

Geometric proof of the equality between entanglement and edge spectraBrian Swingle^{1,2} and T. Senthil¹¹*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*²*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*

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The bulk-edge correspondence for topological quantum liquids states that the spectrum of the reduced density matrix of a large subregion reproduces the thermal state of a physical edge. This correspondence suggests an intricate connection between ground state entanglement and physical edge dynamics. We give a simple geometric proof of the bulk-edge correspondence for a wide variety of physical systems. Our unified proof relies on geometric techniques available in Lorentz invariant and conformally invariant quantum field theories. These methods were originally developed in part to understand the physics of black holes, and we now apply them to determine the local structure of entanglement in quantum many-body systems.

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I. INTRODUCTION

Since the discovery of the fractional quantum Hall fluids¹ and the subsequent elucidation of their topological structure,^{2,3} it has become clear that entanglement plays a crucial role in a wide variety of zero-temperature quantum phases of matter. To describe such phases of matter, it is necessary to understand their pattern of long-range entanglement since they fail to be distinguished by any symmetry breaking pattern. Studies of entanglement entropy have provided a window onto various patterns of entanglement. The boundary law for entanglement entropy states that the entropy of a subsystem of linear size L in d dimensions in the ground state typically scales like L^{d-1} . Reference 4 provides a comprehensive review of this fundamental result. Entanglement entropy has since been used in Refs. 5 and 6 to give meaning to the notion of long-range entanglement in topological fluids. Entanglement considerations have also led to a revolution in our understanding of 1d physics,^{7,8} promising variational states in higher dimensions,^{9,10} and a classification of 1d phases.^{11,12}

More recently, Li and Haldane in Ref. 13 drew attention to interesting physical information encoded in the full spectrum of the reduced density matrix in fractional quantum Hall fluids. Of course, the full spectrum is not universal, but what Li and Haldane argued was that the entanglement spectrum contained a universal part that was characteristic of the phase of interest. Define the “entanglement Hamiltonian” H_R of region R via $\rho_R = e^{-H_R}$. The boundary law implies that H_R behaves, at a very crude level, like the Hamiltonian of a lower dimensional system. Reference 13 argued that for quantum Hall systems the universal part of H_R was actually given by a dynamical Hamiltonian for a physical edge. Thus the entanglement cut becomes a physical cut, and the ground state informs us about quantum dynamics at a physical edge.

Since the Li-Haldane proposal, known as a bulk-edge correspondence, Refs. 14–17 have offered a variety of proofs of the conjecture and variations on the theme. There were even hints of a bulk-edge correspondence in the early work of Ref. 5. In this paper, we present a proof of the bulk-edge correspondence for a wide variety of physical systems that may be approximated as Lorentz invariant at low energies. Our proof has the advantage of simplicity, physical transparency, and generality. We emphasize that the existing arguments for

this relation are restricted to special cases such as quantum Hall fluids in two dimensions and noninteracting topological insulators. Our proof covers all these cases and a great deal more, allowing us to treat interactions and different dimensions all in the same unified geometric framework. We believe our results provide considerable evidence for the claim that the bulk-edge correspondence is a *generic* feature of quantum many-body systems with protected edge states. As an added bonus, our method provides access not only to the spectrum but also the eigenstates of the entanglement Hamiltonian.

Our main technical tools are a set of powerful geometrical constructions in Lorentz invariant and conformally invariant field theories that relate the entanglement spectrum of special subsystems to thermal spectra in appropriate spacetimes. For the interested reader, we note that these tools were originally developed to understand the physics of black holes and our analysis is reminiscent of the black hole membrane paradigm (see Ref. 18). A key difference is that we do not permit the geometry to fluctuate. Of course, the systems we are interested in are not exactly Lorentz invariant, but we may recover the physical situation by adding irrelevant (or marginal) operators that break Lorentz invariance at high energies. Such local perturbations, provided they respect the relevant symmetries, cannot modify universal features of the entanglement spectrum.

The geometric tools described in Refs. 19–22 enable us to rigorously establish a bulk-edge correspondence for topological liquids. If at a real boundary the system has gapless edge modes, then we argue that the entanglement spectrum also contains signatures of these gapless modes. Thus we establish very generally a powerful link between entanglement and quantum dynamics, connecting ground state properties with edge dynamics. Our results treat systems in a variety of dimensions with and without interactions in a completely unified framework. We provide another derivation of topological entanglement entropy in $2+1$ dimensions, a proof of the bulk-edge correspondence for entanglement spectra in fractional quantum Hall states, a proof of the bulk-edge correspondence for topological insulators in $3+1$ dimensions, and a proof of the bulk-edge correspondence for fractionalized topological insulators in $3+1$ dimensions.

This paper is organized as follows. The first section contains an introduction and motivation. Section two describes our geometric tools and the intuition behind them. We show that these tools can reproduce familiar results by rederiving the universal topological entanglement entropy for all two-dimensional topological theories. We then turn to the bulk-edge correspondence in its full form. Section three treats the original Li-Haldane situation of fractional quantum Hall fluids followed by analogous calculations for topological band insulators and a class of recently proposed fractional topological insulators. Finally, we conclude with some comments on future directions.

II. ENTANGLEMENT HAMILTONIANS FROM GEOMETRIC FLOWS

Our geometric techniques permit us to address two kinds of subsystems in d dimensions, half spaces $R = \{x^1 \geq 0, x^2 \dots x^d \in (-\infty, \infty)\}$ and disks $R = B^d = \{\sqrt{\sum_i (x^i)^2} \leq b\}$. Because we work with gapped topological fluids, the precise details of the region shape are not expected to be important. We begin with the half space.

It has been known for some time, motivated by studies of Unruh radiation, that the spectrum of the reduced density matrix of a half space in a relativistic quantum field theory is related to that of the Lorentz boost generator that preserves the so-called Rindler wedge (see Refs. 19 and 20). Focusing on the coordinates x^1 and t mixed by the boost and using light cone variables $x^+ = t + x^1$ and $x^- = t - x^1$, boosts by velocity v send

$$x^\pm \rightarrow e^{\pm\lambda} x^\pm \left(e^\lambda = \sqrt{\frac{1+v}{1-v}} \right). \quad (1)$$

The boundaries of the Rindler wedge are the light rays $x^+ = 0$ and $x^- = 0$ and boosted observers asymptotically approach these lines as $\lambda \rightarrow \pm\infty$. Precisely because a boost can never lead to an observer moving faster than light, the boost transformation only moves points around within the Rindler wedge. An alternate characterization of the Rindler wedge following from relativistic causality is as the region of spacetime where the physics is totally controlled by the state of the half space $x > 0$ at $t = 0$; it is the ‘‘causal development’’ D of the half space at $t = 0$. In concrete terms, no information bearing signal emitted from $x < 0$ at $t = 0$ can enter the Rindler wedge since this would require faster than light information propagation. Having introduced the Rindler wedge and boost, we can state the key result of Refs. 19 and 20: The state of the half space is thermal with respect to the generator of Rindler boosts (defined below) with inverse temperature 2π . We will shortly give two elementary proofs of this statement. Note that this is a formal mathematical result valid for any Lorentz invariant field theory and does not directly depend on the physics of detectors, etc., involved in the Unruh effect.

Nevertheless, this result does account for the radiation seen by accelerated observers, since their effective time evolution is generated by the Rindler boost. In other words, since the Minkowski vacuum looks like a thermal state for the Rindler boost generator, accelerated observers experience such a state as an ordinary thermal bath with respect to their internal clock.

To explain this more fully, note first that the boost generator is

$$-iK \sim x^1 \partial_t + t \partial_{x^1}, \quad (2)$$

so $e^{\lambda(-iK)} x^\pm = e^{\pm\lambda} x^\pm$. Now a uniformly accelerated observer with acceleration a follows the trajectory

$$x^1(\tau) = a^{-1} \cosh(a\tau), \quad t(\tau) = a^{-1} \sinh(a\tau) \quad (3)$$

in terms of the observer’s proper time τ . Indeed, the proper time interval for such a trajectory is

$$d\tau^2 - dx^2 = [\cosh(a\tau)d\tau]^2 - [\sinh(a\tau)d\tau]^2 = d\tau^2 \quad (4)$$

as claimed. Returning to light cone coordinates we find the trajectory

$$x^\pm = \pm a^{-1} e^{\pm a\tau}. \quad (5)$$

This last formulation makes it clear that boosting by parameter λ is equivalent to sending $\tau \rightarrow \tau + \lambda/a$, but this is simply the statement that $H_\tau = K/a$ generates time evolution for the accelerated observer’s internal clock. Now we have already stated that the half-space density matrix is thermal with respect to K with the precise relation being $\rho_{x^1>0} = \exp(-2\pi K)$. From this it immediately follows that $\rho_{x^1>0} = \exp[-(2\pi/a)H_\tau]$ is thermal with temperature $a/(2\pi)$ with respect to the proper time translation generator H_τ (the Hamiltonian) of the accelerated observer.

Now we move away from the physics of accelerated observers and return to our main line of development. We need an operator version of the boost generator to apply these results to quantum field theories. This operator form may be easily obtained by sending $i\partial_t$ to an energy density and $-i\partial_{x^1}$ to a momentum density in the usual way. The operator form acting on degrees of freedom in the half space is

$$K = \int_{x^1>0} x^1 \mathcal{H} - t \mathcal{P}, \quad (6)$$

where $\mathcal{H} = T^{tt}$ and $\mathcal{P} = T^{tx^1}$ are the energy and momentum density components of the stress tensor $T^{\mu\nu}$. Evaluating this generator at $t = 0$ provides an immediate connection between the entanglement Hamiltonian (and boost generator) $2\pi K$ and the energy density \mathcal{H} . There remains the important question of boundary conditions at $x^1 = 0$.

These results may be proven in two ways (see Ref. 18 for a nice discussion). We may obtain the state $\Psi[\phi_{x^1<0}, \phi_{x^1>0}]$ as a Euclidean path integral over the upper half plane. Instead of slicing the path integral at constant imaginary time, we may re-slice the path integral at constant Euclidean angle. Just as the Hamiltonian generates translation in imaginary time, the boost generator K generates rotation in Euclidean angle, and we have

$$\Psi[\phi_{x^1<0}, \phi_{x^1>0}] = \langle \phi_{x^1<0} | e^{-\pi K} | \phi_{x^1>0} \rangle \quad (7)$$

as in Ref. 20. Tracing out $\phi_{x^1<0}$ immediately gives a density matrix $\exp(-2\pi K)$. We have

$$\begin{aligned} & \rho_{x^1>0}(\phi_{x^1>0}, \phi'_{x^1>0}) \\ &= \int D\phi_{x^1<0} \langle \phi_{x^1>0} | e^{-\pi K} | \phi_{x^1<0} \rangle \langle \phi_{x^1<0} | e^{-\pi K} | \phi'_{x^1>0} \rangle \\ &= \langle \phi_{x^1>0} | e^{-2\pi K} | \phi'_{x^1>0} \rangle, \end{aligned} \quad (8)$$

where we have used

$$\int D\phi_{x^1 < 0} |\phi_{x^1 < 0}\rangle \langle \phi_{x^1 < 0}| = 1_{x^1 < 0}. \quad (9)$$

Alternatively, since the Minkowski vacuum $|\Omega\rangle$ is invariant under boosts $K|\Omega\rangle = 0$ and since a complexified boost by $\lambda = 2\pi i$ leaves the coordinates in the Rindler wedge unchanged, we have a kind of periodicity in imaginary boost parameter. This is completely analogous to imaginary time periodicity of observables in the thermal state $e^{-\beta H}$. Identical logic leads to the conclusion that the half-space state is $\exp(-2\pi K)$, although we omit the details here (see Ref. 19 for a mathematical discussion).

Remarkably, the entanglement Hamiltonian is the generator of a geometric flow in spacetime which may be interpreted as time evolution in Rindler space. The reduced density matrix of the half space is then a simple thermal state with respect to time evolution in Rindler space. If we change coordinates to $x^1 = a^{-1}e^u \cosh a\eta$ and $t = a^{-1}e^u \sinh a\eta$, then the metric on the Rindler wedge takes the form $ds^2 = a^{-2}e^{2u}(-d\eta^2 + du^2)$ and the curve $u = 0$ has constant acceleration a . The spectrum of the entanglement Hamiltonian can then be found directly in the continuum by quantizing the low-energy theory in this spacetime. For example, a free scalar field ϕ of mass m obeys the wave equation

$$\square\phi - m^2\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) - m^2\phi = 0 \quad (10)$$

(note that this is not an interesting topological theory). In terms of our coordinates above the wave equation is

$$-a^2e^{-2u}\partial_\eta^2\phi + a^2e^{-2u}\partial_u^2\phi + (\nabla_\perp^2 - m^2)\phi = 0 \quad (11)$$

and introducing Fourier components $\phi = \phi_{\omega k}e^{ikx_\perp - i\omega\eta}$ we have

$$\omega^2\phi_{\omega k} = -\partial_u^2\phi_{\omega k} + [(k^2 + m^2)/a^2]e^{2u}\phi_{\omega k}. \quad (12)$$

A cutoff is necessary for large negative u ; otherwise the local effective temperature, given by $1/\sqrt{g_{\eta\eta}}$, diverges as $u \rightarrow -\infty$, a fact familiar from the study of black hole thermodynamics. Together with the steep e^{2u} potential this cutoff leads to infinite square well like ‘‘energy levels’’ ω^2 for the scalar field example. In particular, note that there no gapless ‘‘edge states’’ for the massive scalar. When considering true topological theories with physical edge states the Rindler spectrum will include edge states.

The results for the Rindler wedge are already quite powerful, but to work with compact subsystems we must introduce a little more technology. If we further restrict ourselves to conformal field theories, then we have an additional result about the reduced density matrix of the d ball as described in Refs. 21 and 22. Those authors showed that the spectrum of the reduced density matrix of a disk is that of a thermal state of the conformal field theory but defined on the hyperbolic space $H^d \times R$. The proof is very similar to Rindler wedge result, and actually follows from the result for the Rindler wedge by a conformal transformation. Indeed, if we look carefully at the Rindler wedge metric in terms of coordinates η and $\rho = a^{-1}e^u$,

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 + dx_\perp^2, \quad (13)$$

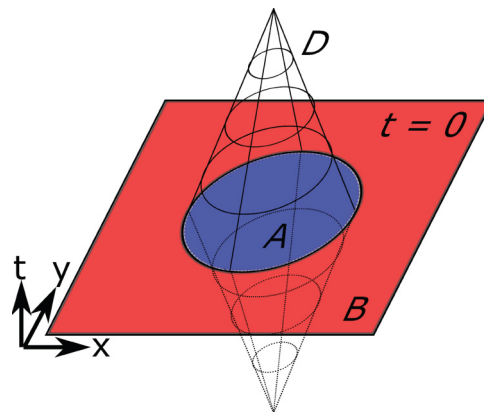


FIG. 1. (Color online) The disk A is colored blue. The rest of the system at fixed time is colored red and denoted B . The boundary causal development D is depicted with the wire outline cones; the solid cone is the forward boundary while the dotted cone is the past boundary. Everything inside the causal development D is determined by the state on A at time $t = 0$ since no other region of spacetime can communicate with D . Figure adapted from Ref. 22.

then a simple manipulation gives

$$ds^2 = \rho^2 \left(-d\eta^2 + \frac{d\rho^2 + dx_\perp^2}{\rho^2} \right), \quad (14)$$

which is conformally equivalent (the overall ρ^2 factor) to $R \times H^d$ where we model H^d as the Poincaré half space. The half-space result is related to the disk result by a process of zooming in on the boundary, a procedure which sends the Poincaré ball to the Poincaré half space. Reference 22 contains details of the geometrical flow in the causal development D_{disk} (shown in Fig. 1) that is mapped to ordinary time evolution in the hyperbolic space.

An important check of this relation is that it correctly reproduces the divergences inherent in continuum entanglement entropy. In the hyperbolic setting, infinities arise because the hyperbolic space is noncompact. To regulate the theory, we must introduce a boundary into the space, and to compute the partition function properly, we must establish boundary conditions for the fields. For example, boundary conditions must be considered to ensure that the partition function is gauge invariant. The metric of d -dimensional hyperbolic space (the Poincaré ball) may be taken to be

$$ds^2 = dw^2 + \sinh^2 w d\Omega_{d-1}^2 \quad (15)$$

with $d\Omega_{d-1}^2$ the metric of the $d - 1$ sphere. Recalling that the conformal field theory on hyperbolic space is at finite temperature, we can compute the total entropy by integrating an entropy density s over the cutoff hyperbolic space. The entropy is thus

$$S_{\text{disk}} = s \text{vol}(S^{d-1}) \int_0^{w_c} dw \sinh^{d-1} w, \quad (16)$$

which does indeed diverge as the cutoff w_c goes to infinity. The cutoff w_c is related to the radius b of the original ball and the ultraviolet cutoff ϵ via $w_c \sim \ln(b/\epsilon)$. Plugging this into our expression for the entropy immediately gives a boundary law $S \sim (b/\epsilon)^{d-1}$ as expected.

Finally, we note that a related construction allows us to obtain the entanglement entropy of the disk (or d ball B^d) directly in terms of a Euclidean partition function on S^3 (on S^{d+1}). This result is also proven in Ref. 22. The proof proceeds as in the previous case by a judicious choice of coordinate transformation and conformal transformation which maps the causal development of the disk into a spacetime, de Sitter space, whose Euclidean section is a sphere. Since $\ln Z$ gives the free energy, and since we will be interested in topological phases where the Hamiltonian is zero, the free energy directly determines the entropy.

To summarize, we introduced three powerful tools that provide access to the reduced density matrix of special subsystems. First, the entanglement Hamiltonian of a half space in any relativistic field theory is given by a certain boost generator. Second, the entanglement Hamiltonian of a ball in any conformal field theory is given by the generator of time translations in hyperbolic space. Third, the entanglement entropy of a ball in any conformal field theory is given by the partition function of the Euclidean theory on S^{d+1} .

The simplest calculation that illustrates the use of these tools is a computation of the entanglement entropy of a disk in topological liquids in $2 + 1$ dimensions. As we already noted, topological field theories are conformal field theories since they insensitive to the metric. The inevitable regulator will introduce nonconformal elements to any topological theory, but these elements will only alter nonuniversal aspects of our results. To compute $Z(S^3)$ we begin with $Z(S^2 \times S^1) = 1$ which indicates a unique ground state on the sphere [in general $Z(\Sigma \times S^1)$ computes the dimension of the Hilbert space of the theory on space Σ]. We want to use surgery on the manifold $S^2 \times S^1$ to convert it into a three sphere S^3 . A similar surgery computation was performed for Chern-Simons theories in Ref. 23, but our perspective is different since we are using results typically associated with conformal field theories and since our computation is for all topological theories.

The space $S^2 \times S^1$ may be cut open along the equator of S^2 to yield two copies of the solid torus $B^2 \times S^1$. Since $\partial B^2 \times S^1 = S^1 \times S^1$, $Z(B^2 \times S^1) = |\Psi\rangle$ is a state in the Hilbert space of the torus generated by imaginary time evolution. This state is normalized since $1 = Z(S^2 \times S^1) = \langle \Psi | \Psi \rangle$. Now instead of gluing the tori back together directly, we make an S modular transformation of one of the boundary tori which exchanges the two noncontractible surface loops. Gluing then yields S^3 as shown in Fig. 2. The modular transformation is

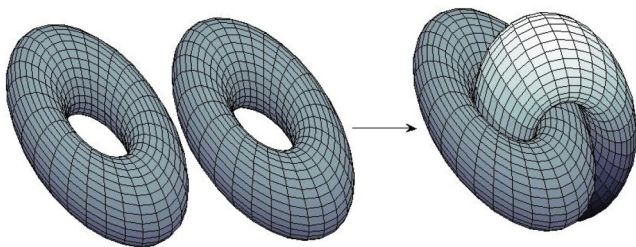


FIG. 2. (Color online) Visualizing the modular transformation that maps two copies of the solid torus $B^2 \times S^1$ to the pair of interlocking solid tori on the right. Expanding the interlocking tori to fill all of space and gluing them along their mutual boundary produces S^3 .

implemented using the modular S matrix S_b^a , and a calculation gives $Z(S^3) = \langle \Psi | \mathcal{S} | \Psi \rangle = S_0^0$ (see Ref. 29 for a review). The 0 or identity components appear because the imaginary time evolution that generated $|\Psi\rangle$ has no Wilson lines inserted. In terms of the total quantum dimension \mathcal{D} of the topological field theory we have $S_0^0 = 1/\mathcal{D}$. Thus $S(B^2) = \ln Z(S^3) = -\ln \mathcal{D}$ (the nonuniversal part has effectively been subtracted away) as shown in Refs. 5,6, and 24.

III. BULK-EDGE CORRESPONDENCE

Here we state our general result that establishes the bulk-edge correspondence. We focus on the case of the Rindler wedge where we have shown that the entanglement Hamiltonian is

$$K = \int_{x^1 > \epsilon} x^1 \mathcal{H}. \tag{17}$$

If we interpret this operator locally then it describes a gap that goes to infinity as we move away from the boundary. More generally, we know how to compute the spectrum of this operator by studying the theory in the Rindler wedge (with a cutoff). How does this operator lead to the bulk-edge correspondence?

Our argument is very simple: The boost operator is equivalent, up to operators localized at the edge, to a Hamiltonian with a sharp edge. Thus, if the Hamiltonian with a sharp edge has protected edge states then the boost generator must possess the same states since the bulk gap is not closed. Formally, let us introduce the operator

$$K_p = \int_{x^1 > \epsilon} p(x^1) \mathcal{H} \tag{18}$$

with $p(x) = \xi$ for $x < \xi$ and $p(x) = x$ for $x > \xi$. Now K_p differs from K only within a shell of thickness ξ near $x^1 = \epsilon$ and hence they possess the same universal edge features. K_p clearly has an edge state since it has a finite gap everywhere and a sharp edge at $x^1 = \epsilon$, so K must also possess the same protected edge states. Of course, it is crucial that we consider only truly protected edge states, i.e., those states that cannot be removed without closing the bulk gap or breaking a symmetry. Perturbative stability, that is a finite basin of attraction under the renormalization group, while quite interesting physically, cannot be used to guarantee the bulk-edge correspondence since we must entertain large perturbations of the edge in our proof.

Let us also address explicitly the role of Lorentz invariance. This is a powerful symmetry that underlies our entire geometric approach; on the other hand it is at best an approximate symmetry at low energies in condensed matter systems. For example, at the edge of a $\nu = 2/3$ fractional quantum Hall state there are two counterpropagating modes (corresponding to $k = 1$ and $k = -3$ in the Chern-Simons language), and these modes need not have identical velocities. However, the important point is that this system may be tuned to a point where the velocities are equal, or equivalently, that the generic case of interest in condensed matter differs from the Lorentz invariant system by operators that are marginal or irrelevant.

Formally speaking, we can consider an adiabatic transformation U that maps the gapped Lorentz invariant ground state $|\Omega\rangle$, say on a sphere, to the true non-Lorentz invariant ground state $|\text{true}\rangle$,

$$U|\Omega\rangle = |\text{true}\rangle. \quad (19)$$

Physically speaking, U adds all kinds of irrelevant operators to the Lorentz invariant model which break Lorentz invariance at high energies and short distances. However, the crucial point is that U is a local transformation because the ground state is gapped. U also provides a local map from the Lorentz invariant entanglement Hamiltonian H_R of region R to the true entanglement Hamiltonian H_R^U . That is $H_R^U = H_R + \Delta H_R$ with ΔH_R a local operator that does not close the bulk “entanglement gap” of H_R . Under these conditions it follows as before that the universal protected features of the edge spectrum of H_R are not perturbed by the addition of ΔH_R . For example, in the case of fractional quantum Hall liquids, the chiral central charge as measured by the physical thermal Hall effect is such a universal number. Indeed, it is roughly the number of left moving modes minus the number of right moving modes. Furthermore, the universality of this number does not depend on all modes having the same velocity and thus is robust to Lorentz symmetry breaking. It can be computed from our results since we know the full entanglement Hamiltonian—eigenstates and eigenvalues.

IV. EXAMPLES IN $d = 2$

We now apply the technology introduced in the previous section to the problem of the entanglement spectrum in fractional quantum Hall fluids. Consider the simplest class of topological fluids at filling fraction $\nu = 1/m$ as described by Laughlin in Ref. 25. These states are described at low energy by an effective Chern-Simons theory for an emergent gauge field that can be used to compute ground state degeneracy and quasiparticle statistics (see Ref. 26).

To access the disk spectrum we study the Chern-Simons theory on hyperbolic space. We must cut off the hyperbolic space at some fixed size and impose boundary conditions to regularize the path integral. Chern-Simons theory on a manifold with boundary is not gauge invariant unless we add extra edge degrees of freedom,²⁴ and because the original electron model was gauge invariant, the reduced density matrix must also be. The edge degrees of freedom are those of a chiral $c = 1$ gapless boson in $1 + 1$ dimensions, and because the bulk is still fully gapped, these edge modes dominate the thermal physics. Hence the “low energy” part of the entanglement spectrum is simply given by the thermal spectrum of the corresponding $1 + 1$ dimensional conformal field theory. Of course, we can similarly conclude that the entanglement spectrum of a half space in the Chern-Simons theory is given by the thermal spectrum of the same conformal field theory on an infinite line.

To be concrete, consider the case of the infinite line in more detail. Recall that we must quantize the theory on the Rindler wedge to obtain the entanglement spectrum. The action for $U(1)_k$ Chern-Simons theory on the Rindler wedge (using

coordinates $\{\eta, \rho, x^2\}$) is

$$\begin{aligned} S_{CS} &= \frac{k}{4\pi} \int ada \\ &= \frac{k}{4\pi} \int_{\rho>0} d\eta d\rho dx^2 (a_\rho \partial_\eta a_2 - a_2 \partial_\eta a_\rho), \end{aligned} \quad (20)$$

where in the last equality we have chosen the $a_\eta = 0$ gauge. Of course, because the Chern-Simons theory is topological this action on the Rindler wedge is identical to that of the Chern-Simons theory on a half plane. We emphasize that the analysis from here on is essentially identical to the standard argument for edge states on a disk, so we only briefly review it. Because of the boundary at $\rho = 0$ we must choose boundary conditions to ensure gauge invariance. Having chosen the gauge $a_\eta = 0$, the bulk equation of motion (no magnetic field) implies that $a_\rho = \partial_\rho f$ and $a_2 = \partial_2 f$; i.e., a is a gradient. Now in the bulk f represents a pure gauge degree of freedom; i.e., $f \rightarrow f + \Lambda$ under a gauge transformation in which $a_\mu \rightarrow a_\mu + \partial_\mu \Lambda$ with $\partial_\eta \Lambda = 0$. However, we must restrict $\Lambda(\rho = 0) = 0$ to have a gauge invariant action and hence the field $f(\eta, \rho = 0, x^2)$ is a physical degree of freedom. The action for f may be obtained from the bulk by performing the ρ integral to obtain

$$S_f = -\frac{k}{4\pi} \int d\eta dx^2 \partial_\eta f \partial_{x^2} f, \quad (21)$$

which is the edge theory of a gapless chiral boson with zero velocity. Adding a nonuniversal term of the form $v(\partial_{x^2} f)^2$ gives the mode a nonzero velocity v . Thus we have shown that gauge invariance requires the Rindler spectrum to contain states appropriate to a gapless chiral one-dimensional mode.

We wish to emphasize that the velocity of these modes is nonuniversal and depends on the regulator. It is formally zero in the topological limit just as the gap is formally infinite. Keeping a finite gap, for example a mass term for Dirac fermions generating the Chern-Simons term, spoils the conformal invariance. We also have to cut off the hyperbolic space at a size set (via the conformal transformation) by the physical cutoff. Recall that the half-space result does not require the topological limit, but a short distance cutoff is still required. With Lorentz invariance the edge speed in the half space must be $v = c$, but this speed is susceptible to nonuniversal corrections once we deform the symmetry. Most importantly, while the velocities and overall energy scales of the edge modes are not protected, they cannot actually be removed by any local perturbation (such as Lorentz breaking irrelevant perturbations).

We can also obtain the entanglement spectrum of integer Hall states in a simple manner by studying Dirac fermions ψ in Rindler space. To compute the spectrum of a massive Dirac fermion in $2 + 1$ dimensional Rindler space we must first introduce the vierbein e_μ^a which maps from local orthonormal coordinates labeled by a to the conventional coordinates labeled by μ . For the Rindler wedge (setting $x^2 = y$) we have

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 + dy^2, \quad (22)$$

which using $g_{\mu\nu} = e_\mu^a e_\nu^b g_{ab}^M$ (g^M is the Minkowski metric) gives $e_\eta^{\hat{\eta}} = \rho$, $e_\rho^{\hat{\rho}} = 1$, and $e_y^{\hat{y}} = 1$. The spin connection ω_b^a is defined via

$$de^a + \omega_b^a \wedge e^b = 0 \quad (23)$$

from which we find

$$d\rho d\eta + \omega_{\rho}^{\hat{\eta}} d\rho \quad (24)$$

or $\omega_{\rho}^{\hat{\eta}} = d\eta$. Using the spin connection along with the flat space gamma matrices γ^a and the Lorentz generators $\gamma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$ we can write the Dirac equation. The final equation is

$$\gamma^a e_a^{\mu} \left(\partial_{\mu} - \frac{i}{2} \omega_{\mu}^{ab} \gamma_{ab} \right) \psi - m\psi = 0. \quad (25)$$

To make further progress we introduce the explicit gamma matrices $\gamma^{\hat{\eta}} = iZ$, $\gamma^{\hat{\rho}} = X$, $\gamma^{\hat{y}} = Y$, where X, Y, Z are the Pauli matrices. Most of the spectrum is gapped due to the mass term, but we can look for a zero-mode solution satisfying $\partial_{\eta}\psi = \partial_y\psi = 0$. The Dirac equation reduces to

$$iZ\rho^{-1}[-i(iY/2)]\psi + X\partial_{\rho}\psi - m\psi = 0, \quad (26)$$

which using $ZY = -iX$ gives

$$(\rho^{-1} + \partial_{\rho})X\psi = m\psi. \quad (27)$$

Choosing $\psi = g\psi_0$ with $X\psi_0 = -\psi_0$ (assuming $m > 0$) we finally obtain

$$\partial_{\rho}g = (-m - \rho^{-1})g, \quad (28)$$

which has the solution

$$g(\rho) = \exp(-m\rho - \ln \rho). \quad (29)$$

Since the Rindler space is anyway cutoff at small ρ this is indeed a valid zero-mode solution of the Dirac equation. A sensible boundary condition is that the current through the cutoff be zero, i.e., $J^{\rho} = \bar{\psi}\gamma^{\rho}\psi = 0$ at the cutoff $\rho = \rho_c$, and our zero-mode solution satisfies this since $\bar{\psi} \propto \psi^+ iZ$ and $\psi^+ iZX\psi = 0$ when $X\psi = -\psi$. In fact, to have an integer Hall response we must have at least two Dirac fermions, and from these two zero modes we can build an edge mode that is actually regular at $\rho = 0$. Upon introducing a momentum in the y direction this zero mode becomes a chiral fermion with directionality determined by the sign of m [the sign enters when we choose $X\psi_0 = -\text{sgn}(m)\psi_0$ for normalizability at large ρ]. Thus we have demonstrated that the entanglement spectrum of the massive Dirac fermion (which is well known to have a Hall response) does indeed exhibit an edge state.

V. EXAMPLES IN $d = 3$

These techniques can also be applied to topological insulators in three dimensions. Consider the time reversal invariant Z_2 topological insulator in $3 + 1$ dimensions with effective theory given by $\theta\epsilon_{abcd}F^{ab}F^{cd}$ with $\theta = 0, \pi$. This is a simple field theory, but the θ term is topological, so we may apply our procedure. We want to determine the entanglement spectrum of a ball in this system, so we map it to the effective thermal problem in hyperbolic space. Because we are asking about the reduced density matrix deep inside the bulk of a time reversal invariant system, the only sensible boundary conditions are those that preserve time reversal.

We now appeal to the robustness of the $\theta = \pi$ insulator to argue that if we do not break time reversal at the surface in the cutoff 3d hyperbolic space, then there will be gapless surface

states. All our caveats about nonuniversal velocities still apply. These surface states will not be seriously perturbed by the fact that the system lives not in flat space but in hyperbolic space as long as the curvature is smooth and the bulk gap persists. Microscopic interactions can be freely included so long as the system remains in the same phase as characterized by the θ term. Thus the entanglement spectrum of a large ball in a Z_2 nontrivial insulator is necessarily gapless just like the spectrum of a physical edge.

Note that we can also easily solve the $3 + 1$ dimensional Dirac equation in hyperbolic space or in the Rindler wedge and depending on the sign of the mass (and a choice about the regulator) we find surface states in the entanglement spectrum provided the boundary condition respects time reversal. The question of regulator can be partly avoided by considering two Dirac fermions of opposite mass so that one fermion is always in the topologically nontrivial phase (recall that the sign of the mass can be changed by a γ^5 transformation which adds a $\theta = \pi$ term to the action due to the chiral anomaly). Indeed, a similar situation occurs in the $2 + 1$ dimensional case since we must always have an even number of Dirac cones to get an integer Hall response.

A word of caution is appropriate here, since in the presence of interactions we cannot rule out the possibility of a surface phase transition. Nevertheless, a version of the bulk-edge correspondence still applies: The entanglement spectrum will always have low-lying states either due to the gapless edge or due to spontaneous breaking of time reversal. Of course, whatever the surface physics, the θ term tells us that it must be gapless and must reproduce the $1/2$ Hall response when time reversal is broken at the surface. Lorentz invariance permits us to seriously constrain the theory, but we can only really argue for surface Dirac cones in a weakly coupled description. If we have further information about the field theory, as in the standard noninteracting Dirac fermion model of topological insulators in $3 + 1$ dimensions, then we can be quite precise about the half-space entanglement Hamiltonian.

A similar argument applies for the $3 + 1$ dimensional fractional topological insulators described in Refs. 27 and 28. These insulators have a θ term in their low-energy effective action with fractional θ/π as well as a topological BF term. In more familiar language, the low-energy theory is deconfined Z_n gauge theory in $3 + 1$ dimensions coupled to external electromagnetic fields via gapped fractionalized fermions that fill a topological band. Since these phases have protected edge states so long as time reversal is preserved, we will have gapless surface modes in the cutoff hyperbolic space or Rindler space. These edge modes will dominate the low-energy thermal spectrum and hence the universal part of the entanglement spectrum. The same caveats concerning the precise form of the edge modes applies here as well, but given a relativistic realization of the low-energy effective theory, such as the one described in Refs. 27–29, we can again be quite precise about the nature of the entanglement Hamiltonian.

VI. CONCLUSIONS

We have established the bulk-edge correspondence for a wide variety of topological quantum fluids. This correspondence relates the spectrum of the reduced density matrix of

a spatial subsystem in the bulk to the thermal spectrum of a physical edge. The entanglement cut becomes a physical cut. In addition to reproducing some old results in a unified framework, we have offered the first proof of the bulk-edge correspondence for fractional topological insulators in 3d. Although we have addressed a wide variety of systems, we believe that our technique has not been exhausted. As an exact nonperturbative relationship between the entanglement spectrum of simple subsystems and the thermal spectrum in simple spacetimes, this technique has much to offer the study of entanglement in quantum many-body systems.

It is possible that the mapping to hyperbolic space or Rindler space might be useful to numerically compute the entanglement spectrum of certain critical points. We would need to study a lattice model which realizes the continuous quantum phase transition of interest on a lattice that mimics

the appropriate geometry. We may be limited by our ability to numerically simulate such a model, but it should be possible for some models and would give direct access to the entanglement spectrum. Using these tools, many interesting entanglement properties of $O(N)$ critical points can be computed using a large N expansion as in Refs. 30 and 31. We end by noting that there are many other partitions of interest besides the spatial ones considered here, so there is still a great deal to understand about the structure of entanglement in quantum many-body systems.

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