

Effect of fermionic impurities on the quantum coherence of a few-boson system

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The impurity effects on the quantum coherence of a few-boson system are studied within the two-site Hubbard model. Periodical collapses and revivals of coherence occur in the presence of either polarized or unpolarized fermionic impurities. The relative strength U_{BF}/U_{BB} of the boson-fermion interaction versus the boson-boson interaction plays a key role in the coherence revivals. As the average filling of the impurity increases, the coherence revivals remain nearly unaffected for $U_{BF}/U_{BB} = z$ ($z \in \mathbb{Z}$ is an integer) while the odd revival peaks are damped for $U_{BF}/U_{BB} = z + 1/2$, in agreement with the experimental observations. For unpolarized fermionic impurities, the coherence revivals are irrelevant to the strength of the fermion-fermion interactions.

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I. INTRODUCTION

The ultracold atoms, trapped either by a magnetic potential or an optical lattice, present an advantageous phase of matter for the investigation of fundamental quantum physics.¹⁻⁷ Since the external potentials, the dimensions, the effective interactions, as well as the atomic components can be well-controlled and precisely measured, the ultracold atoms have been providing versatilely and experimentally feasible means to explore a variety of problems such as quantum simulation, information processing, strongly correlated systems, dynamical evolution, and so on. In particular, atomic interferometers have been realized experimentally by loading Bose-Einstein condensates (BECs) into double-well potentials. Atomic Josephson oscillations⁸⁻¹⁰ and macroscopic quantum self-trapping^{11,12} were predicted and observed experimentally. The phenomena of quantum tunneling,¹³⁻¹⁵ the collapse and revival of quantum coherence,¹⁶⁻¹⁸ disorder effects,¹⁹⁻²² and paired or counterflow superfluidity^{23,24} are extensively explored.

The problem of impurity embedded in a quantum environment poses another interesting topic.^{25,26} In a recent experiment, the authors studied the quantum dynamics of bosons in an optical lattice with a fraction of polarized fermionic impurities.²⁷ The absolute strength of the intraspecies and interspecies interactions are measured as a function of the interspecies scattering length, tuned by means of a Feshbach resonance. The collapses and revivals of the quantum coherence exhibit distinct features in comparison to the pure bosonic system.^{17,27} One of the most prominent facts are the odd revival peaks are damped as the impurity fillings increase. When such impurity systems are scaled down to the few-body regime, the mean-field theory becomes invalid.^{27,28} The double-well model provides the direct ways to go beyond the Gross-Pitaevskii paradigm of gaseous BECs. It is also one of the simplest prototypes for finite lattices and the most instructive means to study quantum dynamics.

In this work, we explore the impurity effects on the quantum coherence of a few-boson system. For bosonic atoms comprising fermionic impurities in the optical lattices, the system can be properly treated as a mixed ensemble of the pure bosonic system and the bosonic system with a single fermionic

impurity in the double-well model. We deal with N interacting bosons with a small fraction of fermionic impurities. Under the approximation of tight binding, the double-well is simplified as a two-site single-band Hubbard model.^{29,30} The boson (fermion) creation \hat{b}_i^\dagger (\hat{f}_i^\dagger) and annihilation \hat{b}_i (\hat{f}_i) operators are constructed for atoms localized in either side of the well. In the limit of strong interactions $U_{BB}/t_B \gg 1$, the system which starts from a initial coherence state experiences periodical collapses and revivals of coherence. The relative strength of the boson-boson interaction versus the boson-fermion interaction U_{BF}/U_{BB} plays an important role on the dynamical revival and collapse of macroscopic matter waves. The results agree with the experimental observations in Ref. 27. It is discovered that symmetrical and antisymmetrical coherent states alternatively occur at the revival peaks. We further investigate the effects of unpolarized fermionic impurities on the quantum coherence. The coherence revivals are irrelevant to the interaction strength between the impurities.

The paper is organized as follows. In Sec. II, we illustrate the model and the formalism. In Secs. III and IV we explore the effects of polarized and unpolarized fermionic impurities on the coherence, respectively. A brief summary is included in Sec. V.

II. MODEL

We consider a mixture of a pure bosonic system and a bosonic system with a single fermionic impurity in the double well. The usual two-site Bose-Hubbard model for the pure bosons is written as²⁹⁻³²

$$\hat{H}_0 = -t_B(\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1) + \frac{1}{2}U_{BB} \sum_{i=1,2} \hat{n}_i(\hat{n}_i - 1), \quad (1)$$

and the Hamiltonian including fermionic atoms should be the Bose-Fermi-Hubbard model which is written as^{29,30}

$$\begin{aligned} \hat{H}_1 = & -t_B(\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1) + \frac{1}{2}U_{BB} \sum_{i=1,2} \hat{n}_i(\hat{n}_i - 1) \\ & - t_F(\hat{f}_1^\dagger \hat{f}_2 + \hat{f}_2^\dagger \hat{f}_1) + U_{BF} \sum_{i=1,2} \hat{n}_i \hat{m}_i. \end{aligned} \quad (2)$$

Here U_{BB} and U_{BF} are the boson-boson and boson-fermion interactions, respectively. t_B and t_F are the bosonic and fermionic

hopping coefficients, respectively. \hat{n}_i and \hat{m}_i ($i = 1, 2$) are the bosonic and fermionic number operators, respectively. Since in the single-band Hubbard model the double occupation of polarized fermions is prohibited, the average site-filling of the fermionic impurities is restricted to $\bar{m} \leq 0.5$.

The Hamiltonian (1) is represented in the Fock basis set $\{|N, 0\rangle, |N-1, 1\rangle, \dots, |0, N\rangle\}$. The eigenstates can be expressed as linear combinations of the Fock bases, $|\psi_j\rangle = \sum_{k=0}^N c_{jk} |N-k, k\rangle$ ($j = 0, 1, 2, \dots, N$), which correspond to the eigenvalues ω_j . The coefficients c_{jk} satisfy the recursive relation¹⁸

$$-t_B \sqrt{(N-k)(k+1)} c_{j(k+1)} - t_B \sqrt{(N-k+1)k} c_{j(k-1)} + \left[\frac{U}{2} (N^2 - 2Nk - N + 2k^2) - \omega_j \right] c_{jk} = 0. \quad (3)$$

The temporal evolution of the state is governed by the Heisenberg's equation for a given initial state $|\psi(0)\rangle$

$$|\psi(\tau)\rangle = \sum_{j=0}^N f_j(\tau) |\psi_j\rangle \equiv \sum_{k=0}^N g_k(\tau) |N-k, k\rangle, \quad (4)$$

where $f_j(0) = \langle \psi_j | \psi(0) \rangle$ and $g_k(\tau) = \sum_{j=0}^N f_j(0) c_{jk} e^{-i\omega_j \tau}$.

$$\hat{H}_2 = \begin{pmatrix} h_{00} & \cdots & 0 & -t_F & \cdots & 0 \\ -t_B \sqrt{(N-1)} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{2} U_{BB} (N-1)(N-2) & 0 & \cdots & -t_F \\ -t_F & \cdots & 0 & \frac{1}{2} U_{BB} (N-1)(N-2) & \cdots & 0 \\ 0 & \cdots & 0 & -t_B \sqrt{(N-1)} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -t_F & 0 & \cdots & h_{00} \end{pmatrix}, \quad (6)$$

where $h_{00} = U_{BB}(N-1)(N-2)/2 + U_{BF}(N-1)$. By solving the Heisenberg's equation of motion, we can also obtain the temporal evolution of the state. The coherence α_2 of the bosons is defined in the same way as for the definition of α_1 in a pure bosonic system.

For the ensemble of mixed systems of pure bosons and bosons with polarized fermionic impurities in the double well, or bosons with fermionic impurities in the optical lattice, the average coherence in accordance to the fermion-filling \bar{m} is

$$\alpha = (1 - 2\bar{m})\alpha_1 + 2\bar{m}\alpha_2. \quad (7)$$

In the following calculations, we choose $N = 10$ bosons as example and set the units of $t_B = t_F = 1$.

III. POLARIZED FERMIONIC IMPURITIES

For the quantum dynamics, we are concerned with a delocalized Bose-Fermi mixture in a shallow optical lattice. When the system parameters are swiftly changed, the atoms

To depict the coherence degree of the system, we introduce a characteristic parameter¹⁸

$$\alpha_1 = \frac{|\lambda_1 - \lambda_2|}{\lambda_1 + \lambda_2}, \quad (5)$$

where λ_1 and λ_2 are the two eigenvalues of the single-particle density $\rho_{\mu\nu}(\tau) = \langle \psi(\tau) | \hat{a}_\mu^\dagger \hat{a}_\nu | \psi(\tau) \rangle$ ($\mu, \nu = 1, 2$) (Refs. 33 and 34). When $\alpha \rightarrow 1$, the system is in the coherent (quasi-coherent) state since in this case there is only one large eigenvalue of matrix $\rho_{\mu\nu}$. Accordingly, $\alpha \rightarrow 0$ indicates the system is in the decoherent or fragmented state because there are two densely populated natural orbits. In the weak-interaction, strong-tunneling limit ($U_{BB}/t_B \ll 1$), each atom is in a coherent superposition of the left-well and right-well states. In the strong interactions or weak tunneling ($U_{BB}/t_B \gg 1$), the tunneling term is negligible. The Hamiltonian is the product of the number operators for the left and right wells. The eigenstates are products of Fock states and are referred to as decoherent states. This regime is analogous to the Mott insulator (MI) phase in optical lattices.

On the other hand, for the boson-fermion mixed system, the Hamiltonian (2) is represented in the Fock set $\{|N, 0\rangle|0, 1\rangle, |N-1, 1\rangle|0, 1\rangle, \dots, |0, N\rangle|0, 1\rangle, |N, 0\rangle|1, 0\rangle, |N-1, 1\rangle|1, 0\rangle, \dots, |0, N\rangle|1, 0\rangle\}$ as

collectively experience a dynamical evolution. In our double-well model, we focus on the typical case in which the initial state is a symmetric coherent state. For the pure bosonic system, the initial state is written as

$$|\psi_B(0)\rangle = \left(\frac{\hat{b}_1^\dagger + \hat{b}_2^\dagger}{\sqrt{2}} \right)^N |0\rangle. \quad (8)$$

For the bosonic system with a polarized fermionic impurity, the initial state is written as

$$|\psi_B(0)\rangle |\psi_F(0)\rangle = \left(\frac{\hat{b}_1^\dagger + \hat{b}_2^\dagger}{\sqrt{2}} \right)^N \left(\frac{\hat{f}_1^\dagger + \hat{f}_2^\dagger}{\sqrt{2}} \right) |0\rangle. \quad (9)$$

When the lattice depth is swiftly changed to $U_{BB}/t_B = 0.2$, which is still in the SF regime, the coherence parameter α fluctuates around 1, as shown in Figs. 1(a) through (d). It indicates that the bosons preserve coherence or quasicohherence and is nearly unaffected by the impurities.

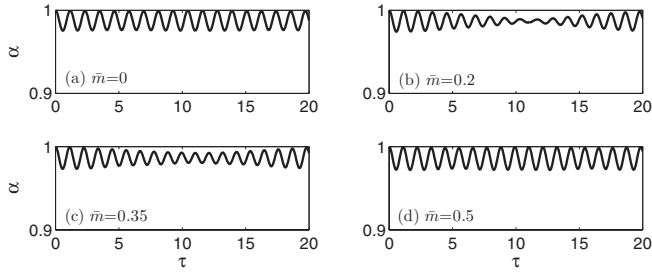


FIG. 1. Temporal evolution of $\alpha(\tau)$ for the $N = 10$ system starting from a coherent state. $U_{BB} = 0.2$ and $U_{BF}/U_{BB} = 0.5$. From (a) to (d), $\bar{m} = 0, 0.2, 0.35, 0.5$. For all cases $\alpha \rightarrow 1$.

On the other hand, when the lattice depth is swiftly changed to the deep MI regimes ($U_{BB}/t_B \gg 1$), the value of α oscillates between 1 and 0 with a period of $T \approx 0.031$, as shown in Figs. 2(a) through 2(c). It implies that the system experiences collapses and revivals of the coherence.¹⁸ Notably, the odd revival peaks are gradually damped as the impurity filling increases for $U_{BF}/U_{BB} = z + 0.5$ [Figs. 2(a) and 2(c)]. At $\bar{m} = 0.5$, these revival peaks are ultimately destroyed. In comparison, as $U_{BF}/U_{BB} = 1$, no revival peaks are damped. The coherence revivals are not affected by the impurities regardless of the variations of the impurity fillings [Fig. 2(b)]. This phenomenon was recently observed in the experiment.²⁷

This phenomenon can be understood by the following single-site evolution model. As the system is swiftly changed to the deep MI regimes, both the bosonic and fermionic tunnelings are suppressed and the delocalized distributions of bosons and fermions are frozen. The eigenstates at a lattice site are given by product atom number states, containing

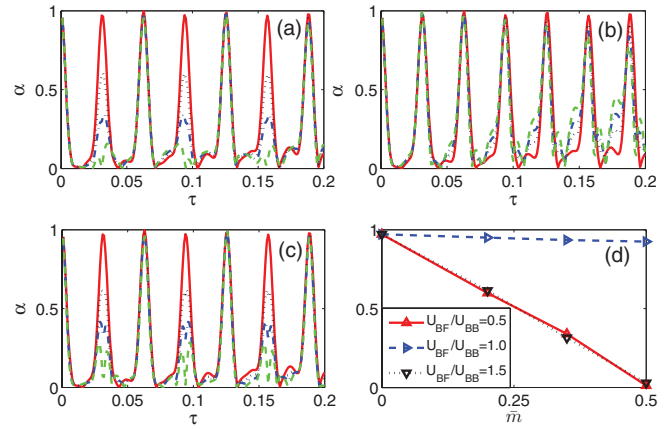


FIG. 2. (Color online) Temporal evolution of $\alpha(\tau)$ for the $N = 10$ system starting from a coherent state. $U_{BB} = 100$. From (a) to (c), and $U_{BF}/U_{BB} = 0.5, 1.0, 1.5$. The solid (red) line, dotted (black) line, dash-dotted (blue) line, and dashed (green) line represent $\bar{m} = 0, \bar{m} = 0.2, \bar{m} = 0.35$, and $\bar{m} = 0.5$, respectively. (d) The dependence of the height of the first revival peak on the impurity filling \bar{m} is illustrated.

k bosons ($k = 0, 1, 2, \dots, N$) and m fermions ($m = 0, 1$). The eigenenergies corresponding to the eigenstates $|\psi_{km}\rangle = (\hat{b}_j^\dagger)^k (\hat{f}_j^\dagger)^m |0\rangle$ are $E_{km} = U_{BB}k(k-1)/2 + U_{BF}km$. The initial state can be expanded as

$$|\Psi(\tau)\rangle = \frac{1}{2^{(N+1)/2}} \sum_{k=0}^N C_N^k (\hat{b}_1^\dagger)^{N-k} (\hat{b}_2^\dagger)^k (\hat{f}_1^\dagger + \hat{f}_2^\dagger) |0\rangle, \quad (10)$$

which is a linear superposition of products of the eigenstates. It evolves independently according to

$$\begin{aligned} |\Psi(\tau)\rangle &= \frac{1}{2^{(N+1)/2}} \sum_{k=0}^N \{ C_N^k e^{-i[U_{BB}(N-k)(N-k-1)/2 + U_{BF}(N-k)]\tau} (\hat{b}_1^\dagger)^{N-k} \times e^{-i[U_{BB}k(k-1)/2]\tau} (\hat{b}_2^\dagger)^k \hat{f}_1^\dagger |0\rangle \\ &\quad + C_N^k e^{-i[U_{BB}(N-k)(N-k-1)/2]\tau} (\hat{b}_1^\dagger)^{N-k} \times e^{-i[U_{BB}k(k-1)/2 + U_{BF}k]\tau} (\hat{b}_2^\dagger)^k \hat{f}_2^\dagger |0\rangle \} \\ &= \frac{e^{-iU_{BB}N(N-1)\tau/2}}{2^{(N+1)/2}} \sum_{k=0}^N C_N^k e^{iU_{BB}(N-k)\tau} (\hat{b}_1^\dagger)^{N-k} (\hat{b}_2^\dagger)^k [e^{-iU_{BF}(N-k)\tau} \hat{f}_1^\dagger + e^{-iU_{BF}k\tau} \hat{f}_2^\dagger] |0\rangle. \end{aligned} \quad (11)$$

It follows that, except for a global phase factor, there is a time-dependent phase factor attached to each term in Eq. (11). At time interval $T = \pi/U_{BB}$, the phases $e^{iU_{BB}(N-k)k\tau} = (-1)^k$. In the meantime, if $U_{BF}/U_{BB} = z$ ($z \in \mathbb{Z}$) the fermionic terms also give rise to a factor of $(-1)^k$ as $U_{BF}T = z\pi$. Hence the state evolves to the initial coherent states. At other times the superposition from various terms in Eq. (11) cancels and the coherence is destroyed. These are the coherence collapses and revivals, with the revival period $T = \pi/U_{BB}$. On the other hand, if $U_{BF}/U_{BB} = (z + 0.5)$, the fermionic terms recover its initial state only after a time interval $T_F = \pi/U_{BF}$. Hence the revival period should be $T = 2\pi/U_{BB}$, implying that the odd revival peaks are damped.

Since the revival period is doubled when $U_{BF}/U_{BB} = (z + 0.5)$, all phase factors in Eq. (11) equal 1. We conclude that the state revives alternatively to the antisymmetric and symmetric

coherent states at odd and even peaks, respectively. Here the antisymmetric coherent state means $|\Psi_B\rangle = (\frac{\hat{b}_1^\dagger - \hat{b}_2^\dagger}{\sqrt{2}})^N |0\rangle$. To further identify the difference of the odd-even revival peaks, we examine the average-value of the pseudospin $\langle S_x \rangle$, which is defined by

$$\langle S_x \rangle = \frac{1}{2} (\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1). \quad (12)$$

Figure 3 displays the temporal evolution of $\langle S_x(\tau) \rangle$. Figure 3(a) explicitly shows that $\langle S_x \rangle$ alternatively equals -1 and $+1$ at the odd and even revival peaks, respectively. It demonstrates the difference of coherent states at each revival period. In Fig. 3(b), which corresponds to $U_{BF}/U_{BB} = 1$, $\langle S_x \rangle$ varies almost synchronously with α . No revival peaks are damped in this case. We suggest an experimental measurement of the physical quantity $\langle S_x \rangle$ to verify our conclusion.

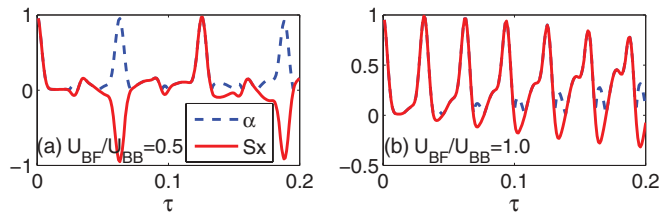


FIG. 3. (Color online) Temporal evolution of $\langle S_x \rangle$ for (a) $U_{BF}/U_{BB} = 0.5$ and (b) $U_{BF}/U_{BB} = 1$. The dashed line is $\alpha(\tau)$.

From Eq. (7), we approximately obtain a linear relation between the damp of the odd revival peaks and the impurity fillings \bar{m} as $\alpha_{\text{odd peaks}} \approx 1 - 2\bar{m}$, providing that α_2 tends to zero at the odd revival peaks. The data points indicate the average of the first three odd revival peaks for $\bar{m} = 0, 0.2, 0.35, 0.5$. The result qualitatively agrees with the numerical results plotted in Fig. 2(d).

IV. UNPOLARIZED FERMIONIC IMPURITIES

We next examine the case of unpolarized fermionic impurities. The Hamiltonian is

$$\begin{aligned} \hat{H}_2 = & -t_B(\hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1) + \frac{1}{2}U_{BB} \sum_{i=1,2} \hat{n}_i(\hat{n}_i - 1) \\ & - t_F \sum_{\sigma=\uparrow,\downarrow} (\hat{f}_{1\sigma}^\dagger \hat{f}_{2\sigma} + \hat{f}_{2\sigma}^\dagger \hat{f}_{1\sigma}) + U_{BF} \sum_{i,\sigma} \hat{n}_i \hat{m}_{i\sigma} \\ & + U_{FF} \sum_i \hat{m}_{i\uparrow} \hat{m}_{i\downarrow}, \end{aligned} \quad (13)$$

where U_{FF} is the fermion-fermion interaction and $\hat{m}_{i\uparrow}$ ($\hat{m}_{i\downarrow}$) are the spin-up (spin-down) fermionic number operators at the i th site. We have chosen $t_{F\uparrow} = t_{F\downarrow} \equiv t_F$ and $U_{BF\uparrow} = U_{BF\downarrow} \equiv U_{BF}$. The average coherence degree is a mixture of the pure bosons, bosons with spin-up or spin-down fermions, and bosons with both spin-up and spin-down fermions, which is expressed as

$$\begin{aligned} \alpha = & (1 - 2\bar{m}_\uparrow)(1 - 2\bar{m}_\downarrow)\alpha_1 + 2\bar{m}_\uparrow(1 - 2\bar{m}_\downarrow)\alpha_2 \\ & + 2\bar{m}_\downarrow(1 - 2\bar{m}_\uparrow)\alpha_3 + 2\bar{m}_\uparrow \times 2\bar{m}_\downarrow \alpha_4, \end{aligned} \quad (14)$$

where $\alpha_1, \alpha_2, \alpha_3$, and α_4 refer to the coherence degree of the pure bosons, bosons with a spin-up fermion, bosons with a spin-down fermion, and bosons with a spin-up and a spin-down fermions system, respectively. \bar{m}_\uparrow (\bar{m}_\downarrow) represent the average fillings of spin-up (spin-down) fermionic impurities.

Figure 4 shows the temporal evolution of coherence degree α for $U_{BF}/U_{BB} = 0.5$ (left column) and $U_{BF}/U_{BB} = 1$ (right column), respectively. The coherence collapses and revives as in the case of the polarized fermionic impurities. Intriguingly, the revivals are irrelevant to the strength of the

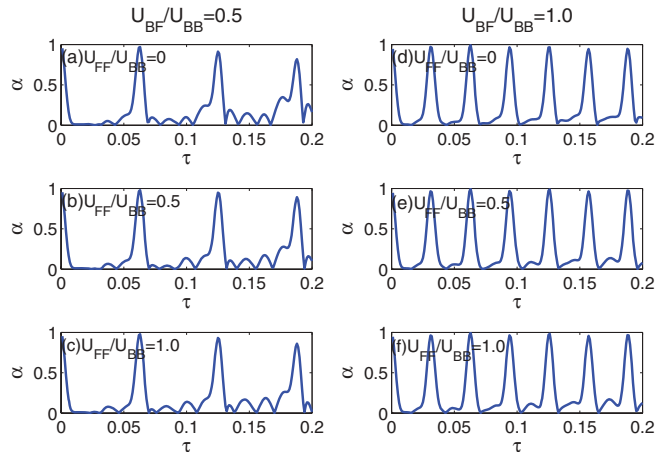


FIG. 4. (Color online) Temporal evolution of α for $U_{BF}/U_{BB} = 0.5$ (left column) and $U_{BF}/U_{BB} = 1$ (right column) for fermion-fermion interactions $U_{FF}/U_{BB} = 0.0, 0.5, 1.0$, respectively. In all cases $U_{BB} = 100$ and $\bar{m}_\uparrow = \bar{m}_\downarrow = 0.5$.

fermion-fermion interaction U_{FF} . This effect can also be interpreted by the independent evolution model as described above.

As to the damp of the odd revival peaks, we can deduce a relation with the relative fillings of the unpolarized fermionic impurities $\gamma = \bar{m}_\downarrow/\bar{m}_\uparrow$ from Eq. (14). Since $\alpha_2 = \alpha_3 \approx \alpha_4$, we have

$$\alpha \approx (1 - 2\bar{m}_{\text{eff}})\alpha_1 + 2\bar{m}_{\text{eff}}\alpha_2, \quad (15)$$

where $\bar{m}_{\text{eff}} = \bar{m}_\uparrow + \bar{m}_\downarrow - 2\bar{m}_\uparrow\bar{m}_\downarrow = (1 + \gamma)\bar{m}_\uparrow - 2\gamma\bar{m}_\uparrow^2$. We conclude $\alpha_{\text{odd peaks}} \approx 1 - 2\bar{m}_{\text{eff}}$. Our results may be demonstrated by the experimental device used by the authors of Ref. 27.

V. SUMMARY

In summary, we have studied the quantum coherence of a few-boson system with fermionic impurities in a double-well potential. The damp of the odd revival peaks is closely related to the relative strength of the boson-fermion and boson-boson interactions. We identify that the symmetric and the antisymmetric coherent states appear alternatively at the revival peaks. When the system comprises unpolarized fermionic impurities, we found that the coherence revivals are irrelevant to the fermion-fermion interaction.

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