Hall effect in superconducting films

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Near the superconducting phase transition, fluctuations significantly modify the electronic transport properties. Here we study the fluctuation corrections to the Hall conductivity in disordered films, extending previous derivations to a broader range of temperatures and magnetic fields, including the vicinity of the magnetic field induced quantum critical point. In the process, we found a new contribution to the Hall conductivity that was not considered before. Recently, our theory has been used to fit measurements of the Hall resistance in amorphous TaN films.

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Measurements of the Hall effect in classically weak magnetic fields provide useful information about the density of the current carriers as well as the sign of their charge. According to the Drude formulas, the ratio between the Hall (σ_{xy}) and longitudinal (σ_{xx}) conductivities is $\omega_c\tau$, where $\omega_c = |eH/m^*c|$ is the cyclotron frequency of the quasiparticles (electrons or holes) and τ is the elastic scattering time. The appearance of the cyclotron frequency in the expression for σ_{xy} manifests the fact that, for the Hall effect to be finite, particle-hole asymmetry is required. As is well known, within the Drude model the Hall coefficient is independent of τ and ω_c, and is only a function of the charge carrier density n, $R_H \equiv \rho_{xy}/H = 1/nec$. Weak localization corrections arising due to the interference effects, although modifying both σ_{xy} and σ_{xx}, leave R_H unchanged. In contrast, electron-electron interactions affect the transverse and longitudinal components of the conductivity tensor in a way that violates the delicate balance between them and, therefore, R_H is no longer universal. In particular, a significant change in the Hall coefficient occurs near the superconducting transition as a result of the fluctuations induced by electron-electron interaction in the Cooper channel. As we show here, the corrections to the Hall conductivity due to superconducting fluctuations diverge more strongly than the longitudinal ones. Furthermore, the particle-hole asymmetry factor $\omega_c\tau$ is multiplied by $\varsigma\mu$, which makes it parametrically larger. The parameter ς is proportional to the derivative of the density of states with respect to the energy at the chemical potential μ. The only other transport property that is sensitive to this quantity is the thermoelectric coefficient.[1]

Close to the superconducting phase transition, yet in the normal metallic phase, the fluctuations of the superconducting order parameter form a new branch of collective excitations. Since these excitations are charged, they create a new channel for the electric current. As a result, the electric conductivity is determined not only by the single-particle excitations (quasiparticles), but also by the current carried by the fluctuations. The direct contribution of the superconducting fluctuations to the longitudinal electric conductivity is described by the Aslamazov-Larkin term.[2] In the vicinity of the transition, this contribution can be interpreted as the Drude conductivity of the fluctuating Cooper pairs. Besides, the fluctuations affect strongly the quasiparticles, and thereby influence the conductivity. The scattering of the current-carrying quasiparticles by the superconducting fluctuations is described by the Maki-Thompson term.[3,4] Another effect can be attributed to the modification of the quasiparticle density of states by the long-living superconducting fluctuations.[5]

Similar to the Hall conductivity of free electrons, the corrections to σ_{xy} generated by the superconducting fluctuations vanish in the absence of particle-hole asymmetry. To demonstrate the dependence of the conductivity on the particle-hole asymmetry, we shall use the Aslamazov-Larkin corrections as an example. Close to T_c, the superconducting fluctuations can be described using the time-dependent Ginzburg-Landau (TDGL)[6–8] equation:

$$-\frac{a}{T_c}\left(\frac{\partial}{\partial t}+2ie\varphi\right)\Delta(\mathbf{r},t) = \left[\frac{T-T_c}{T_c}+\frac{\pi D}{8T_c}(-i\nabla-2e\mathbf{A})^2\right]\Delta(\mathbf{r},t). \quad (1)$$

Here $\Delta(\mathbf{r},t)$ is a complex field describing the order parameter fluctuating in time and space (a detailed discussion of the TDGL theory can be found in Ref. 5). The coefficient a is known from microscopic calculations to be equal to $\pi/8$, and $e = -|e|$ is the electron charge. The first term on the right-hand side corresponds to the finite energy needed to create a fluctuation of the superconducting order parameter above the transition temperature. We can look at the semi-phenomenological equation presented in Eq. (1) as describing $2e$-charged particles with a lifetime $\tau_\Delta \sim (T-T_c)^{-1}$. The conductivity associated with these particles is simply their

Drude conductivity, $\sigma_{xx} = (2e)^2 n_\Delta \tau_\Delta / m_{GL} \sim e^2 T/(T - T_c)$. A comparison with the microscopic calculations shows that the Aslamazov-Larkin contribution to the longitudinal conductivity coincides with the one obtained using the semiphenomenological equation. However, the correction to the Hall conductivity cannot be captured by Eq. (1).

The TDGL equation can be derived directly from the microscopic theory by integrating out the single-particle degrees of freedom. Then, under the assumption that the quasiparticles have a constant density of states, one arrives at Eq. (1). Since no particle-hole asymmetry has been introduced, the excitations associated with the superconducting fluctuations, as described by Eq. (1), are invariant under particle-hole transformation. Therefore, it should not be surprising that the contribution of the superconducting fluctuations to the Hall conductivity vanishes in the framework of this equation. It was first pointed out by Fukuyama *et al.*[9] that the Aslamazov-Larkin correction vanishes unless the derivative of the density of states with respect to the energy is taken into account. In other words, this contribution to the Hall conductivity depends on the particle-hole asymmetry. This important observation was the basis for subsequent studies of the Hall effect in conventional and high-T_c superconductors[10–14] as well as in superfluid Fermi systems.[15,16]

Aronov *et al.*[17,18] incorporated the particle-hole asymmetry into the TDGL equation by adding a new term:

$$-\left(\frac{\partial}{\partial t} + 2ie\varphi\right)\left(\frac{a}{T_c} + i\varsigma\right)\Delta(\mathbf{r},t) = \left[\frac{T - T_c}{T_c} + \frac{\pi D}{8T_c}(-i\nabla - 2e\mathbf{A})^2\right]\Delta(\mathbf{r},t). \qquad (2)$$

This equation was used to derive the Aslamazov-Larkin correction to the Hall conductivity. The authors of Ref. 18 claimed that the new parameter can be related to the derivative of the critical temperature with respect to the chemical potential, $\varsigma = -0.5d \ln T_c/d\mu \sim -\lambda^{-1}\nu'(\mu)/\nu(\mu)$. Here λ is the dimensional coupling constant determining $T_c = \omega_D \exp(-1/\lambda)$, and $\nu(\mu)$ is the density of states at the Fermi energy while $\nu'(\mu)$ is its derivative with respect to the energy. Hence, the corrections to the Hall conductivity, being proportional to ς, can provide information on the dependence of the density of states on the energy. Microscopic calculation presented in Appendix A confirms that for three-dimensional electrons ς is proportional to $1/(\lambda\varepsilon_F)$. (Throughout the entire paper we consider a not-too-thin film in which the electrons are three-dimensional while the superconducting fluctuations are two-dimensional.) Analysis of Eq. (2) reveals that in the diffusive regime the cyclotron frequency corresponding to the charged field Δ is equal to $\Omega_c = |4eHD/c|$, where $\Omega_c \propto (\varepsilon_F\tau)\omega_c \gg \omega_c$. In Ω_c, the effective charge is equal to $2e$ and the diffusion coefficient D replaces $1/2m$, because in the fluctuation propagators the kinetic energy $p^2/2m$ is replaced with Dq^2. Consequently, the Drude-like contribution of the superconducting fluctuations to the Hall conductivity is proportional to $\varsigma\Omega_c$.

In this paper we extend previous theoretical analysis[9,17,18] of the corrections to the Hall conductivity for various temperatures and magnetic fields. We concentrate on the superconducting fluctuation corrections in the normal state where the fluctuations can be treated in the Gaussian approximation. (In the superconducting phase the Hall effect is caused by the motion of vortices rather than by fluctuations of the order parameter.) Although the diagonal component of the magnetoresistance has been studied for the entire phase diagram including the vicinity of the quantum critical point, induced by magnetic field,[19] to date there has been no similar systematic analysis of the Hall resistance. The results for the leading corrections to the Hall conductivity generated by the superconducting fluctuations are summarized in Fig. 3. This work was inspired by recent measurements of the Hall conductivity in disordered tantalum nitride (TaN_x) films.[20] Some of the results presented here were used in Ref. 20 for the analysis of Hall conductivity measurements.

As we explained above, the particle-hole asymmetry enters the Hall conductivity either via the quasiparticle mass (or equivalently, the cyclotron frequency ω_c) or the derivative of the density of states. While the former appears when the Lorentz force acts on the quasiparticles in order to turn the current from the longitudinal to the transverse direction, the latter appears when the Lorentz force acts on the superconducting fluctuations. Thus, in general, there are two distinct types of corrections to the Hall conductivity: one proportional to $\omega_c\tau$ and the other to $\varsigma\Omega_c \sim \omega_c\tau/\lambda$. Since the coupling constant for the superconducting interaction is usually much smaller than unity, one may expect only the second kind of contributions to be important. However, the two contributions also differ in their dependence on the distance from the superconducting transition, $\ln T/T_c(H)$ or $\ln H/H_{c2}(T)$. Moreover, we have found a new term which, although it is not enhanced by the inverse coupling constant $1/\lambda$, contributes to the transverse conductivity in a broad range of temperatures and magnetic fields. In particular, this contribution, unique to the Hall conductivity, gives the most dominant fluctuation correction to σ_{xy} far from the transition at $T \gg T_c$.

The rest of the paper is organized as follows: The derivation of the Hall conductivity using the quantum kinetic equation is discussed in Sec. I and Appendix B. The results of the calculation for the different regions of the T-H phase diagram are given in Sec. II.

I. DERIVATION OF THE HALL CONDUCTIVITY

For the derivation of the Hall conductivity we apply the quantum kinetic technique,[21–23] but the same result can be obtained using the Kubo formula. The details of the derivation are described in Appendix B. For the purpose of illustration, we use diagrammatic representation for the different contributions to the transport coefficient. The well known set of diagrams corresponding to the fluctuation corrections to the longitudinal conductivity is presented in Fig. 1. In general all these diagrams may contribute to the leading correction to the transverse conductivity, but actually this is not the case. It was shown in Ref. 9 that the anomalous Maki-Thompson correction [illustrated in Fig. 1(a)] is simply equal to $\delta\sigma_{xy}^{\mathrm{AMT}} = -2\omega_c\tau\delta\sigma_{xx}^{\mathrm{AMT}}(H,T)$. Therefore, we do not have to dwell on the derivation of this contribution. Furthermore, we found that, out of the remaining ten diagrams contributing to $\delta\sigma_{xx}$, only a few give nonzero contribution to $\delta\sigma_{xy}$. These are the

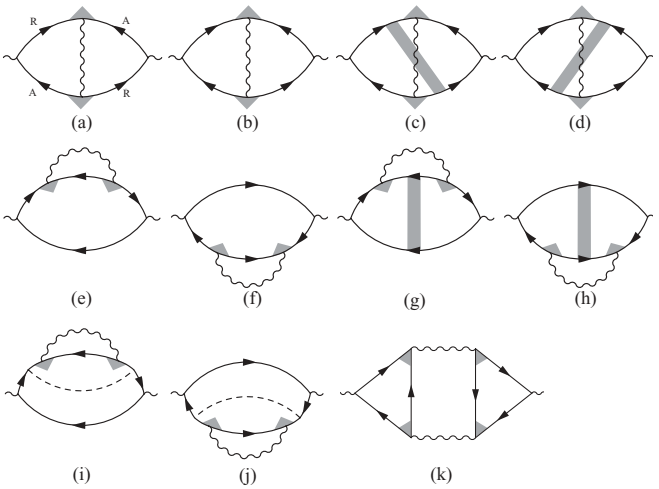

FIG. 1. The eleven diagrams contributing to the superconducting fluctuations corrections to the longitudinal conductivity $\delta\sigma_{xx}$. (a) The anomalous Maki-Thompson corrections. The analytical structure of the Green's functions are indicated by R (retarded) and A (advanced). (b)–(d) The regular Maki-Thompson corrections. (e)–(j) The density of states corrections. (k) The Aslamazov-Larkin term.

Aslamazov-Larkin term, Fig. 1(k), and two of the density of states terms, Figs. 1(g) and 1(h). Although each of the other diagrams gives a nonzero contribution to $\delta\sigma_{xy}$, their sum vanishes. In addition, we have discovered a new contribution to the Hall current, which is presented in Fig. 2. The contribution of this term to σ_{xx} is smaller by a factor of $T\tau$ than those from the set of ten diagrams in Fig. 1. In contrast, its contribution to the Hall conductivity is of the same order as the rest of the terms.

The entire dependence on the magnetic field is incorporated through the propagators of the quasiparticles, superconducting fluctuations, and Cooperons (which describe the multiple scattering of two quasiparticles by impurities). Since we are interested in the linear response to the electric field, all propagators entering the diagrams are calculated at thermal equilibrium. The equation for the quasiparticles Green's function at equilibrium in the presence of a magnetic field is

$$\left[\epsilon+\frac{1}{2m}\left(\nabla-\frac{ie}{c}\mathbf{A}(\mathbf{r},t)\right)^2-V_{\rm imp}(\mathbf{r})+\mu\right]g^{R,A}(\mathbf{r},\mathbf{r}';\epsilon)-\int d\mathbf{r}_1\,\Sigma^{R,A}_{eq}(\mathbf{r},\mathbf{r}_1;\epsilon)g^{R,A}(\mathbf{r}_1,\mathbf{r}',\epsilon)=\delta(\mathbf{r}-\mathbf{r}'). \tag{3}$$

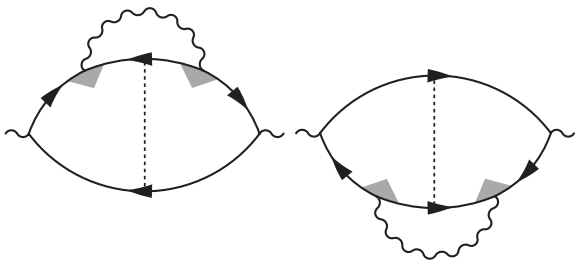

FIG. 2. The new contribution to the Hall conductivity.

Here, Σ_{eq} is the quasiparticle self-energy at equilibrium. The Green's function depends on the two spatial coordinates and not only on the relative one due to the impurities potential $V_{\rm imp}(\mathbf{r})$ and the vector potential $\mathbf{A}(\mathbf{r},t)$. The equilibrium Green's function can be written as a product of the phase factor, $\exp\{ie\int_{\mathbf{r}'}^{\mathbf{r}}\mathbf{A}(\mathbf{r}_1)d\mathbf{r}_1/c\}$, and the gauge invariant Green's function, $\hat{\tilde{G}}_{eq}$. In the presence of a uniform (and constant in time) magnetic field, this representation of the Green's function takes the following simple form:

$$\hat{g}(\mathbf{r},\mathbf{r}';\epsilon)=\hat{\tilde{g}}(\mathbf{r},\mathbf{r}';\epsilon)e^{-ie\mathbf{B}\cdot[(\mathbf{r}-\mathbf{r}')\times(\mathbf{r}+\mathbf{r}')]/4c}. \tag{4}$$

Then, the retarded and advanced components of $\hat{\tilde{g}}$ satisfy the equation

$$\left[\epsilon+\frac{1}{2m}\left(\nabla-i\frac{e\mathbf{B}}{2c}\times(\mathbf{r}-\mathbf{r}')\right)^2-V_{\rm imp}(\mathbf{r})-\tilde{\Sigma}^{R,A}_{eq}+\mu\right]\times\tilde{g}^{R,A}(\mathbf{r},\mathbf{r}';\epsilon)=\delta(\mathbf{r}-\mathbf{r}'), \tag{5}$$

where the product of the Green's function and the self-energy should be understood as a convolution in real space. Now the entire dependence of the gauge invariant Green's function on the center-of-mass coordinate is due to the impurities. After averaging over disorder, the gauge-invariant part of the Green's function $\hat{g}$ becomes translational invariant, i.e., it is a function of the relative coordinate $\boldsymbol{\rho}=\mathbf{r}-\mathbf{r}'$ alone (see Ref. 24 and references therein):

$$\left[\epsilon+\frac{1}{2m}\left(\frac{\partial^2}{\partial\boldsymbol{\rho}^2}-\frac{(e\mathbf{B}\times\boldsymbol{\rho})^2}{4c^2}\right)-\tilde{\Sigma}^{R,A}_{eq}+\mu\pm\frac{i}{2\tau}\right]\times\tilde{g}^{R,A}(\boldsymbol{\rho},\epsilon)=\delta(\boldsymbol{\rho}). \tag{6}$$

We restrict the calculation to the limit $\omega_c\tau\ll1$. Therefore, we may neglect the dependence of $\tilde{G}$ on the magnetic field entering through the Landau quantization of the quasiparticles states. Then, the only dependence of the quasiparticle Green's functions on the magnetic field is through the phase as described in Eq. (4). We wish to point out that, in the normal state, the permeability is close to unity and, correspondingly, we do not distinguish between B (the magnetic flux density) and the magnetic field H.

Unlike the quasiparticles, the Landau quantization of the collective modes (the fluctuations of the superconducting order parameter) cannot be neglected. The equilibrium propagator of the superconducting fluctuations, like the quasiparticle Green's functions, can be separated into the phase factor $\exp\{2ie\int_{\mathbf{r}'}^{\mathbf{r}}\mathbf{A}(\mathbf{r}_1)d\mathbf{r}_1/c\}$ and the gauge invariant part $\tilde{L}$. The gauge invariant part, $\tilde{L}$, can be written using the Landau level quantization, $\tilde{L}^{R,A}(\mathbf{r},\mathbf{r}';\omega)=\sum_N\varphi_{N,0}(\mathbf{r}-\mathbf{r}')\tilde{L}_N(\omega)$, where

$$\tilde{L}^{R,A}_N(\omega)=-\frac{1}{\nu}\left[\ln\left(\frac{T}{T_c}\right)+\psi_{R,A}(\omega,N)-\psi\left(\frac{1}{2}\right)+\varsigma\omega\right]^{-1}, \tag{7a}$$

$$\psi_{R,A}(\omega,N)=\psi\left(\frac{1}{2}\mp\frac{i\omega}{4\pi T}+\frac{\Omega_c(N+1/2)}{4\pi T}\right). \tag{7b}$$

Here, $\psi(x)$ is the digamma function, N is the index of the Landau level, and $\varphi_{N,n}(\mathbf{r})$ is the wave function of a particle

in the Nth Landau level solved in the symmetric gauge. As we have already discussed, the appearance of the parameter ς in Eq. (7a) introduces the particle-hole asymmetry into the propagator of the superconducting fluctuations. In a similar way, the gauge invariant part of the Cooperon can be written in terms of the Landau levels:

$$\tilde{C}_N^{R,A}(\epsilon,\omega-\epsilon)=\frac{1}{\mp i(2\epsilon-\omega)\tau+\Omega_c\tau(N+1/2)}, \quad (8)$$

In the derivation of the Aslamazov-Larkin, Fig. 1(k), and density of states diagrams, Figs. 1(g) and 1(h), we can neglect the dependence of the quasiparticles on the magnetic field. This is because the contributions from the phase associated with the quasiparticle Green's functions [see Eq. (4)] add to zero.

Then the integration over the quasiparticle degrees of freedom is trivial. As a result, the Aslamazov-Larkin term becomes ($e<0$)

$$\begin{aligned} j^y_{AL}=&\, i\frac{e^2E_x}{8\pi^2}\nu^2\mathrm{sgn}(H)\int d\omega\sum_{N\geqslant 0}(N+1)\frac{\partial n_P(\omega)}{\partial\omega}[\psi_R(\omega,N)+\psi_A(\omega,N)-\psi_R(\omega,N+1)-\psi_A(\omega,N+1)]\\ &\times[\psi_R(\omega,N)-\psi_R(\omega,N+1)]\left[\tilde{L}^R_N(\omega)\tilde{L}^A_{N+1}(\omega)-\tilde{L}^R_{N+1}(\omega)\tilde{L}^A_N(\omega)\right]+i\frac{e^2E_y}{2\pi^2}\nu^2\mathrm{sgn}(H)\int d\omega\sum_{N=0}^{\infty}(N+1)n_P(\omega)\\ &\times[\psi_R(\omega,N)-\psi_R(\omega,N+1)]^2\left[\frac{\partial\tilde{L}^R_N(\omega)}{\partial\omega}\tilde{L}^R_{N+1}(\omega)-\frac{\partial\tilde{L}^R_{N+1}(\omega)}{\partial\omega}\tilde{L}^R_N(\omega)\right]+\text{c.c.} \end{aligned} \quad (9)$$

and the density of states contribution is

$$\begin{aligned} j^y_{DOS}=&-\frac{e^2E_x}{4\pi^2}\nu\,\mathrm{sgn}(H)\int d\omega\sum_{N\geqslant 0}(N+1)\left\{\frac{-i}{2}\frac{\partial n_P(\omega)}{\partial\omega}\tilde{L}^R_N(\omega)\left[\frac{\Omega_c(N+1)-\Omega_cN}{4\pi T}\psi'_R(\omega,N)\right.\right.\\ &\left.+\psi_R(\omega,N)-\psi_R(\omega,N+1)-\frac{\Omega_c(N+1)-\Omega_cN}{4\pi T}\psi'_A(\omega,N)-\psi_A(\omega,N)+\psi_A(\omega,N+1)\right]\\ &\left.-\frac{n_P(\omega)}{4\pi T}\tilde{L}^A_N(\omega)\left[\frac{\Omega_c(N+1)-\Omega_cN}{4\pi T}\psi''_A(\omega,N)+\psi'_A(\omega,N)-\psi'_A(\omega,N+1)\right]-(N\leftrightarrow N+1)\right\}+\text{c.c.} \end{aligned} \quad (10)$$

In the above equations we denote the Bose distribution function by $n_P(\omega)$. The notation $N\leftrightarrow N+1$ means that N is replaced by $N+1$ and the other way around in all the terms inside the curly brackets. In both terms some of the propagators of the collective modes (the superconducting fluctuations and Cooperons) are functions of the Nth Landau level while the index for the others propagators is $N+1$. This is due to the Lorentz force turning the collective modes from the x into the y direction. For more details of the derivation see Appendix B. At low H for which $\Omega_c\ll 4\pi T$, the discrete sum over the Landau levels can be replaced by an integral (the continuum limit).

In contrast to the Aslamazov-Larkin and the density of states corrections, in the derivation of the new contribution illustrated in Fig. 2 the Lorentz force acts on the quasiparticles in order to turn the current. Hence, we cannot ignore the magnetic field entering their phase. Consequently, the integration over the quasiparticle degrees of freedom is more subtle than in the derivation of the previous terms; see Appendix B for details. The result of integrating out the quasiparticles is

$$\begin{aligned} j^y_{\text{new}}=&-i\frac{e^2E_x}{32\pi^2}\nu\Omega_c^2\left[\frac{1}{\varepsilon_F}+\frac{\nu'(\mu)}{\nu(\mu)}\right]\mathrm{sgn}(H)\int d\omega\sum_{N\geqslant 0}\left\{\left(\frac{1}{4\pi T}\right)^2 n_P(\omega)\tilde{L}^A_N(\omega)\psi''_A(\omega,N)\right.\\ &\left.+\frac{i}{8\pi T}\frac{\partial n_P(\omega)}{\partial\omega}\tilde{L}^R_N(\omega)[\psi'_R(\omega,N)-\psi'_A(\omega,N)]\right\}+\text{c.c.} \end{aligned} \quad (11)$$

In the above expression all collective mode propagators have the same Landau level index. Although it is not evident, this contribution is proportional to the cyclotron frequency of the quasiparticles. Comparison with the correction to the longitudinal conductivity arising from the modification of the tunneling density of states by the fluctuations[25,26] shows that the new term describes how the tunneling density of states reveals itself in the transverse conductivity.

II. FLUCTUATION CORRECTIONS TO THE HALL EFFECT

We now present the leading corrections to the Hall conductivity in the different regions of the phase diagram plotted in Fig. 3. A similar phase diagram has been previously discussed in a study of the Nernst effect in amorphous superconducting films.[21,22] As shown in Fig. 3, the phase diagram is divided into many subregions. This is because the magnetic field plays

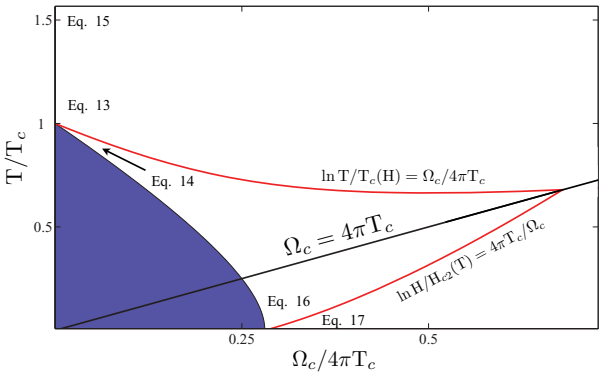

FIG. 3. (Color online) The phase diagram for the corrections to the Hall conductivity $\delta\sigma_{xy}$. The equations indicated on the phase diagram correspond to the expressions for $\delta\sigma_{xy}$ written in the text. $\Omega_c = 4eHD/c$ is the cyclotron frequency corresponding to the superconducting fluctuations in the diffusive regime.

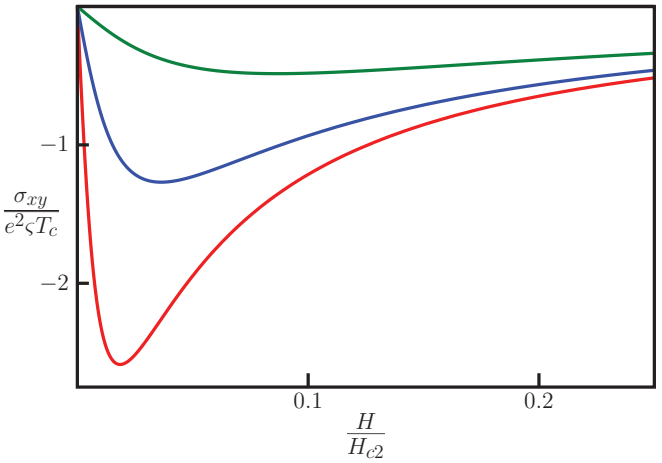

FIG. 4. (Color online) Corrections to the Hall conductivity $\delta\sigma_{xy}$ as described by Eq. (12) for $T = 1.01T_c$ (red curve), $T = 1.02T_c$ (blue curve), and $T = 1.05T_c$ (green curve). The Hall conductivity is given in units of $e^2|\varsigma|T_c$.

a double role: not only does it drive the transition between the metallic normal state and the superconducting one, it also quantizes the collective modes in the Cooper channel (both the superconducting fluctuations and Cooperons). The shaded area corresponds to the superconducting phase which is bounded by the line $T = T_c(H)$. There are two crossover lines in the vicinity of the transition. In the area below the line $\ln T/T_c(H) = \Omega_c/4\pi T$ the Landau level quantization of the superconducting fluctuations becomes essential. The other line, $\ln H/H_{c2}(T) = 4\pi T/\Omega_c$, separates the regions of classical and quantum fluctuations at low temperatures. The low-H and high-T region is separated from the high-H and low-T region by the line $\Omega_c = 4\pi T$.

As we explained in the previous section, different contributions to the Hall conductivity are characterized by the way the magnetic field deflects the current to the transverse direction. The magnetic field can turn the current via the collective modes or the quasiparticles. The first case yields contributions proportional to $\varsigma\Omega_c \sim \omega_c\tau/\lambda$, where λ is the dimensionless coupling constant of the attractive electron-electron interaction in the Cooper channel. The other possibility results in corrections that do not contain the large factor $1/\lambda$.

Close to the line of phase transition, $T \gtrsim T_c(H)$, and for a small magnetic field, $\Omega_c \ll 4\pi T$, the leading correction to σ_{xy} is given by the Aslamazov-Larkin term

$$\delta\sigma_{xy} = \frac{2e^2\varsigma T\nu}{\pi}\mathrm{sgn}(H)\sum_n (n+1)\frac{[\tilde{L}_n(0) - \tilde{L}_{n+1}(0)]^3}{[\tilde{L}_{n+1}(0) + \tilde{L}_n(0)]^2}. \tag{12}$$

The above equation is derived from Eq. (9) by expanding to the first order in ςT. In addition, we integrated over the frequency ω only up to T (accounting for the classical fluctuations alone). This correction to the Hall conductivity, just like the Drude term, is negative, because $\varsigma < 0$. Note that here, and in what follows, we consider negative charge carriers $e < 0$. As we show in Fig. 4, for $T > T_c(H = 0)$, the correction to the Hall conductivity is a nonmonotonic function of the magnetic field. In the close vicinity of $T_c(H = 0)$, $\delta\sigma_{xy}$ has a peak at $H^* = 1.3\frac{\phi_0}{2\pi}\frac{\ln T/T_c}{\xi^2}$, which up to a factor of 1.3 coincides with the ghost field observed in measurements of the Nernst effect[27] (here $\xi^2 = \pi D/8T_c$). The above expression has been successfully used to fit the data obtained in recent measurements of the Hall conductivity in amorphous tantalum nitride films (see Ref. 20).

As the magnetic field goes to zero and $T > T_c(H = 0)$, the discrete sum over the Landau levels can be replaced by a continuous integral. Then the correction to the Hall conductivity from Eq. (12) becomes

$$\delta\sigma_{xy} = e^2\frac{\varsigma\Omega_c}{96}\mathrm{sgn}(H)\left(\frac{1}{\ln T/T_c(H)}\right)^2. \tag{13}$$

Curiously, close to the transition the divergence of the Hall conductivity, $\delta\sigma_{xy} \sim 1/\ln^2(T/T_c)$, is stronger than the one known for the longitudinal conductivity,[2] $\delta\sigma_{xx} \sim 1/\ln(T/T_c)$. When $T < T_c(H = 0)$, the Landau level quantization is essential. Moreover, below the line $\ln T/T_c(H) = \Omega_c/4\pi T$ only the lowest Landau level contributes to the sum, and one gets

$$\delta\sigma_{xy} = \frac{2e^2\varsigma T_c}{\pi}\mathrm{sgn}(H)\frac{1}{\ln T/T_c(H)}. \tag{14}$$

Note that this expression does not contain the magnetic field as a prefactor.

At $T \gg T_c$ but still at a small magnetic field, the process described by the new contribution introduced in this paper (see Fig. 2) dominates:

$$\delta\sigma_{xy} \approx \frac{e^2\omega_c\tau}{4\pi^2}\mathrm{sgn}H\ln\left(\frac{\ln 1/T\tau}{\ln T/T_c}\right). \tag{15}$$

The new term, and therefore also the leading correction to the Hall conductivity at $T \gg T_c$, is proportional to ω_c, because in this case the Lorentz force turning the current from the longitudinal to the transverse direction acts on the quasiparticles rather than the superconducting fluctuations. Comparing Eq. (15) with the correction to the longitudinal conductivity in this region,[28] one may observe that $\delta\sigma_{xy} = -\frac{\omega_c\tau}{2}\delta\sigma_{xx}$.

In the vicinity of the magnetic field driven quantum critical point, $H \approx H_{c2}(T = 0)$, all three terms discussed in the previous section as well as the anomalous Maki-Thompson

term contribute comparably to the Hall conductivity. In the classical regime where $\ln H/H_{c2}(T) < 4\pi T/\Omega_c \ll 1$, the Hall conductivity is

$$\delta\sigma_{xy} \approx \frac{2e^2}{\pi \ln H/H_{c2}} \mathrm{sgn} H \left(\varsigma T - \frac{21T}{8\varepsilon_F}\right). \quad (16)$$

Here the Hall conductivity depends on the magnetic field only via $\ln H/H_{c2}$, which measures the distance to the phase transition. In the quantum regime, $4\pi T/\Omega_c < \ln H/H_{c2}(T) \ll 1$, the Hall conductivity acquires the form

$$\delta\sigma_{xy} \approx \frac{e^2 \mathrm{sgn} H}{2\pi^2} \left(\omega_c\tau - \frac{2\varsigma\Omega_c}{3}\right) \ln \frac{1}{\ln H/H_{c2}}. \quad (17)$$

Note that the correction to the Hall conductivity changes its sign at low magnetic field when the temperature is raised from $T \sim T_c$ to $T \gg T_c$ [i.e., when the dominant correction switches between Eqs. (13) and (15)]. Similarly, we expect a change of sign when the magnetic field is increased [see Eqs. (17) and (16)]. This change in sign of the corrections cannot explain the change in sign of the Hall coefficient observed in various superconductors in the mixed state[29–33] as the analysis in terms of Gaussian fluctuations is not applicable in this regime.

Finally, we wish to emphasize how the Landau quantization of the collective modes enters the Hall conductivity. In general, to obtain the fluctuation corrections to σ_{xy} one must sum over all Landau levels. However, there are limiting cases in which the sum can be simplified: (i) $H \to 0$ and (ii) $\ln T/T_c(H) \ll \Omega_c/4\pi T$. In the first case, the sum over N can be replaced by an integral. This simplification has been used to obtain Eqs. (13) and (15). In the second case, the critical behavior is determined by the contribution from the lowest Landau level. Consequently, in deriving Eqs. (14), (16), and (17) we neglected terms with $N > 0$.

In conclusion, we extended the previous calculations of the Hall conductivity[9,18] to a broader range of temperatures and magnetic fields. The fluctuations corrections can be divided into two groups. The first contains terms proportional to $\varsigma\Omega_c$ and includes the Aslamazov-Larkin contribution [Fig. 1(k)] and part of the density of states corrections [Figs. 1(g) and 1(h)]. The other group includes the new contribution $\delta\sigma_{xy}^{\mathrm{NEW}}$ (Fig. 2) that was not considered before, and the anomalous Maki-Thompson term [Fig. 1(a)]. These corrections are proportional to $\omega_c\tau$. Unlike the anomalous Maki-Thompson correction, the new contribution modifies the Hall resistivity. This becomes obvious if we rewrite the Hall resistivity in terms of the two components of the conductivity tensor, $\rho_{xy} = -\sigma_{xy}/(\sigma_{xx}^2 + \sigma_{xy}^2) \approx -\sigma_{xy}/\sigma_{xx}^2$, and extract the fluctuation correction to the resistivity, $\delta\rho_{xy} = -\delta\sigma_{xy}/\sigma_{xx}^2 + 2\sigma_{xy}\delta\sigma_{xx}/\sigma_{xx}^3$, with $\sigma_{xy} = -\omega_c\tau\sigma_{xx}$. Since $\delta\sigma_{xy}^{\mathrm{AMT}} = -2\omega_c\tau\delta\sigma_{xx}^{\mathrm{AMT}}$, the anomalous Maki-Thompson correction to ρ_{xy} vanishes, while the correction from $\delta\sigma_{xy}^{\mathrm{NEW}}$ remains. Recently, the part of our result which is proportional to $\varsigma\Omega_c$ has been reproduced using the Usadel equation.[26]

Our results for the different regimes of the phase diagram are summarized in Fig. 3.

ACKNOWLEDGMENTS

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APPENDIX A: PARTICLE-HOLE ASYMMETRY AND SUPERCONDUCTING FLUCTUATIONS

Here we will explain the mechanism of appearance of the parameter ς in the propagator of superconducting fluctuations given in Eq. (7a). For that we calculate $\hat{L}$, taking into account the dependence of the density of states and velocity of the quasiparticles on energy. In the normal state, the quasiparticles are described in terms of the Fermi liquid theory, where the standard approximation is to consider the density of states and velocity in the vicinity of the Fermi energy as constants. The dependence of the Fermi liquid parameters on energy leads only to small corrections and can be usually ignored. However, under this approximation the propagator of superconducting fluctuations satisfies $L^R(\omega) = L^A(-\omega)$ and, consequently, the fluctuation corrections to the Hall effect vanish. Therefore, when studying the Hall effect, we have to go beyond the Fermi liquid approximation. Note that although the fluctuations in superconducting films are effectively two-dimensional, the quasiparticles in a not-too-thin film are still three-dimensional and, hence, the density of states ν is not a constant.

The propagator of superconducting fluctuations at equilibrium satisfies the following equation:

$$L^{R,A}(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{\nu_0}[-\lambda^{-1} + \Pi^{R,A}(\mathbf{r},t;\mathbf{r}',t')]^{-1}. \quad (A1)$$

In this work we study effects of superconducting fluctuations in the Gaussian approximation. After averaging over disorder, the polarization operator can be written in terms of the Cooperon and the quasiparticle Green's functions:

$$\hat{\Pi}(\mathbf{r},t;\mathbf{r}',t') = \frac{1}{\nu_0}\int d\mathbf{r}_1 dt_1 \hat{g}(\mathbf{r},t;\mathbf{r}_1,t_1)\hat{g}(\mathbf{r},t;\mathbf{r}_1,t_1)\hat{C}(\mathbf{r}_1,t_1;\mathbf{r}',t'). \quad (A2)$$

It will be enough to find Π in the absence of magnetic field, and reintroduce the magnetic field in the end. Then, the calculation can be done in momentum and frequency space, and the Cooperon becomes

$$C^R(\mathbf{q},\epsilon,\omega-\epsilon) = \left[1 - V_{\mathrm{imp}}^2 \int \frac{d\mathbf{k}}{(2\pi)^3} g^R(\mathbf{k},\epsilon)g^A(\mathbf{q}-\mathbf{k},\omega-\epsilon)\right]^{-1}. \quad (A3)$$

The particle-hole asymmetry enters the calculation of the Cooperon in numerous ways. First of all, the nonconstant density of states affects the elastic scattering time, and hence modifies the quasiparticle Green's function:

$$g^{R,A}(\mathbf{k},\epsilon) = \left[\epsilon - \xi_{\mathbf{k}} \pm i\pi V_{\mathrm{imp}}^2 \nu(\epsilon)\right]^{-1}. \quad (A4)$$

For a parabolic spectrum of three-dimensional quasiparticles, $\nu(\epsilon) \approx \nu_0(1+\epsilon/2\varepsilon_F)$. Similarly, the integration over the momentum in Eq. (A3) is sensitive to the energy dependence of the density of states and velocity. In practice, however, the analysis of the leading contribution shows that only the modification of the quasiparticle Green's functions is important. Then, expanding the density of states in the Green's functions, one gets

$$C^{R,A}(\mathbf{q},\epsilon,\omega-\epsilon) = \frac{1+\omega/4\varepsilon_F}{\mp i(2\epsilon-\omega)\tau + Dq^2\tau}, \tag{A5}$$

where $\tau = (2\pi V_{\rm imp}^2 \nu_0)^{-1}$ is the elastic scattering time at the Fermi energy calculated in the Born approximation.

We can see that the particle-hole asymmetry modifies the Cooperon by the factor $(1+\omega/4\varepsilon_F)$. Correspondingly, the polarization operator becomes

$$\Pi^{R,A}(\mathbf{q},\omega) = -\left(1+\frac{\omega}{4\varepsilon_F}\right)\left[\psi\left(\frac{1}{2}+\frac{\mp i\omega+Dq^2}{4\pi T}\right) - \psi\left(\frac{1}{2}\right)+\ln\frac{T}{T_c}-\frac{1}{\lambda}\right]. \tag{A6}$$

Not too far from the superconducting transition, e.g., when $T \gtrsim T_c$, we can write the propagator $L^{R,A}(\mathbf{q},\omega)$ to the leading corrections due to the particle-hole asymmetry as

$$\begin{aligned} L^{R,A}(\mathbf{q},\omega) &= -\frac{1}{\nu_0}\left\{\frac{1}{\lambda}+\left(1+\frac{\omega}{4\varepsilon}\right)\left[\ln\frac{T}{T_c} + \psi\left(\frac{1}{2}+\frac{\mp i\omega+Dq^2}{4\pi T}\right)-\psi\left(\frac{1}{2}\right)-\frac{1}{\lambda}\right]\right\}^{-1} \\ &\approx \frac{-1}{\nu_0}\left[\ln\frac{T}{T_c}+\psi\left(\frac{1}{2}+\frac{\mp i\omega+Dq^2}{4\pi T}\right) - \psi\left(\frac{1}{2}\right)-\frac{\omega}{4\varepsilon_F\lambda}\right]^{-1}. \end{aligned} \tag{A7}$$

Defining $\varsigma = -1/4\varepsilon_F\lambda$, we get the expression for the propagator of the superconducting fluctuations used in the main text [see Eq. (7a)]. The asymmetry parameter ς can be rewritten as $\varsigma = -0.5d\ln T_c/d\ln\mu$, in accordance with Ref. 18. Furthermore, in the presence of magnetic field, the term Dq^2 in the propagator L (as well as in the Cooperon) is quantized into the Landau levels, $Dq^2 \to \Omega_c(N+1/2)$. One may still use the obtained value for the parameter ς in the propagator L as given in Eq. (7a) for the analysis of fluctuation effects in the Hall conductivity in the whole region T-H of the superconducting transition, $T = T_c(H)$.

Finally, let us remark that, although the asymmetry affects also the Cooperon, in the derivation of the corrections to the Hall conductivity we neglected it. Including the dependence of the Cooperon on the particle-hole asymmetry leads to corrections which are smaller by a factor $T\tau \ll 1$ or $1/\varepsilon_F\tau \ll 1$ than the terms discussed in this paper.

APPENDIX B: DERIVATION OF THE HALL CONDUCTIVITY

We apply here the quantum kinetic technique as described in Refs. 21–23. In the presence of superconducting fluctuations we describe the system using two fields: the quasiparticle field and the fluctuations of the superconducting order parameter. The matrix functions $\hat{G}(\mathbf{r},\mathbf{r}',\epsilon)$ and $\hat{\mathcal{L}}(\mathbf{r},\mathbf{r}',\omega)$ written in the Keldysh form,[34–36]

$$\hat{F}(\mathbf{r},t;\mathbf{r}',t') = \begin{pmatrix} F^R(\mathbf{r},t;\mathbf{r}',t') & F^K(\mathbf{r},t;\mathbf{r}',t') \\ 0 & F^A(\mathbf{r},t;\mathbf{r}',t') \end{pmatrix}, \tag{B1}$$

(where F can be either G or $\mathcal{L}$) describe the propagation of these two fields, respectively. The Keldysh components of the propagators correspond to the generalized distribution functions. According to the quantum kinetic approach the current can be written in terms of the generalized distribution functions. For this purpose, we express the charge density in terms of the propagators of the quasiparticles, $\hat{G}$, and superconducting fluctuations, $\hat{\mathcal{L}}$. Since both the quasiparticles and the superconducting fluctuations carry charge, they both enter the continuity equation. Extracting the electric current from the continuity equation we get

$$\begin{aligned} \mathbf{j}_e^{\rm con}(\mathbf{r},t) &= ie\int d\mathbf{r}'dt'[\hat{\mathbf{v}}(\mathbf{r},t;\mathbf{r}',t')\hat{G}(\mathbf{r}',t';\mathbf{r},t)]^< \\ &\quad + ie\int d\mathbf{r}'dt'[\hat{\boldsymbol{\mathcal{V}}}(\mathbf{r},t;\mathbf{r}',t')\hat{\mathcal{L}}(\mathbf{r}',t';\mathbf{r},t)]^< + \text{h.c.} \end{aligned} \tag{B2}$$

Each of the terms in the current is a product of the renormalized velocity and propagator. The matrix $\hat{\mathbf{v}}(\mathbf{r},t;\mathbf{r}',t')$ is the velocity of the quasiparticles renormalized by the self-energy $\hat{\Sigma}(\mathbf{r},t;\mathbf{r}',t')$:

$$\begin{aligned} \hat{\mathbf{v}}(\mathbf{r},t;\mathbf{r}',t') &= -\frac{i}{2m}\left(\nabla - \frac{ie}{c}\mathbf{A}(\mathbf{r}) - \nabla' - \frac{ie}{c}\mathbf{A}(\mathbf{r}')\right) \\ &\quad \times \delta(\mathbf{r}-\mathbf{r}')\delta(t-t') - i(\mathbf{r}-\mathbf{r}')\hat{\Sigma}(\mathbf{r},t;\mathbf{r}',t'), \end{aligned} \tag{B3}$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential. Similarly, we define $\hat{\boldsymbol{\mathcal{V}}}(\mathbf{r},t;\mathbf{r}',t') = -i(\mathbf{r}-\mathbf{r}')\hat{\Pi}(\mathbf{r},t;\mathbf{r}',t')$ to be the "renormalized velocity" of the superconducting fluctuations. Here $\hat{\Pi}$ is the polarization operator in the Cooper channel (note that in fact $\hat{\boldsymbol{\mathcal{V}}}$ does not have the dimension of velocity). In general, all quantities in the equation for the current depend on the external electric and magnetic fields.

Next, we derive the kinetic equation for the two propagators in the presence of electric and magnetic fields. We consider the linear response to the electric field while keeping the entire dependence on the magnetic field. Then the $\mathbf{E}$-dependent quasiparticle Green's function is

$$\begin{aligned} \hat{G}_{\mathbf{E}}(\mathbf{r},\mathbf{r}',\epsilon) &= \hat{g}(\epsilon)\hat{\Sigma}_{\mathbf{E}}(\epsilon)\hat{g}(\epsilon) \\ &\quad - \frac{ie\mathbf{E}}{2}\left[\frac{\partial\hat{g}(\epsilon)}{\partial\epsilon}\hat{\mathbf{v}}_{eq}(\epsilon)\hat{g}(\epsilon) - \hat{g}(\epsilon)\hat{\mathbf{v}}_{eq}(\epsilon)\frac{\partial\hat{g}(\epsilon)}{\partial\epsilon}\right]. \end{aligned} \tag{B4}$$

The product of matrices should be understood as a convolution of the spatial coordinates. In addition, we used the fact that we are interested in the stationary solution for the Green's function in the presence of a dc electric field. Hence, all Green's functions and self-energies are function of the time difference $t-t'$, and it was possible to Fourier transform the equation from the relative time coordinate to the frequency ϵ. In the above equation $\hat{g}$ is the equilibrium Green's function and $\hat{v}_{eq}$ is

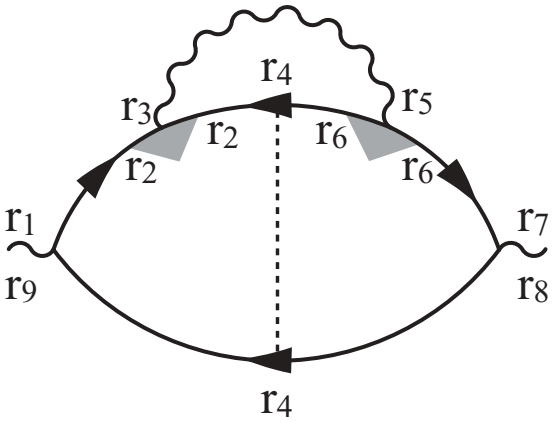

FIG. 5. Detailed description of one of the new terms presented in Fig. 5.

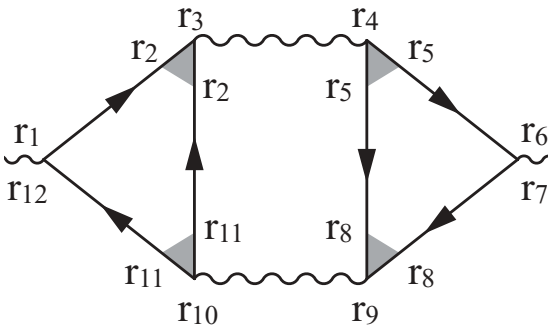

FIG. 6. The Aslamazov-Larkin correction.

the quasiparticle velocity at equilibrium. Note that the equation for the field dependent Green's function is a self-consistent equation, as it contains the **E**-dependent self-energy which is itself a function of $\hat{G}_{\mathbf{E}}$. In addition, $\Sigma_{\mathbf{E}}$ may depend on the electric field through the propagator of the superconducting fluctuations. The equation for the **E**-dependent part of $\hat{\mathcal{L}}$ takes a form similar to Eq. (B4) for $\hat{G}_{\mathbf{E}}$:

$$\hat{L}_{\mathbf{E}}(\omega) = -\hat{L}\hat{\Pi}_{\mathbf{E}}\hat{L} + ie\mathbf{E}\Big[\frac{\partial\hat{L}(\omega)}{\partial\omega}\hat{\mathcal{V}}_{eq}(\omega)\hat{L}(\omega) - \hat{L}(\omega)\hat{\mathcal{V}}_{eq}(\omega)\frac{\partial\hat{L}(\omega)}{\partial\omega}\Big]. \tag{B5}$$

Here $\hat{\mathcal{V}}_{eq}$ is the velocity of the superconducting fluctuations at equilibrium, and $\Pi_{\mathbf{E}}$ is the electric field dependent polarization operator which depends on $G_{\mathbf{E}}$. The discussion of the equilibrium propagators $\hat{g}$ and $\hat{L}$ appears in the main text. In the following, we neglect the particle-hole asymmetry in $\hat{\mathcal{V}}_{eq}$ as well as in all the terms except $\hat{\mathcal{L}}$ since they result in less singular contributions than those discussed here.

The next step in the derivation of the current is to insert the expression for the **E**-dependent propagators and velocities into Eq. (B2). Up to now we have only made two assumptions: (i) we restricted the calculation to the regime of linear response to the electric field, and (ii) we considered a classically weak magnetic field for which the cyclotron frequency of the quasiparticles satisfies $\omega_c\tau \ll 1$. As we are interested in the Gaussian fluctuations, we will make further simplification by expanding with respect to the superconducting fluctuations. Below we give a diagrammatic interpretation for the dominant contributions to the Hall conductivity. The expression for the vertices and the analytical structure of these diagrams have been found from the quantum kinetic equation. The quantum kinetic approach provides a simple and clear derivation of the Hall conductivity; however, one can reach the same result using the standard Kubo formula.

As we already explained, we can classify the contributions to the Hall conductivity according to the way the current is deflected by the Lorentz force. The first group containing the anomalous Maki-Thompson and the new contribution includes terms in which the quasiparticles are used in order to turn the current, while the current in the second group [Figs. 1(g), 1(h), and 1(k)] is deflected using the collective modes. Let us first use, as an example, one of the new terms presented in Fig. 2 and Fig. 5 to demonstrate how the magnetic field enters these kind of contributions. Decomposing all propagators in the diagram shown in Fig. 5 into the phase and gauge invariant parts [see Eqs. (4), (7a), and (8)], we get

$$j^{y}_{\text{New}} = -i\frac{e^2E_x}{4\pi\nu\tau\sqrt{2\pi\ell_H^2}}\int\frac{d\epsilon\,d\omega}{(2\pi)^2}d\mathbf{r}_2\cdots d\mathbf{r}_9\frac{\partial n_F(\epsilon)}{\partial\epsilon}e^{-i\Phi}\tilde{v}_y(\mathbf{r}_9,\mathbf{r}_1)\tilde{v}_x(\mathbf{r}_7,\mathbf{r}_8)\tilde{g}^A(\mathbf{r}_8-\mathbf{r}_4;\epsilon)\tilde{g}^A(\mathbf{r}_4-\mathbf{r}_9;\epsilon)$$
$$\times\,\tilde{g}^R(\mathbf{r}_1-\mathbf{r}_2;\epsilon)\tilde{g}^A(\mathbf{r}_4-\mathbf{r}_2;\omega-\epsilon)\tilde{g}^A(\mathbf{r}_6-\mathbf{r}_4;\omega-\epsilon)\tilde{g}^R(\mathbf{r}_6-\mathbf{r}_7;\epsilon)\sum_N\varphi_{N,0}(\mathbf{r}_2-\mathbf{r}_6)\big[\tilde{C}^R_N(\epsilon,\omega-\epsilon)\big]^2$$
$$\times\,\big[\tilde{L}^R_N(\omega)[n_P(\omega)+n_F(\omega-\epsilon)]+\tilde{L}^A_N(\omega)n_P(\omega)\big]+\text{c.c.} \tag{B6}$$

Here $\ell_H = \sqrt{c/2eH}$ is the magnetic length for the $2e$ excitations in the Cooper channel,

$$\tilde{v}_x(\mathbf{r}_9,\mathbf{r}_1) = \lim_{\mathbf{r}_9\to\mathbf{r}_1}\big[\nabla^x_1/2m + ieH(y_1-y_2)/4mc - \nabla^x_9/2m - ieH(y_4-y_9)/4mc\big]$$

is the velocity written in its gauge invariant form, and $n_P(\omega)$ is the Bose distribution function. The phase Φ is the flux enclosed by the paths of all charged excitations:

$$\Phi = e\mathbf{H}[(\mathbf{r_4}-\mathbf{r_1})\times(\mathbf{r_1}-\mathbf{r_2})+(\mathbf{r_6}-\mathbf{r_7})\times(\mathbf{r_7}-\mathbf{r_4})+2(\mathbf{r_6}-\mathbf{r_4})\times(\mathbf{r_4}-\mathbf{r_2})]/2c. \tag{B7}$$

All propagators of the collective modes ($\tilde{L}$ as well as $\tilde{C}$) have the same Landau level index. As we show later, this is not always the case. Since all terms in the above equation are translational invariant (functions of the relative coordinates alone), we can rewrite the integral in terms of the relative momenta. Then, the spatial coordinates appearing in the flux Φ and diamagnetic term become derivatives with respect to the momenta:

$$
\begin{aligned}
j^y_{\rm NEW} = &-i\frac{e^2E_x}{4\pi\nu\tau\sqrt{2\pi\ell_H^2}}\sum_N\int\frac{d\epsilon\,d\omega}{(2\pi)^2}\frac{d\mathbf{k}_1\cdots d\mathbf{k}_6 d\mathbf{q}}{(2\pi)^{3d}}\delta(\mathbf{k}_2-\mathbf{k}_3)\delta(\mathbf{k}_1-\mathbf{k}_6)\delta(\mathbf{k}_3+\mathbf{k}_4-\mathbf{q})\delta(\mathbf{k}_5+\mathbf{k}_6-\mathbf{q})\frac{\partial n_F(\epsilon)}{\partial\epsilon}\\
&\times\exp\left\{-i\frac{eH}{2c}\left[\frac{\partial}{\partial\mathbf{k}_2}\times\frac{\partial}{\partial\mathbf{k}_3}+\frac{\partial}{\partial\mathbf{k}_6}\times\frac{\partial}{\partial\mathbf{k}_1}+2\frac{\partial}{\partial\mathbf{k}_5}\times\frac{\partial}{\partial\mathbf{k}_4}\right]\right\}\left(\frac{ik_3^y}{2m}-\frac{eH}{4mc}\frac{\partial}{\partial k_3^x}+\frac{ik_2^y}{2m}+\frac{eH}{4mc}\frac{\partial}{\partial k_2^x}\right)\\
&\times\left(\frac{ik_1^x}{2m}-\frac{eH}{4mc}\frac{\partial}{\partial k_1^y}+\frac{ik_6^x}{2m}+\frac{eH}{4mc}\frac{\partial}{\partial k_6^y}\right)\tilde{g}^A(\mathbf{k}_1;\epsilon)\tilde{g}^A(\mathbf{k}_2;\epsilon)\tilde{g}^R(\mathbf{k}_3;\epsilon)\tilde{g}^A(\mathbf{k}_4;\omega-\epsilon)\tilde{g}^A(\mathbf{k}_5;\omega-\epsilon)\tilde{g}^R(\mathbf{k}_6;\epsilon)\\
&\times\sum_N\varphi_{N,0}(\mathbf{q})[\tilde{C}^R_N(\epsilon,\omega-\epsilon)]^2\left[\tilde{L}^R_N(\omega)[n_P(\omega)+n_F(\omega-\epsilon)]+\tilde{L}^A_N(\omega)n_P(\omega)\right]+\text{c.c.}
\end{aligned}
\tag{B8}
$$

The magnetic field enters $\hat{L}$ and $\hat{C}$ as Ω_c/T, which is not necessarily small and, hence, we cannot expand in this parameter. In contrast, the flux can be expanded in powers of the magnetic field. Since each power introduces an additional derivative with respect to the quasiparticle momentum which can act either on the velocity vertex or the Green's functions, the small parameter emerging from the expansion is $\omega_c\tau\ll 1$. Similar smallness is associated with the diamagnetic term. Nevertheless, the magnetic field entering via Φ cannot be neglected, as the zero-order term vanishes. Actually, extracting the magnetic field from the flux is the reason why the new contribution is of the same order as the contribution corresponding to the diagram in Fig. 1. In contrast, the contribution from Fig. 5 to the longitudinal conductivity (obtained by replacing v_y by v_x in the vertex) is smaller by a factor of $T\tau$ than all other terms described in Fig. 1. Following Ref. 24, we can obtain all nonzero contributions arising from expansion of the flux to the first order in H. Then, we can integrate over the quasiparticle momenta $\mathbf{k}_i$ and frequency ϵ. Under the approximation of constant density of states and velocity in the vicinity of the Fermi energy the integral vanishes. Keeping corrections to this approximation, $\nu(\xi)\approx\nu+\nu'(\varepsilon_F)\xi$ and $v(\xi)=v_F(1+\xi/\varepsilon_F)$, results in a nonvanishing contribution to the Hall conductivity. Despite the smallness usually associated with these corrections, here it gives a contribution to $\delta\sigma_{xy}$ comparable to all others:

$$
\begin{aligned}
j^y_{\rm NEW} = &-i\frac{e^3E_xH}{4\pi\ell_H^2c}\nu D\tau^3\sum_{N\geqslant 0}\int\frac{d\epsilon\,d\omega}{2\pi}\frac{v_F^2}{d}\left(\frac{1}{\varepsilon_F}+\frac{\nu'}{\nu_0}\right)\\
&\times\left\{\left[\tilde{C}^R_N(\epsilon,\omega-\epsilon)\right]^2\left[[n_F(\epsilon-\omega)-n_F(\epsilon)]\frac{\partial n_P(\omega)}{\partial\omega}\tilde{L}^R_N(\omega)-n_P(\omega)\frac{\partial n_F(\epsilon)}{\partial\epsilon}\tilde{L}^A_N(\omega)\right]\right\}-\text{c.c.}
\end{aligned}
\tag{B9}
$$

Further integration over the Bosonic frequency ω and summation over the Landau level N is standard, and analytical solutions can be obtained in several limiting cases. The other part of the new contribution presented in Fig. 2 gives exactly the same result. In the same way, we can derive the contributions from the two-Cooperon diagrams shown in Figs. 1(b), 1(e), 1(f), 1(i), and 1(j). While the expression corresponding to Fig. 1(b) is identically zero, the rest of the terms are nonzero and their contributions are proportional to $\omega_c\tau$. However, the sum of these four diagrams vanishes.

As a representative example of the terms in the second group, we present the derivation of the Aslamazov-Larkin correction (see Fig. 6). To keep our demonstration as simple as possible, we consider only part of the term (only contributions in which one propagator L is retarded and the other is advanced):

$$
\begin{aligned}
j^y_{AL}(\mathbf{r}_1) = &-\frac{e^2E_x}{4\pi\ell_H^2}\int\frac{d\epsilon\,d\epsilon'd\omega}{(2\pi)^3}\sum_{N,M}\int d\mathbf{r}_2\cdots d\mathbf{r}_{12}e^{-i\Phi}\tilde{v}_y(\mathbf{r}_{12},\mathbf{r}_1)\tilde{v}_x(\mathbf{r}_6,\mathbf{r}_7)\tilde{g}^R(\mathbf{r}_1-\mathbf{r}_2,\epsilon)\tilde{g}^A(\mathbf{r}_{11}-\mathbf{r}_2,\omega-\epsilon)\\
&\times\tilde{g}^R(\mathbf{r}_{11}-\mathbf{r}_{12},\epsilon)\tilde{g}^R(\mathbf{r}_5-\mathbf{r}_6,\epsilon')\tilde{g}^A(\mathbf{r}_5-\mathbf{r}_8,\epsilon')\tilde{g}^R(\mathbf{r}_7-\mathbf{r}_8,\epsilon')\varphi_{N,0}(\mathbf{r}_2-\mathbf{r}_5)\varphi_{M,0}(\mathbf{r}_8-\mathbf{r}_{11})\\
&\times\tilde{C}^R_N(\epsilon,\omega-\epsilon)\tilde{C}^R_M(\epsilon,\omega-\epsilon)\tilde{L}^R_N(\omega)\tilde{L}^A_M(\omega)\tilde{C}^R_N(\epsilon',\omega-\epsilon')\tilde{C}^R_M(\epsilon',\omega-\epsilon')F(\epsilon,\epsilon',\omega).
\end{aligned}
\tag{B10}
$$

Here, $F(\epsilon,\epsilon',\omega)=[\tanh(\epsilon/2T)-\tanh((\epsilon-\omega)/2T)]\tanh((\omega-\epsilon')/2T)\partial n_P(\omega)/\partial\omega$, and the gauge invariant velocity $\tilde{v}$ was already defined in the previous example. The phase Φ is

$$
\Phi=\frac{e\mathbf{H}}{2c}\left[(\mathbf{r}_{11}-\mathbf{r}_1)\times(\mathbf{r}_1-\mathbf{r}_2)+(\mathbf{r}_5-\mathbf{r}_6)\times(\mathbf{r}_6-\mathbf{r}_8)+2(\mathbf{r}_2-\mathbf{r}_5)\times(\mathbf{r}_5-\mathbf{r}_8)+2(\mathbf{r}_8-\mathbf{r}_{11})\times(\mathbf{r}_{11}-\mathbf{r}_2)\right].
\tag{B11}
$$

The first two terms in Eq. (B11) correspond to the magnetic fluxes accumulated in the triangles $(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_{11})$ and $(\mathbf{r}_5,\mathbf{r}_6,\mathbf{r}_8)$, respectively. One may check that the contributions to the transverse current obtained by expanding the fluxes from these two triangles or the diamagnetic terms vanish. Therefore, the integration over the coordinates of the two triangles can be done with the quasiparticle Green's functions taken at $\mathbf{H}=0$. After integrating over the quasiparticles degrees of freedom, the triangles $(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_{11})$ and $(\mathbf{r}_5,\mathbf{r}_6,\mathbf{r}_8)$ become proportional to gradients acting on the propagators in the particle-particle channel. Using the remaining two fluxes, corresponding to the triangles $(\mathbf{r}_2,\mathbf{r}_5,\mathbf{r}_8)$ and $(\mathbf{r}_2,\mathbf{r}_8,\mathbf{r}_{11})$, the expression for the current can be written in the

following way:

$$j^y_{AL} = -\frac{e^2E_x}{8\pi^2\ell_H^2}\nu^2\tau^4\int d\epsilon\, d\epsilon' d\omega \int d\mathbf{r}\sum_{N,M}\left[2D\left(\frac{\partial}{\partial y}-\frac{ieHx}{c}\right)\varphi_{N,0}(\mathbf{r})\right]\left[2D\left(\frac{\partial}{\partial x}+\frac{ieHy}{c}\right)\varphi_{M,0}(\mathbf{r})\right]$$
$$\times\,\tilde{C}^R_N(\epsilon,\omega-\epsilon)\tilde{C}^R_M(\epsilon,\omega-\epsilon)\tilde{L}^R_N(\omega)\tilde{L}^A_M(\omega)\tilde{C}^R_N(\epsilon',\omega-\epsilon')\tilde{C}^R_M(\epsilon',\omega-\epsilon')F(\epsilon,\epsilon',\omega). \tag{B12}$$

Let us define the velocity operator, $V_i = 2D[\nabla_i - ie(\mathbf{H}\times\mathbf{r})_i/c]$, of an auxiliary particle with a mass equal to $1/2D$. The integral over the coordinate corresponds to the matrix element of the velocity operators $\langle N,0|V_iV_j|M,0\rangle$, where $|M,0\rangle = \varphi_{M,0}$ is the quantum state of the particle in the M Landau level and with zero angular momentum in the z direction. Using the known properties of the Laguerre polynomials, the matrix element can be written as

$$\langle N,0|V_iV_j|M,0\rangle = 4ieD^2H[(N+1)\delta_{N,M-1}+(-1)^{i+j}(M+1)\delta_{M,N-1}]/c. \tag{B13}$$

Finally, the contribution to the current acquires the form

$$j^y_{AL} = i\frac{e^3E_xH}{2\pi^2\ell_H^2c}\nu^2D^2\tau^4\int d\epsilon\, d\epsilon' d\omega\sum_{N=0}^{\infty}(N+1)\tilde{C}^R_N(\epsilon,\omega-\epsilon)\tilde{C}^R_{N+1}(\epsilon,\omega-\epsilon)$$
$$\times\,\tilde{C}^R_N(\epsilon',\omega-\epsilon')\tilde{C}^R_{N+1}(\epsilon',\omega-\epsilon')\left[\tilde{L}^R_N(\omega)\tilde{L}^A_{N+1}(\omega)-\tilde{L}^R_{N+1}(\omega)\tilde{L}^A_N(\omega)\right]F(\epsilon,\epsilon',\omega). \tag{B14}$$

In the derivation of the new contribution discussed previously, we had to keep corrections to the constant density of states but could set the other small parameter $\varsigma = 0$. Here we must keep ς nonzero, while assuming $\nu(\epsilon)$ to be constant. The vanishing of $\delta\sigma_{xy}$ when both $\nu(\epsilon) = \text{const}$ and $\varsigma = 0$ occurs because the Hall conductivity is zero in a particle-hole symmetric system. Consequently, we found that the Aslamazov-Larkin contribution to $\delta\sigma_{xy}$ is proportional to $\Omega_c\varsigma$.

In the same way, we can derive the remaining parts of the Aslamazov-Larkin term, as well as the three-Cooperon diagrams presented in Figs. 1(c) and 1(d). The contributions of the first two to the Hall conductivity are given in Eqs. (9) and (10). Examining Fig. 1(c), one can see that it is a mirror image of Fig. 1(d). Therefore, they acquire opposite signs as a result of turning the current using the magnetic field, and their sum is identical to zero.

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