Conformal field theory approach to Fermi liquids and other highly entangled states

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The Fermi surface may be usefully viewed as a collection of $(1+1)$-dimensional chiral conformal field theories. This approach permits straightforward calculation of many anomalous ground-state properties of the Fermi gas, including entanglement entropy and number fluctuations. The $(1+1)$-dimensional picture also generalizes to finite temperature and the presence of interactions. We argue that the low-energy entanglement structure of Fermi liquid theory is universal, depending only on the geometry of the interacting Fermi surface. We also describe three additional systems in $3+1$ dimensions where a similar mechanism leads to a violation of the boundary law for entanglement entropy.

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I. INTRODUCTION

Fermi liquid theory forms the core of our theory of metals, and many materials are well described by Fermi liquid theory over at least some portion of their phase diagram. The experimental prominence and simplicity of Fermi liquids give them permanent appeal, so the community has expended much effort understanding their underlying structure. The most physical way to understand the universality of Fermi liquids is in terms of renormalization-group approaches that involve scaling toward the Fermi surface.[1–3] These approaches have given us a physical picture of Fermi liquids as mostly attractive fixed points. To this mostly attractive fixed point, we must add an infinite set of marginal deformations labeled by Landau parameters and the marginally relevant BCS instability. The apparent stability of Fermi liquids has a dark side, however, as it has proved challenging to find simple systems exhibiting non-Fermi liquid behavior. Indeed, Fermi liquids are continually surprising us with their versatility.

The latest surprise comes from the study of many-body entanglement entropy. Entanglement entropy is an attempt to characterize the real-space entanglement properties of quantum ground states. It is defined as the von Neumann entropy of the reduced density matrix of a spatial subsystem of the full system. Most systems in $d > 1$ spatial dimensions satisfy a boundary law for the entanglement entropy of a spatial region,[4] but free fermions violate this boundary law with a logarithmic correction.[5–10] For a region of linear size L, the entanglement entropy of most known critical and noncritical systems is nonuniversal and scales as $S_L \sim L^{d-1}$.[4] However, the entanglement entropy for free fermions scales as $S_L \sim L^{d-1} \ln L$ with the Fermi momentum k_F making up the extra units where needed. Furthermore, there is a precise conjecture for the form of this term known as the Widom formula.[6] However, a full physical proof of the Widom formula and the extension to interacting fermions remain open questions.

The only other systems known to violate the boundary law are conformal field theories (or, more generally, scale invariant theories) in $d = 1$ spatial dimensions.[11] We have suggested that these two violations of the boundary law are related because the Fermi surface in any dimension may be regarded as a collection of $(1+1)$-dimensional chiral conformal field theories.[12] However, it should be understood that this correspondence, at least, in its simplest form, is only expected to hold in the low-energy limit. Below, we provide additional evidence for the patch formulation and the Widom formula by constructing a suitable generalization that can exactly compute the low-energy thermal entropy of a free Fermi gas. We also wish to emphasise that this $(1+1)$-dimensional construction is not an attempt to bosonize the Fermi surface in higher dimensions, although these points of view are related.[13,14] Indeed, the bosonization point of view has recently been pursued in Ref. 15, which agrees with our results. Note that a more complicated patching procedure has recently been employed to discuss a Fermi surface coupled to a gapless boson, but we do not address these issues here.[16,17]

To further support these ideas, we also give three other systems, all in $3+1$ dimensions, that violate the boundary law for entanglement. We argue that, in all these cases, the common mechanism is the presence of many gapless one-dimensional modes. The systems we consider include Weyl fermions in a magnetic field, a strong coupling holographic generalization of the Weyl fermion problem,[18] and dislocations in certain kinds of topological insulators.[19–25] These systems provide further evidence for the claim that the quasi-one-dimensional picture of anomalous entanglement is a ubiquitous mechanism, although we certainly do not claim that this is the only way to have highly entangled states. Our results in the holographic context also provide a useful clue into the nature of the strongly coupled system described by the holographic geometry, i.e., that it may be similar to a collection of fermion zero modes.

In this paper, we give a more complete formulation of a Fermi surface as a collection of $(1+1)$-dimensional chiral conformal field theories. We will first describe the basic setup for free fermions and will sketch the arguments leading to the anomalous entanglement entropy of the free Fermi gas. As examples of the formalism, we compute number fluctuations and heat capacity of a Fermi gas from the $(1+1)$-dimensional point of view. Next, we include interactions and argue that interacting Fermi liquids violate the boundary law for entanglement entropy in a universal way. The formalism is also applied to several other systems in three dimensions that are shown to violate the boundary law. We conclude with some comments about the relevance of these results to other systems and about future work.

II. CHIRAL FERMIONS ON THE FERMI SURFACE

We begin by reviewing the arguments of Ref. 12. Consider fermions on a lattice (much of what we say is independent of the details of the ultraviolet regulator). Given a generic band structure and filling fraction, a finite density of fermions will form a metallic state with a Fermi surface. For simplicity, let us assume that this Fermi surface is nearly spherical. The low-energy degrees of freedom are particle-hole fluctuations near the Fermi surface in momentum space. Note that quasiparticles exist only above the Fermi surface, whereas quasiholes exist only below. Both particles and holes move with the same group velocity set by the local Fermi velocity.

Each such patch on the Fermi surface is equivalent to a single gapless chiral fermion in $1+1$ dimensions, the dimensions being the radial direction and time. The patch is chiral because all the excitations have a common velocity. For illustrative purposes, let us specialize to the case of $d=2$ so that the Fermi surface is one dimensional. We can formalize these statements by saying that the low-energy effective action of the free Fermi gas is

$$\mathcal{S}_\psi = \int d\theta \int dk\, dt\, \psi_\theta^+(k,t)[i\partial_t - v_F k]\psi_\theta(k,t), \quad (1)$$

where θ labels the patch on the Fermi surface. Each operator $\psi_\theta(k,t)$ should be regarded as a free fermion in one spatial dimension, the local radial direction, as specified by θ. The basic approach will be to compute physical properties of the Fermi gas by appropriate sums over the $(1+1)$-dimensional degrees of freedom labeled by θ.

Starting from this free low-energy effective action, we can trace the effects of interactions using a renormalization-group flow, but we will continue to focus on the case of free fermions. Each patch, labeled by θ, is equivalent to the chiral half of a one-dimensional free relativistic fermion. The nonchiral relativistic fermion has a total central charge $c=2$ as a conformal field theory. This central charge is a sum of left and right moving pieces $c = c_L + c_R$ where, for the free relativistic fermion, we have $c_L = c_R = 1$. This assignment should not be confused with the Majorana fermion, which has central charges $c_L = c_R = 1/2$. The basic fact we use about such a one-dimensional conformal field theory is that the entanglement entropy on an interval of length L is given by $S = \frac{c_L+c_R}{6}\ln(L/\epsilon)$, where ϵ is an ultraviolet cutoff. We also use the generalization of this formula to finite temperature $T = 1/\beta$ given by

$$S = \frac{c_L + c_R}{6}\ln\left[\frac{\beta v}{\pi\epsilon}\sinh\left(\frac{\pi L}{\beta v}\right)\right], \quad (2)$$

where v is the characteristic velocity in the conformal field theory.[11,26] Later, v will be identified with the local renormalized Fermi velocity v_F. This formula represents a crossover function between thermal and entanglement entropy, and we generalize this result to Fermi liquids in any dimension.

The formulation sketched above gives an intuitive understanding of the anomalous entanglement properties of the Fermi surface. If we ask about the entanglement entropy in a region A of linear size L, then, we naturally coarse grain the Fermi surface into coarse patches of typical size $1/L^{d-1}$ for a total of $(k_F L)^{d-1}$ coarse patches. Each patch contributes roughly $\ln L$ as is appropriate for a one-dimensional conformal field theory, and the total entanglement entropy scales as $L^{d-1}\ln L$ as observed.[5–9] The Widom formula for the entanglement entropy is obtained from a more precise counting of patches.[12] The fundamental interpretation of this counting procedure and the Widom formula is in terms of an effective central charge coming from the amalgamation of many $(1+1)$-dimensional degrees of freedom.[12] We will now use similar counting arguments to compute a number of observables for free fermions.

In addition to anomalous entanglement entropy, the free Fermi gas has anomalously large fluctuations of some conserved quantities. Many systems in $d>1$ spatial dimensions satisfy a boundary law for ground-state fluctuations of various physical quantities. One notable exception to this intuition occurs in symmetry-broken phases. There, the fluctuations of conserved quantities scale with the volume of the subregion. As an example, let us consider the number operator N_A, corresponding to the number of fermions in region A. Typically, one would expect to find $\langle (N_A - \langle N_A\rangle)^2\rangle \sim L^{d-1}$ where L is the linear size of region A. This boundary law for fluctuations in the ground state appears to be relatively general, however, it is violated in the case of free fermions.[6,10] Just like their entanglement properties, free fermions have anomalously large number fluctuations scaling, such as $L^{d-1}\ln L$. It is natural to ask if we can account for these fluctuations by viewing the Fermi surface as a collection of chiral one-dimensional conformal field theories.

The leading logarithmically corrected boundary law behavior can again be traced to the presence of numerous gapless modes at the Fermi surface. Consider first the problem of computing number fluctuations in a one-dimensional gas of free nonrelativistic fermions at finite density. We wish to calculate the fluctuations $\Delta N_A^2 = \langle (N_A - \langle N_A\rangle)^2\rangle$ in the fermionic ground state with A an interval of length L. By writing the operator N_A as a restricted integral over the fermion density operator, the calculation can be reduced to an integral using Wick's theorem. To leading order in L, the fluctuations scale as $\Delta N_A^2 \sim \frac{1}{\pi^2}\ln(L/\epsilon)$. Note that this result generalizes for Luttinger liquids in one dimension.[27] There are two Fermi points or patches, and each Fermi point contributes $\frac{1}{2}\frac{1}{\pi^2}\ln L$ to the answer. Returning to $d>1$ dimensions, the number fluctuations can again be written in integral form using Wick's theorem. However, the analysis of the integral is considerably more complex.[10,28] Instead, we can obtain the exact expression for the asymptotic behavior of the number fluctuations indirectly using the one-dimensional picture.

To perform the mode counting, let us return for a moment to the entanglement entropy. We choose a spatial region A of linear size L and ask about the entanglement entropy of this region. The Widom formula takes the form of an integral over the boundary of A and the Fermi surface,

$$S = \frac{1}{(2\pi)^{d_s-1}}\frac{\ln L}{12}\int_k\int_x dA_k dA_x |n_x \cdot n_k|, \quad (3)$$

where n_x and n_k stand for unit normals to the boundary of A and the Fermi surface, respectively. The detailed choice of linear size L in the logarithm is immaterial in this formula since it only corrects the ultraviolet-sensitive boundary law piece of the entanglement entropy. Once more, this formula should be interpreted as counting the effective number of chiral one-dimensional modes contributing ln L to the entropy. Each patch has $c_L = 1$ and $c_R = 0$ where left and right movers are defined by the local radial direction. Thus, each patch should contribute $\frac{c_L + c_R}{6} \ln L = \frac{1}{6} \ln L$ to the entropy. The Widom formula is essentially this entropy per patch times the number of such patches,

$$N_{\text{modes}} = \frac{1}{(2\pi)^{d-1}} \frac{1}{2} \int_k \int_x dA_k dA_x |n_x \cdot n_k|, \tag{4}$$

where the $|n_x \cdot n_k|$ factor arises from computing the projected area when performing the mode counting.[12]

Returning to the number fluctuations, each patch contributes $\frac{1}{2}\frac{1}{\pi^2} \ln L$ following the one-dimensional result, and the asymptotic form of the number fluctuations for a region A in higher dimensions is, thus,

$$\Delta N_A^2 = \frac{1}{(2\pi)^{d-1}} \frac{\ln L}{4\pi^2} \int_k \int_x dA_k dA_x |n_x \cdot n_k|. \tag{5}$$

Precisely this formula has been obtained previously by a lengthier and rigorous analysis (note that some confusion can arise in the comparison because of different bases for the logarithms).[6,10]

III. GENERALIZATION TO FINITE TEMPERATURE AND INTERACTIONS

So far, we have worked at zero temperature and without interactions, but these restrictions are not essential. For example, the heuristic picture of the Fermi surface as a collection of one-dimensional chiral patches gives roughly the correct finite temperature entropy for a region A of linear size L. Each patch contributes roughly LT to the entropy, and the number of patches is proportional $(k_F L)^{d-1}$ giving roughly the correct entropy for free fermions. There is a reason to be skeptical, however. The details of the patch counting depend on the boundary of region A, but any such dependence should disappear in thermodynamic quantities. Fortunately, the physically correct choice of linear size L naturally cancels this detailed boundary dependence.

Returning to the case of $d = 2$, let us assume that the Fermi surface is a circle of radius k_F. The real-space region A is taken to be a circle of radius R. Consider a specific patch on the Fermi surface, and choose coordinates so that this patch has velocity equal to $v_F \hat{x}$ as in Fig. 1. To compute the integral over the boundary of region A, we must specify what the effective linear size L is. We parametrize the boundary of A by an angle θ defined relative to the x axis. For a mode with velocity $v_F \hat{x}$ and a point on the boundary of A labeled by θ, the physically correct linear size is $L_{\text{eff}} = 2L|\cos\theta|$, that is, the chordal distance across the circle parallel to the x axis as in Fig. 1. The high-temperature limit of Eq. (2) with $L = L_{\text{eff}}$ is

$$S_{1+1} = \frac{c_L + c_R}{6} \frac{\pi L_{\text{eff}}}{\beta v_F}. \tag{6}$$

We use this expression for the entropy contribution of each patch to compute the entropy of the entire Fermi surface,

$$S = \frac{1}{2\pi} \frac{1}{2} \int_k \int_x dA_k dA_x |n_x \cdot n_k| \frac{1}{6} \frac{\pi L_{\text{eff}}}{\beta v_F}. \tag{7}$$

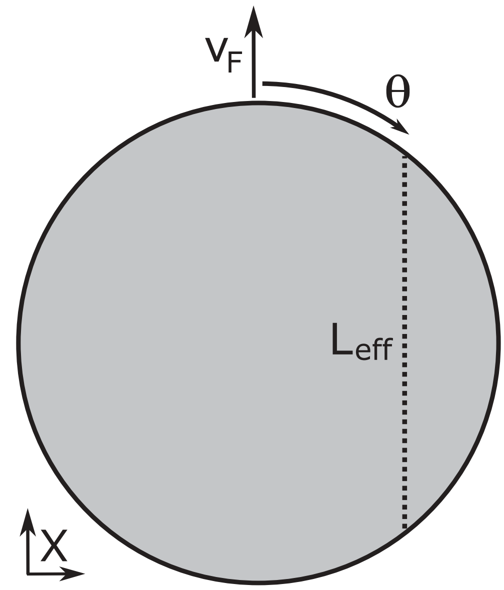

FIG. 1. A sketch of a circular real-space region A of radius L. The Fermi velocity of a particular patch on the Fermi surface is shown superimposed on the circle. The effective length $L_{\text{eff}} = 2L\cos\theta$, which is just the chordal distance across the circle parallel to the Fermi velocity, is a function of angle θ relative to v_F.

The integral over the real-space boundary can be carried out as described above. What remains is an integral of $1/v_F$ over the Fermi surface, a familiar result giving the density of states. The final result is

$$S = \frac{\pi}{6} m T (\pi L^2), \tag{8}$$

where m is the fermion mass and πL^2 is the area of region A. This is nothing but the usual thermal entropy of a gas of free fermions in a box of area πL^2 to leading order in T/T_F. A similar calculation gives the correct finite temperature number fluctuations to leading order in T/T_F. The fact that the Widom formula and the patch counting it embodies can be used to compute exact low-temperature thermal properties is strong evidence in its favor.

It is also possible to include interactions. In the renormalization-group treatment of Fermi liquids, most interactions are irrelevant. The exceptions are forward-scattering interactions, but these interactions do not drastically modify the $(1+1)$-dimensional picture. The low-energy effective action Eq. (1) has an emergent $U(1)^\infty$ symmetry, corresponding to number conservation on each patch, and this symmetry survives in the low-energy limit of Fermi liquids because only forward-scattering remains.[13,14] Sticking with the simplest possible situation, let us assume that approximate rotational symmetry is preserved as we turn on interactions. With the assumption of rotational symmetry, Luttinger's theorem that the area (in $d = 2$ dimensions) enclosed by the Fermi surface remains constant implies that k_F is not renormalized by interactions. Indeed, the content of Fermi liquid theory is that the effects of interactions may be subsumed entirely in terms of a renormalized Fermi velocity and a set of Landau parameters.

To see that interactions can be automatically included, let us compute the heat capacity of a Fermi liquid. The standard result of Fermi liquid theory is that the heat capacity depends on interactions only through the renormalized mass (assuming k_F remains at its free value). Remarkably, we can describe the thermodynamics of an interacting Fermi liquid, say, in $d = 2$, using Eq. (7) so long as we replace the bare Fermi velocity by the physical renormalized Fermi velocity. In particular, the mode counting remains unchanged despite the presence of interactions. Given that we can reproduce thermodynamics exactly using the patch construction, we have a strong hint that the interacting Fermi liquid does indeed have an entanglement entropy described by the Widom formula. Dropping the assumption of spherical symmetry, the formalism predicts that the entanglement entropy is still highly universal, it is totally insensitive to the Landau parameters, for example, and depends only on the shape of the interacting Fermi surface.

IV. OTHER HIGHLY ENTANGLED STATES

A. Weyl fermion in a magnetic field

Consider a single Weyl fermion ψ charged under a gauge field A with charge q in $3+1$ dimensions. The equation of motion for this fermion is $\gamma^\mu D_\mu \psi = 0$ with $\gamma^5\psi = -\psi$ where $D_\mu = \partial_\mu - iqA_\mu$ is the covariant derivative. Let there be a finite magnetic field, say, in the z direction: $F_{12} = \partial_1 A_2 - \partial_2 A_1 = B$. The magnetic field defines a length scale called the magnetic length $\ell_B^2 = 1/B$ (the units are made up by the flux quantum). On length scales much less than ℓ_B, the theory looks like a $(3+1)$-dimensional conformal field theory. On length scales much bigger than ℓ_B, the theory becomes effectively $1+1$ dimensional. Indeed, the Weyl fermion is special because it possesses zero modes that avoid being gapped by the magnetic field.

We take the γ matrices to satisfy $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta^{\mu\nu}$ mostly minus. The chiral γ matrix is defined to be $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and I work in the chiral basis where Dirac spinors decompose as $\psi^T = (\psi_L \psi_R)^T$ with

$$\gamma^5 = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix}. \tag{9}$$

The Weyl equation for a left-handed spinor is

$$(i\partial_t - i\sigma^i D_i)\psi_L = 0, \tag{10}$$

with σ^i as the usual Pauli matrices. The vector potential in Landau gauge is $A_y = Bx$ for a constant magnetic field B in the z direction. Most solutions of the Weyl equation in a finite magnetic field have a gap coming from the cyclotron motion, but there are also zero-mode solutions. These solutions may be heuristically understood as arising from a balance between the Zeeman energy and the orbital cyclotron energy.

Zero-mode solutions may be found by setting $\partial_t\psi_L = \partial_z\psi_L = 0$ to obtain

$$\sigma^x\partial_x\psi_L + \sigma^y(\partial_y - iqBx)\psi_L = 0. \tag{11}$$

Landau gauge maintains translation invariance in the y direction, so we try a solution of the form $\psi_L(x,y) = \psi_L(x)e^{iky}$. The Weyl equation reduces to

$$\partial_x\psi_L = -\sigma^z(qBx - k)\psi_L, \tag{12}$$

with solution,

$$\psi_L(x) = \exp\left[-\frac{qB}{2}\left(x - \frac{k}{qB}\right)^2\sigma^z\right]\psi_L(0). \tag{13}$$

In order for this solution to be normalizable, we must have $\sigma^z\psi_L(0) = \psi_L(0)$ (assuming $qB > 0$) leaving only 1 degree of freedom. The spacing of k is determined by the length of the system in the y direction to be $\Delta k = \frac{2\pi}{L_y}$. We have one zero for each value of k such that $\psi_L(x)$ sits inside the system in the x direction. The degeneracy g of zero modes is, thus, $g = \frac{qBL_x}{\Delta k} = \frac{qBL_xL_y}{2\pi}$. More generally, these zero modes and their degeneracies are protected by an index theorem relating the number of zero modes to the magnetic flux penetrating the system: $N_{\text{zero modes}} = \frac{q}{2\pi}\int F_{12}\,dx\,dy$.

So far, we have ignored the z direction, but these zero modes actually disperse in the z direction. Assuming a more general solution of the form $\psi_L(x,y,z,t) = e^{ip_z z + ip_y y - iEt}\psi_L(x)$, the full Weyl equation becomes

$$E\psi_L - p_z\sigma^z\psi_L - i[\sigma^x\partial_x + \sigma^y(\partial_y - iqBx)]\psi_L = 0. \tag{14}$$

The second half of this equation is solved with the same zero-mode profile as above. The first half reduces to the equation $E = p_z$ using the fact that $\sigma^z\psi_L = \psi_L$ follows from the normalization condition. Thus, each zero mode is actually a relativistic chiral fermion in one spatial dimension. The low-energy physics is controlled entirely by these zero modes as all other modes are gapped by the cyclotron motion.

Using the one-dimensional structure, we can compute the entanglement entropy of the Weyl fermion. Consider a box of linear size L. The entanglement entropy S_L is defined as the von Neumann entropy of the reduced density matrix corresponding to the box: $S_L = -\text{Tr}(\rho_L \ln \rho_L)$. For one-dimensional conformal field theories, the entanglement entropy is known to have the form

$$S_L = \frac{c_L + c_R}{6}\ln\left(\frac{L}{\epsilon}\right), \tag{15}$$

where c_L and c_R are the left and right central charges and ϵ is an ultraviolet cutoff.[11] Weyl fermions in a magnetic field may be described by a large number of one-dimensional gapless modes, and these modes are each equivalent to a chiral $(1+1)$-dimensional conformal field theory, the dimensions being z and t. Each chiral fermion mode has $c_L = 1$ and $c_R = 0$ and, hence, contributes $(1/6)\ln L$ to the entanglement entropy. For a cube of side length L aligned with the z direction, we have $qBL^2/(2\pi)$ zero modes for a total entanglement entropy,

$$S_L = \left(\frac{qBL^2}{2\pi}\right)\frac{1}{6}\ln\left(\frac{L}{\epsilon}\right). \tag{16}$$

This formula may be checked using the generalization of one-dimensional entanglement entropy to finite temperature,

$$S_L = \frac{c_L + c_R}{6}\ln\left(\frac{\beta}{\pi\epsilon}\sinh\frac{\pi L}{\beta}\right). \tag{17}$$

Note that, unlike the case of the Fermi surface, here, all the modes point in the same direction, so the resulting integral over one-dimensional modes is trivial. The thermal entropy of these zero modes in a cube of size L is, thus,

$$S = \left(\frac{qBL^2}{2\pi}\right)\frac{\pi LT}{6}, \tag{18}$$

which agrees with the direct thermodynamic calculation.

Before moving on, let me note that a single charged Weyl fermion does not give a consistent quantum theory. This is due to the presence of a gauge anomaly proportional to $\mathrm{Tr}(Q^3)$ where Q is the charge matrix. This anomaly must vanish for a completely well-defined chiral gauge theory, but this can be accomplished by adding Weyl fermions with compensating charges. The boundary law violating behavior remains, and thus, there are consistent configurations of Weyl fermions that violate the boundary law for entanglement entropy.

B. Holographic generalization

We have computed the entanglement entropy for a single free Weyl fermion and found a term that violates the boundary law for entanglement entropy. A useful choice for incorporating interactions is $\mathcal{N}=4$ SU(N) Yang-Mills theory, which includes $4N^2$ Weyl fermions as part of the field content. These fermions sit in the adjoint of the non-Abelian gauge group SU(N), whereas the magnetic field B corresponds to a weakly gauged $U(1)$ subgroup of the R symmetry. In zero magnetic field, this theory is conformal at all values of the t'Hooft coupling $\lambda = g_{YM}^2 N$, but it is particularly amenable to study at strong coupling because of holographic duality. This duality relates the $\mathcal{N}=4$ theory to a theory of quantum gravity, type IIB string theory, in an asymptotically five-dimensional anti-de Sitter space-time (AdS_5). The limits $\lambda \to \infty$ and $N \to \infty$ in the field theory give classical supergravity in anti-de Sitter space on the gravity side.

In the strong coupling and large N limits, configurations of the super-Yang-Mills theory have an emergent geometric interpretation in terms of classical gravitational field configurations. The ground state of the field theory is dual to pure anti-de Sitter space, and the field theory at finite temperature is accessed via a bulk black hole. The field theory in a background magnetic field at zero temperature is obtained from a magnetically charged extremal black hole in the bulk. Given the bulk geometric configuration, the leading large N contribution to the entanglement entropy can be determined holographically by computing the area of certain minimal surfaces in the bulk.[29,30]

Consider extremal magnetic brane solutions in Einstein-Maxwell theory with negative cosmological constants in five dimensions.[18] These solutions interpolate between an asymptotically AdS_5 region and a near horizon $AdS_3 \times T^2$ region (assuming the xy plane is compactified). The asymptotic AdS_5 region corresponds to the unperturbed $\mathcal{N}=4$ theory at high energies. The near-horizon region appears as a result of turning on a magnetic field in the gauge theory. The radial evolution represents a renormalization-group flow from a $(3+1)$-dimensional conformal field theory at high energies to an effectively $(1+1)$-dimensional conformal field theory at low energies. This is qualitatively similar to the physics of free Weyl fermions, and even at strong coupling, the crossover scale is determined by the magnetic length. At zero temperature, the metric may be written in the form

$$ds^2 = -U(r)\,dt^2 + \frac{dr^2}{U(r)} + U(r)\,dz^2 + e^{2V}(dx^2+dy^2), \tag{19}$$

with r as the radial coordinate ($r \to \infty$ is the boundary) and z as the direction of the magnetic field on the boundary.[18] We use bulk units with the AdS radius set to 1. In addition to the metric, the gauge field has a profile given by $F = B\,dx \wedge dy$. The asymptotic AdS_5 region is described by $U = e^{2V} = r^2$, whereas the near horizon $AdS_3 \times T^2$ region corresponds to $U = 3r^2$ and $e^{2V} = B/3$. Notice that, in the near-horizon region, the xy plane has decoupled from the radial coordinate and has a fixed size given by the magnetic length.

The entanglement entropy of a region in the dual field theory is determined by the area of the minimal surface in the bulk that terminates on the boundary of the region in the field theory. The entanglement entropy is just this minimal area divided by $4G_N^{(5)}$. We will focus on the entanglement entropy of a rectangular region in boundary theory of size $L \times L \times L_z$. Assuming $L \gg L_z$ gives approximate translation invariance in the xy plane. The minimal surface calculation reduces to a two-dimensional problem involving only the variables z and r. The zero-temperature geometry is only known numerically, and the minimal surface calculation can also only be performed numerically. However, the important physics can be extracted without the numerical details. For cubic regions with all dimensions less than the magnetic length, the minimal surface only probes the AdS_5 region and gives the usual ultraviolet divergent boundary law for entanglement entropy.

For boundary regions of linear size much larger than the magnetic length, the minimal surface passes right through the asymptotic AdS_5 region toward the near-horizon region. Once in the near-horizon region, the x and y directions freeze out, and the minimal surface behaves exactly as in AdS_3. In particular, we find the characteristic $\ln(L_z/\ell)$ dependence familiar from $(1+1)$-dimensional conformal field theory with the magnetic length providing the cutoff. The entanglement entropy, thus, consists of two pieces, a nonuniversal boundary law contribution from the asymptotically AdS_5 region and a universal low-energy piece $S_L \sim N^2 B L^2 \ln(L_z/\ell_B)$. The appearance of the magnetic field can be understood because the effective $(1+1)$-dimensional central charge is related to $1/G_N^{(3)}$, which is enhanced relative to $1/G_N^{(5)}$ by a factor of BL^2 from the freeze-out of the xy plane. This strong-coupling version of the free Weyl fermion system, thus, also violates the boundary law for entanglement entropy at low energies.

C. Topological insulators

In both cases considered above, the appearance of many gapless one-dimensional modes was responsible for the highly entangled nature of the quantum state. This intuition can be applied to more experimentally relevant systems known as topological insulators. These systems are time-reversal invariant electronic band insulators that are not smoothly connected to trivial band insulators. In particular, they possess interesting topological structures that give rise to protected edge modes. These edge modes are robust so long as time-reversal invariance is preserved.[19,21,25] Topological insulators in three spatial dimensions have gapless surface states living in two spatial dimensions, but these modes do not lead to a violation of the boundary law for entanglement entropy. Similarly, the bulk of a topological insulator is gapped in a perfect crystal and certainly satisfies a boundary law for entanglement entropy.

However, experimentally realized topological insulators are not perfect crystals; they possess topological defects, including dislocations in the crystalline bulk. Remarkably, for certain kinds of topological insulators and dislocation types, the dislocations have been shown to support gapless fermionic modes.[31] These effectively one-dimensional modes make the dislocations into gapless quantum wires threading the otherwise gapped bulk. The one-dimensional modes in the quantum wires are analogous to the Weyl zero modes considered above with the dislocations playing the role of magnetic field lines. In the presence of a finite density of dislocations supporting gapless modes, the bulk of a strong topological insulator violates the boundary law for entanglement entropy.

To estimate the size of the violation, consider the artificial situation of a dilute array of topologically nontrivial dislocations all aligned. Let these dislocations have an areal density ρ [a typical value of ρ might be 10^{12} m^{-2} (Ref. 31)]. A region in the bulk of size $L \times L \times L_z$ with the z axis chosen parallel to the dislocations effectively contains ρL^2 gapless one-dimensional fermionic modes. These modes should each contribute roughly $\ln(L_z/\epsilon)$ to the entanglement entropy. The boundary law violating component of the entanglement entropy is, thus, on the order of $S_L \sim \rho L^2 \ln(L_z/\epsilon)$. This estimate is crude, but it should suffice for a reasonably uniform and collimated set of dislocations. Note that, despite the enhanced L dependence relative to the usual boundary law, this term may be much smaller than the boundary law term for experimentally accessible system sizes and dislocation densities. We also wish to emphasize that this is a statement about the zero-temperature quantum state. The helical modes are protected from elastic scattering (such scattering might otherwise localize a one-dimensional gapless mode), but at finite temperature or in the presence of inelastic processes, the boundary law violating behavior will be disrupted.

V. DISCUSSION

We have elaborated on a view of the Fermi surface as a collection of $(1+1)$-dimensional chiral conformal field theories. We focused mostly on circular real-space regions and Fermi surfaces in $d = 2$ spatial dimensions, but the generalization for other geometries and higher dimensions is straightforward. So long as region A is convex, the proper effective length is physically well defined. To define this length, draw a line from a fixed point on the spatial boundary such that the line is parallel to the Fermi velocity of a fixed point on the Fermi surface. For convex A, this line will intersect the boundary of A in one other place. The length of the line segment between these two intersections is $L_{\rm eff}$ for the chosen points. Using this definition, the thermal entropy calculation can be shown to produce the volume of region A irrespective of the particular shape chosen. The ultimate message is that the anomalous entanglement entropy depends only on the geometry of the interacting Fermi surface in any dimension. This result is part of a growing body of work that strongly links entanglement and geometry in many-body physics.

The present formulation was originally motivated by renormalization-group treatments of the Fermi liquid and by attempts to understand intuitively the origin of the anomalous entanglement properties of fermions. However, it seems to be giving us much more. It can handle both finite temperature and some kinds of interactions, and it provides strong evidence both for the still unproven Widom formula for free fermions and for its extension to interacting fermions. On the practical side, it provides a simple and unified formalism for many calculations that previously appeared quite complex in the free Fermi gas.

We have also described three other systems special to three dimensions that exhibit anomalous entanglement properties following from a plethora of gapless one-dimensional modes. One of systems was a holographic strong-coupling generalization of the Weyl fermion problem, which provides evidence that the picture developed here does not break down at strong coupling. In all these cases, the patch argument was essentially unnecessary because the systems were already strongly one dimensional due either to a magnetic field or because they were hosted on one-dimensional defects.

These results also support the prediction in Ref. 12 that other exotic phases of matter, including non-Fermi liquid metals,[32] spin liquids with a spinon Fermi surface, holographic non-Fermi liquids,[33] and Bose metals[34] violate the boundary law for entanglement entropy. Numerical evidence supporting this prediction has recently been provided in Ref. 35. To truly extend the patch argument given here to these systems, one must account for nontrivial coupling between patches, such as that provided by a gapless boson, but we leave this to future papers. It is also likely that these exotic phases have anomalously large fluctuations analogous to the fermion number fluctuations in a Fermi liquid, although these statements are more model dependent. One hope is that anomalous properties, such as entanglement entropy and charge fluctuations may provide useful numerical and experimental handles for identifying such exotic phases.[36]

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[1] R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).

[2] J. Polchinski, arXiv:hep-th/9210046.

[3] G. Benfatto and G. Gallavotti, J. Stat. Phys. **59**, 541 (1990).

[4] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010).

[5]M. M. Wolf, Phys. Rev. Lett. **96**, 010404 (2006).
[6]D. Gioev and I. Klich, Phys. Rev. Lett. **96**, 100503 (2006).
[7]T. Barthel, M.-C. Chung, and U. Schollwock, Phys. Rev. A **74**, 022329 (2006).
[8]L. Ding, N. Bray-Ali, R. Yu, and S. Haas, Phys. Rev. Lett. **100**, 215701 (2008).
[9]W. Li, L. Ding, R. Yu, T. Roscilde, and S. Haas, Phys. Rev. B **74**, 073103 (2006).
[10]R. Helling, H. Leschke, and W. Spitzer, Int. Math. Res. Not. (2011) 1451.
[11]P. Calabrese and J. Cardy, J. Stat. Mech.: Theory Exp. (2004) P06002.
[12]B. Swingle, Phys. Rev. Lett. **105**, 050502 (2010).
[13]A. Luther, Phys. Rev. B **19**, 320 (1979).
[14]F. D. M. Haldane, in *Proceedings of the International School of Physics "Enrico Fermi Course CXXI "Perspectives in Many-Particle Physics, Varenna, Italy, 1992*, edited by R. A. Broglia and J. R. Schrieffer (North-Holland, Amsterdam, 1994), pp. 5–29.
[15]W. Ding, A. Seidel, and K. Yang, Phys. Rev. X **2**, 011012 (2012).
[16]S.-S. Lee, Phys. Rev. B **80**, 165102 (2009).
[17]M. A. Metlitski and S. Sachdev, Phys. Rev. B **82**, 075127 (2010).
[18]E. D'Hoker and P. Kraus, J. High Energy Phys. 10 (2009) 088.
[19]C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
[20]B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).
[21]L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007).
[22]J. E. Moore and L. Balents, Phys. Rev. B **75**, 121306(R) (2007).
[23]R. Roy, Phys. Rev. B **79**, 195322 (2009).
[24]B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science **314**, 1757 (2006).
[25]C. Xu and J. E. Moore, Phys. Rev. B **73**, 045322 (2006).
[26]V. E. Korepin, Phys. Rev. Lett. **92**, 096402 (2004).
[27]H. F. Song, S. Rachel, and K. Le Hur, Phys. Rev. B **82**, 012405 (2010).
[28]D. Gioev, Int. Math. Res. Not. (2006) 095181.
[29]S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006).
[30]D. V. Fursaev, J. High Energy Phys. 09 (2006) 018.
[31]Y. Ran, Y. Zhang, and A. Vishwanath, Nature Phys. **5**, 298 (2009).
[32]T. Senthil, Phys. Rev. B **78**, 035103 (2008).
[33]H. Liu, J. McGreevy, and D. Vegh, Phys. Rev. D **83**, 065029 (2011).
[34]O. I. Motrunich and M. P. A. Fisher, Phys. Rev. B **75**, 235116 (2007).
[35]Y. Zhang, T. Grover, and A. Vishwanath, Phys. Rev. Lett. **107**, 067202 (2011).
[36]I. Klich and L. Levitov, Phys. Rev. Lett. **102**, 100502 (2009).