

**Landau-Lifshitz-Bloch equation for ferrimagnetic materials**

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We derive the Landau-Lifshitz-Bloch (LLB) equation for a two-component magnetic system valid up to the Curie temperature. As an example, we consider disordered GdFeCo ferrimagnet where the ultrafast optically induced magnetization switching under the action of heat alone has been recently reported. The two-component LLB equation contains the longitudinal relaxation terms responding to the exchange fields from the proper and the neighboring sublattices. We show that the sign of the longitudinal relaxation rate at high temperatures can change depending on the dynamical magnetization value and a dynamical polarization of one material by another can occur. We discuss the differences between the LLB and the Baryakhtar equations, recently used to explain the ultrafast switching in ferrimagnets. The two-component LLB equation forms the basis for the large-scale micromagnetic modeling of nanostructures at high temperatures and ultrashort time scales.

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**I. INTRODUCTION**

The Landau-Lifshitz-Bloch (LLB) dynamical equation of motion for macroscopic magnetization vector<sup>1</sup> has recommended itself as a valid micromagnetic approach at elevated temperatures,<sup>2</sup> especially useful for temperatures  $T$  close to the Curie temperature  $T_C$  ( $T > 3T_C/4$ ) and ultrafast time scales. In several exciting novel magnetic phenomena, this approach has been shown to be a necessary tool. These phenomena include laser-induced ultrafast demagnetization,<sup>3–6</sup> thermally driven domain wall motion via the spin-Seebeck effect,<sup>7</sup> spin-torque effect at elevated temperatures,<sup>8,9</sup> or heat-assisted magnetic recording.<sup>10</sup>

In the area of laser-induced ultrafast demagnetization, the LLB equation has been shown to describe adequately the dynamics in Ni<sup>5</sup> and Gd.<sup>6</sup> The main feature of the LLB equation allowing its suitability for the ultrafast magnetization dynamics is the presence of longitudinal relaxation term coming from the strong exchange interaction between atomic spins. Because the exchange fields are large (10–100 T), the corresponding characteristic longitudinal relaxation timescale is of the order of 10–100 femtoseconds and thus manifests itself in the ultrafast processes. The predictions of the LLB equations related to the linear reversal path for the magnetization dynamics<sup>4</sup> as well as to the critical slowing down of the relaxation times at high laser pump fluency<sup>5</sup> have been confirmed experimentally.

In ferrimagnetic GdFeCo alloys, not only the longitudinal change of magnetization but also a controllable optical magnetization switching has been observed, and this has stimulated a great deal of effort to attempt on many levels to explain this process, see review in Ref. 11. The ferrimagnetic materials consist of at least two antiferromagnetically coupled magnetic sublattices. The magnetic moments of each sublattice are different, leading to a net macroscopic magnetization  $M(T)$  defined as the sum of magnetization coming from each sublattice. The main feature of the ferrimagnetic materials is that at some temperature, called magnetization compensation temperature  $T_M$ , the macroscopic magnetization is zero  $M(T_M) = 0$ , although the magnetization of each sublattice is not. The angular momentum compensation temperature at which the total angular momentum  $T_A$  is zero is also of interest.

Simplified considerations of the ferromagnetic resonance of two-sublattice magnets<sup>12</sup> predict that at this temperature the effective damping is infinite and this stimulated investigation of the magnetization reversal when going through an angular-momentum compensation point.<sup>13,14</sup>

Recently, K. Vahaplar *et al.*,<sup>4</sup> suggested that the optically induced ultrafast switching in GdFeCo involves a linear reversal mechanism, proposed theoretically in Ref. 15. This is an especially fast mechanism since it is governed by the longitudinal relaxation time, which can be two orders of magnitude faster than the transverse relaxation time governing precessional switching. The modeling of Ref. 4 was based on macrospin LLB approach, essentially treating a ferrimagnet as a ferromagnet. The model showed that in order to have the magnetization switching, a strong field around 20 T was necessary. This field can, in principle, come in the experiment with circularly polarized light from the inverse Faraday effect. More recently, T. Ostler *et al.*<sup>16</sup> used a multispin atomistic approach based on the Heisenberg model showing that the switching occurs without any applied field or even with the field up to 40 T applied in the opposite direction. The predictions for the heat-driven reversal were confirmed in several experiments in magnetic thin films and dots using linearly polarized pulses. Moreover, I. Radu *et al.*<sup>17</sup> used the same atomistic model for the magnetization dynamics to simulate GdFeCo and compared the simulation results to the experimental data measured by the element-specific x-ray magnetic circular dichroism (XMCD). They unexpectedly found that the ultrafast magnetization reversal in this material, where spins are coupled antiferromagnetically, occurs by way of a transient ferromagnetic-like state.

The latter experiments demonstrate the deficiency in application of the macrospin ferromagnetic LLB model to the description of the ultrafast dynamics in a ferrimagnetic material GdFeCo. It is clear that the situation of a ferromagnetic-like state in a ferrimagnetic material cannot be described in terms of a macrospin LLB equation in which a ferrimagnet is essentially treated as a ferromagnet. In a ferromagnetic LLB equation, the sublattices cannot have their own dynamics and thus the processes such as the angular momentum transfer between them are essentially ignored. In this situation,

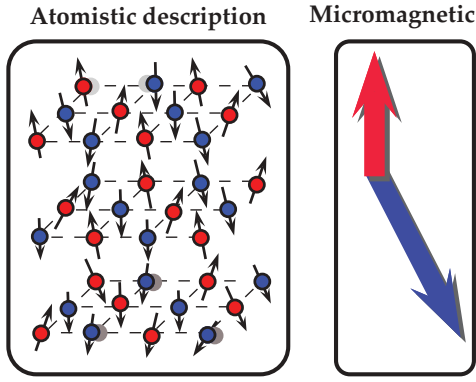


FIG. 1. (Color online) (Left) Sketch of atomistic regular ferrimagnetic lattice. Each point represents a magnetic moment associated with an atomic site. Magnetic moments of blue points are pointing downwards and red ones upwards. (Right) A macroscopic view of partial average magnetization  $m_A = \langle s_A \rangle$  and  $m_B = \langle s_B \rangle$  by two macrospins in each sublattice as described by the Landau-Lifshitz-Bloch equation.

the only possible reversal mode is the linear relaxation requiring a strong applied magnetic field as was the case of Ref. 4.

On a general basis, atomistic models are convenient to model ferrimagnetic materials but for modeling of larger spatial scales, a macroscopic equation similar to ferromagnetic LLB equation is desirable. This will open a possibility to a correct micromagnetic modeling of ferri- and antiferromagnetic nano and microstructures at ultrafast time scales and/or high temperatures. Additionally, this can also allow more correct understanding of longitudinal relaxation in two-component (for example, ferrimagnetic) compounds, taking into account the inter-sub-lattice exchange.

In this paper, we derive a macroscopic equation for the magnetization dynamics of a two-component system valid at elevated temperatures in the classical case. As a concrete example, we consider the disordered GdFeCo alloy, the cases of two-component ferromagnets as well as ordered ferrimagnets and antiferromagnets can be easily deduced. Figure 1 shows a sketch of an atomistic model for a ferrimagnetic material and the corresponding micromagnetic approximation. The atomistic model is based on the classical Heisenberg model for a crystallographically amorphous ferrimagnetic alloy<sup>18</sup> and the Langevin dynamics simulations of a set of the Landau-Lifshitz-Gilbert (LLG) equations for localized atomistic spins. In the macroscopic approach, each sublattice is represented by a macrospin with variable length and direction. We use the mean-field approximation (MFA) to derive a macroscopic equation of motion for the magnetization of each sublattice. It contains both transverse and longitudinal relaxation terms and interpolates between the Landau-Lifshitz equation at low temperatures and the Bloch equation at high temperatures. We investigate the signs of the relaxation rates of both transition (TM) and rare-earth (RE) metals as a function of temperature. We conclude that it is a good starting point for performing large scale simulations in multilattice magnetic systems as the LLB equation is for ferromagnetic materials.<sup>3,19</sup>

## II. ATOMISTIC MODEL FOR A DISORDERED FERRIMAGNET

The models for binary ferrimagnetic alloys of the type  $A_x B_{1-x}$ , randomly occupied by two different species ( $A$  and  $B$ ) of magnetic ions have been previously extensively investigated theoretically.<sup>20-22</sup> In such models,  $A$  and  $B$  ions have different atomic quantum spin values  $S_A$  and  $S_B$  ( $S_A \neq S_B$ ). In the present paper, we use the classical counterpart of these models by considering the classical spins with magnetic moments  $\mu_A \neq \mu_B$ . We denote  $A$  specie as TM and  $B$  specie as RE. A further but nonessential simplification is to assume that the interactions between spins in the disordered binary alloy are of the Heisenberg form with the exchange interactions different for different pairs of spins ( $AA$ ,  $BB$ , or  $AB$ ).

Let us start with the model for a ferrimagnet described by the classical Hamiltonian of the type

$$\mathcal{H} = - \sum_i^{\mathcal{N}} \mu_i \mathbf{H} \cdot \mathbf{s}_i - \sum_i^{\mathcal{N}} D_i (s_i^z)^2 - \sum_{(ij)} J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j, \quad (1)$$

where we have considered unit length classical vectors for all lattice sites  $\mathbf{s}_i = \boldsymbol{\mu}_i / \mu_i$ . Here  $\mathcal{N}$  is the total number of spins,  $(i, j)$  are lattice sites,  $\mu_i$  is the magnetic moment located at lattice site  $i$ . The external applied field is expressed by  $\mathbf{H}$ . The anisotropy is considered as uniaxial with  $D_i$  being the anisotropy constant of site  $i$ . The third sum is over all nearest and next-to-nearest neighbor pairs. The Heisenberg exchange interaction parameter between adjacent sites is  $J_{ij} = J_{AA(BB)} > 0$  if both sites  $(i, j)$  are occupied by  $A(B)$  type magnetic moments and  $J_{ij} = J_{AB} < 0$  if the sites  $(i, j)$  are occupied by  $A$  and  $B$ , respectively. We consider that the ordered TM alloy is represented by the fcc-type lattice. To simulate the amorphous character of the TM-RE alloy,  $x \cdot 100\%$  lattice sites are substituted randomly with RE magnetic moments.

The magnetization dynamics of this model interacting with the bath is described by the stochastic Landau-Lifshitz-Gilbert (LLG) equation

$$\dot{\mathbf{s}}_i = \gamma_i [\mathbf{s}_i \times \mathbf{H}_{i,\text{tot}} + \boldsymbol{\zeta}_i] - \gamma_i \lambda_i [\mathbf{s}_i \times [\mathbf{s}_i \times \mathbf{H}_{i,\text{tot}}]], \quad (2)$$

where  $\lambda_i$  is the coupling to the heat bath parameter and  $\gamma_i$  is the gyromagnetic ratio. In what follows and for simplicity, we use the same values for TM and RE,  $\gamma_{\text{TM}} = \gamma_{\text{RE}} = \gamma = 1.76 \times 10^7 \text{ rad s}^{-1} \text{ Oe}^{-1}$ ,  $\lambda_{\text{TM}} = \lambda_{\text{RE}} = \lambda = 0.1$ . The stochastic thermal fields  $\boldsymbol{\zeta}_i$  are uncorrelated in time and on different lattice sites. They can be coupled to different heat baths (via temperature of electrons or phonons) and could have different strength of coupling (via  $\lambda_i$  and  $\mu_i$ ) for each atom type ( $A$  or  $B$ ). The correlators of different components of thermal field can be written as

$$\langle \zeta_{i,\alpha}(t) \zeta_{j,\beta}(t') \rangle = \frac{2\lambda_i k_B T}{\mu_i \gamma_i} \delta_{ij} \delta_{\alpha\beta} \delta(t - t'), \quad (3)$$

where  $\alpha, \beta$  are Cartesian components,  $k_B$  is the Boltzmann constant, and  $T$  is the temperature of the heat bath to which the spins are coupled. The effective fields are given by

$$\mathbf{H}_{i,\text{tot}} \equiv - \frac{1}{\mu_i} \frac{\partial \mathcal{H}}{\partial \mathbf{s}_i} = \mathbf{H} + \frac{2D_i}{\mu_i} s_i^z \mathbf{e}_z + \frac{1}{\mu_i} \sum_{j \in \text{neig}(i)} J_{ij} \mathbf{s}_{ij}.$$

TABLE I. Table with parameters of transition metal (TM) and rare-earth (RE) compounds. Anisotropy constant  $D_{\text{TM(RE)}}$  is taken equal for both lattices. Exchange parameters  $J_{\text{TM(RE)}}$ /per link are taken in order to give correct Curie temperature of pure compounds ( $x = 0$  pure TM or  $x = 1$  pure RE). Antiferromagnetic exchange parameter  $J_{\text{RE-TM}}$  is chosen so that the temperature dependence of the TM and RE sublattices agrees qualitatively with results of XMCD measurements of static magnetization.<sup>18</sup>

	$\mu/\mu_B$	$D$ (Joule)	$J$ (Joule)
Transition metal (TM)	2.217	$8.0725 \times 10^{-24}$	$4.5 \times 10^{-21}$
Rare-earth (RE)	7.63	$8.0725 \times 10^{-24}$	$1.26 \times 10^{-21}$
TM-RE	...	...	$-1.09 \times 10^{-21}$

The particular values for exchange parameters and the anisotropy constants (see Table I) are chosen in such a way that the static properties coincide with experimental measurements in GdFeCo.<sup>18</sup>

### III. LLB EQUATION FOR CLASSICAL FERRIMAGNET

#### A. Equation derivation

The idea of the two-component LLB model is presented in Fig. 1. Namely, our aim is to evaluate the dynamics of the macroscopic classical polarization  $\mathbf{m} = \langle \mathbf{s} \rangle^{\text{conf}}$ , where the average is performed over temperature as well as the microscopic disorder configurations.

The dynamics of the mean magnetization can be obtained through the Fokker-Planck equation (FPE) for noninteracting spins.<sup>1,23</sup> The FPE for the distribution function of an ensemble of interacting spins can be derived in the same way as in the ferromagnetic case.<sup>1</sup> The FPE has as the static solution the Boltzmann distribution function  $f_0(\{\mathbf{s}_i\}) \propto \exp[-\beta\mathcal{H}(\{\mathbf{s}_i\})]$ , where  $\mathcal{H}$  is given by Eq. (1) and  $\beta = 1/(k_B T)$ . Since the exact solution is impossible even in the simple ferromagnetic case, then, we resort to the mean-field approximation (MFA) with respect to spin-spin interactions and random average with respect to disorder configurations. In the MFA, the distribution function is multiplicative and we can use the same strategy as in the ferromagnetic case,<sup>1</sup> we take the distribution function  $f_i$  of each lattice site  $i$ , which satisfy the FPE for a noninteracting spin and perform the substitution  $\mathbf{H} \Rightarrow \langle \mathbf{H}_v^{\text{MFA}} \rangle^{\text{conf}}$ , where  $v = \text{TM}$  or  $\text{RE}$  indicates the sublattices. Thus we start with the paramagnetic LLB equation that was derived in the original article by D. Garanin<sup>1</sup> and is equally valid for the present purpose and substitute the external field by the MFA one in each sublattice. The corresponding set of coupled LLB equations for each sublattice magnetization  $\mathbf{m}_v$  has the following form:

$$\dot{\mathbf{m}}_v = \gamma_v [\mathbf{m}_v \times \langle \mathbf{H}_v^{\text{MFA}} \rangle^{\text{conf}}] - \Gamma_{v,\parallel} \left( 1 - \frac{\mathbf{m}_v \cdot \mathbf{m}_{0,v}}{m_v^2} \right) \mathbf{m}_v - \Gamma_{v,\perp} \frac{[\mathbf{m}_v \times [\mathbf{m}_v \times \mathbf{m}_{0,v}]]}{m_v^2}, \quad (4)$$

where

$$\mathbf{m}_{0,v} = B(\xi_{0,v}) \frac{\xi_{0,v}}{\xi_{0,v}}, \quad \xi_{0,v} \equiv \beta \mu_v \langle \mathbf{H}_v^{\text{MFA}} \rangle^{\text{conf}}. \quad (5)$$

Here,  $\xi_{0,v} \equiv |\xi_{0,v}|$ ,  $B(\xi) = \coth(\xi) - 1/\xi$  is the Langevin function,

$$\Gamma_{v,\parallel} = \Lambda_{v,N} \frac{B(\xi_{0,v})}{\xi_{0,v} B'(\xi_{0,v})}, \quad \Gamma_{v,\perp} = \frac{\Lambda_{v,N}}{2} \left[ \frac{\xi_{0,v}}{B(\xi_{0,v})} - 1 \right] \quad (6)$$

describe parallel and perpendicular relaxation, respectively,  $\Lambda_{v,N} = 2\gamma_v \lambda_v / \beta \mu_v$  is the characteristic diffusion relaxation rate or, for the thermoactivation escape problem, the Néel attempt frequency.

Next step is to use in Eqs. (4) and (5) the MFA expressions. The MFA treatment for the disordered ferrimagnet has been presented in Ref. 18. The resulting expressions for the fields have the following forms:

$$\langle \mathbf{H}_{\text{RE}}^{\text{MFA}} \rangle^{\text{conf}} = \mathbf{H}'_{\text{eff,RE}} + \frac{J_{0,\text{RE}}}{\mu_{\text{RE}}} \mathbf{m}_{\text{RE}} + \frac{J_{0,\text{RE-TM}}}{\mu_{\text{RE}}} \mathbf{m}_{\text{TM}}, \quad (7)$$

$$\langle \mathbf{H}_{\text{TM}}^{\text{MFA}} \rangle^{\text{conf}} = \mathbf{H}'_{\text{eff,TM}} + \frac{J_{0,\text{TM}}}{\mu_{\text{TM}}} \mathbf{m}_{\text{TM}} + \frac{J_{0,\text{TM-RE}}}{\mu_{\text{TM}}} \mathbf{m}_{\text{RE}}, \quad (8)$$

where  $J_{0,\text{TM}} = qzJ_{\text{TM-TM}}$ ,  $J_{0,\text{RE}} = xzJ_{\text{TM-TM}}$ ,  $J_{0,\text{RE-TM}} = qzJ_{\text{TM-RE}}$ ,  $J_{0,\text{TM-RE}} = xzJ_{\text{TM-RE}}$ ,  $z$  is the number of nearest neighbors between TM moments in the ordered lattice,  $x$  and  $q = 1 - x$  are the RE and TM concentrations. The field  $\mathbf{H}'_{\text{eff},v}$  contains the external applied and the anisotropy fields acting on the sublattice  $v = \text{TM}, \text{RE}$ .

The equilibrium magnetization of each sublattice  $m_{e,v}$  within the MFA approach can be obtained via the self-consistent solution of the Curie-Weiss equations:

$$\mathbf{m}_{\text{RE}} = B(\xi_{\text{RE}}) \frac{\xi_{\text{RE}}}{\xi_{\text{RE}}}, \quad \mathbf{m}_{\text{TM}} = B(\xi_{\text{TM}}) \frac{\xi_{\text{TM}}}{\xi_{\text{TM}}}. \quad (9)$$

The resulting equation (4) with expressions (7) and (8) constitutes the LLB equation for a ferrimagnet and can be already used for numerical modeling at large scale since in what follows some approximations will be used. The use of these approximations is necessary for understanding the relaxation of a ferrimagnetic system from theoretical point of view. We will also get the LLB equation in a more explicit and compact form.

We treat the most general case where the continuous approximation in each sublattice can be used. Basically, in the spirit of the MFA approximation, in each sublattice, we treat the  $k = 0$  mode. In order to handle the problem analytically, we decompose the magnetization vector  $\mathbf{m}_v$  into two components  $\mathbf{m}_v = \mathbf{\Pi}_v + \boldsymbol{\tau}_v$ , where  $\mathbf{\Pi}_v$  is perpendicular to  $\mathbf{m}_\kappa$ , so that it can be expressed as  $\mathbf{\Pi}_v = -[\mathbf{m}_\kappa \times [\mathbf{m}_\kappa \times \mathbf{m}_v]]/m_\kappa^2$ , and  $\boldsymbol{\tau}_v$  is parallel to  $\mathbf{m}_\kappa$ , and it can be expressed as  $\boldsymbol{\tau}_v = \mathbf{m}_\kappa (\mathbf{m}_v \cdot \mathbf{m}_\kappa) / m_\kappa^2$ , where  $\kappa \neq v$ .

We can shorten the notation by definition of the following new variable  $\Theta_{v\kappa}$ :

$$\Theta_{v\kappa} = \frac{\mathbf{m}_v \cdot \mathbf{m}_\kappa}{m_\kappa^2} \implies \mathbf{m}_v = \mathbf{\Pi}_v + \Theta_{v\kappa} \mathbf{m}_\kappa. \quad (10)$$

As a consequence, the MFA exchange field  $\langle \mathbf{H}_{\text{EX},v}^{\text{MFA}} \rangle^{\text{conf}}$  in Eqs. (7) and (8) can be written as the sum of the exchange fields parallel and perpendicular to magnetization of the

sublattice  $\nu$ :

$$\begin{aligned} \langle \mathbf{H}_{\text{EX},\nu}^{\text{MFA}} \rangle^{\text{conf}} &= \left( \frac{J_{0,\nu}}{\mu_\nu} + \frac{J_{0,\nu\kappa}}{\mu_\nu} \Theta_{\kappa\nu} \right) \mathbf{m}_\nu + \frac{J_{0,\nu\kappa}}{\mu_\nu} \boldsymbol{\Pi}_\kappa \\ &= \frac{\tilde{J}_{0,\nu}}{\mu_\nu} \mathbf{m}_\nu + \frac{J_{0,\nu\kappa}}{\mu_\nu} \boldsymbol{\Pi}_\kappa \\ &= \mathbf{H}_{\text{EX},\nu}^{\parallel} + \mathbf{H}_{\text{EX},\nu}^{\perp}, \end{aligned} \quad (11)$$

where we have defined a new function  $\tilde{J}_{0,\nu}(\mathbf{m}_\kappa, \mathbf{m}_\nu)$  as  $\tilde{J}_{0,\nu} = J_{0,\nu} + J_{0,\nu\kappa} \Theta_{\kappa\nu}(\mathbf{m}_\kappa, \mathbf{m}_\nu)$ , we remark that  $\tilde{J}_{0,\nu}$  is not a constant but a function of both sublattice magnetization. The exchange field is, therefore, separated in two contributions, a longitudinal one,  $\mathbf{H}_{\text{EX},\nu}^{\parallel} = (\tilde{J}_{0,\nu}/\mu_\nu) \mathbf{m}_\nu$ , and a transverse one,  $\mathbf{H}_{\text{EX},\nu}^{\perp} = (J_{0,\nu\kappa}/\mu_\nu) \boldsymbol{\Pi}_\kappa$ .

In the following, we will consider that the transverse contribution is small in comparison to longitudinal one, i.e.,  $|\mathbf{H}_{\text{EX},\nu}^{\parallel}| \gg |\mathbf{H}_{\text{EX},\nu}^{\perp}|$ . This is a good approximation because differently to the ferromagnetic case, where the transverse fluctuations are defined by the anisotropy and Zeeman fields (of the order of 1 T strength), in the case of ferrimagnets, the fluctuations included in  $\mathbf{H}_{\text{EX},\nu}^{\perp}$  are defined by the interlattice exchange field (of the order of 10 T), which is usually very large in comparison to the anisotropy or Zeeman fields. Thus the transverse magnetization fluctuations are small in comparison to the longitudinal component. Finally,  $\langle \mathbf{H}_\nu^{\text{MFA}} \rangle^{\text{conf}} \simeq \mathbf{H}_{\text{EX},\nu}^{\parallel} + \mathbf{H}_{\text{eff},\nu}''$  where  $\mathbf{H}_{\text{eff},\nu}'' = \mathbf{H} + \mathbf{H}_{\text{A},\nu} + \mathbf{H}_{\text{EX},\nu}^{\perp}$  and  $\mathbf{H}_{\text{A},\nu}$  is the anisotropy field.

We now expand  $\mathbf{m}_{0,\nu}$  up to the first order in  $\mathbf{H}_{\text{eff},\nu}''$ , under the assumption  $|\mathbf{H}_{\text{EX},\nu}^{\parallel}| \gg |\mathbf{H}_{\text{eff},\nu}''|$ . Similar to the ferromagnetic case, from Eq. (5), we get (see details in Appendix):

$$\begin{aligned} \mathbf{m}_{0,\nu} &\simeq \frac{B_\nu}{m_\nu} \mathbf{m}_\nu + B'_\nu \beta \mu_\nu \frac{(\mathbf{m}_\nu \cdot \mathbf{H}_{\text{eff},\nu}'') \mathbf{m}_\nu}{m_\nu^2} \\ &\quad - \frac{B_\nu \mu_\nu}{m_\nu \tilde{J}_{0,\nu}} \frac{[(\mathbf{H}_{\text{eff},\nu}'' \times \mathbf{m}_\nu) \times \mathbf{m}_\nu]}{m_\nu^2}, \end{aligned} \quad (12)$$

substituting this into Eq. (4) and repeating the same calculations as in the ferromagnetic case we get the following equation of motion:

$$\begin{aligned} \dot{\mathbf{m}}_\nu &= \gamma_\nu [\mathbf{m}_\nu \times \mathbf{H}_{\text{eff},\nu}''] \\ &\quad - \gamma_\nu \alpha_\parallel^\nu \left( \frac{1 - B_\nu/m_\nu}{\mu_\nu \beta B'_\nu} - \frac{\mathbf{m}_\nu \cdot \mathbf{H}_{\text{eff},\nu}''}{m_\nu^2} \right) \mathbf{m}_\nu \\ &\quad - \gamma_\nu \alpha_\perp^\nu \frac{[\mathbf{m}_\nu \times [\mathbf{m}_\nu \times \mathbf{H}_{\text{eff},\nu}'']] }{m_\nu^2}, \end{aligned} \quad (13)$$

where  $B_\nu = B_\nu(\beta \tilde{J}_{0,\nu}(\mathbf{m}_\nu, \mathbf{m}_\kappa) m_\nu)$  depends on the sublattice magnetizations  $(\mathbf{m}_\nu, \mathbf{m}_\kappa)$  and the damping parameters are

$$\alpha_\parallel^\nu = \frac{2\lambda_\nu}{\beta \tilde{J}_{0,\nu}}, \quad \alpha_\perp^\nu = \lambda_\nu \left( 1 - \frac{1}{\beta \tilde{J}_{0,\nu}} \right). \quad (14)$$

### B. Temperature dependence of damping parameters

The temperature dependence of the damping parameters is obtained in the first order in deviations of magnetization from their equilibrium value. Note that in Eq. (13), all terms are of the first order in the parameter  $H_{\text{eff},\nu}''/H_{\text{EX},\nu}$  so that the damping parameters should be evaluated in the zero order

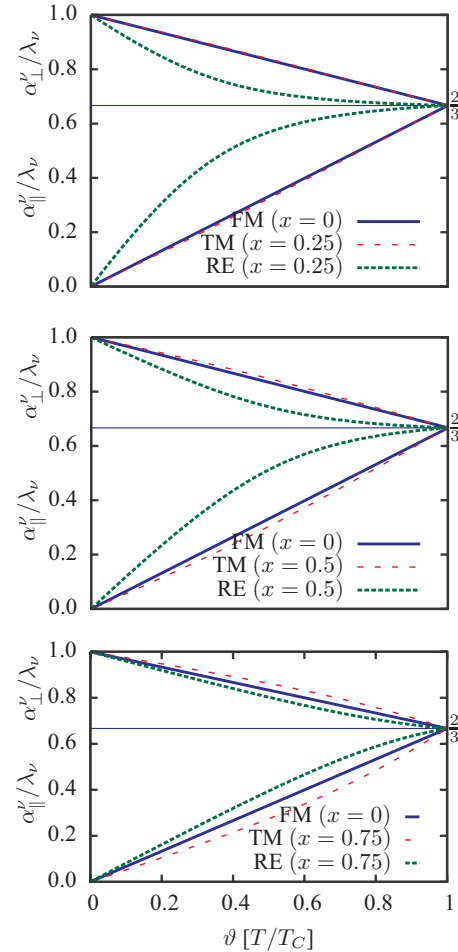


FIG. 2. (Color online) Damping parameters  $\alpha_{\parallel(\perp)}^\nu(\vartheta)$  (normalized to the corresponding intrinsic values) for a pure ferromagnet (FM), rare-earth (RE) component in a GdFeCo ferrimagnet and a transition metal (TM) in a ferrimagnet as a function of reduced temperature  $\vartheta = T/T_C$  for three different rare earth (RE) concentrations  $x$ . The blue solid line represents the  $x = 0$  limit that corresponds to a pure ferromagnet (FM). (Up) The corresponding curves for a 25% concentration of RE. (Middle) The corresponding damping parameters for a 50% alloy. (Bottom) Damping values for 75% RE amount. It can be also seen as a RE doped with a 25% of transition metal (TM).

in this parameter. Consequently, we can use the following equilibrium expression:

$$\tilde{J}_{0,\nu} \simeq \frac{J_{0,\nu} m_{e,\nu} + |J_{0,\nu\kappa}| m_{e,\kappa}}{m_{e,\nu}}, \quad (15)$$

where the sign of the second term does not depend on the sign of the interlattice exchange interaction,  $J_{0,\nu\kappa}$ . The effective damping parameters depend on temperature  $T$  via temperature-dependent equilibrium magnetization. The temperature dependence of damping parameters (14), normalized to the intrinsic coupling parameter, are presented in Fig. 2 for a GdFeCo RE-TM ferrimagnet and for various concentrations of RE impurities.

Let us consider some limiting cases. First, we consider the simplest case of a completely symmetric antiferromagnet (AFM). In the AFM, all the relevant parameters are equal

for both lattices, they have the same magnetic moments  $\mu_A = \mu_B$  and the same intralattice exchange parameters  $J_{0,v}$ , the interlattice exchange parameter is also the same  $J_{0,v\kappa} = J_{0,\kappa v}$  in contrast to our disordered ferrimagnet. In this case, the equilibrium magnetizations as a function of temperature are the same  $m_{e,v}(T) = m_{e,\kappa}(T)$  and the effective exchange parameter reduces to  $\tilde{J}_{0,v} = J_{0,v} + |J_{0,v\kappa}|$ , i.e., the sum of the two interactions coming from the intralattice and interlattice exchanges. The Néel temperature in the MFA reads  $k_B T_N = \tilde{J}_{0,v}/3$  and the damping parameters recover the ferromagnetic type expression

$$\alpha_{\parallel}^{v(\text{AFM})} = \lambda_v \frac{2T}{3T_N}, \quad \alpha_{\perp}^{v(\text{AFM})} = \lambda_v \left(1 - \frac{T}{3T_N}\right). \quad (16)$$

The use of the critical temperature provides an expression in which the damping parameters do not depend explicitly on the interlattice exchange, the implicit dependence comes from the change of the Néel temperature as the exchange parameter  $J_{0,v\kappa}$  varies. There is a more simple AFM, with nearest-neighbor interactions only and one interlattice exchange parameter  $J_{0,v\kappa}$ , it gives the same result as above and exactly the same as for the ferromagnet.

Next interesting case is when one of the three exchange parameters can be neglected. We can consider, for example, a negligible exchange between the rare-earth magnetic moments, it is a good approximation if the impurity content is low. Then we can write the effective exchange as

$$\tilde{J}_{0,\text{TM}} = \frac{J_{0,\text{TM}} m_{e,\text{TM}} + |J_{0,\text{TM-RE}}| m_{e,\text{RE}}}{m_{e,\text{TM}}} \simeq J_{0,\text{TM}}, \quad (17)$$

$$\tilde{J}_{0,\text{RE}} = |J_{0,\text{RE-TM}}| \frac{m_{e,\text{TM}}}{m_{e,\text{RE}}}. \quad (18)$$

In this case, the TM damping parameters can be approximately expressed with the antiferromagnetic or ferromagnetic ( $T_N \rightarrow T_C$ ) formula (16) because in the limit  $x \rightarrow 0$  the Curie temperature of the disordered ferrimagnet is close to  $k_B T_C = J_{0,\text{TM}}/3$ .<sup>16</sup> The damping parameter for the RE lattice, however, is different. It strongly depends on the polarization effect of the TM lattice on the RE magnetization. In this case close to  $T_C$ , where  $m_{\text{RE(TM)}}$  are small, the polarization effect can be expressed using the expansion of the Langevin function  $B$  for  $\xi \rightarrow 0$ , i.e.  $B \approx \xi/3$  [note that  $B'(0) = 1/3$ ]. In this case, we have  $B(\xi_{\text{RE}}) = m_{e,\text{RE}} \approx \beta J_{0,\text{RE-TM}} m_{e,\text{TM}}$ , thus, Eq. (18) can be approximated as  $\tilde{J}_{0,\text{RE}} \approx 1/(3\beta)$ . Therefore we have the following expressions:

$$\alpha_{\parallel}^{\text{TM}} = \lambda_{\text{TM}} \frac{2T}{3T_C}, \quad \alpha_{\parallel}^{\text{RE}} = \frac{2}{3} \lambda_{\text{RE}}. \quad (19)$$

$$\alpha_{\perp}^{\text{TM}} = \lambda_{\text{TM}} \left(1 - \frac{T}{3T_C}\right), \quad \alpha_{\perp}^{\text{RE}} = \frac{2}{3} \lambda_{\text{RE}}. \quad (20)$$

We observe in Fig. 2 that even for quite large amounts of RE of 25% and 50%, the above approximation holds quite well. These relations become quite important above  $T_C$  since they can be used to extend the damping parameters for the region  $T > T_C$ , similar to the ferromagnetic case. However, the correct description of the damping parameters above  $T_C$  turned out to be more complicated for ferrimagnet than for ferromagnet.

If the interlattice exchange is large in comparison to the intra-lattice one then the equilibrium magnetization of both

lattices is similar and the damping parameters behave similar to those of the FM damping parameters, presented above. This case is in agreement with a concentration of 75% of RE in Fig. 2 (down). As predicted, we observe that the damping parameters are very similar for both sublattices.

Note that these damping parameters should be distinguished from those of the normal modes (FMR and exchange)<sup>28</sup> with more complicated expressions that can be obtained via linearization of the set of two-coupled LLB equations, similar to the LLG approach.

### C. Longitudinal relaxation parameters

The function  $1 - B_v/m_v$  in Eq. (13) is a small quantity proportional to the deviation from the equilibrium in both sublattices. It can be further simplified as a function of the equilibrium parameters after some algebra. Similar to the ferromagnetic case, the ferrimagnetic LLB equation can be put in a compact form using the notion of the longitudinal susceptibility.

The initial longitudinal susceptibility can be evaluated on the basis of the Curie-Weiss equations (9). Let us assume that in the absence of an external field, the equilibrium sublattice magnetizations  $\mathbf{m}_{\text{TM}}$  and  $\mathbf{m}_{\text{RE}}$  are, respectively, parallel and antiparallel to the  $z$  axis (a stronger condition of the smallness of the perpendicular components can be also applied). The  $z$  axis is chosen such that it is the easy axis of the magnetic crystal. To evaluate the longitudinal susceptibility, the field should be applied parallel to the easy direction, then in the approximation of small perpendicular components (large longitudinal exchange field) we can neglect in the first approximation the possible change of directions of  $\mathbf{m}_{\text{RE}}$  and  $\mathbf{m}_{\text{TM}}$ . In order to calculate the susceptibility, we expand the right-hand side of Eq. (9) in terms of the external field:

$$m_v(T, H_z) \approx m_v(T, 0) + \mu_v H_z \beta B'_v \left(1 + \frac{\partial H_{\text{EX},v}^z}{\partial H_z}\right), \quad (21)$$

where  $B_v = B_v(\beta \mu_v H_{\text{EX},v})$  and its derivative  $B'_v = B'_v(\beta \mu_v H_{\text{EX},v})$  are evaluated in absence of applied and anisotropy fields. Then,

$$\tilde{\chi}_{v,\parallel} = \left[ \frac{\partial m_v(T, H_z)}{\partial H_z} \right]_{H_z=0} = \mu_v \beta B'_v \left[ 1 + \frac{\partial H_{\text{EX},v}^z}{\partial H_z} \right], \quad (22)$$

where

$$\frac{\partial H_{\text{EX},v}^z}{\partial H_z} = \beta J_{0,v} \tilde{\chi}_{v,\parallel} + \beta |J_{0,v\kappa}| \tilde{\chi}_{\kappa,\parallel}.$$

Thus the longitudinal susceptibility of one sublattice is expressed in terms of another:

$$\tilde{\chi}_{v,\parallel} = \frac{\mu_v}{J_{0,v}} \frac{J_{0,v} \beta B'_v}{1 - J_{0,v} \beta B'_v} \left( \frac{|J_{0,v\kappa}|}{\mu_v} \tilde{\chi}_{\kappa,\parallel} + 1 \right). \quad (23)$$

Finally, we obtain two coupled equations for  $\tilde{\chi}_{\text{RE},\parallel}$  and  $\tilde{\chi}_{\text{TM},\parallel}$ , solving them, we get the MFA expression for the

susceptibilities:

$$\tilde{\chi}_{v,\parallel} = \left( \frac{\mu_\kappa}{|J_{0,\kappa v}|} \right) \frac{|J_{0,\kappa v}| \beta B'_v |J_{0,\nu\kappa}| \beta B'_\kappa + (\mu_\nu/\mu_\kappa) |J_{0,\kappa v}| \beta B'_v (1 - J_{0,\kappa} \beta B'_\kappa)}{(1 - J_{0,\nu} \beta B'_\nu)(1 - J_{0,\kappa} \beta B'_\kappa) - (|J_{0,\kappa v}| \beta B'_v)(|J_{0,\nu\kappa}| \beta B'_\kappa)} = \left( \frac{\mu_\kappa}{|J_{0,\kappa v}|} \right) G_\nu(T). \quad (24)$$

The longitudinal susceptibility  $\tilde{\chi}_{v,\parallel}$  is, therefore, a function of temperature that we have called  $G_\nu(T)$ . It tends to zero at low temperature and diverges approaching Curie temperature  $T_C$  of the magnetic system, similar to the ferromagnetic case. The function  $G_\nu = (|J_{0,\nu\kappa}|/\mu_\nu) \tilde{\chi}_{v,\parallel}$  can be seen as a reduced longitudinal susceptibility.

Now we derive an approximate expression for the small quantity  $1 - B_\nu/m_\nu$  as a function of equilibrium quantities and the deviation of each sublattice magnetization from its equilibrium. In the first approximation, we expand the function  $B_\nu/m_\nu$  near the equilibrium, as was done for the ferromagnet. The function  $B_\nu$  in the zero order in perpendicular field components,  $H''_{\text{eff},\nu}/H_{\text{EX},\nu}$ , can be written as a function of  $m_\nu$  and  $m_\kappa$  as follows:

$$B_\nu \approx B_\nu(\beta[J_{0,\nu} m_\nu + |J_{0,\nu\kappa}| \tau_\kappa]), \quad (25)$$

where  $\tau_\kappa = |(\mathbf{m}_\nu \cdot \mathbf{m}_\kappa)|/m_\nu$  is the length of the projection of the magnetization of the sublattice  $\kappa$  onto the sublattice  $\nu$ . We expand the function  $B_\nu/m_\nu$  in the variables  $m_\nu$  and  $m_\kappa$  near the equilibrium:

$$\begin{aligned} \frac{B_\nu}{m_\nu} &\approx \frac{B_{e,\nu}}{m_{e,\nu}} + \left[ \frac{1}{m_\nu} \left( \frac{\partial B_\nu}{\partial m_\nu} \right) - \frac{1}{m_\nu^2} B_\nu \right]_{\text{eq}} \delta m_\nu + \left[ \frac{1}{m_\nu} \frac{\partial B_\nu}{\partial \tau_\kappa} \right]_{\text{eq}} \delta \tau_\kappa \\ &= 1 - [1 - \beta J_{0,\nu} B'_\nu]_{\text{eq}} \frac{\delta m_\nu}{m_{e,\nu}} + [\beta |J_{0,\nu\kappa}| B'_\nu]_{\text{eq}} \frac{\delta \tau_\kappa}{m_{e,\nu}}, \end{aligned} \quad (26)$$

here,  $\delta m_\nu = m_\nu - m_{e,\nu}$ , with  $m_{e,\nu} = B_\nu(\beta \mu_\nu H_{\text{EX},\nu})$ , where  $H_{\text{EX},\nu}$  is evaluated at the equilibrium, and  $\delta \tau_\kappa = \tau_\kappa - \tau_{e,\kappa}$ , where  $\tau_{e,\kappa} = |(\mathbf{m}_{e,\nu} \cdot \mathbf{m}_{e,\kappa})|/m_{e,\nu}$  and it corresponds to the projection of the equilibrium magnetization  $\mathbf{m}_{e,\kappa}$  onto the other sublattice magnetization direction. It is easy to show that  $\partial \tau_\kappa / \partial m_\nu = 0$ . Similar to the ferromagnetic case, we would like to arrive to a simplified expression as a function of sublattice susceptibilities. For this purpose, we divide the above expression by  $\mu_\nu \beta B'_\nu$ :

$$\frac{1 - B_\nu/m_\nu}{\mu_\nu \beta B'_\nu} = \frac{1}{\tilde{\chi}_{v,\parallel}} \frac{\delta m_\nu}{m_{e,\nu}} + G_\kappa \left( \frac{1}{\tilde{\chi}_{v,\parallel}} \frac{\delta m_\nu}{m_{e,\nu}} - \frac{1}{\tilde{\chi}_{\kappa,\parallel}} \frac{\delta \tau_\kappa}{m_{e,\nu}} \right), \quad (27)$$

where we have used Eq. (23) and the function  $G_\kappa = |J_{0,\nu\kappa}| \tilde{\chi}_{\kappa,\parallel} / \mu_\nu$  has now more sense. Thus the contribution to the dynamical equation (4) of the exchange interaction (the LLB equation with longitudinal relaxation only) given by Eq. (27) reads

$$\frac{\dot{\mathbf{m}}_\nu}{\gamma_\nu} \Big|_{\text{EX}} = - \frac{\alpha_\parallel^v}{m_{e,\nu}} \left( \frac{1 + G_\kappa}{\tilde{\chi}_{v,\parallel}} \delta m_\nu - \frac{|J_{0,\nu\kappa}|}{\mu_\nu} \delta \tau_\kappa \right) \mathbf{m}_\nu. \quad (28)$$

Note that the first term defines the intralattice relaxation of the sublattice (for example, TM) to its own direction. The second term describes the angular momenta transfer between sublattices driven by the temperature. This equation has the

form

$$\frac{\dot{\mathbf{m}}_\nu}{\gamma_\nu} = \tilde{\Gamma}_\nu \mathbf{m}_\nu \quad (29)$$

and it gives the LLB equation for the case when the average magnetization of the two sublattices remains always parallel or antiparallel.

#### D. Final forms of the LLB equation

In order to be consistent with the ferromagnetic LLB equation (and the Landau theory of phase transitions), we expand the deviations  $\delta m_\nu$  ( $\delta \tau_\kappa$ ) around  $m_{e,\nu}^2$  ( $\tau_{e,\nu}^2$ ) up to the quadratic terms. Similar to FM case, we write

$$\frac{\delta m_\nu}{m_{e,\nu}} \approx \frac{1}{2m_{e,\nu}^2} (m_\nu^2 - m_{e,\nu}^2). \quad (30)$$

Therefore we can write the effective longitudinal fields as

$$\mathbf{H}_{\text{eff},\parallel}^v = \left[ \frac{1}{2\Lambda_{\nu\nu}} \left( \frac{m_\nu^2}{m_{e,\nu}^2} - 1 \right) - \frac{1}{2\Lambda_{\nu\kappa}} \left( \frac{\tau_\kappa^2}{\tau_{e,\kappa}^2} - 1 \right) \right] \mathbf{m}_\nu, \quad (31)$$

where in order to shorten the notations we have defined the longitudinal rates as

$$\Lambda_{\nu\nu}^{-1} = \frac{1}{\tilde{\chi}_{v,\parallel}} (1 + G_\kappa), \quad \Lambda_{\nu\kappa}^{-1} = \frac{\tau_{e,\kappa}}{m_{e,\nu}} \frac{|J_{0,\nu\kappa}|}{\mu_\nu} \quad \text{with } \nu \neq \kappa, \quad (32)$$

where  $G_\kappa$  is also expressed in terms of the longitudinal susceptibility via Eq. (24).

#### 1. Form I

Finally, we collect all the above derived approximate expressions and we finish up with the compact form of the LLB equation for the reduced magnetization vector,  $\mathbf{m}_\nu = \mathbf{M}_\nu/M_\nu(T = 0 \text{ K})$ :

$$\begin{aligned} \dot{\mathbf{m}}_\nu &= \gamma_\nu [\mathbf{m}_\nu \times \mathbf{H}_{\text{eff},\nu}] - \gamma_\nu \alpha_\parallel^v \frac{(\mathbf{m}_\nu \cdot \mathbf{H}_{\text{eff},\nu})}{m_\nu^2} \mathbf{m}_\nu \\ &\quad - \gamma_\nu \alpha_\perp^v \frac{[\mathbf{m}_\nu \times [\mathbf{m}_\nu \times \mathbf{H}_{\text{eff},\nu}]]}{m_\nu^2}, \end{aligned} \quad (33)$$

where the effective field  $\mathbf{H}_{\text{eff},\nu}$  for sublattice  $\nu$  is defined as

$$\begin{aligned} \mathbf{H}_{\text{eff},\nu} &= \mathbf{H} + \mathbf{H}_{A,\nu} + \frac{J_{0,\nu\kappa}}{\mu_\nu} \boldsymbol{\Pi}_\kappa \\ &\quad + \left[ \frac{1}{2\Lambda_{\nu\nu}} \left( \frac{m_\nu^2}{m_{e,\nu}^2} - 1 \right) - \frac{1}{2\Lambda_{\nu\kappa}} \left( \frac{\tau_\kappa^2}{\tau_{e,\kappa}^2} - 1 \right) \right] \mathbf{m}_\nu \end{aligned} \quad (34)$$

and the relaxation parameters  $\alpha_\parallel^v$  and  $\alpha_\perp^v$  are given by Eq. (14).

Or in a more explicit form, as a function of sublattice magnetizations  $\mathbf{m}_v$  and its values at the equilibrium  $\mathbf{m}_{e,v}$ :

$$\begin{aligned} \dot{\mathbf{m}}_v = & \gamma_v [\mathbf{m}_v \times \mathbf{H}_{\text{eff},v}] - \gamma_v \alpha_v^{\parallel} \frac{(\mathbf{m}_v \cdot \mathbf{H}_{\text{eff},v}^{\parallel})}{m_v^2} \mathbf{m}_v \\ & - \gamma_v \alpha_v^{\perp} \frac{[\mathbf{m}_v \times [\mathbf{m}_v \times \mathbf{H}_{\text{eff},v}^{\parallel}]]}{m_v^2}, \end{aligned} \quad (35)$$

where we have defined the longitudinal field  $\mathbf{H}_{\text{eff},v}^{\parallel}$  as

$$\begin{aligned} \mathbf{H}_{\text{eff},v}^{\parallel} = & \left\{ \frac{1}{2\Lambda_{v\nu}} \left( \frac{m_v^2}{m_{e,v}^2} - 1 \right) \right. \\ & \left. - \frac{1}{2\Lambda_{v\kappa}} \left[ \left( \frac{\mathbf{m}_v \cdot \mathbf{m}_{e,\kappa}}{\mathbf{m}_{e,v} \cdot \mathbf{m}_{e,\kappa}} \right)^2 - 1 \right] \right\} \mathbf{m}_v \end{aligned} \quad (36)$$

and the effective field  $\mathbf{H}_{\text{eff},v}$  reads

$$\mathbf{H}_{\text{eff},v} = \mathbf{H} + \mathbf{H}_{A,v} + \frac{J_{0,v\kappa}}{\mu_v} \mathbf{m}_{\kappa}.$$

In Eq. (35), also the temperature-dependent damping parameters are given by Eq. (14).

## 2. Form 2

It is also interesting to put the LLB equation in a more symmetric form in terms of the macroscopic magnetization,  $\mathbf{M}_v = x_v \mu_v \mathbf{m}_v / \nu_v$ , where  $x_v$  stands for the concentration of sites of type  $v = \text{TM}$  or  $\text{RE}$  ( $x_v = x$  for  $\text{RE}$  and  $x_v = q$  for  $\text{TM}$ ),  $\mu_v$  is the atomic magnetic moment of the lattice  $v$ , and  $\nu_v$  is the atomic volume. We multiply each sublattice LLB equation (35) by the corresponding factor, for example, in the case of  $\text{TM}$  by  $q\mu_{\text{TM}}/\nu_{\text{TM}}$  and we obtain

$$\begin{aligned} \dot{\mathbf{M}}_v = & \gamma_v [\mathbf{M}_v \times \mathbf{H}_{\text{eff},v}] - L_{\parallel,v} \frac{(\mathbf{M}_v \cdot \mathbf{H}_{\text{eff},v}^{\parallel})}{M_v^2} \mathbf{M}_v \\ & - L_{\perp,v} \frac{[\mathbf{M}_v \times [\mathbf{M}_v \times \mathbf{H}_{\text{eff},v}^{\parallel}]]}{M_v^2}, \end{aligned} \quad (37)$$

where the effective fields read

$$\begin{aligned} \mathbf{H}_{\text{eff},v}^{\parallel} = & \left\{ \frac{1}{2\tilde{\Lambda}_{v\nu}} \left( \frac{M_v^2}{M_{e,v}^2} - 1 \right) \right. \\ & \left. - \frac{1}{2\tilde{\Lambda}_{v\kappa}} \left[ \left( \frac{\mathbf{M}_v \cdot \mathbf{M}_{e,\kappa}}{\mathbf{M}_{e,v} \cdot \mathbf{M}_{e,\kappa}} \right)^2 - 1 \right] \right\} \mathbf{M}_v. \end{aligned} \quad (38)$$

The rate parameters are  $\tilde{\Lambda}_{v\kappa} = \nu_v \Lambda_{v\kappa} / \mu_v x_v$  and the effective field  $\mathbf{H}_{\text{eff},v}$  has the following form:

$$\mathbf{H}_{\text{eff},v} = \mathbf{H} + \mathbf{H}_{A,v} + A \mathbf{M}_{\kappa}.$$

Here, the exchange parameter is introduced as  $A = zJ_{\text{TM-RE}}/\mu_{\text{RE}}\mu_{\text{TM}}$ . The damping coefficients  $L_{\parallel,v}$  and  $L_{\perp,v}$  read

$$L_{\parallel,v} = \gamma_v x_v \mu_v \alpha_v^{\parallel} / \nu_v, \quad L_{\perp,v} = \gamma_v x_v \mu_v \alpha_v^{\perp} / \nu_v.$$

## IV. RELAXATION OF MAGNETIC SUBLATTICES

The rate of the longitudinal relaxation is temperature dependent through parameters such as the damping parameters  $\alpha_v^{\parallel}$ , see Eq. (14) and Fig. 2, and the longitudinal susceptibilities.

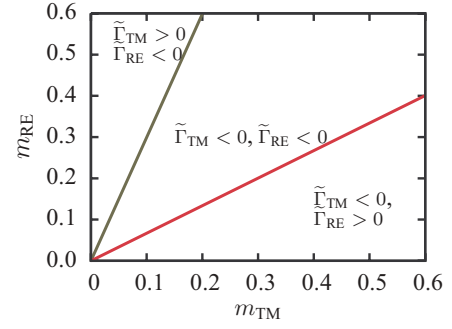


FIG. 3. (Color online) Different longitudinal relaxation regions for  $T/T_C = 0.95$  for parameters of the  $\text{GdFeCo}$  alloy with  $x = 0.25$ .

The sign of the rate,  $\tilde{\Gamma}_v \lesseqgtr 0$ , depends on the instantaneous magnetization values. From Eq. (28), we can consider the following lines separating different relaxation signs:

$$\delta m_v = \frac{|J_{0,v\kappa}|}{\mu_v} \frac{\tilde{\chi}_{v,\parallel}}{G_{\kappa} + 1} \delta \tau_{\kappa} = \tilde{\chi}_{v\kappa,\parallel} \delta \tau_{\kappa}, \quad (39)$$

where we have defined the dimensionless variable  $\tilde{\chi}_{v\kappa,\parallel}$ , which describes the effect of the change in one sublattice on the other. This variable can be interpreted as a susceptibility  $\tilde{\chi}_{v\kappa,\parallel} = \delta m_v / \delta m_{\kappa}$ . Indeed, we can expand

$$\begin{aligned} m_v(T, \delta m_v, \delta m_{\kappa}) \approx & m_v(T, 0, 0) + \beta J_{0,v} B'_v \delta m_v \\ & + \beta |J_{0,v\kappa}| B'_v \delta m_{\kappa}. \end{aligned} \quad (40)$$

Now using that by definition  $\delta m_v = m_v(T, \delta m_v, \delta m_{\kappa}) - m_v(T, 0, 0)$ , we obtain

$$\tilde{\chi}_{v\kappa,\parallel} = |J_{0,v\kappa}| \left( \frac{\beta B'_v}{1 - J_{0,v} \beta B'_v} \right). \quad (41)$$

Next, we substitute Eq. (23) into Eq. (41) and we get the relation between the susceptibilities, exactly described by Eq. (39).

The problem of relaxation sign is, therefore, reduced to the study of the sign of the function  $\delta m_v - \tilde{\chi}_{v\kappa,\parallel} \delta \tau_{\kappa}$ . Let us assume the equilibrium state that is close to  $T_C$ , describing the situation during the ultrafast laser-induced demagnetization.<sup>17</sup> Figure 3 shows three possible instantaneous rates for  $T = 0.95T_C$ , depending on the relative state of both sublattice magnetizations. The lines separating different relaxation types are straight lines with the slope  $\tilde{\chi}_{v\kappa,\parallel}(T)$ .

In the following, we use atomistic LLG Langevin simulations described in Sec. II as well as the integration of the LLB equation (4) for the same material parameters, see Table I. In order to compare MFA based LLB equation and the atomistic simulations, we have renormalized exchange parameters, as described in Ref. 18. In the atomistic simulation, the system size is taken as  $\mathcal{N} = 60^3$ , i.e.,  $3\mathcal{N}$  coupled differential equations has to be solved simultaneously within this approach, whereas only six (two sublattices and three components for each) are used in the macrospin LLB approach. We compare the different relaxation regions depending on the instantaneous magnetic state with those predicted by the LLB equation and depicted in Fig. 3. The initial conditions in the simulations are the following: in all three cases we start from an equilibrium state at  $T = 600$  K (for the considered concentration  $x = 0.25$ , we get  $T_C = 800$  K). After that, for

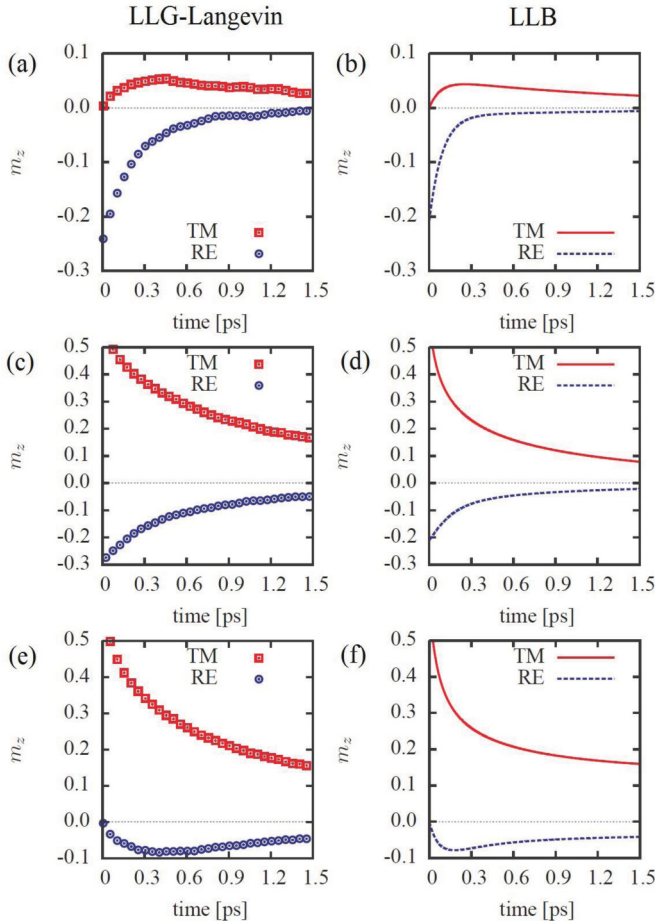


FIG. 4. (Color online) Comparison between atomistic LLG-Langevin and macrospin LLB calculations of the longitudinal relaxation of the GdFeCo alloy ( $x = 0.25$ ) corresponding to the three different relaxation cases in Fig. 3. In the left column, we show atomistic LLG-Langevin multispin simulations and in the right one, the LLB macrospin calculations. The graphs (a) and (b) correspond to the region with  $\tilde{\Gamma}_{\text{TM}} > 0$  and  $\tilde{\Gamma}_{\text{RE}} < 0$ . The graphs (c) and (d) correspond to the region with  $\tilde{\Gamma}_{\text{TM}} < 0$  and  $\tilde{\Gamma}_{\text{RE}} < 0$ . The graphs (e) and (f) correspond to the region with  $\tilde{\Gamma}_{\text{TM}} < 0$  and  $\tilde{\Gamma}_{\text{RE}} > 0$ .

the situations of Figs. 4(a) and 4(e) we put one of the sublattice magnetizations equal to zero,  $m_{\text{TM(RE)}} = 0$ . In the atomistic approach this is done by totally disordering the system. Finally, the temperature is set to  $T = 0.95T_C$  and the relaxation of both sublattices is visualized. The results are presented in Fig. 4.

For the region  $m_{\text{RE}} \gg m_{\text{TM}}$  above the gray line in Fig. 3, the rate for the TM is positive,  $\tilde{\Gamma}_{\text{TM}} > 0$ , thus the TM magnetization will increase while  $\tilde{\Gamma}_{\text{RE}} < 0$  and the RE magnetization will decrease. Thus we have initially a dynamical polarization of TM by RE. As it can be seen in Figs. 4(a) and 4(b) initially the TM magnetic order increases from a totally disordered state, while the RE relaxes directly to the equilibrium, i.e., the sign of the RE rate is always the same. In the central region of Fig. 3, between gray and red lines, both magnetizations go to the equilibrium by decreasing their value, see Figs. 4(c) and 4(d). Finally, in the low region of Fig. 3, the situation is symmetric to the upper region but now TM magnetization decreases and the RE magnetization increases initially, see Figs. 4(e) and 4(f). Thus the predictions of the LLB equation

are in agreement with full atomistic simulations which also provides a validation for our analytic derivation.

As a representative example, in GdFeCo near the magnetization reversal the situation is the following:<sup>17</sup> the TM magnetization is almost zero,  $m_{\text{TM}} \approx 0$  and the RE has finite magnetization value  $m_{\text{TM}} > 0$ . This happens due to the fact that the Gd sublattice is intrinsically slower than the FeCo one due to a larger magnetic moment. This situation corresponds to the upper region in Fig. 3 where the rates are  $\tilde{\Gamma}_{\text{TM}} > 0$  and  $\tilde{\Gamma}_{\text{RE}} < 0$ . Under these circumstances, the RE magnetization dynamically polarizes the TM sublattice magnetization through the interlattice exchange interaction  $H_{\text{EX, TM-RE}} \approx |J_{0, \text{TM-RE}}| m_{\text{RE}} > 0$ . Consequently, the TM magnetization goes opposite to its equilibrium position  $m_e^{\text{TM}} = 0$  [see Figs. 4(a) and 4(b)]. The existence of opposite relaxation signs in TM and RE is consistent with a recently reported ferromagnetic state in a ferrimagnetic materials Ref. 17, however, it does not necessary lead to it, nor it necessary means the switching of the TM magnetization, as was suggested in Ref. 24. To have a switching one should cross the line  $m_z^{\text{TM}} = 0$ , which cannot be done within the approach of longitudinal relaxation only which only describes the relaxation to the equilibrium. The crossing of the line  $m_z^{\text{TM}} = 0$  can be only provided by a stochastic kick which is always present in the modeling using stochastic atomistic approach.<sup>16,17</sup> This topic will be the subject of future work.

## V. THE LLB EQUATION AND THE BARYAKHTAR EQUATION

In this section, we would like to discuss the differences between the LLB equation and the equation derived by V. Baryakhtar<sup>25</sup> and used in Ref. 24 to explain the ultrafast magnetization reversal and the transient ferromagnetic-like state in ferrimagnets. The Baryakhtar equation was derived from the Onsager principle which in general is valid near the thermodynamic equilibrium only. The general derivation is based on the symmetry approach. Another strong supposition made in its derivation is the separation of the time scales: the exchange interaction time scale and the relativistic interaction time scale (defined in our case by the parameter  $\lambda$ ) are assumed to be separated. The resulting equation has the following form:

$$\frac{1}{\gamma_\nu} \frac{dM_\nu}{dt} = \lambda_e (H_\nu - H_k) + \lambda_\nu H_\nu. \quad (42)$$

Here,  $\nu = \text{TM, RE}$ ,  $\lambda_\nu$  describes transfer of the angular momentum from sublattices to the environment,  $\lambda_e$  is of the exchange origin and stems from spin-spin interactions, conserving the total angular momentum but allowing for the transfer of angular momentum between the sublattices. The effective fields defined as  $H_\nu = -\delta W / \delta M_\nu$  are derived from the magnetic energy  $W$ . In Ref. 24, the authors used the Landau type free energy expansion near the critical temperature, corresponding to the form Eq. (30).

In comparison to the Baryakhtar equation, the LLB equation, derived here includes the transverse exchange mode and allows the transfer of the energy or momentum between the longitudinal and transverse motion. The ferrimagnetic LLB equation has three terms among which it is the precession term that conserves the total angular momentum. The precession in



the interlattice exchange field given by  $[\mathbf{m}_{\text{TM}} \times \mathbf{m}_{\text{RE}}]$  allows the transfer of angular momentum between sublattices. The longitudinal and transverse relaxation terms, which are related to the coupling to the heat bath, are both proportional to  $\lambda$ . Differently to ferromagnets, both the transverse motion given by precession and transverse relaxation terms are not negligible on the femtosecond time scale in comparison to longitudinal motion because in both cases the field acting on both motions is of the exchange origin.

In principle, the ferrimagnetic LLB equation can be cast in a form, similar to the Baryakhtar equation if we restrict ourselves to longitudinal motion only, considering the antiparallel sublattices alignment. For the longitudinal relaxation only [see Eq. (28)], we have the following expression:

$$\frac{\dot{m}_z^v}{\gamma_v} = \alpha_{\parallel}^v H'_v + \alpha_{v\kappa}^{\parallel} (H'_v + H'_\kappa), \quad (43)$$

where  $H'_v = -(\frac{\delta m_v}{\tilde{\chi}_{v,\parallel}})m_z^v/m_v$ , stands for the fields coming from interaction of each lattice with itself and  $H'_\kappa$ —with the opposite sublattice. One can see that the sign of the effective field coming from the other sublattice is opposite for the LLB Eq. (43) and the Baryakhtar equation Eq. (42). In order to illustrate the consequence of this, we can compare the equations for the limiting case close to  $T_C$ . In this case, the Baryakhtar equation [see Eq. (1.33) in Ref. 25] reads

$$\frac{\dot{m}_z^v}{\gamma_v} = -\lambda^v \frac{m_z^v}{\tilde{\chi}_{v,\parallel}} - \lambda_e \left( \frac{m_z^v}{\tilde{\chi}_{v,\parallel}} + \frac{m_z^\kappa}{\tilde{\chi}_{\kappa,\parallel}} \right), \quad (44)$$

where  $m_z^v$  is the absolute value of the  $z$  component of the magnetization in the sublattice  $v$  and we explicitly considered that the sign of  $z$  components is opposite for the sublattice  $v$  and  $\kappa$ . In the same limit, considering  $m_{\text{TM(RE)}} = m_{e,\text{TM(RE)}} + \delta m_{\text{TM(RE)}}$ , and following Eq. (28), the LLB equation takes a similar form:

$$\frac{\dot{m}_z^v}{\gamma_v} = -\alpha_{\parallel}^v \frac{m_z^v}{\tilde{\chi}_{v,\parallel}} - \alpha_{\parallel}^v \frac{|J_{0,v\kappa}|}{\mu_v} \left( \frac{\tilde{\chi}_{\kappa,\parallel}}{\tilde{\chi}_{v,\parallel}} m_z^v - m_z^\kappa \right). \quad (45)$$

Note that for the LLB equation the contribution of the opposite sublattice is negative while for the Baryakhtar equation it is positive. This has important consequences in the longitudinal interlattice relaxation of the sublattices, changing the results of Fig. 3. Particularly, in Fig. 3 the straight lines (grey and red) could have negative slopes for the Baryakhtar approach. Then, in Fig. 4, the different sign would give a contrary relaxation direction of the magnetization recovery of TM in the first 0.3 ps as compared to Fig. 4(a) and of RE as compared to Fig. 4(e).

Furthermore, in Fig. 5, we show the temperature dependence of the ratio of partial susceptibilities,  $\tilde{\chi}_{\kappa,\parallel}/\tilde{\chi}_{v,\parallel}$  appearing in Eq. (45). We can see that at temperatures not very close to  $T_C$ :  $\tilde{\chi}_{\text{TM},\parallel}/\tilde{\chi}_{\text{RE},\parallel} \ll 1$  and the contrary behavior close to  $T_C$ . Thus for the TM and temperatures close to  $T_C$  the second term in the right-hand side of Eq. (45) could be neglected and the third term with the opposite sign can compete with the first one, leading either to slowing down of the relaxation rate or even to changing its sign, as presented in Figs. 4(a) and 4(b). The behavior of RE on the contrary is dominated by this term and the sign of relaxation cannot be changed, as is seen in the same figure. Obviously, this behavior cannot be described by Eq. (44) where all terms have the same sign. In

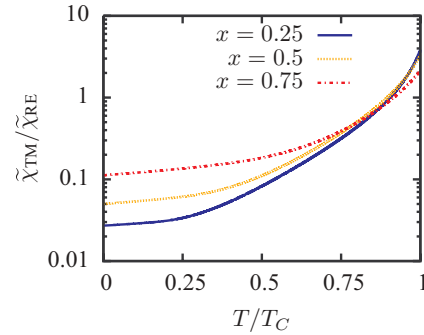


FIG. 5. (Color online) Temperature dependence of the ratio between longitudinal susceptibilities for parameters of the GdFeCo alloy.

order to have the opposite relaxation sign, one has to assume for this equation a priori that the signs of the  $z$  components of magnetization in both sublattices are the same, i.e., to start with the ferromagnetic-like state without specifying its origin.

Finally, we would like to note that because we have treated the spin-spin interaction in MFA we have lost correlation contribution. Consequently, both LLB and Baryakhtar equations do not describe the energy transfer from the uniform modes into nonlinear spin waves and vice versa. In ferromagnets,<sup>26</sup> this contribution is usually two or three orders of magnitude smaller than the contribution to relaxation through the coupling to the bath. At this stage we do not know how large this contribution can be in ferrimagnets. In Ref. 26, the contribution of nonlinear spin waves was artificially incremented by using a random anisotropy to cause noncolinearities. In principle, in ferrimagnets one can see a small amount of RE as precursor of noncolinearities, with the strength of the order of interlattice exchange parameter  $J_{\text{TM-RE}}$ . For completeness, a microscopic treatment of the spin wave contribution would be desirable, we let this task for the future. Particularly, this should take into account magnetisation relaxation in the absence of any external coupling to the bath, not described at present time by the LLB equation.

## VI. CONCLUSIONS

We have derived the Landau-Lifshitz-Bloch equation for a two-sublattice system such as a GdFeCo ferrimagnet for which an ultrafast switching has been reported.<sup>14,17</sup> Although in our derivation we refer to a TM-RE alloy, it is equally valid for a two-component ferromagnet, as well as for an antiferromagnet. The generalization to more components is straightforward. The new equation constitutes an important step forward in classical description of the dynamics of ferrimagnets, which is traditionally based on two-coupled macroscopic LLG equations. For example, the FMR and exchange modes have recently attracted attention due to possibility to optically excite them.<sup>13,27</sup> Their temperature dependence can be now correctly understood in terms of our approach.<sup>28</sup> Furthermore, recent ultrafast dynamics experiments using XMCD showed different sublattice dynamics on ultrafast time scale in a two-sublattice magnets such as GdFeCo<sup>17</sup> or FeNi,<sup>29</sup> which can be modeled using this new approach. Finally, this equation can serve in the future as a basis for multiscale modeling in two-component

systems at high temperatures and/or ultrafast time scales, the same way as the LLB equation for ferromagnets.<sup>19</sup> This also opens a possibility for micromagnetic modeling of ultrafast dynamics in large structures, such as sub-micron and micron-size ferrimagnetic dots, whose dimensions do not allow modeling by atomistic approach. Similarly, it will be useful for micromagnetic modeling at high temperatures, such as thermally-driven domain wall motion in nanostructures.

The LLB equation correctly shows the possibility to reverse the sign of relaxation at high temperatures and, therefore, is consistent with the existence of a recently reported ferromagnetic state in a ferrimagnet.<sup>17</sup> The validity of the approach has been checked against full-scale atomistic simulations presented in Fig. 4. However, unlike the equation, derived by Baryakhtar and used recently to describe the GdFeCo switching,<sup>24</sup> it is not based on the separation of time scales and on the Onsager principle. Instead, both the coupling to the external bath and the exchange interaction form part of the same longitudinal and transverse relaxation terms. We show important differences in the resulting form of the equation.

Unfortunately, at the present time, the compact derivation was possible only under some assumptions. The employed conditions certainly allow to describe the normal modes such as ferromagnetic resonance and antiferromagnetic exchange precessional modes in ferrimagnets.<sup>28</sup> The same way the approximation is sufficient to describe the switching of ferrimagnet if it occurs through a linear reversal path<sup>4,24</sup> or if sublattices noncollinearities are not too large. Whether the applied approximation completely describes the situation of the ultrafast reversal is an open question which we will investigate in the future. For modeling, the initial paramagnetic equation (4) with the MFA field (7) and (8) can always be used, providing the check for the approximation. Finally, up to now, we were not able to derive a compact expression for the equation above  $T_C$  which is also a necessary step for the full modeling of the ultrafast switching.

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#### APPENDIX

In this appendix, we present detailed derivation of the LLB equation in the form of Eq. (13). Our starting point is the LLB equation (4). First, we obtain an approximate expression for  $\mathbf{m}_{0,v}$  as given by Eq. (12), and later we simplify expressions for the relaxation rates  $\Gamma_{v,\parallel}$  and  $\Gamma_{v,\perp}$  [see Eq. (6)] to express the damping parameters in the form of Eqs. (14).

We start from Eq. (5),

$$\mathbf{m}_{0,v} = B(\xi_{0,v})\hat{\mathbf{u}}_v, \quad \xi_{0,v} \equiv \beta\mu_v \langle \mathbf{H}_v^{\text{MFA}} \rangle^{\text{conf}}, \quad (\text{A1})$$

where  $\hat{\mathbf{u}}_v = \xi_{0,v}/\xi_{0,v}$  and  $\langle \mathbf{H}_v^{\text{MFA}} \rangle^{\text{conf}} = \mathbf{H}_{\text{EX},v}^{\parallel} + \mathbf{H}_{\text{eff},v}^{\prime\prime}$ . Here,  $\mathbf{H}_{\text{eff}}^{\prime\prime}$  contains the anisotropy, applied and the perpendicular component of the exchange field (see Sec. III A). In the case of a strong homogeneous exchange field  $|\mathbf{H}_{\text{EX},v}^{\parallel}| \gg |\mathbf{H}_{\text{eff},v}^{\prime\prime}|$ ,

the MFA field can be expanded up to first order in  $H_{\text{eff},v}^{\prime\prime}$  as

$$|\langle \mathbf{H}_v^{\text{MFA}} \rangle^{\text{conf}}| \simeq H_{\text{EX},v}^{\parallel} + \frac{\mathbf{H}_{\text{EX},v}^{\parallel} \cdot \mathbf{H}_{\text{eff},v}^{\prime\prime}}{H_{\text{EX},v}^{\parallel}}. \quad (\text{A2})$$

Therefore,  $\xi_{0,v} = \beta\mu_v |\langle \mathbf{H}_v^{\text{MFA}} \rangle^{\text{conf}}|$  can be written as  $\xi_{0,v} = \xi_{\text{EX},v} + \delta\xi_v$  with  $\xi_{\text{EX},v} \gg \delta\xi_v$ , where we identify  $\xi_{\text{EX},v} = \beta\mu_v H_{\text{EX},v}^{\parallel}$  and  $\delta\xi_v = \beta\mu_v (\mathbf{H}_{\text{EX},v}^{\parallel} \cdot \mathbf{H}_{\text{eff},v}^{\prime\prime})/H_{\text{EX},v}^{\parallel}$ . Expanding the Langevin function around  $\xi_{\text{EX},v}$  we get

$$B(\xi_{0,v}) \simeq B_v + B'_v \delta\xi_v \quad (\text{A3})$$

and

$$\hat{\mathbf{u}}_v \simeq \frac{\mathbf{H}_{\text{EX},v}^{\parallel} + \mathbf{H}_{\text{eff},v}^{\prime\prime}}{H_{\text{EX},v}^{\parallel}} \left[ 1 - \frac{\mathbf{H}_{\text{EX},v}^{\parallel} \cdot \mathbf{H}_{\text{eff},v}^{\prime\prime}}{(H_{\text{EX},v}^{\parallel})^2} \right], \quad (\text{A4})$$

where  $B_v = B(\xi_{\text{EX},v})$  and  $B'_v = B'(\xi_{\text{EX},v})$ . Substituting Eqs. (A3) and (A4) in Eq. (A1) and neglecting the terms quadratic in  $H_{\text{eff},v}^{\prime\prime}/H_{\text{EX},v}$ , we get

$$\mathbf{m}_{0,v} \simeq B_v \left[ \frac{\mathbf{H}_{\text{EX},v}^{\parallel} + \mathbf{H}_{\text{eff},v}^{\prime\prime}}{H_{\text{EX},v}^{\parallel}} - \frac{(\mathbf{H}_{\text{EX},v}^{\parallel} \cdot \mathbf{H}_{\text{eff},v}^{\prime\prime})\mathbf{H}_{\text{EX},v}^{\parallel}}{(H_{\text{EX},v}^{\parallel})^3} \right] + B'_v \beta\mu_v \frac{(\mathbf{H}_{\text{EX},v}^{\parallel} \cdot \mathbf{H}_{\text{eff},v}^{\prime\prime})\mathbf{H}_{\text{EX},v}^{\parallel}}{(H_{\text{EX},v}^{\parallel})^2}. \quad (\text{A5})$$

Using the vector calculus identity  $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$ , Eq. (A5) can be written as

$$\mathbf{m}_{0,v} \simeq B_v \frac{\mathbf{H}_{\text{EX},v}^{\parallel}}{H_{\text{EX},v}^{\parallel}} + B'_v \beta\mu_v \frac{(\mathbf{H}_{\text{EX},v}^{\parallel} \cdot \mathbf{H}_{\text{eff},v}^{\prime\prime})\mathbf{H}_{\text{EX},v}^{\parallel}}{(H_{\text{EX},v}^{\parallel})^2} - \frac{B_v}{H_{\text{EX},v}^{\parallel}} \frac{[[\mathbf{H}_{\text{eff},v}^{\prime\prime} \times \mathbf{H}_{\text{EX},v}^{\parallel}] \times \mathbf{H}_{\text{EX},v}^{\parallel}]}{(H_{\text{EX},v}^{\parallel})^2}. \quad (\text{A6})$$

Finally, we use  $\mathbf{H}_{\text{EX},v}^{\parallel} = (\tilde{J}_{0,v}/\mu_v)\mathbf{m}_v$  [see Eq. (11)] in Eq. (A6), and we obtain

$$\mathbf{m}_{0,v} \simeq \frac{B_v}{m_v} \mathbf{m}_v + B'_v \beta\mu_v \frac{(\mathbf{m}_v \cdot \mathbf{H}_{\text{eff},v}^{\prime\prime})\mathbf{m}_v}{m_v^2} - \frac{B_v \mu_v}{m_v \tilde{J}_{0,v}} \frac{[[\mathbf{H}_{\text{eff},v}^{\prime\prime} \times \mathbf{m}_v] \times \mathbf{m}_v]}{m_v^2}, \quad (\text{A7})$$

which corresponds to Eq. (12).

In order to get the final LLB equation (13), which is valid in the first order in the small parameter  $H_{\text{eff},v}^{\prime\prime}/H_{\text{EX},v}$ , it is enough to evaluate  $\Gamma_{v,\parallel}$  and  $\Gamma_{v,\perp}$  in the zero order. Therefore the term  $B(\xi_{0,v})/\xi_{0,v}$  appearing in both relaxation rates in Eq. (6) can easily be approximated considering the facts that  $B(\xi_{0,v}) = m_{0,v}$  and  $\xi_{0,v} \simeq \xi_{\text{EX},v} = \beta\mu_v H_{\text{EX},v}^{\parallel} = \beta\tilde{J}_{0,v} m_{0,v}$ , thus we get  $B(\xi_{0,v})/\xi_{0,v} = \beta\tilde{J}_{0,v}$  and also  $B'_v \equiv B'(\xi_{0,v}) \simeq B'(\xi_{\text{EX},v})$ . Thus the relaxation rates can be written as

$$\Gamma_{v,\parallel} = \frac{2\gamma_v \lambda_v \beta \tilde{J}_{0,v}}{\beta \mu_v B'_v}, \quad \Gamma_{v,\perp} = \frac{\gamma_v \lambda_v}{\beta \mu_v} \left( \frac{1}{\beta \tilde{J}_{0,v}} - 1 \right). \quad (\text{A8})$$

Finally, we substitute Eqs. (A7) and (A8) into Eq. (4). From there, we get the equation of motion [see Eq. (13)]:

$$\dot{\mathbf{m}}_v = \gamma_v [\mathbf{m}_v \times \mathbf{H}_{\text{eff},v}''] - \gamma_v \alpha_{\parallel}^v \left( \frac{1 - B_v/m_v}{\mu_v \beta B_v'} - \frac{\mathbf{m}_v \cdot \mathbf{H}_{\text{eff},v}''}{m_v^2} \right) \mathbf{m}_v - \gamma_v \alpha_{\perp}^v \frac{[\mathbf{m}_v \times [\mathbf{m}_v \times \mathbf{H}_{\text{eff},v}'']]}{m_v^2},$$

and the transverse and longitudinal damping parameters [see Eqs. (14)],

$$\alpha_{\parallel}^v = \frac{2\lambda_v}{\beta \tilde{J}_{0,v}},$$

$$\alpha_{\perp}^v = \lambda_v \left( 1 - \frac{1}{\beta \tilde{J}_{0,v}} \right).$$

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