-M})$ is an $N-M$ component integer vector, and 0_M denotes an M component vector of 0's.

It is convenient to divide our analysis into two cases: either (a) the vector w has at least one odd entry or (b) the vector w only has even entries. We begin with case (a). First, we introduce some notation. We define $N-M$ vectors $e_1,\ldots,e_{N-M}$, each with $N-M$ components, by

$$\begin{aligned} e_1&=(1,0,0,\ldots,0),\\ e_2&=(0,1,0,\ldots,0),\\ e_3&=(0,0,1,\ldots,0),\\ &\vdots \end{aligned} \quad (52)$$

and M vectors $f_1,\ldots,f_M$, each with M components, by

$$\begin{aligned} f_1&=(1,0,0,\ldots,0),\\ f_2&=(1,-1,0,\ldots,0),\\ f_3&=(1,0,-1,\ldots,0).\\ &\vdots \end{aligned} \quad (53)$$

We then define $N-1$ vectors $\Xi_1,\ldots,\Xi_{N-1}$, each with $2N$ components by

$$\begin{aligned} \Xi_i&=(0_M,f_{i+1},0_{N-M},0_{N-M}), \quad 1\leqslant i\leqslant M-1,\\ \Xi_i&=(0_M,f_1,-e_{i-M+1},-e_{i-M+1}), \quad M\leqslant i\leqslant N-1. \end{aligned} \quad (54)$$

These vectors obey the conditions $T\Xi_i=\Xi_i$, and $\Xi_i^T\mathcal{K}\Xi_j=0$. Furthermore, since these vectors are charge conserving, we have $\Xi_i^T\mathcal{K}\Lambda_c=\frac{1}{e^*}\Xi^T\tau=0$. Therefore, if we define $\Lambda_1=\Lambda_c,\Lambda_2=\Xi_1,\Lambda_3=\Xi_2$, etc., we will have a set of N vectors satisfying condition (49).

To complete the argument, we need to check that these perturbations do not spontaneously break time-reversal

symmetry. Like the example discussed in the previous section, it is sufficient to show that for any set of a_i, which have no common factor, the linear combination $\sum_i a_i \Lambda_i$ is always primitive. This is turn reduces to the question of showing that the $\binom{2N}{N}$ $N \times N$ minors of the matrix with columns Λ_i have no common factors (see Appendix D). This property can be established straightforwardly, using the fact that the greatest common divisor of $\{\Lambda_{ci}\}$ is 1. For a simple example, see Eqs. (47) and (48).

We now consider case (b): the vector w has only even entries. In this case, we define a vector

$$\Lambda_n^T = (0_M, \alpha_{\text{tot}} \cdot f_1, -\alpha, -\alpha), \tag{55}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{N-M})$, $\alpha_{\text{tot}} = \sum_i \alpha_i$, and α_i is 0 or 1 depending on whether u_i (50) is even or odd.

We then let

$$\Lambda_1 = \tfrac{1}{2}(\Lambda_c + \Lambda_n), \quad \Lambda_2 = \tfrac{1}{2}(\Lambda_c - \Lambda_n). \tag{56}$$

Just as before, one can check that these two vectors obey the condition $\Lambda_1^T \mathcal{K} \Lambda_1 = \Lambda_2^T \mathcal{K} \Lambda_2 = \Lambda_1^T \mathcal{K} \Lambda_2 = 0$. Thus we now have two vectors satisfying condition (49).

To fully gap out the edge, we need to find $N-2$ more vectors. These can be obtained as follows. Consider the $N-1$ vectors Ξ_i (54). These vectors obey the conditions $T\Xi_i = \Xi_i$, and $\Xi_i^T \mathcal{K} \Xi_j = 0$. Furthermore, by the same reasoning as above, we have $\Xi_i^T \mathcal{K} \Lambda_1 = \Xi_i^T \mathcal{K} \Lambda_2 = 0$. Therefore, if we define $\Lambda_3 = \Xi_1, \Lambda_4 = \Xi_2, \ldots$, we will have a set of $N+1$ vectors satisfying condition (49). The only problem is that these vectors are not all linearly independent since $\Lambda_n = \sum_i \alpha_i \Xi_{i+M-1}$. Hence, we need to drop an appropriate one of the Ξ_i's. Once we do this, we will have the required set of N vectors. As in case (a), we can verify that these perturbations do not spontaneously break time-reversal symmetry by showing that the $\binom{2N}{N}$ $N \times N$ minors of the matrix with columns Λ_i have no common factors.

Strictly speaking, we are not quite finished, since we implicitly assumed that $M \neq 0$. If $M = 0$, our construction of the scattering terms Λ_i needs to be slightly modified. In this case, we define

$$\Lambda_n^T = \left(\sum_{i=2}^{N} \alpha_i, -\alpha_2, \ldots, -\alpha_N, \sum_{i=2}^{N-1} \alpha_i, -\alpha_2, \ldots, -\alpha_N \right). \tag{57}$$

Also, we define $\Xi_1, \ldots \Xi_{N-1}$ by

$$\begin{aligned} \Xi_1 &= (1,-1,0,0,\ldots,1,-1,0,0,\ldots), \\ \Xi_2 &= (1,0,-1,0,\ldots,1,0,-1,0,\ldots), \\ \Xi_3 &= (1,0,0,-1,\ldots,1,0,0,-1,\ldots). \\ &\vdots \end{aligned} \tag{58}$$

We then define Λ_1 and Λ_2 as in Eq. (56) and take $\Lambda_3 = \Xi_1, \Lambda_4 = \Xi_2, \ldots$ as above. The remainder of the analysis is identical to case (b), studied above.

IV. STABILITY OF THE EDGE MODES: FLUX INSERTION ARGUMENT

In the previous section, we found that when $\frac{1}{e^*} \chi^T \mathcal{K}^{-1} \tau$ was odd, there was no simple way to gap out the edge modes without breaking time-reversal or charge-conservation symmetry. However, that analysis does not rule out the possibility of gapping out the edge using other, more complicated, perturbations. In this section, we fill in this hole: we give a general argument proving that it is impossible to gap the edge when $\frac{1}{e^*} \chi^T \mathcal{K}^{-1} \tau$ is odd. Combining this result with the conclusions of Sec. III completes our proof that Abelian time-reversal invariant insulators have protected gapless edge modes if and only if the quantity $\frac{1}{e^*} \chi^T \mathcal{K}^{-1} \tau$ is odd.

The argument we present is very similar to the one given in Ref. 6, which is in turn a generalization of the flux insertion argument of Ref. 5. The statement we prove is as follows: we consider one of the insulators (19) and (20) in a cylindrical geometry with an even number of electrons and zero flux through the cylinder. We also assume that the ground state is time-reversal invariant and has short range correlations. Assuming that the quantity $\frac{1}{e^*} \chi^T \mathcal{K}^{-1} \tau$ is odd, we prove that the system always contains at least one low lying excited state, that is, an excited state whose energy gap vanishes in the thermodynamic limit. Furthermore, this excited state has the important property that it is in the same "topological sector" as the ground state (as will be explained below). We interpret this low lying state as evidence for a protected gapless edge mode.

A. Integer case

We begin by explaining the argument in a simple case. To be specific, we consider a noninteracting toy model where spin-up and spin-down electrons each form $\nu = k$ integer quantum Hall states with opposite chiralities. [This example corresponds to the quantum spin Hall state (21) with $K = \mathbf{1}_{\mathbf{k}}$.] As mentioned above, we consider this model in a cylindrical geometry with an even number of electrons and zero flux through the cylinder. We assume that the ground state is time-reversal invariant and has short range correlations. We will show that if k is odd there is always at least one low lying excited state, that is a state whose energy gap vanishes in the thermodynamic limit. Furthermore, this state is robust if we add arbitrary time-reversal invariant, charge conserving local perturbations to the Hamiltonian, as long as we do not close the bulk gap or spontaneously break one of the symmetries.

To begin, we consider the ground state $|\Psi_0\rangle$ of the toy model at zero flux and imagine adiabatically inserting $\Phi_0/2 = hc/2e$ flux through the cylinder. Let us call the resulting state $|\Psi_1\rangle$. Similarly, we let $|\Psi_2\rangle$ be the state obtained by adiabatically inserting $-\Phi_0/2$ flux (see Fig. 1). We note that $|\Psi_1\rangle, |\Psi_2\rangle$ are time reversed partners: $|\Psi_2\rangle = \mathcal{T}|\Psi_1\rangle$. The remainder of the argument can be divided into two parts. In the first part, we show that when k is odd, $|\Psi_1\rangle$ has a Kramers degeneracy at the two ends of the cylinder (a precise definition of this notion of "local Kramers degeneracy" is given in Sec. IV D). In the second part, we use this Kramers degeneracy to prove that there is a robust low-lying excited state at zero flux.

We begin with the first part—establishing that $|\Psi_1\rangle$ has a Kramers degeneracy at the two ends of the cylinder. Here, we

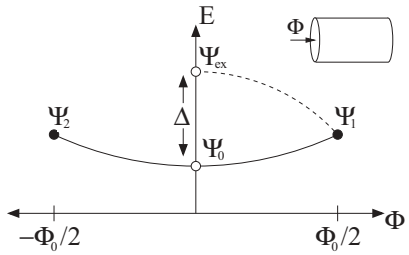

FIG. 1. The flux insertion argument for the $\nu = k$ toy model: we start with the ground state Ψ_0 and adiabatically inset $\pm\Phi_0/2$ flux through the cylinder, obtaining two states Ψ_1 and Ψ_2. If k is odd, then Ψ_1 has a Kramers degeneracy at the two ends of the cylinder, and is therefore degenerate in energy with three other states, one of which is Ψ_2. If we start in one of these three degenerate states and then adiabatically reduce the flux to 0, we obtain an excited state Ψ_{ex} whose energy gap Δ vanishes in the thermodynamic limit.

give an intuitive argument; we derive this claim rigorously (and in greater generality) in Sec. IV D. It is useful to start with the $\nu = 1$ case. Consider the $\nu = 1$ toy model in the Landau gauge. The single particle eigenstates for spin-up and spin-down electrons are then Landau orbitals, localized in the longitudinal direction and delocalized in the periodic direction. Each spin-up orbital has a corresponding time-reversed spin-down partner. These pairs are equally occupied with spin-up and spin-down electrons in the ground state $|\Psi_0\rangle$, since by assumption, $|\Psi_0\rangle$ is time-reversal invariant [see Fig. 2(a)]. If we now adiabatically insert $\Phi_0/2$ flux, each spin-up orbital shifts to the right by half of the interorbital spacing, while each spin-down orbital shifts to the left. The resulting state $|\Psi_1\rangle$ has one unpaired spin-up electron on the right edge and one unpaired spin-down electron on the left edge [see Fig. 2(b)]. More generally, for the $\nu = k$ toy model, one finds that $|\Psi_1\rangle$ has k unpaired spins on the two edges. We can now see that

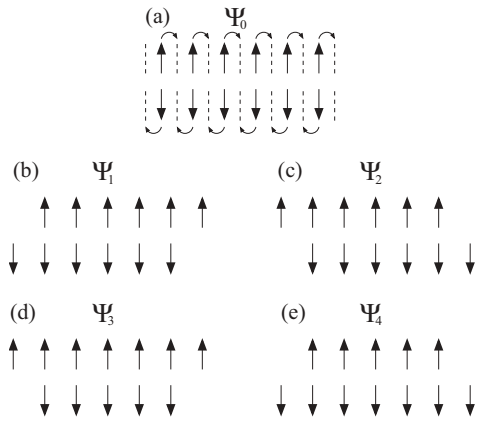

FIG. 2. (a) A schematic portrait of the ground state Ψ_0 of the $\nu = 1$ toy model. Working in the Landau gauge, the single-particle states consist of spin-up and spin-down Landau orbitals. In the time-reversal invariant ground state, the spin-up and spin-down orbitals are equally occupied. After inserting $\Phi_0/2$ flux, the spin-up and spin-down orbitals shift in opposite directions, resulting in the state Ψ_1. (b) The state Ψ_1 has a Kramers degeneracy on both ends of the cylinder and is therefore degenerate with its time-reversed partner (c) Ψ_2 as well as two other states, (d) Ψ_3, and (e) Ψ_4.

when k is odd, $|\Psi_1\rangle$ has an odd number of electrons localized near the two edges. On an intuitive level, this property implies our result: $|\Psi_1\rangle$ has a Kramers degeneracy at each of the two ends of the cylinder.

We now explain the second part of the argument: why the Kramers degeneracy at half a flux quantum implies that there is a robust low-lying excited state at zero flux. The basic point is very simple: as long as the two ends of the cylinder are well separated, Kramers theorem guarantees that $|\Psi_1\rangle$ is part of a multiplet of four states, which are nearly degenerate in energy [see Figs. 2(b)–2(e)]. We note that $|\Psi_2\rangle$ is one of these degenerate states, as it is the time reversed partner of $|\Psi_1\rangle$. We now imagine starting at $\Phi_0/2$ flux and then adiabatically reducing the flux to 0. If we start with the state $|\Psi_1\rangle$, then adiabatic flux removal takes us to the ground state $|\Psi_0\rangle$. However, if we start with $|\Psi_2\rangle$, or one of the other two states degenerate with $|\Psi_1\rangle$, the result is an eigenstate $|\Psi_{\rm ex}\rangle$ of the zero flux Hamiltonian which is distinct from $|\Psi_0\rangle$ (see Fig. 1). At the same time, it necessarily has low energy since the energy change Δ associated with an adiabatic insertion of flux through a cylinder must vanish in the thermodynamic limit (assuming charge conservation is not broken spontaneously). In this way, we can construct a low-lying excited state $|\Psi_{\rm ex}\rangle$ whose energy gap vanishes in the thermodynamic limit.

To complete the argument, we now imagine adding an arbitrary local, time reversal invariant, charge conserving perturbation to the system (for example, we could add short-ranged interactions between the electrons). As long as the perturbation does not close the bulk gap, the above picture must stay the same: the local Kramers degeneracy between $|\Psi_1\rangle$, $|\Psi_2\rangle$ must remain intact, and hence $|\Psi_{\rm ex}\rangle$ must continue to be low in energy. We conclude that the system always contains at least one low-lying excited state $|\Psi_{\rm ex}\rangle$ whose energy gap vanishes in the thermodynamic limit.

B. Topological order and ground-state degeneracy on a cylinder

One of the main complications in extending the flux insertion argument to the general case is that the more general states (19) and (20) have "topological order,"[11,12] that is, they support quasiparticle excitations with fractional statistics. An important consequence of this is that these systems typically have multiple degenerate ground states when defined in a topologically nontrivial geometry, such as a torus.[11,12,19] This ground-state degeneracy is very robust and cannot be split by any local perturbation (in the thermodynamic limit).

Similar low-lying states can occur in a cylindrical geometry: the geometry of interest here.[20] As an example, consider a toy model where spin-up and spin-down electrons each form $\nu = 1/3$ Laughlin states with opposite chiralities (this model corresponds to the case (21) with $K = 3$). This model has three protected low-lying states in a cylindrical geometry. The first low-lying state is the ground state $|\Psi_0\rangle$. The other two states can be obtained by starting with $|\Psi_0\rangle$ and adiabatically inserting either Φ_0 flux, yielding a state $|\Psi_0'\rangle$, or $2\Phi_0$ flux, yielding $|\Psi_0''\rangle$. We know that these states are (nearly) degenerate in energy, since the energy change Δ associated with inserting a fixed amount of flux must vanish in the thermodynamic limit (assuming that charge conservation is not broken). At the same time, we can see that they are

orthogonal to one another since the insertion of a flux quantum transfers a spin-up $e/3$ quasiparticle from the left edge to the right edge, and a spin-down $e/3$ quasiparticle from the right edge to the left edge. Given that the states differ in the amount of spin on the two edges, they must be orthogonal.

In fact, the three states are not only orthogonal, but they belong to different "topological sectors." That is, they cannot be coupled together by any $\mathcal{O}$, which is a finite product of local operators:

$$\langle\Psi_0|\mathcal{O}|\Psi_0'\rangle = \langle\Psi_0|\mathcal{O}|\Psi_0''\rangle = \langle\Psi_0'|\mathcal{O}|\Psi_0''\rangle = 0. \tag{59}$$

On an intuitive level, these matrix elements vanish because an $e/3$ quasiparticle is a fractionalized excitation with nontrivial statistics, so no local operator composed out of electron creation and annihilation operators can create or destroy such an excitation. (Instead, a "stringlike" operator is required to move such an excitation from one edge to another.) A more precise way to argue this is to note that if one takes a pair of states, say $|\Psi_0\rangle$ and $|\Psi_0'\rangle$, and measures the Berry phase associated with moving an $e/3$ spin-up quasiparticle around the cylinder, one gets a different result for the two states (with the difference given by the statistical phase $2\theta_{\text{ex}} = 2\pi/3$). No local operator (or finite product of local operators) can change this relative Berry phase and hence no term of this kind can connect the two states.

C. General case: Outline of the argument

As explained above, when a topologically ordered system is defined in a cylindrical geometry, one typically finds multiple low-lying states, each one belonging to a different topological sector. Because of this phenomenon, we will raise our standards for the flux insertion argument. It is not enough to just show that there are low-lying states at zero flux; this is (nearly) always the case in topologically ordered systems and does not constitute evidence for a protected gapless edge mode, even at a heuristic level. Instead, we will show that there is a low-lying state *in the same topological sector* as the ground state. This will establish the existence of an "unexpected" low-lying state, which can plausibly be taken as evidence for a gapless edge mode.

Because we want to establish this stronger claim, the generalized flux insertion argument begins by inserting not $\pm\Phi_0/2$ flux but $\pm\Phi_0/2e^*$ flux where e^* is the smallest charged excitation, in units of e. By inserting this (larger) amount of flux, we guarantee that the resulting states, $|\Psi_1\rangle, |\Psi_2\rangle$ lie in the same topological sector.[6,20] To see this, note that $|\Psi_1\rangle$ can be obtained from $|\Psi_2\rangle$ by inserting Φ_0/e^* flux. This flux insertion process changes the Berry phase associated with braiding a quasiparticle around the cylinder by

$$\Delta\theta = 2\pi q \frac{1}{e^*}, \tag{60}$$

where q is the charge of the quasiparticle (in units of e). By construction $\frac{q}{e^*}$ is always an integer so that $\Delta\theta$ is a multiple of 2π for every quasiparticle. Hence $|\Psi_1\rangle$ and $|\Psi_2\rangle$ have the same Berry phases with respect to all quasiparticles, implying that they belong to the same topological sector.

Other than this small modification, the argument proceeds as in the integer case. In the first step, we show that the state $|\Psi_1\rangle$ has a Kramers degeneracy near the two ends of the cylinder as long as $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is odd. In the second step, we use this Kramers degeneracy to construct a protected low-lying excited state $|\Psi_{\text{ex}}\rangle$ at zero flux. Similarly to the integer case, we construct $|\Psi_{\text{ex}}\rangle$ by starting with $|\Psi_2\rangle$ and adiabatically inserting $-\Phi_0/2e^*$ flux. Importantly, $|\Psi_{\text{ex}}\rangle$ is guaranteed to be in the same topological sector as the ground state $|\Psi_0\rangle$, since $|\Psi_1\rangle, |\Psi_2\rangle$ are in the same topological sector. In this way, we see that if $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is odd, then the system has a protected low-lying state in the same topological sector as the ground state. This is exactly what we wanted to show.

The only piece of the argument which is missing is the proof that $|\Psi_1\rangle$ has a Kramers degeneracy near the two ends of the cylinder whenever $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is odd. We now establish this fact. The first step is to explain more precisely what it means to have a Kramers degeneracy near the two ends of the cylinder.

D. Local Kramers degeneracy and a local analog of Kramers theorem

In this section, we give a precise definition of "local Kramers degeneracy." We also use this definition to state and prove a local analog of Kramers theorem. We start by reviewing the usual (global) notion of Kramers degeneracy. Recall that a quantum many-body state $|v\rangle$ is said to be "Kramers degenerate" with its time reversed partner $|v'\rangle = \mathcal{T}|v\rangle$ if $|v\rangle$ contains an odd number of electrons, or equivalently $\mathcal{T}^2|v\rangle = -|v\rangle$. The motivation for this terminology is that, if v, v' are of this form, then it follows that

$$\langle v'|\mathcal{O}|v\rangle = 0, \quad \langle v|\mathcal{O}|v\rangle = \langle v'|\mathcal{O}|v'\rangle \tag{61}$$

for any Hermitian time-reversal invariant operator $\mathcal{O}$. In particular, $|v\rangle$ and $|v'\rangle$ are guaranteed to be orthogonal and degenerate in energy for any time-reversal invariant Hamiltonian H. The result (61) is known as Kramers theorem.

Now suppose that $|v\rangle$ is a quantum many body state with an *even* number of electrons. In this case, $|v\rangle$ does not satisfy the requirements for the usual global Kramers degeneracy. However, $|v\rangle$ may still exhibit a "local Kramers degeneracy." Imagine, for example, we take a time-reversal invariant insulator and insert two additional electrons, trapping them in widely separated potential wells. Let us denote the position of the two potential wells by a, b. We expect intuitively that we can treat the two regions near a, b as two separate systems, each with an odd number of electrons and a corresponding local Kramers degeneracy. Moreover, we expect that this local Kramers degeneracy guarantees that the ground state is part of a multiplet of four degenerate states; two states coming from each region.

We now formalize this intuition. First, we give a definition of local Kramers degeneracy; afterwards we state a local analog of Kramers theorem. All proofs are given in the Appendixes.

First, we need to define the notion of a "local operator": we will say that an operator is local if it can be written as a sum of terms, each of which is a product of an even number of electron creation and annihilation operators acting in some finite sized region. One implication of this definition is that

local operators always commute if they act on nonoverlapping regions.

We are now ready to define "local Kramers degeneracy". Let $|v\rangle$ be a quantum many body state with an even number of electrons. Suppose v has short range correlations. That is,

$$\langle v|\mathcal{O}_1\mathcal{O}_2|v\rangle = \langle v|\mathcal{O}_1|v\rangle\langle v|\mathcal{O}_2|v\rangle \tag{62}$$

for any two widely separated local operators, $\mathcal{O}_1$ and $\mathcal{O}_2$. We will say that $|v\rangle$ has a local Kramers degeneracy in regions a and b if:

(1) The state $|v\rangle$ satisfies

$$\mathcal{T}|v\rangle = S_a S_b|v\rangle \tag{63}$$

for some local operators S_a and S_b acting on regions a and b. We will assume S_a, S_b are normalized so that $\|S_a v\| = \|S_b v\| = 1$.

(2) The state $|v\rangle$ satisfies

$$\mathcal{T}_a^2|v\rangle = \mathcal{T}_b^2|v\rangle = -|v\rangle, \tag{64}$$

where $\mathcal{T}_a \equiv \mathcal{T} S_b$, $\mathcal{T}_b \equiv \mathcal{T} S_a$.

We now explain the physical meaning of these conditions. The first condition (63) is essentially the statement that $|v\rangle$ is time-reversal invariant away from regions a and b. The second condition (64) formalizes the notion of having an odd number of electrons localized in regions a and b. The basic idea is that the two antilinear operators $\mathcal{T}_a$ and $\mathcal{T}_b$ implement a local time-reversal symmetry transformation in the two regions a and b. The condition $\mathcal{T}_a^2|v\rangle = -|v\rangle$ is then analogous to the usual requirement for Kramers theorem, $\mathcal{T}^2|v\rangle = -|v\rangle$.

A useful fact, proved in Appendix E, is that if $|v\rangle$ satisfies Eqs. (62) and (63), then either

$$\mathcal{T}_a^2|v\rangle = \mathcal{T}_b^2|v\rangle = +|v\rangle \text{ or } \mathcal{T}_a^2|v\rangle = \mathcal{T}_b^2|v\rangle = -|v\rangle. \tag{65}$$

The two cases correspond (roughly speaking) to having either an even or odd number of electrons localized near regions a and b. (Not surprisingly, local Kramers degeneracies always come in pairs, since we have assumed that the total number of electrons is even.)

To get a feeling for this definition, it is useful to think about the above example of a time reversal invariant insulator state $|v\rangle$, with two additional electrons localized near points a and b. Intuitively, $|v\rangle$ has a local Kramers degeneracy near a and b. At the same time, $|v\rangle$ satisfies the above conditions, as we now show. To establish the first condition (63), let us assume without loss of generality that the two electron spins point in the $+\hat{z}$ direction. Then, we have $\mathcal{T}|v\rangle = \sigma_a^x\sigma_b^x|v\rangle$, where σ_a^x, σ_b^x are spin-flip operators acting on the spins localized near a and b. Hence, we can take $S_a = \sigma_a^x$, $S_b = \sigma_b^x$. As for the second condition (64), this relation follows from the fact that $\mathcal{T}\sigma_b^x\mathcal{T} = -\sigma_b^x$ so that $\mathcal{T}_a^2|v\rangle = (\mathcal{T}\sigma_b^x)^2|v\rangle = -|v\rangle$ (and similarly for $\mathcal{T}_b^2|v\rangle$). Finally, we note that $\mathcal{T}_a$ does in fact implement a local time-reversal transformation near region a: $\mathcal{T}_a|v\rangle$ is a state where the spin near b points in the $+\hat{z}$ direction and the spin near a points in the $-\hat{z}$ direction—the "local time reverse" of $|v\rangle$.

To complete our discussion, we now state a local analog of Kramers theorem (61) based on this definition. The proof of this result is given in Appendix F.

Local Kramers theorem: *Let $|v\rangle$ be a quantum many-body state satisfying conditions (62)–(64) so that $|v\rangle$ has a local Kramers degeneracy in regions a and b. Let $|v'\rangle = \mathcal{T}_a|v\rangle$. Then $|v\rangle$ and $|v'\rangle$ satisfy*

$$\langle v'|\mathcal{O}|v\rangle = 0, \quad \langle v|\mathcal{O}|v\rangle = \langle v'|\mathcal{O}|v'\rangle \tag{66}$$

for any $\mathcal{O}$ which is a finite product of local, Hermitian, time-reversal invariant operators.

In order to understand the implications of this result, suppose that $|v\rangle$ is a ground state of a time-reversal invariant Hamiltonian H with a finite energy gap. We can see from Eq. (66) that $|v'\rangle = \mathcal{T}_a|v\rangle$ is orthogonal to $|v\rangle$ and degenerate in energy. In other words, Eq. (66) guarantees that H has a twofold ground-state degeneracy. In addition, Eq. (66) implies that if we add an arbitrary local time-reversal invariant perturbation to H then the degeneracy between $|v\rangle$ and $|v'\rangle$ does not split to any finite order in perturbation theory: at each order, the off-diagonal matrix elements vanish, while the two diagonal elements are identical. In this way, Eq. (66) implies the existence of a robust ground state degeneracy which is analogous to the usual Kramers degeneracy.

It is worth mentioning that while the above theorem focuses entirely on the Kramers degeneracy in region a, there is an identical twofold degeneracy coming from region b. Thus there are four degenerate states altogether: $|v\rangle, |v'\rangle = \mathcal{T}_a|v\rangle, |v''\rangle = \mathcal{T}_b|v\rangle, |v'''\rangle = \mathcal{T}|v\rangle$. This entire multiplet of four states obeys the analog of Eq. (66), as can be shown using arguments similar to the ones given in Appendix F.

E. Establishing that $|\Psi_1\rangle$ has a local Kramers degeneracy

We now fill in the missing piece of the flux insertion argument from Sec. IV C: we show that if $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is odd, then $|\Psi_1\rangle$ has a local Kramers degeneracy near the two ends of the cylinder. The first step is to understand the relationship between the two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$. Recall that these states are obtained by starting in the state $|\Psi_0\rangle$ and adiabatically inserting $\pm\Phi_0/2e^*$ flux. Therefore, if we start in $|\Psi_1\rangle$ and then insert $-\Phi_0/e^*$ flux, we obtain $|\Psi_2\rangle$.

We next use the edge theory (10) for the two edges of the cylinder to analyze the effect of this flux insertion process. A simple calculation (see Appendix G) shows that the effect of the flux insertion process is given by applying an operator $S_l\, S_r$ to $|\Psi_1\rangle$, where S_l acts on the left edge and S_r acts on the right edge:

$$|\Psi_2\rangle = S_l S_r|\Psi_1\rangle. \tag{67}$$

Here, S_l and S_r are given by

$$S_l = \Gamma_l(\Lambda_c), \quad S_r = \Gamma_r(-\Lambda_c) \tag{68}$$

with

$$\Gamma_l(\Lambda) \equiv \int \frac{dx}{\sqrt{L}} e^{i\Theta_l(\Lambda)} \tag{69}$$

and similarly for Γ_r. The vector Λ_c is defined by $\Lambda_c = \frac{1}{e^*}\mathcal{K}^{-1}\tau$.

We can now identify S_l and S_r with the S_a and S_b operators from the definition of local Kramers degeneracy. With this identification, Eq. (67) immediately establishes one of the conditions (63) for local Kramers degeneracy. All that remains is to prove the relation (64). In other words, we need to show $\mathcal{T} S_l \mathcal{T} S_l|\Psi_1\rangle = -|\Psi_1\rangle$. To this end, note that according to

Eq. (29), $\Gamma_l(\Lambda_c)$ transforms under time reversal as

$$\mathcal{T}^{-1}\Gamma_l(\Lambda_c)\mathcal{T} = \Gamma_l(-\Lambda_c)e^{i\pi\chi^T\Lambda_c}. \tag{70}$$

Then, since $\chi^T\Lambda_c = \frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is odd, we have

$$\mathcal{T}^{-1}\Gamma_l(\Lambda_c)\mathcal{T} = -\Gamma_l(-\Lambda_c). \tag{71}$$

It follows that

$$\begin{aligned}\mathcal{T}S_l\mathcal{T}S_l|\Psi_1\rangle &= -\Gamma_l(-\Lambda_c)\Gamma_l(\Lambda_c)|\Psi_1\rangle \\ &= -\Psi_1\end{aligned} \tag{72}$$

as required.

V. CONCLUSION

In this work, we have investigated the properties of general time-reversal invariant insulators with Abelian quasiparticle statistics. First, we constructed all possible Chern-Simons theories that can describe such insulators. Second, we derived a general criterion for when such states have protected edge modes—that is, edge modes that cannot be gapped without breaking time-reversal or charge-conservation symmetry, explicitly or spontaneously. Finally, we gave a precise definition of "local Kramers degeneracy" and we proved a local analog of Kramers theorem, the important concepts in the theory of topological insulators.

A number of questions remain open. First, we do not yet have microscopic realizations of all the Chern-Simons theories discussed here. In particular, we do not have any examples where the vector x in the definition of χ (20) is nonzero. Such examples would be particularly interesting because, in their simplest form, they would give microscopic models for bosonic topological insulators without fractionalization (see Sec. II D for a brief discussion). These bosonic topological insulators were conjectured to exist in Ref. 16, but have not yet been studied in the context of concrete microscopic models.

Another issue is that we have focused entirely on insulators with *Abelian* statistics. Yet it is not hard to construct microscopic models for non-Abelian time-reversal invariant insulators. For example, one can imagine toy models, similar to those discussed in Ref. 6, where the spin-up and spin-down electrons each form *non-Abelian* fractional quantum Hall states with opposite chiralities. It would be interesting to analyze the stability of the edge modes in this case. This stability question was partially addressed by the flux insertion argument in Ref. 6. However, the analysis in Ref. 6 is not yet complete since the flux insertion argument only allows us to prove that the edge modes are protected when σ_{sH}/e^* is odd; it does not prove that the edge modes can be gapped out when σ_{sH}/e^* is even. Completing this analysis requires a microscopic investigation of the stability of the edge modes, and is an interesting question for future research.

Finally, it would be interesting to generalize the approach presented here to the three dimensional case. We now have a number of examples[10,21,22] of three dimensional fractional topological insulators, but these constructions almost certainly do not exhaust all the possibilities. It would be interesting to develop a general classification scheme for 3D fractional topological insulators analogous to the one discussed here.

ACKNOWLEDGMENTS

M.L. was supported in part by an Alfred P. Sloan Research Fellowship. A.S. thanks the US-Israel Binational Science Foundation, the Minerva foundation and Microsoft Station Q for financial support.

APPENDIX A: GENERAL TIME-REVERSAL INVARIANT CHERN-SIMONS THEORIES

In this section, we find the most general $p \times p$ integer matrices, $\mathcal{K}$, T, and p component vectors τ, χ satisfying

$$T^T\mathcal{K}T = -\mathcal{K}, \tag{A1}$$

$$T^2 = 1, \tag{A2}$$

$$T\tau = \tau, \tag{A3}$$

$$(1 - T^T)\chi \equiv \tau \ (\mathrm{mod}\ 2). \tag{A4}$$

We show that the most general solutions to these consistency conditions are of the form Eqs. (19) and (20), up to the equivalence transformations (15)–(17).

To begin, we note that $\mathrm{Tr}(T) = 0$. We can derive this fact by multiplying both sides of (A1) on the right by $T\mathcal{K}^{-1}$ and taking the trace. Next, we note that T has eigenvalues ± 1 since $T^2 = 1$. Combining this with the fact that $\mathrm{Tr}(T) = 0$, we conclude that T has an equal number of $+1$ and -1 eigenvalues. Let the number of $+1$ eigenvalues be N, so that $p = 2N$. Interestingly, we can already see that p must be even.

In the third step, we choose a basis $\{v_1, \ldots, v_N\}$ for the $+1$ eigenspace of T. We choose this basis so that the v_i are integer vectors. This is always possible since this eigenspace is spanned by the columns of $1 + T$, a matrix with integer entries. In fact, we will go a step further and choose the basis so that the $N \times N$ minors of the matrix with columns $\{v_1, \ldots, v_N\}$ have no common factor. This property guarantees that we can extend $\{v_1, \ldots, v_N\}$ to an integer basis $\{v_1, \ldots, v_N, w_1, \ldots, w_N\}$ for the whole $p = 2N$ dimensional space, such that the matrix with columns $\{v_1, \ldots, v_N, w_1, \ldots, w_N\}$ has determinant ± 1.

We next make a change of basis to $\{v_1, \ldots, v_N, w_1, \ldots, w_N\}$. Equivalently, we make a transformation $T \to UTU^{-1}$ (15), where U^{-1} is the matrix with columns $\{v_1, \ldots, v_N, w_1, \ldots, w_N\}$. After this change of basis, T is of the form

$$T = \begin{pmatrix} \mathbf{1_N} & F \\ 0 & G \end{pmatrix}, \tag{A5}$$

where F and G are of dimensions $N \times N$.

In the fifth step, we use $T^2 = 1$ to deduce that $G^2 = 1$. We also have $\mathrm{Tr}(G) = \mathrm{Tr}(T) - N = -N$. Combining these two facts, we conclude that all the eigenvalues of G are equal to -1 so that $G = -\mathbf{1_N}$. Hence T is of the form

$$T = \begin{pmatrix} \mathbf{1_N} & F \\ 0 & -\mathbf{1_N} \end{pmatrix}. \tag{A6}$$

The sixth step is to make another transformation $T \to UTU^{-1}$ (15), where U is an integer matrix of the form

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \tag{A7}$$

and $\det(U_1) = \det(U_2) = \pm 1$. Under this transformation, $F \to U_1 F U_2^{-1}$. By choosing U_1 and U_2 appropriately we can make F a diagonal matrix: this follows from the Smith normal form[23] for integer matrices.

The seventh step is to make another transformation $T \to UTU^{-1}$ (15), where U is of the form

$$U = \begin{pmatrix} \mathbf{1_N} & Y \\ 0 & \mathbf{1_N} \end{pmatrix}. \tag{A8}$$

Under this transformation, $F \to F - 2Y$. By choosing Y appropriately, we can guarantee that F has only 0's and 1's on the diagonal. Hence, we can assume without loss of generality that F is of the form

$$F = \begin{pmatrix} \mathbf{1_{N-M}} & 0 \\ 0 & 0 \end{pmatrix}, \tag{A9}$$

where $M \leqslant N$. Putting this all together, we conclude that T can be written in the form

$$T = \begin{pmatrix} \mathbf{1_{N-M}} & 0 & \mathbf{1_{N-M}} & 0 \\ 0 & \mathbf{1_M} & 0 & 0 \\ 0 & 0 & \mathbf{-1_{N-M}} & 0 \\ 0 & 0 & 0 & \mathbf{-1_M} \end{pmatrix}. \tag{A10}$$

The final step is to make yet another transformation $T \to UTU^{-1}$ (15), where U is of the form

$$U = \begin{pmatrix} \mathbf{1_{N-M}} & 0 & 0 & 0 \\ 0 & \mathbf{1_M} & 0 & 0 \\ \mathbf{1_{N-M}} & 0 & \mathbf{1_{N-M}} & 0 \\ 0 & 0 & 0 & \mathbf{1_M} \end{pmatrix}. \tag{A11}$$

This transformation changes T to

$$T = \begin{pmatrix} 0 & 0 & \mathbf{1_{N-M}} & 0 \\ 0 & \mathbf{1_M} & 0 & 0 \\ \mathbf{1_{N-M}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{-1_M} \end{pmatrix}. \tag{A12}$$

After reordering the rows and columns, we arrive at

$$T = \begin{pmatrix} \mathbf{-1_M} & 0 & 0 & 0 \\ 0 & \mathbf{1_M} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1_{N-M}} \\ 0 & 0 & \mathbf{1_{N-M}} & 0 \end{pmatrix}. \tag{A13}$$

Using the relations (A1) and (A3), we then deduce that $\mathcal{K}, \tau$ must be of the form

$$\mathcal{K} = \begin{pmatrix} 0 & A & B & B \\ A^T & 0 & C & -C \\ B^T & C^T & K & W \\ B^T & -C^T & W^T & -K \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 \\ t' \\ t \\ t \end{pmatrix}, \tag{A14}$$

where matrix A is of dimension $M \times M$, $\mathcal{K}$ and $W = -W^T$ are of dimension $(N-M) \times (N-M)$ and the matrices B, C are of dimension $M \times (N-M)$. Also, t' is an M component integer vector and t is an $(N-M)$ component integer vector. In fact, it follows from condition (4) that t' is an *even* vector.

As for χ, it is not hard to see that we can put χ in the form

$$\chi = \begin{pmatrix} y_1 \\ 0 \\ 0 \\ y_2 \end{pmatrix} \tag{A15}$$

using an appropriate transformation $\chi \to \chi + \frac{1}{\pi}\mathcal{K}(1-T)\xi$ (16). Then, using the relation (A4), we deduce that

$$y_2 \equiv t \pmod 2 \tag{A16}$$

and y_1 is an integer vector. Finally, we use the equivalence $\chi \to \chi + 2v$ (17) to set $y_2 = t$, and to change y_1 into a vector of 1's and 0's. This completes our derivation of Eqs. (19) and (20).

APPENDIX B: REFORMULATION OF EDGE MODE CRITERION IN TERMS OF SPIN-HALL CONDUCTIVITY

According to the results in Secs. III and IV, the insulators described by Eqs. (19) and (20) have protected edge modes if and only if $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ is odd. In this section, we show that this criterion can be reformulated in simpler language for a large class of s^z conserving insulators. To be specific, we consider insulators such that $\chi \equiv \tau_\downarrow \pmod 2$, where $\tau_\uparrow$ and $\tau_\downarrow$ are the "charge vectors," which keep track of the (separately conserved) spin-up and spin-down electrons. For this class of insulators, we show that the edge mode criterion can be equivalently phrased in terms of the parity of σ_{sH}/e^* where σ_{sH} is the spin-Hall conductivity in units of $e/2\pi$.

To see this, define

$$\Lambda_c = \frac{1}{e^*}\mathcal{K}^{-1}\tau. \tag{B1}$$

Then, we can write

$$\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau = \chi^T\Lambda_c. \tag{B2}$$

At the same time, we have

$$\sigma_{\mathrm{sH}} = \frac{1}{2}(\tau_\uparrow - \tau_\downarrow)^T\mathcal{K}^{-1}\tau = -\tau_\downarrow^T\mathcal{K}^{-1}\tau, \tag{B3}$$

where the second equality follows from time-reversal symmetry. Hence,

$$\frac{\sigma_{\mathrm{sH}}}{e^*} = -\frac{1}{e^*}\tau_\downarrow^T\mathcal{K}^{-1}\tau = -\tau_\downarrow^T\Lambda_c. \tag{B4}$$

Comparing Eqs. (B2) and (B4), the claim follows immediately: if $\chi \equiv \tau_\downarrow \pmod 2$, then $\frac{1}{e^*}\chi^T\mathcal{K}^{-1}\tau$ has the same parity as σ_{sH}/e^*. Thus, in this case, the criterion for protected edge modes can be equivalently formulated as the condition that σ_{sH}/e^* is odd.

APPENDIX C: THE NULL VECTOR CRITERION

In this section, we explain why the null vector condition (37) guarantees that the perturbation (36) can gap out the

two pairs of edge modes in example 2 (see Sec. III C). The idea is as follows: we consider a change of variables of the form $\Phi' = U\Phi$, where U is a 4×4 matrix whose first two rows are $\Lambda_1^T \mathcal{K}, \Lambda_2^T \mathcal{K}$. The condition (37) guarantees that, if we choose the other two rows of U appropriately, then the resulting $\mathcal{K}' = (U^{-1})^T \mathcal{K} U^{-1}, \tau' = (U^{-1})^T \tau$ can be put in the form

$$\mathcal{K}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \tau' = \begin{pmatrix} t_1 \\ t_2 \\ 0 \\ 0 \end{pmatrix}. \tag{C1}$$

(The reason that τ' takes the above form is that Λ_1 and Λ_2 are *charge conserving* vectors.)

We next assume that interactions on the edge are such that $\mathcal{V}' = (U^{-1})^T \mathcal{V} U^{-1}) = v\delta_{IJ}$ (we can make this assumption without loss of generality since the velocity matrix is nonuniversal and can be modified by appropriate perturbations at the edge). In this case, the edge theory can be written as the sum of two decoupled actions:

$$L = L_{13} + L_{24}, \tag{C2}$$

where

$$L_{13} = \frac{1}{4\pi}[2\partial_x \Phi_1' \partial_t \Phi_3' - v(\partial_x \Phi_1')^2 - v(\partial_x \Phi_3')^2] + \frac{t_1}{2\pi} \epsilon^{\mu\nu} \partial_\mu \Phi_1' A_\nu \tag{C3}$$

and

$$L_{24} = \frac{1}{4\pi}\left[2\partial_x \Phi_2' \partial_t \Phi_4' - v(\partial_x \Phi_2')^2 - v(\partial_x \Phi_4')^2\right] + \frac{t_2}{2\pi} \epsilon^{\mu\nu} \partial_\mu \Phi_2' A_\nu. \tag{C4}$$

In the new Φ' variables, the perturbation (36) becomes

$$U(x)\{\cos[\Phi_1' - \alpha(x)] + (-1)^{Q(\Lambda_1)} \cos[\Phi_2' - \alpha(x)]\}. \tag{C5}$$

It is now easy to analyze the effect of the perturbation (C5): clearly, this term will gap out the two Luttinger liquids by freezing the values of Φ_1' and Φ_2'. Note that the relevance or irrelevance of Eq. (C5) is not important here, since the system will always be gapped out for large U. Alternatively, it is not hard to see that we can always make Eq. (C5) relevant if we tune the velocity matrix $\mathcal{V}'$ so that the coefficients of $(\partial_x \Phi_1')^2$ and $(\partial_x \Phi_2')^2$ are much larger than the coefficients of $(\partial_x \Phi_3')^2$ and $(\partial_x \Phi_4')^2$.

APPENDIX D: PRIMITIVITY CONDITION

Let $\Lambda_1, \ldots, \Lambda_N$ be N integer vectors with M components, $M \geqslant N$. Let B denote the matrix with columns $\Lambda_1, \ldots, \Lambda_N$. In this section, we show that there exist integers $a_1, \ldots, a_N$ with no common divisor such that the linear combination $\sum_i a_i \Lambda_i$ is nonprimitive, if and only if the set of $\binom{2N}{N}$ $N \times N$ minors of the matrix B have a common factor.

To begin, we make use of the Smith normal form[23] for integer matrices. According to this result, we can always find an $M \times M$ integer matrix S and an $N \times N$ integer matrix T, both with determinant ± 1, such that $B = S\,D\,T$, where D is an $M \times N$ integer matrix of the form

$$D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_N \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{D1}$$

Next, we compare the $N \times N$ minors of B and D. Let b denote the greatest common factor of the $N \times N$ minors of B, and let d denote the greatest common factor of the $N \times N$ minors of D. Given that S and T are integer matrices, it follows from $B = S\,D\,T$ that the minors of B are integer linear combinations of the minors of D. Hence d divides into b. On the other hand, since S and T have determinant ± 1, we can equally well write $D = S^{-1}\,B\,T^{-1}$, implying that the minors of D are integer linear combinations of the minors of B. Thus, we also know that b divides into d. Combining these two observations gives $b = d$. At the same time, the explicit expression for D (D1) immediately implies $d = |d_1\, d_2 \cdots d_N|$. We conclude that the greatest common factor b is larger than 1 if and only if one of the d_j's is different from ± 1.

To complete the argument, we need to show that having one of the d_j's different from ± 1 is necessary and sufficient for having a nonprimitive linear combination $\sum_i a_i \Lambda_i$. To see that this condition is sufficient, suppose that one of the d_j's is different from ± 1. We then define $a_1, \ldots, a_N$ to be the elements of the jth column of T^{-1}. Using $B = S\,D\,T$, we can see that $\sum_i a_i \Lambda_i = d_j\, \Lambda$, where Λ is the jth column of S. Also, we can see that the a_i have no common divisor since T^{-1} has determinant ± 1. Hence we have constructed a nonprimitive linear combination. Conversely, suppose $d_i = \pm 1$ for all i, and let $a_1, \ldots, a_N$ be integers with no common factor. Defining $a = (a_1, \ldots, a_N)$, we have $\sum_i a_i \Lambda_i = B\,a = S\,D\,T\,a$. We then note that since S, T have determinant ± 1, the vector $S\,D\,T\,a$ must have no common factor. We conclude that $\sum_i a_i \Lambda_i$ is primitive for any choice of a_i. This completes the proof.

APPENDIX E: PROOF OF EQ. (65) REGARDING LOCAL KRAMERS DEGENERACIES

Let $|v\rangle$ be a quantum many-body state with an even number of electrons and short-range correlations. In this section, we show that if $|v\rangle$ satisfies Eq. (63), then either

$$\mathcal{T}_a^2 |v\rangle = \mathcal{T}_b^2 |v\rangle = |v\rangle \;\text{ or }\; \mathcal{T}_a^2 |v\rangle = \mathcal{T}_b^2 |v\rangle = -|v\rangle. \tag{E1}$$

The first step is to show that $\mathcal{T}_a^2 |v\rangle = \zeta |v\rangle$ and $\mathcal{T}_b^2 |v\rangle = \frac{1}{\zeta} |v\rangle$ for some complex number ζ. To see this, we define $A = \mathcal{T}_a^2$ and $B = \mathcal{T}_b^2$ and note that

$$\begin{aligned} BA|v\rangle &= (\mathcal{T} S_a \mathcal{T} S_a)(\mathcal{T} S_b \mathcal{T} S_b)|v\rangle \\ &= (\mathcal{T} S_a \mathcal{T}) S_a (\mathcal{T} S_b \mathcal{T}) S_b |v\rangle \\ &= (\mathcal{T} S_a \mathcal{T})(\mathcal{T} S_b \mathcal{T}) S_a S_b |v\rangle \\ &= \mathcal{T} S_a S_b \mathcal{T} S_a S_b |v\rangle \\ &= |v\rangle. \end{aligned} \tag{E2}$$

Here, the third equality follows from the fact that S_a,$(\mathcal{T} S_b \mathcal{T})$ are local operators acting in nonoverlapping regions and therefore commute. Next we note

$$\begin{aligned}\langle v|B^\dagger B|v\rangle\langle v|A^\dagger A|v\rangle &= \langle v|(B^\dagger B)(A^\dagger A)|v\rangle \\ &= \langle v|A^\dagger B^\dagger B A|v\rangle \\ &= \langle v|v\rangle \\ &= 1, \end{aligned} \tag{E3}$$

where the first equality comes from the fact that v has short-range correlations and A, B are local operators acting in distant locations: $A = \mathcal{T}_a^2 = (\mathcal{T} S_b \mathcal{T}) S_b$ acts in region b, while $B = \mathcal{T}_b^2 = (\mathcal{T} S_a \mathcal{T}) S_a$ acts in region a.

On the other hand, if we insert a complete set of states $\Sigma_i |v_i\rangle\langle v_i|$ into $\langle v|B^\dagger B|v\rangle, \langle v|A^\dagger A|v\rangle$ and we choose this set of states so that one of the v_i's is v, we deduce the inequalities

$$\begin{aligned}\langle v|B^\dagger B|v\rangle &\geqslant |\langle v|B|v\rangle|^2, \\ \langle v|A^\dagger A|v\rangle &\geqslant |\langle v|A|v\rangle|^2. \end{aligned} \tag{E4}$$

Multiplying these two inequalities gives

$$\begin{aligned}\langle v|B^\dagger B|v\rangle\langle v|A^\dagger A|v\rangle &\geqslant |\langle v|B|v\rangle|^2\, |\langle v|A|v\rangle|^2 \\ &= |\langle v|BA|v\rangle|^2 \\ &= |\langle v|v\rangle|^2 \\ &= 1, \end{aligned} \tag{E5}$$

where the first equality again follows from the fact that v has short range correlations. Comparing Eqs. (E3) and (E5), we deduce that v must be an eigenvector of both A and B, since this is the only way the inequality can be saturated. In other words, $\mathcal{T}_a^2|v\rangle = \zeta|v\rangle, \mathcal{T}_b^2|v\rangle = \frac{1}{\zeta}|v\rangle$ for some complex ζ.

Now we show that $\zeta = \pm 1$. First, we note that

$$\begin{aligned}\mathcal{T}_a|v\rangle &= \mathcal{T} S_b|v\rangle \\ &= \mathcal{T} S_b(\mathcal{T} S_a S_b|v\rangle) \\ &= (\mathcal{T} S_b \mathcal{T}) S_a S_b|v\rangle \\ &= S_a(\mathcal{T} S_b \mathcal{T}) S_b|v\rangle \\ &= S_a \mathcal{T}_a^2|v\rangle, \end{aligned} \tag{E6}$$

so that

$$\mathcal{T}_a|v\rangle = \zeta S_a|v\rangle. \tag{E7}$$

This relation, together with the fact that $\|\mathcal{T}_a v\| = \|\mathcal{T} S_b v\| = \|S_b v\| = 1$ and $\|S_a v\| = 1$, implies that $|\zeta| = 1$. To see that $\zeta = \pm 1$, we note that

$$\mathcal{T}_a^2(\mathcal{T}_a|v\rangle) = \mathcal{T}_a(\mathcal{T}_a^2|v\rangle) = \zeta^* \mathcal{T}_a|v\rangle \tag{E8}$$

by antilinearity. At the same time,

$$\mathcal{T}_a^2(S_a|v\rangle) = S_a \mathcal{T}_a^2|v\rangle = \zeta S_a|v\rangle, \tag{E9}$$

where the first equality follows from the fact that $\mathcal{T}_a^2$ and S_a act in different regions and hence commute. Given that $\mathcal{T}_a|v\rangle \propto S_a|v\rangle$ by Eq. (E7), we must have $\zeta = \zeta^*$. We conclude that $\zeta = \pm 1$.

APPENDIX F: PROOF OF THE LOCAL ANALOG OF KRAMERS THEOREM

In this Appendix, we prove the local analog of Kramers theorem described in Sec. IV D. Let $|v\rangle$ be a quantum many body state with an even number of electrons, short range correlations, and with a local Kramers degeneracy in regions a and b [i.e., suppose $|v\rangle$ satisfies conditions (62)–(64)]. Define $|v'\rangle = \mathcal{T}_a|v\rangle$, where $\mathcal{T}_a = \mathcal{T} S_b$. The result we wish to establish is that

$$\langle v'|\mathcal{O}|v\rangle = 0, \tag{F1}$$

$$\langle v|\mathcal{O}|v\rangle = \langle v'|\mathcal{O}|v'\rangle, \tag{F2}$$

for any $\mathcal{O}$, which is a finite product of local, Hermitian, time-reversal invariant operators.

To begin, we write $\mathcal{O} = \mathcal{O}_1 \cdots \mathcal{O}_k$, where the $\mathcal{O}_i$ are local, Hermitian, time-reversal invariant operators acting on widely separated regions. We first consider the special case where all of the $\mathcal{O}_i$ act on regions that are far from b. (Here, when we say "far," we mean much farther than the correlation length). The proof in this case is based on three observations:

(1) The operator $\mathcal{O}$ commutes with $\mathcal{T}_a$.

(2) For any two operators $\mathcal{O}$ and $\mathcal{O}'$ that act on regions far from b,

$$\langle \mathcal{T}_a \mathcal{O}' v|\mathcal{T}_a \mathcal{O} v\rangle = \langle \mathcal{O} v|\mathcal{O}' v\rangle. \tag{F3}$$

In other words, $\mathcal{T}_a$ behaves like an antiunitary operator within the subspace of states of the form $\{\mathcal{O}|v\rangle\}$.

(3) The state $|v'\rangle$ can be written as

$$|v'\rangle = \mathcal{T}_a|v\rangle = -S_a|v\rangle. \tag{F4}$$

The first observation follows from the fact that $\mathcal{O}$ commutes with S_b, and therefore also commutes with $\mathcal{T}_a = \mathcal{T} S_b$. The second observation (F3) follows from

$$\begin{aligned}\langle \mathcal{T}_a \mathcal{O}' v|\mathcal{T}_a \mathcal{O} v\rangle &= \langle \mathcal{T} S_b \mathcal{O}' v|\mathcal{T} S_b \mathcal{O} v\rangle \\ &= \langle S_b \mathcal{O} v|S_b \mathcal{O}' v\rangle \\ &= \langle v|\mathcal{O}^\dagger S_b^\dagger S_b \mathcal{O}' v\rangle \\ &= \langle v|(\mathcal{O}^\dagger \mathcal{O}')(S_b^\dagger S_b)|v\rangle \\ &= \langle v|\mathcal{O}^\dagger \mathcal{O}'|v\rangle\, \langle v|S_b^\dagger S_b|v\rangle \\ &= \langle v|\mathcal{O}^\dagger \mathcal{O}'|v\rangle \\ &= \langle \mathcal{O} v|\mathcal{O}' v\rangle, \end{aligned} \tag{F5}$$

where we are using the fact that $|v\rangle$ has short range correlations in the fifth equality. As for the third observation (F4), this is equivalent to the identity derived in Eq. (E6).

Given these three observations, our argument proceeds just like the proof of the usual Kramers theorem. To derive (F1), we note that

$$\begin{aligned}\langle v'|\mathcal{O} v\rangle &= \langle \mathcal{T}_a \mathcal{O} v|\mathcal{T}_a v'\rangle \\ &= \langle \mathcal{O} \mathcal{T}_a v|\mathcal{T}_a v'\rangle \\ &= \langle \mathcal{O} v'|\mathcal{T}_a^2 v\rangle \\ &= -\langle \mathcal{O} v'|v\rangle \\ &= -\langle v'|\mathcal{O} v\rangle, \end{aligned} \tag{F6}$$

where the first equality follows from Eqs. (F3) and (F4). It then follows that $\langle v'|\mathcal{O} v\rangle = 0$. As for Eq. (F2), we have

$$\begin{aligned}\langle v|\mathcal{O} v\rangle &= \langle \mathcal{T}_a \mathcal{O} v|\mathcal{T}_a v\rangle \\ &= \langle \mathcal{O} \mathcal{T}_a v|\mathcal{T}_a v\rangle \\ &= \langle \mathcal{O} v'|v'\rangle \\ &= \langle v'|\mathcal{O} v'\rangle. \end{aligned} \tag{F7}$$

This completes the argument for the special case where all the $\mathcal{O}_i$ act far from b. We now consider the general case, where some of the $\mathcal{O}_i$ operators may act in the vicinity of b. We define $\mathcal{O}_\alpha$ to be the product of all the $\mathcal{O}_i$ that act far from b, and $\mathcal{O}_\beta$ to be the product of all the other $\mathcal{O}_i$. We can then express $\mathcal{O}$ as a product $\mathcal{O} = \mathcal{O}_\alpha \mathcal{O}_\beta$. We note that the definition of $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ guarantees that $\mathcal{O}_\alpha$ acts on regions far from b, while $\mathcal{O}_\beta$ acts on regions far from a. Also, $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ act on regions that are far from one another.

The factorization $\mathcal{O} = \mathcal{O}_\alpha \mathcal{O}_\beta$ is useful because it allows us to reduce the general problem to the special case treated above. Indeed, consider the relation (F1). Simple manipulations give

$$\begin{aligned}\langle v'|\mathcal{O}v\rangle &= \langle v'|\mathcal{O}_\alpha\mathcal{O}_\beta v\rangle \\ &= -\langle S_a v|\mathcal{O}_\alpha\mathcal{O}_\beta v\rangle \\ &= -\langle v|S_a^\dagger\mathcal{O}_\alpha\mathcal{O}_\beta|v\rangle \\ &= -\langle v|S_a^\dagger\mathcal{O}_\alpha|v\rangle\,\langle v|\mathcal{O}_\beta|v\rangle \\ &= \langle v'|\mathcal{O}_\alpha v\rangle\,\langle v|\mathcal{O}_\beta|v\rangle, \end{aligned} \tag{F8}$$

where the fourth equality follows from the fact that v has short range correlations. On the other hand, we know that $\langle v'|\mathcal{O}_\alpha|v\rangle = 0$ by the special case discussed above. The relation (F1) follows immediately.

Similarly, to derive Eq. (F2), we note that

$$\begin{aligned}\langle v'|\mathcal{O}v'\rangle &= \langle v'|\mathcal{O}_\alpha\mathcal{O}_\beta v'\rangle \\ &= \langle S_a v|\mathcal{O}_\alpha\mathcal{O}_\beta S_a v\rangle \\ &= \langle v|S_a^\dagger\mathcal{O}_\alpha\mathcal{O}_\beta S_a v\rangle \\ &= \langle v|(S_a^\dagger\mathcal{O}_\alpha S_a)(\mathcal{O}_\beta)|v\rangle \\ &= \langle v|S_a^\dagger\mathcal{O}_\alpha S_a|v\rangle\,\langle v|\mathcal{O}_\beta|v\rangle \\ &= \langle v'|\mathcal{O}_\alpha|v'\rangle\,\langle v|\mathcal{O}_\beta|v\rangle. \end{aligned} \tag{F9}$$

Then, using the fact that $\langle v'|\mathcal{O}_\alpha|v'\rangle = \langle v|\mathcal{O}_\alpha|v\rangle$ we deduce

$$\begin{aligned}\langle v'|\mathcal{O}v'\rangle &= \langle v|\mathcal{O}_\alpha|v\rangle\,\langle v|\mathcal{O}_\beta|v\rangle \\ &= \langle v|\mathcal{O}_\alpha\mathcal{O}_\beta|v\rangle \\ &= \langle v|\mathcal{O}v\rangle. \end{aligned} \tag{F10}$$

This completes our proof of the local analog of Kramers theorem.

APPENDIX G: ADIABATIC FLUX INSERTION AND THE EDGE THEORY

In this Appendix, we analyze the effect of the adiabatic flux insertion of $-1/e^*$ flux quanta using the edge theory (10) to describe the two edges of the cylinder. We show that the effect of this flux insertion process is given by Eq. (67).

For now, we focus on one of the edges, say the left edge. The adiabatic insertion of $-1/e^*$ flux quanta can be implemented by applying a slowly varying vector potential $(A_t, A_x) = [0, f(t)/L]$, where $f(-\infty) = 0$ and $f(\infty) = -2\pi/e^*$. Since the edge theory is quadratic, we can analyze the effect of this flux insertion using the classical equations of motion. We have

$$-\mathcal{K}\partial_t\partial_x\Phi + \mathcal{V}\partial_x^2\Phi = \tau(\partial_t A_x - \partial_x A_t). \tag{G1}$$

Integrating over x and using the above expression for A, we derive

$$\frac{d\rho}{dt} = -\mathcal{K}^{-1}\frac{\tau}{2\pi}\frac{df}{dt}. \tag{G2}$$

where $\rho_I \equiv \frac{1}{2\pi}\int dx\,\partial_x\Phi_I$. The net effect of the flux insertion is therefore

$$\rho \to \rho + \frac{1}{e^*}\mathcal{K}^{-1}\tau. \tag{G3}$$

On the other hand, if one quantizes (10), one derives the canonical commutation relations

$$[\Phi_I(y), \partial_x\Phi_J(x)] = 2\pi i\mathcal{K}^{-1}_{IJ}\delta(x-y). \tag{G4}$$

Letting $\Gamma(\Lambda) = \int \frac{dx}{\sqrt{L}} e^{i\Theta(\Lambda)}$, where $\Theta(\Lambda) = \Lambda^T\mathcal{K}\Phi$, we derive the commutation relation

$$[\rho, \Gamma(\Lambda)] = \Lambda\,\Gamma(\Lambda). \tag{G5}$$

Comparing Eqs. (G3) and (G5), we conclude that the adiabatic flux insertion is implemented by the operator $\Gamma(\frac{1}{e^*}\mathcal{K}^{-1}\tau) = \Gamma(\Lambda_c)$.

Including the effect on the right edge as well, we have

$$|\Psi_2\rangle = \Gamma_l(\Lambda_c)\Gamma_r(-\Lambda_c)|\Psi_1\rangle, \tag{G6}$$

where Γ_l and Γ_r denote the operators acting on the left and right edges, respectively. This completes our derivation of Eq. (67).

[1] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
[2] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).
[3] B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).
[4] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[5] L. Fu and C. L. Kane, Phys. Rev. B **74**, 195312 (2006).
[6] M. Levin and A. Stern, Phys. Rev. Lett. **103**, 196803 (2009).
[7] M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, Ann. Phys. **310**, 428 (2004).
[8] T. Neupert, L. Santos, S. Ryu, C. Chamon, and C. Mudry, Phys. Rev. B **84**, 165107 (2011).
[9] L. Santos, T. Neupert, S. Ryu, C. Chamon, and C. Mudry, Phys. Rev. B **84**, 165138 (2011).
[10] M. Levin, F. J. Burnell, M. Koch-Janusz, and A. Stern, Phys. Rev. B **84**, 235145 (2011).
[11] X.-G. Wen, *Quantum Field Theory of Many-body Systems: From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, Oxford, 2007).
[12] X.-G. Wen, Adv. Phys. **44**, 405 (1995).
[13] X. G. Wen and A. Zee, Phys. Rev. B **46**, 2290 (1992).
[14] B. Scharfenberger, R. Thomale, and M. Greiter, Phys. Rev. B **84**, 140404 (2011).
[15] A. Y. Kitaev, Ann. Phys. **303**, 2 (2003).
[16] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, arXiv:1106.4772.

[17]It is not hard to show that $e^{i\Theta(\Lambda)}$ and $e^{i\Theta(\Lambda')}$ commute if $\Lambda^T\mathcal{K}\Lambda'$ is even and anticommute if $\Lambda^T\mathcal{K}\Lambda'$ is odd, using $[\phi_I(x),\phi_J(y)] = \pi i\mathcal{K}^{-1}_{IJ}\mathrm{sgn}(x-y)$.

[18]F. D. M. Haldane, Phys. Rev. Lett. **74**, 2090 (1995).

[19]T. Einarsson, Phys. Rev. Lett. **64**, 1995 (1990).

[20]Y. Gefen and D. J. Thouless, Phys. Rev. B **47**, 10423 (1993).

[21]B. Swingle, M. Barkeshli, J. McGreevy, and T. Senthil, Phys. Rev. B **83**, 195139 (2011).

[22]J. Maciejko, X.-L. Qi, A. Karch, and S.-C. Zhang, Phys. Rev. Lett. **105**, 246809 (2010).

[23]V. V. Prasolov, *Problems and Theorems in Linear Algebra* (American Mathematical Society, Providence, RI, 1994).