Deconfined quantum criticality and logarithmic violations of scaling from emergent gauge symmetry

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We demonstrate that the low-energy effective theory for a deconfined quantum critical point in $d = 2 + 1$ dimensions contains a leading-order contribution given by the Faddeev-Skyrme model. The Faddeev-Skyrme term is shown to give rise to the crucial Maxwell term in the CP^1 field theory governing the deconfined quantum critical point. We derive the leading contribution to the spin stiffness near the quantum critical point and show that it exhibits a logarithmic correction to scaling of the same type as recently observed numerically in low-dimensional models of quantum spin systems featuring a quantum critical point separating an antiferromagnetically ordered state from a valence bond solid state. These corrections, appearing away from upper or lower critical dimensions, reflect an emergent gauge symmetry of low-dimensional antiferromagnetic quantum spin systems.

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Much of our understanding of phase transitions is based on the concept of spontaneous symmetry breaking,[1] which provides a mechanism for the spontaneous generation of an ordered state as one or more parameters of a many-body system are varied. In Abelian systems the ordered state can be related to the disordered symmetric state by a duality transformation mapping a strongly coupled regime onto a weakly coupled one.[2–4] In the case of a U(1) symmetry, the symmetric phase is described in the dual picture by a *disorder* parameter,[3] as opposed to the order parameter describing the broken symmetry state in the original picture. The disorder parameter is nonzero when topological defects of the U(1) theory (vortices) are condensed. The U(1) symmetry of the dual theory is then spontaneously broken. The superfluid phase corresponds to the U(1) symmetric state. The vortex condensation of the dual theory reflects the nontriviality of the first homotopy group of U(1), namely, $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$, the group of integers. This leads, for instance, to flux quantization in superconductors.

When the order field is composed of more elementary constituents, the disordered phase exhibits nontrivial features that do not follow from standard spontaneous symmetry-breaking arguments. This is so, for instance, in two-dimensional quantum spin systems featuring a paramagnetic phase where the symmetries of the underlying lattice is broken. On a square lattice where SU(2)-invariant spin interactions compete, a valence-bond solid (VBS) state emerges in the paramagnetic phase.[5] An example of this is the $J-Q$ model,[6] with a four-spin exchange around the plaquette of a square lattice in addition to the usual Heisenberg term:

$$H = J\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j - Q\sum_{\langle ijkl\rangle}\left(\mathbf{S}_i\cdot\mathbf{S}_j-\frac{1}{4}\right)\left(\mathbf{S}_k\cdot\mathbf{S}_l-\frac{1}{4}\right). \tag{1}$$

When $J \gg Q$ the Heisenberg term dominates the physics and the ground state is antiferromagnetic (AF), while for $Q \gg J$ the four-spin term favors a VBS state. Numerical works[6–9] show that the $J-Q$ model has an emergent U(1) symmetry. This had previously been predicted in models with the same phase structure by introducing a new paradigm for phase transitions,[10] the so-called deconfined quantum criticality (DQC) scenario. Introducing two different order parameters to describe a phase transition, one for the AF phase and another one for the VBS phase, it was argued that near the phase transition more fundamental building blocks, namely, elementary excitations known as spinons, constitute both order parameters. In this scenario staggered Berry phases interfere destructively with the hedgehogs (magnetic monopoles in spin space), leading to spinon deconfinement at the phase transition.[10] This mechanism has been confirmed by large-scale Monte Carlo (MC) simulations of an easy-plane antiferromagnet.[11]

Previous analyses of spinon deconfinement considered a model for easy-plane antiferromagnets exhibiting a U(1) symmetry.[12] The resulting theory is described by a CP^1 model, which due to the easy-plane anisotropy has a global U(1) symmetry in addition to the local one.[10,12] This property makes the model self-dual, and it was argued that this self-duality would imply a second-order quantum phase transition at zero temperature.[10] However, MC simulations[11,13] revealed a first-order phase transition, also found by a subsequent renormalization group (RG) analysis of the model.[14] Thus, one may ask if the same would also be found in the globally SU(2) invariant case supposed to be described by an isotropic CP^1 model with a noncompact Maxwell term. MC simulations on such a model on a lattice have found a first-order phase transition.[15] This contradicts previous MC results[16] obtaining a second-order phase transition. Numerical work on the $J-Q$ model appears to show a second-order phase transition;[6,7,9,17] see, however, Ref. 8.

Recently,[9,17] a feature of the $J-Q$ model that does not seem to follow from DQC was observed. Namely, logarithmic violations of scaling in the zero-temperature spin stiffness and in the finite-temperature uniform susceptibility were found. In systems with continuous symmetries, this normally occurs either at the upper or lower critical dimensions. In Refs. 9 and 17, the systems considered are $(2+1)$-dimensional. Thus, the corresponding field theory prescribed by DQC would be neither at the upper nor the lower critical dimension. Moreover, no logarithms are expected to occur in the zero-temperature spin stiffness or in the finite-temperature uniform susceptibility, since both these quantities can be derived from a current correlation function. For this reason, it has been suggested[9] that the DQC scenario should be revised in order to accommodate this new aspect. A phenomenological theory at finite temperature involving a gas of free spinons has been proposed recently[18] to fit the logarithmic behavior of the simulations. A nonstandard power behavior for the thermal gap at criticality was introduced to make a logarithm appear in the uniform susceptibility. Since the free spinon gas has the usual spectrum at zero temperature, no quantum critical logarithmic behavior can be derived in this way for the spin stiffness at zero temperature. Moreover, the origin of the anomalous scaling of the thermal gap was not addressed. In this paper, we show that the logarithmic violation of scaling in the $J-Q$ model is actually encoded in the DQC scenario and that this is a direct consequence of the emergent U(1) gauge symmetry.

If quantum criticality in the $J-Q$ model follows from DQC, it should be governed by an effective lattice gauge theory where a staggered Berry phase suppresses its magnetic monopoles.[10] In the absence of this staggered Berry phase this lattice gauge theory is given by the CP^1 model with a compact Maxwell term,[10]

$$S = -\frac{1}{g}\sum_{j,\mu,a} z_{aj}^* e^{-iA_{j\mu}} z_{aj+\hat{\mu}} + \text{H.c.} - \frac{1}{e^2}\sum_{j,\mu,\nu,\lambda}\cos(\epsilon_{\mu\nu\lambda}\Delta_\nu A_{j\lambda}), \tag{2}$$

where $a = 1,2$, Δ_μ denotes the μth component of the lattice gradient and the complex scalar fields satisfy the local constraint $|z_{1,j}|^2 + |z_{2,j}|^2 = 1$. The action (2) has been studied numerically,[19] and a second-order phase transition was found. Also, the field theory of Eq. (2) has been studied via a renormalization group analysis.[20] The resulting flow diagram interpolates between the quantum critical points of the O(3) and O(4) nonlinear σ models; see Fig. 1. We thus expect the universality class of the phase transition of Eq. (2) to be O(3), which obviously features a quantum critical point, in agreement with Ref. 19.

Within the DQC paradigm, the effect of the staggered Berry phase is essential. This can conveniently be accounted for by rewriting the Maxwell term in Eq. (2) in Villain form, $S_{\text{Maxwell}} = \frac{1}{2e^2}\sum_j(\epsilon_{\mu\nu\lambda}\Delta_\nu A_{j\lambda} - 2\pi n_{j\mu})^2$, and coupling the integer fields $n_{j\mu}$ to fixed, time-independent fields ζ_j,

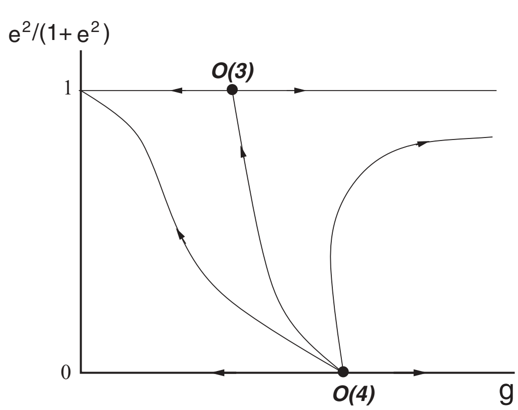

FIG. 1. Schematic flow diagram of the CP^1 model with a compact Maxwell term. The Maxwell term allows for an interpolation between the O(3) and O(4) nonlinear σ model fixed points.

taking the values 0, 1, 2, 3 on the dual lattice, as follows: $i(\pi/2)\sum_j \zeta_j \Delta_\mu n_{j\mu}$. This defines the Berry phase of a compact CP^1 model believed to describe the essential physics at deconfined quantum critical points.[21] A partial dualization of the model yields[21]

$$S_{\text{SJ}} = \frac{1}{4}\sum_j\left[e^2\left(N_{j\mu} - \frac{1}{4}\Delta_\mu\zeta_j\right)^2 + g(\epsilon_{\mu\nu\lambda}\Delta_\nu N_{j\lambda})^2\right] + S_{\mathbf{n}} - 2\pi i\sum_j N_{j\mu}k_{j\mu}(\mathbf{n}). \tag{3}$$

Here, $S_{\mathbf{n}}$ is the action of the nonlinear σ model and $k_{j\mu}$ is the topological current. In the continuum limit, we have

$$k_\mu \approx \frac{1}{4\pi}\epsilon_{\mu\nu\lambda}\mathbf{n}\cdot(\partial_\nu\mathbf{n}\times\partial_\lambda\mathbf{n}). \tag{4}$$

The lattice fields $N_{j\mu}$ are integer-valued and ζ_j are fixed fields arising from the Berry phase in the original model having specific values on the dual lattice (details can be found in Refs. 21 and 22). Using the Poisson formula to promote the integer fields $N_{j\mu}$ to real fields $B_{j\mu}$, performing the shift $B_{j\mu} \to B_{j\mu} + \Delta_\mu\zeta_j/4$, and integrating over $B_{j\mu}$, one obtains

$$\tilde{S}_{\text{SJ}} = 2\pi^2\sum_{i,j}D_{\mu\nu}(x_i - x_j)[k_{i\mu}(\mathbf{n}) + m_{i\mu}][k_{j\nu}(\mathbf{n}) + m_{j\nu}] + S_{\mathbf{n}} - \frac{i\pi}{2}\sum_j m_{j\mu}\Delta_\mu\zeta_j. \tag{5}$$

Here $D_{\mu\nu}(x_i - x_j)$ satisfies the equation $[(-g\Delta^2 + e^2)\delta_{\mu\lambda} + 2\Delta_\mu\Delta_\lambda]D_{\lambda\nu}(x_i - x_j) = 2\delta_{\mu\nu}\delta_{ij}$, $m_{j\mu}$ are new integer-valued vector fields arising from the Poisson summation, and we have used $\Delta_\mu k_{j\mu}(\mathbf{n}) = 0$. The first term in Eq. (5) can be approximated by

$$\mathcal{L}_{\text{Skyrme}} = \frac{1}{2e^2}[\epsilon_{\mu\nu\lambda}\mathbf{n}\cdot(\partial_\nu\mathbf{n}\times\partial_\lambda\mathbf{n})]^2, \tag{6}$$

first introduced in Ref. 23. The effective Lagrangian in the continuum limit is thus

$$\mathcal{L} = \frac{1}{2g}(\partial_\mu\mathbf{n})^2 + \mathcal{L}_{\text{Skyrme}} + \cdots, \tag{7}$$

where the three-component direction field $\mathbf{n}$ satisfies the local constraint $\mathbf{n}^2 = 1$ and the ellipses denote other terms related to the Berry phase, which are being neglected in the above approximation. We return to this below. The model in Eq. (7)[24] is known to have a rich geometric and topological structure.[25–27] The emergent U(1) symmetry of the Skyrme term follows from the compact U(1) gauge group in the continuum arising as a subgroup of the SU(2) gauge group.[28] To see this, consider the functional integral

$$Z = \int \mathcal{D}\mathbf{n}\mathcal{D}C_\mu \exp\left(-\frac{1}{4e^2}\int d^3x\mathbf{F}_{\mu\nu}\cdot\mathbf{F}_{\mu\nu}\right), \tag{8}$$

where $\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - \mathbf{W}_\mu\times\mathbf{W}_\nu$ is an SU(2) field strength with an adjoint non-Abelian gauge field of the form $\mathbf{W}_\mu = \mathbf{n}\times\partial_\mu\mathbf{n} + \mathbf{n}C_\mu$ and C_μ is an Abelian gauge field. The Skyrme contribution Eq. (6) follows from integrating out C_μ in Eq. (8). This is reminiscent of the arguments of Ref. 29, where a four-dimensional version of Eq. (7)[30] is argued to be a low-energy description of SU(2) Yang-Mills theory.

The model Eq. (7) has to be modified to account for the destructive interference between the Berry phases and the magnetic monopoles. As discussed in detail in Ref. 10, this interference mechanism suppresses the monopoles, which implies an effective model given by the *noncompact* CP^1 model with a Maxwell term. Such a model may be written in the form[27]

$$\mathcal{L} = \frac{1}{2g}(\partial_\mu \mathbf{n})^2 + \frac{1}{2g}C_\mu^2 + \frac{1}{2e^2}[\epsilon_{\mu\nu\lambda}\partial_\nu C_\lambda + \epsilon_{\mu\nu\lambda}\mathbf{n}\cdot(\partial_\nu \mathbf{n}\times\partial_\lambda \mathbf{n})]^2. \tag{9}$$

Setting $\mathbf{n} = z_a^* \boldsymbol{\sigma}_{ab} z_b$, we obtain the CP^1 realization of Eq. (9):

$$\mathcal{L} = \frac{1}{g}|(\partial_\mu - iA_\mu)z_a|^2 + \frac{1}{2g}C_\mu^2 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu C_\lambda + 2i\epsilon_{\mu\nu\lambda}\partial_\nu z_a^* \partial_\lambda z_a)^2. \tag{10}$$

Classically, by enforcing the constraint $|z_1|^2 + |z_2|^2 = 1$, we have

$$A_\mu = (i/2)(z_a^*\partial_\mu z_a - z_a \partial_\mu z_a^*). \tag{11}$$

We can now perform the *singular* gauge transformation $A_\mu \to A_\mu - C_\mu$, $z_a \to e^{i\int_0^x dx'_\mu C_\mu(x')} z_a$, to obtain

$$\mathcal{L} = \frac{1}{g}|(\partial_\mu - iA_\mu)z_a|^2 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2. \tag{12}$$

This is precisely the standard DQC model.[10] Note that a similar decoupling does not hold in the case of the Ginzburg-Landau theory for two-component superconductors discussed in Ref. 27, since there the sum of the respective Cooper pair densities is not CP^1 constrained. Thus, we see that if we insist on the emergent character of A_μ as expressed in Eq. (11), the Maxwell term in Eq. (12) is just the Skyrme term, which this is shown to be contained in Eq. (3).

The gauge transformation employed to decouple C_μ and thus derive Eq. (12) was performed at the classical level, in which case the equation of motion for A_μ yields $A_\mu = (i/2)(z_a^*\partial_\mu z_a - z_a\partial_\mu z_a^*)$. In some calculations involving higher-order quantum fluctuations, it might be more appropriate to consider the Lagrangian (10), since quantum fluctuations give the gauge field A_μ an independent dynamics. However, in most lowest-order approximations, Eq. (12) is sufficiently accurate. Note that Eq. (11) does not follow from the equation of motion for A_μ derived from the Lagrangian (12).

Next we calculate the spin stiffness, ρ_s. The latter is obtained from the gauge-invariant response to a twist associated to the spin current tensor $\mathbf{J}_\mu = (\mathbf{n}\times\partial_\mu \mathbf{n})/g$.[31] Due to the constraint $\mathbf{n}^2 = 1$, we have

$$\frac{g}{2}\mathbf{J}_\mu^2 = \frac{1}{2g}(\partial_\mu \mathbf{n})^2, \tag{13}$$

which provides some insight into the meaning of $1/g$ as the *bare* stiffness. Note that when $d = 1+0$, $\mathbf{J}_\mu$ is just the angular momentum of a particle having moment of inertia $1/g$ and constrained to move on the surface of the S^2 sphere. By generalizing the mechanics of a particle on a sphere to a field theory in $d = D+1$ space-time dimensions, the twist is realized by the response to an external constant triplet field $\mathbf{S}_\mu$, which amounts to making the replacement $\partial_\mu \mathbf{n} \to \partial_\mu \mathbf{n} + \mathbf{S}_\mu \times \mathbf{n}$. In order to facilitate the calculations, we can assume without loss of generality that $S_\mu^a = \delta_{a=3} S_\mu$. Thus, in the CP^1 representation S_μ will couple to the third component of $\mathbf{J}_\mu$, which is written in terms of spinon fields as $J_\mu^3 = \frac{1}{g}[j_\mu^{(1)} - j_\mu^{(2)}]$, where $j_\mu^{(a)} = i(z_a^*\partial_\mu z_a - z_a\partial_\mu z_a^*) - 2A_\mu|z_a|^2$, with no sum over a. Variation of the Lagrangian Eq. (12) yields $\frac{1}{g}[j_\mu^{(1)} + j_\mu^{(2)}] = \frac{1}{e^2}\partial_\nu F_{\nu\mu}$, which allows us to write $J_\mu^3 = \frac{2}{g}j_\mu^{(1)} - \frac{1}{e^2}\partial_\nu F_{\nu\mu}$. The spin stiffness is calculated by computing the response of J_μ^3 when it is coupled to an external source, S_μ. Differentiating the free energy functional with respect to the source, letting the source vanish at the end, yields the spin stiffness in the form

$$\rho_s = \frac{4}{g}\langle |z_1|^2|z_2|^2\rangle - \frac{1}{d}\int d^d x K_{\mu\mu}(x), \tag{14}$$

where $K_{\mu\nu}(x) = \langle J_\mu^3(x) J_\nu^3(0)\rangle$ is the current correlator. The above formula is a generalization to the O(3) case of the well-known formula for the superfluid density of a globally U(1)-invariant system. Interestingly, the O(3) stiffness involves the correlator of a gauge-invariant U(1) current within a framework associated to a global SU(2) symmetry.

When only global symmetries are involved, conserved currents do not renormalize.[32,33] This fact elegantly provides a foundation for the scaling relation $\rho_s \sim \xi^{2-D}$ for the superfluid stiffness in systems with a global U(1) symmetry.[34] Indeed, current conservation implies that the current correlation function does not exhibit an anomalous dimension, implying that the scaling behavior of the superfluid stiffness is simply determined by dimensional analysis. However, the non-renormalization theorem fails in gauge theories,[35] such as the Lagrangian in Eq. (12). In the absence of the Maxwell term, we have the standard CP^1 model, which is equivalent to the O(3) nonlinear σ model. In this case the non-renormalization theorem is still valid, since the gauge field then is just an auxiliary field. Specifically, if we introduce the dimensionless coupling $\hat{g} = g\Lambda^{d-2}$, where Λ is the ultraviolet cutoff, along with the parameter $r = 1 - \hat{g}/\hat{g}_c$, with $\hat{g}_c$ being the critical coupling, we obtain that for the standard CP^1 model in $d = D+1$ space-time dimensions near criticality, $\rho_s \sim r^{\nu(d-2)}$, corresponding to standard Josephson scaling.[34] This result can be derived using an expansion in $\epsilon = d-2$ or, in the case of the CP^{N-1} model, by means of a $1/N$ expansion.[36] Furthermore, at finite temperature and at $\hat{g} = \hat{g}_c$, scale invariance implies $\rho_s \sim T^{d-2}$, which is essentially the quantum critical spin susceptibility at finite temperature.

For the model in Eq. (12), the presence of the Maxwell term leads to logarithmic corrections in the spin stiffness for $d = 2+1$ when approaching the critical point from the broken symmetry (Higgs) phase, where the gauge field is gapped. More precisely, the exact expression up to order $1/N$ and

$d = 2 + 1$, keeping both Ng and Ne^2 fixed, is given by[37]

$$\frac{\rho_s}{r} \sim 1 - \frac{16}{3\pi^2 N}\ln\left(\frac{16r}{N\hat{g}+16r}\right) + \frac{64}{3\pi^2 N\left(1+\frac{256r}{N^2\hat{f}\hat{g}}\right)}\left[\ln\left(\frac{64r}{N^2\hat{f}\hat{g}}\right) + \frac{3}{\sqrt{1-\frac{2048r}{N^2\hat{f}\hat{g}}}}\ln\left(\frac{1+\sqrt{1-\frac{2048r}{N^2\hat{f}\hat{g}}}}{1-\sqrt{1-\frac{2048r}{N^2\hat{f}\hat{g}}}}\right)\right] + \cdots, \tag{15}$$

where we have introduced the dimensionless coupling $\hat{f} = e^2/\Lambda$. For $\hat{f} \gg 1$, we obtain

$$\frac{\rho_s}{r} \approx 1 - \frac{16}{3\pi^2 N}\ln\left(\frac{16r}{N\hat{g}+16r}\right) - \frac{128}{3\pi^2 N}\ln\left(\frac{128r}{N^2\hat{f}\hat{g}}\right), \tag{16}$$

which implies the critical exponent $\nu = 1 - 48/(\pi^2 N)$ of the standard CP^{N-1} model at large N.[36] The same behavior is obtained at fixed $\hat{f}$ and $r \to 0$, consistent with the result that at the CP^{N-1} fixed point the Maxwell term is (dangerously) irrelevant. For $\hat{f} \to 0$, on the other hand, only the first two terms in Eq. (15) remain, leading to the critical exponent $\nu = 1 - 16/(3\pi^2 N)$ of an $\mathrm{O}(2N)$ nonlinear σ model. Therefore, Eq. (15) interpolates between the fixed points of the CP^{N-1} and $\mathrm{O}(2N)$ models. The logarithmic correction to the Josephson scaling obtained here is in agreement with recent numerical findings.[9] This provides further evidence that the DQC scenario describes the quantum critical regime of the AF-VBS transition in low-dimensional quantum spin models. These log-corrections, appearing away from lower and upper critical dimensions, reflect an emergent gauge symmetry of such systems. They also elucidate the dangerously irrelevant character of the emergent Maxwell term in the CP^{N-1} model and its primary role in the log-correction.

The interpolation between the large N fixed points of the CP^{N-1} and $\mathrm{O}(2N)$ obtained above is of the same type we have found before in our discussion in connection with Fig. 1. For $N = 2$ the standard CP^{N-1} model is equivalent to the O(3) nonlinear σ model. The DQC regime lies on the critical separatrix of Fig. 1. Thus, the logarithmic violation of scaling obtained here is fundamentally different from the one usually encountered at the upper or lower critical dimension of local field theories.

It would be interesting to observe the violation of Josephson scaling in experiments by measuring the spin susceptibility, which would essentially be a measurement of the spin stiffness. Good candidates are low-dimensional quantum antiferromagnets featuring geometric frustration, such as the organic Mott insulating compound $\mathrm{EtMe_3P[Pd(dmit)_2]_2}$ (Ref. 38) or the kagome lattice system $\mathrm{Zn}_x\mathrm{Cu}_{4-x}\mathrm{(OH)_6Cl_2}$.[39] Field theories with emergent U(1) gauge symmetries have been proposed to describe the quantum criticality in these materials.[40,41]

In this paper, we have considered a class of quantum spin systems featuring a breakup of spin-1/2 objects into more fundamental constituents called spinons, accompanied by an emergent massless gauge field and hence an emergent U(1) gauge symmetry. We show that this emergence gives rise to logarithmic violations of Josephson scaling of the spin stiffness of these systems. It originates with a breakdown of the non-renormalization property of the conserved current that underpins Josephson scaling. Similar violations of scaling should appear in the susceptibility of the system. A violation of Josephson scaling has been observed in numerical works on the spin stiffness of quantum antiferromagnets with ring exchange and constitutes an experimental signature of spin-fractionalization and emergence of massless photons in low-dimensional quantum antiferromagnets. We have proposed candidate materials in which to look for this.

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