

Non-Abelian statistics of vortices with multiple Majorana fermions

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We consider the exchange statistics of vortices, each of which traps an odd number (N) of Majorana fermions. We assume that the fermions in a vortex transform in the vector representation of the $SO(N)$ group. Exchange of two vortices turns out to be non-Abelian, and the corresponding operator is further decomposed into two parts: a part that is essentially equivalent to the exchange operator of vortices having a single Majorana fermion in each vortex, and a part representing the Coxeter group. Similar decomposition was already found in the case with $N = 3$, and the result shown here is a generalization to the case with an arbitrary odd N . We can obtain the matrix representation of the exchange operators in the Hilbert space that is constructed by using Dirac fermions nonlocally defined by Majorana fermions trapped in separated vortices. We also show that the decomposition of the exchange operator implies tensor-product structure in its matrix representation.

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I. INTRODUCTION

There has been considerable interest recently in zero-energy fermion modes trapped inside vortices in superconductors.¹ Vortices in a chiral p -wave superconductor are endowed with non-Abelian statistics^{2,3} because of the zero-energy Majorana fermions inside them.⁴ Excitations which obey non-Abelian statistics are called non-Abelian anyons. They are expected to form the basis of topological quantum computations^{5,6} and have been investigated intensively.⁷ A recent classification of topological superconductors clarifies the condition that vortices have zero-energy Majorana (or Dirac) fermions in their cores.^{8,9} One remarkable development is the fact that non-Abelian anyons in three dimensions¹⁰ can be realized by monopoles with Majorana fermions trapped inside their cores, and give a new non-Abelian statistics, the projective ribbon permutation statistics.¹¹

More recently, the non-Abelian statistics of vortices with additional $SO(3)$ symmetry has been investigated and shown to have a novel structure.¹² In this case, the Majorana fermions form the vector representation of the $SO(3)$ group. It is shown that the representation under the exchange of two vortices is written by the tensor product of two matrices. One matrix is identical to the exchange matrix for vortices with a single Majorana fermion in each core, found by Ivanov³ modulo trivial change of basis, and the other matrix is shown to be a generator of the Coxeter group, which is a symmetry group of certain polytopes in high dimensions.¹³

Such vortices can be physically realized in a “color” superconductor. At extremely high densities and low temperatures, which could be achieved in the cores of compact stars, hadronic matter undergoes phase transition into quark matter, and it is expected to exhibit the color superconductivity.¹⁴ Unlike the ordinary superconductivity in a metal, color superconductivity is induced by condensation of diquarks, which are pairs of two quarks (fermions having “colors” and “flavors”). The order parameter of the color-flavor locked (CFL) phase is given by

a 3×3 matrix

$$\Phi_{\alpha i} = \epsilon_{\alpha\beta\gamma} \epsilon_{ijk} \left((q^T)_{\beta}^j C \gamma_5 (q)_{\gamma}^k \right), \quad (1.1)$$

where q is the quark field, $\alpha, \beta, \gamma = r, g, b$ ($i, j, k = u, d, s$) are color (flavor) indices, and the transpose is employed with respect to the spinor index. At asymptotically high densities, the ground states are expected to be in the CFL phase, in which the diquark acquires an expectation value like $\Phi_{\alpha i} = \Delta \delta_{\alpha i}$, where Δ is a BCS gap function. The original color $SU(3)_C$ and flavor $SU(3)_F$ symmetry break down to an $SU(3)_{C+F}$ symmetry, the elements of which are given by “locked” rotations of color and flavor, $\Phi \rightarrow U \Phi U^{-1}$. This structure is similar to the Balian-Werthamer (BW) state of the ^3He superfluids, in which the order parameter is invariant under the locked rotations of spin and orbit states. We can argue that there exist topologically (and dynamically) stable vortices¹⁵ in the CFL phase by examining the symmetry-breaking pattern. The vortices in the CFL phase are color flux tubes as well as superfluid vortices since both local and global symmetries are broken in the ground state. In the presence of a vortex, the order parameter takes the value like $\Phi(r) = \Delta \text{diag}\{f(r)e^{i\theta}, g(r), g(r)\}$ where r is the radial coordinate and $f(r)$ and $g(r)$ are functions of r . This kind of vortex solution breaks the $SU(3)_{C+F}$ symmetry down to $SU(2)_{C+F} \times U(1)$ symmetry inside the core^{16–21} and fermion zero modes belong to representations of $SU(2)_{C+F}$. It has been shown^{22,23} that one CFL vortex has triplet and singlet Majorana fermions inside it.²⁷ Thus, vortices in a color superconductor provide an example of the system with fermion zero modes in the vector representation of $SO(3)$ in their cores since the triplet of $SU(2)$ is equivalent to the vector representation of $SO(3)$. However, it should be emphasized that the results obtained in Ref. 12 do not depend on details of specific models. The only assumption adopted there is that a vortex has Majorana fermions which transform in the vector representation of $SO(3)$, and therefore we expect that such a system can be found

in condensed matter systems such as exotic superconductors or ultracold atomic gasses.

In this article, we generalize the results of Ref. 12 obtained for SO(3) to the case of SO(N) where N is an arbitrary odd integer $N \geq 3$. We discuss the exchange statistics of two vortices having N Majorana fermions trapped in their cores, which are transformed in an SO(N) group. Notice that the SO(N) symmetry is the maximum symmetry in the presence of N Majorana fermions which are real fields. We discuss the exchange statistics in both the operator and representation levels. We find that both the operator and the matrix representation of the exchange operation generally have factorized structures for arbitrary odd N . In particular, the matrix representation is written as a tensor product of two matrices, as previously found in the SO(3) case. One of the two matrices is the one discussed in Ref. 3 in the case that a single fermion is trapped in each vortex. We show that the other is again a generator of the Coxeter group, as in the case of SO(3). When one Majorana fermion is topologically protected in the vortex core such as in class-D topological superconductors, it is robust against perturbations and remains zero energy under the exchange operation. In addition to that, N Majorana fermions remain zero energy as far as the SO(N) symmetry remains intact.

This paper is organized as follows. In Sec. II, we review the non-Abelian statistics in the case of a single Majorana fermion studied in Ref. 3. In Sec. III, after summarizing the previous work on the SO(3) case,¹² we present generalization to the case of multiple (N) Majorana fermions with SO(N) symmetry. We explicitly show factorization of the exchange operators into the known part similar to the case with a single Majorana fermion and the part corresponding to the Coxeter group. We also show an interesting decomposition of the Majorana fermion operators, which clarifies the action of the Coxeter group on Majorana fermions. In Sec. V, we discuss the relation between the decomposition of the exchange operator and its matrix representation. Section VI is devoted to summary and discussion. In the Appendix, we prove the decomposition theorem of the exchange operation of vortices.

II. VORTICES WITH $N = 1$

We briefly review how non-Abelian statistics emerges for a set of n vortices, each of which contains a single Majorana fermion in its core.³ This provides the simplest example of non-Abelian statistics of vortices, but as we will see later, we can always identify the same structure even for the case with multiple Majorana fermions as far as the number of fermions N is odd.

Consider an exchange of two vortices in a system of $n = 2m$ vortices.²⁸ Each vortex has a single Majorana fermion localized at the core of it, and one can specify the position of a vortex on a two-dimensional plane (we label the vortices $k = 1, \dots, n$). Notice that the trapped Majorana fermion has zero energy, which gives rise to degeneracy of the ground (lowest-energy) state. Since the existence of Majorana fermions is topologically guaranteed, the degeneracy is not disturbed by small perturbations, and hence we treat the exchange of vortices as an adiabatic process.

Let an operation T_k be defined as an exchange of the k th and $(k + 1)$ th vortices, in which the $(k + 1)$ th vortex turns around the k th vortex in an anticlockwise way. All the exchanges of two vortices are realized by successive application of the exchanges of adjacent vortices T_k , and they form a braid group B_n . They indeed satisfy the braid relations

$$T_k T_\ell T_k = T_\ell T_k T_\ell \quad \text{for } |k - \ell| = 1, \quad (2.1)$$

$$T_k T_\ell = T_\ell T_k \quad \text{for } |k - \ell| > 1. \quad (2.2)$$

Recall that the vortices are accompanied by Majorana fermions, and one can express the action of T_k on Majorana fermions as a transformation. To this end, we define a Majorana operator γ_k corresponding to the Majorana fermion in the k th vortex,³ satisfying the self-conjugate condition $(\gamma_k)^\dagger = \gamma_k$ and the anticommutation relation $\{\gamma_k, \gamma_\ell\} = 2\delta_{k\ell}$ (the Clifford algebra). Then, the operation T_k induces the following transformation³:

$$T_k : \begin{cases} \gamma_k \rightarrow \gamma_{k+1}, \\ \gamma_{k+1} \rightarrow -\gamma_k \end{cases} \quad (2.3)$$

with the rest γ_ℓ ($\ell \neq k, k + 1$) unchanged. One can explicitly check that the transformation (2.3) satisfies the braid relations (2.1) and (2.2). The minus sign in the second line is essential for the non-Abelian statistics because it gives $T_k^4 = 1$ (the Bose-Einstein or Fermi-Dirac statistics give just $T_k^2 = 1$). The transformation (2.3) is realized by the unitary operator

$$\tau_k \equiv \exp\left(\frac{\pi}{4}\gamma_{k+1}\gamma_k\right) = \frac{1}{\sqrt{2}}(1 + \gamma_{k+1}\gamma_k). \quad (2.4)$$

Indeed, one finds

$$\tau_k \gamma_k \tau_k^{-1} = \gamma_{k+1}, \quad (2.5)$$

$$\tau_k \gamma_{k+1} \tau_k^{-1} = -\gamma_k, \quad (2.6)$$

$$\tau_k \gamma_\ell \tau_k^{-1} = \gamma_\ell \quad (\ell \neq k, k + 1). \quad (2.7)$$

We call the operator τ_k for this single Majorana case the ‘‘Ivanov operator’’ since it was first found by Ivanov.³ One can explicitly see that this transformation is indeed non-Abelian in the matrix representation of τ_k . To construct the Hilbert space on which the operator τ_k acts, we define Dirac fermions²⁹ by using two Majorana fermions at adjacent vortices $\Psi_K = (\gamma_{2K-1} + i\gamma_{2K})/2$ with $K = 1, \dots, m$. These Dirac fermion operators satisfy the usual anticommutation relations

$$\{\Psi_K, \Psi_L^\dagger\} = \delta_{KL}, \quad \{\Psi_K, \Psi_L\} = \{\Psi_K^\dagger, \Psi_L^\dagger\} = 0. \quad (2.8)$$

If one defines Ψ_K and Ψ_K^\dagger as the annihilation and creation operators, respectively, then one can construct the Hilbert space by acting the creation operators Ψ_K^\dagger 's on the ‘‘Fock vacuum state’’ $|0\rangle$ defined by $\Psi_K|0\rangle = 0$ for all K . Within this Hilbert space, the operators τ_k 's are now expressed as matrices which we call the Ivanov matrices. The Ivanov matrices contain off-diagonal elements representing the non-Abelian statistics. Explicit forms of the Ivanov matrices for two and four vortices are given in Ref. 3 and also in Ref. 12 with a different expression, both of which are related to each other by unitary transformations.

III. VORTICES WITH MULTIPLE MAJORANA FERMIONS $N \geq 3$

Now let us turn to the case with vortices having multiple Majorana fermions in their cores. Each vortex traps N Majorana fermions γ_k^a ($a = 1, \dots, N$) which transform in the vector representation of $SO(N)$ symmetry, hence we call them non-Abelian Majorana fermions. We consider N to be an arbitrary odd number, and this is a generalization of the simplest nontrivial case $N = 3$ in Ref. 12.

The non-Abelian Majorana operators γ_k^a satisfy the self-conjugate conditions and the anticommutation relations

$$(\gamma_k^a)^\dagger = \gamma_k^a, \quad \{\gamma_k^a, \gamma_\ell^b\} = 2\delta^{ab}\delta_{k\ell}. \quad (3.1)$$

We define the exchange of the k th and $(k+1)$ th vortices so that the Majorana fermion operator with each a transforms in the same way as the case with a single Majorana fermion [see Eq. (2.3)] (Ref. 30)

$$T_k : \begin{cases} \gamma_k^a & \rightarrow \gamma_{k+1}^a \\ \gamma_{k+1}^a & \rightarrow -\gamma_k^a \end{cases} \quad \text{for all } a, \quad (3.2)$$

with the rest γ_ℓ^a ($\ell \neq k, k+1$) unchanged. The exchange operations T_k 's satisfy the braid relations (2.1) and (2.2). Their action on γ_ℓ^a 's can be represented in terms of non-Abelian Majorana operators γ_k^a 's in the following way. Since the transformation (3.2) for each a is equivalent to the single Majorana case, one can use the same expression for the unitary operator which induces the transformation for each a :

$$\tau_k^a \equiv \exp\left(\frac{\pi}{4}\gamma_{k+1}^a\gamma_k^a\right) = \frac{1}{\sqrt{2}}(1 + \gamma_{k+1}^a\gamma_k^a). \quad (3.3)$$

Thus, the exchange operator for the vortices having multiple fermions should be represented as the product of them:

$$\tau_k^{[N]} \equiv \prod_{a=1}^N \tau_k^a. \quad (3.4)$$

This exchange operator is $SO(N)$ invariant as shown in the next section. One can check that the operator $\tau_k^{[N]}$ applied to γ_ℓ^a indeed generates the desired transformation (3.2):

$$\tau_k^{[N]}\gamma_k^a(\tau_k^{[N]})^{-1} = \gamma_{k+1}^a, \quad (3.5)$$

$$\tau_k^{[N]}\gamma_{k+1}^a(\tau_k^{[N]})^{-1} = -\gamma_k^a, \quad (3.6)$$

$$\tau_k^{[N]}\gamma_\ell^a(\tau_k^{[N]})^{-1} = \gamma_\ell^a \quad (\ell \neq k, k+1). \quad (3.7)$$

This transformation is again non-Abelian, which is explicitly seen in the matrix representation.

To obtain the matrix representation, we can perform the same procedure as in the case with single Majorana fermions. Namely, by defining the Dirac fermion operators

$$\Psi_K^a \equiv \frac{1}{2}(\gamma_{2K-1}^a + i\gamma_{2K}^a), \quad \Psi_K^{a\dagger} \equiv \frac{1}{2}(\gamma_{2K-1}^a - i\gamma_{2K}^a), \quad (3.8)$$

which satisfy ($K, L = 1, \dots, m$)

$$\{\Psi_K^a, \Psi_L^{b\dagger}\} = \delta_{KL}\delta^{ab}, \quad \{\Psi_K^a, \Psi_L^b\} = \{\Psi_K^{a\dagger}, \Psi_L^{b\dagger}\} = 0, \quad (3.9)$$

we can construct the Hilbert space. Then, we can find matrix representation of $\tau_k^{[N]}$. In Ref. 12, three of us explicitly constructed the Hilbert space for the case of $N = 3$ and $n = 2, 4$, and found the matrix expression of $\tau_k^{[3]}$ according to different representations of $SO(3)$. The matrices have off-diagonal elements and thus they are non-Abelian. In principle, one can do the same thing for an arbitrary odd N . In this paper, however, we first look into interesting structure of the operator $\tau_k^{[N]}$, which was also suggested in the previous paper.¹² Namely, the operator $\tau_k^{[N]}$ itself can be decomposed into two parts. Then, we discuss the relation between the decomposition of $\tau_k^{[N]}$ and the matrix representation of $\tau_k^{[N]}$ in Sec. V.

IV. COXETER GROUP FOR MULTIPLE MAJORANA FERMIONS $N \geq 3$

In the previous analysis¹² with $N = 3$, we found that the matrix representation of $\tau_k^{[3]}$ in the four-vortex sector can be decomposed into a tensor product of two matrices: one is the same as the Ivanov matrix for the single Majorana fermion case, and the other is identified with generators of the Coxeter group. We also found that the similar decomposition is possible at the operator level.¹² Namely, one can express $\tau_k^{[3]}$ as a product of two distinct operators which give rise to the corresponding matrices in the matrix representation. In this section, we discuss that such a decomposition at the operator level holds even for an arbitrary odd number of N .

Before we go into details, let us define some useful notations. We first define a composite operator Γ_k^a by

$$\Gamma_k^a \equiv \gamma_{k+1}^a\gamma_k^a, \quad (4.1)$$

which have the following properties:

$$\{\Gamma_k^a, \Gamma_l^b\} = -2\delta_{kl} \quad \text{for } a = b \quad \text{and} \quad |k - l| \leq 1, \quad (4.2)$$

$$[\Gamma_k^a, \Gamma_l^b] = 0 \quad \text{for } a \neq b, \quad \text{or} \quad (a = b \quad \text{and} \quad |k - l| > 1). \quad (4.3)$$

For later convenience, we define another composite operator ($1 \leq n \leq N$)

$$\Gamma_k^{(n)} \equiv \frac{1}{(n!)^2} e^{\frac{\pi}{2}i(n-1)} \epsilon^{a_1 \dots a_n} \epsilon^{b_1 \dots b_n} \delta_{b_{n+1}}^{a_{n+1}} \dots \delta_{b_N}^{a_N} \\ \times \gamma_{k+1}^{a_1} \dots \gamma_{k+1}^{a_n} \gamma_k^{b_1} \dots \gamma_k^{b_n}, \quad (4.4)$$

where $\epsilon^{a_1 \dots a_n}$ is the completely antisymmetric tensor. It is evident that the operators $\Gamma_k^{(n)}$'s are $SO(N)$ invariants for all n . For example, for $N = 3$,

$$\Gamma_k^{(1)} = \Gamma_k^1 + \Gamma_k^2 + \Gamma_k^3, \quad \Gamma_k^{(2)} = \Gamma_k^1\Gamma_k^2 + \Gamma_k^2\Gamma_k^3 + \Gamma_k^3\Gamma_k^1, \\ \Gamma_k^{(3)} = \Gamma_k^1\Gamma_k^2\Gamma_k^3. \quad (4.5)$$

With those composite operators, the exchange operator $\tau_k^{[N]}$ can be represented as

$$\tau_k^{[N]} = \left(\frac{1}{\sqrt{2}}\right)^N \prod_{a=1}^N (1 + \Gamma_k^a) = \exp\left\{\frac{\pi}{4} \sum_a \Gamma_k^a\right\} \\ = \left(\frac{1}{\sqrt{2}}\right)^N \sum_{n=1}^N \Gamma_k^{(n)}. \quad (4.6)$$

In the final expression, we confirm that the operator $\tau_k^{[N]}$ is $\text{SO}(N)$ invariant.

A. The case of $\text{SO}(3)$

The simplest nontrivial case with $N = 3$ provides us with useful information which is helpful in discussing the decomposition for the general case N . Let us first recall that the operator $\tau_k^{[N=3]}$ has been found to be decomposed into two parts (see Appendix B in Ref. 12):

$$\tau_k^{[3]} = \sigma_k^{[3]} h_k^{[3]}, \quad (4.7)$$

where both of the operators $\sigma_k^{[3]}$ and $h_k^{[3]}$ are given in terms of the Majorana operators γ_k^a as

$$\sigma_k^{[3]} = \frac{1}{2} (1 - \gamma_{k+1}^1 \gamma_{k+1}^2 \gamma_k^1 \gamma_k^2 - \gamma_{k+1}^2 \gamma_{k+1}^3 \gamma_k^2 \gamma_k^3 - \gamma_{k+1}^3 \gamma_{k+1}^1 \gamma_k^3 \gamma_k^1) \quad (4.8)$$

and

$$h_k^{[3]} = \frac{1}{\sqrt{2}} (1 - \gamma_{k+1}^1 \gamma_{k+1}^2 \gamma_{k+1}^3 \gamma_k^1 \gamma_k^2 \gamma_k^3). \quad (4.9)$$

One can also rewrite them compactly in new notations as [see Eqs. (4.4) and (4.5)]

$$\sigma_k^{[3]} = \frac{1}{2} (1 + \Gamma_k^{(2)}), \quad h_k^{[3]} = \frac{1}{\sqrt{2}} (1 + \Gamma_k^{(3)}). \quad (4.10)$$

Note that $\sigma_k^{[3]}$ and $h_\ell^{[3]}$ are commutative for any pair of k and ℓ . Thus, $\tau_k^{[3]}$ is the product of $\sigma_k^{[3]}$ and $h_k^{[3]}$. It was shown that the operators $\sigma_k^{[3]}$'s satisfy the Coxeter relations³¹

$$(\sigma_k^{[3]})^2 = 1, \quad (4.11)$$

$$(\sigma_k^{[3]} \sigma_\ell^{[3]})^3 = 1 \quad \text{for } |k - \ell| = 1, \quad (4.12)$$

$$(\sigma_k^{[3]} \sigma_\ell^{[3]})^2 = 1 \quad \text{for } |k - \ell| > 1. \quad (4.13)$$

The Coxeter group found here for the $n = 2m$ vortices is classified as A_{2m-1} . It is also known as the symmetric group S_{2m} , which is the symmetry group of a regular $(2m - 1)$ simplex.

In contrast, the other part $h_k^{[3]}$ works in the same way as the Ivanov operator τ_k defined in Eq. (2.4), although $h_k^{[3]}$ has a more complicated structure. This is naturally understood if one notices that the operator $\gamma_k^1 \gamma_k^2 \gamma_k^3$ is $\text{SO}(3)$ invariant (singlet), and thus it treats three Majorana fermions together as if it does not have any $\text{SO}(3)$ index. This picture is very useful when we consider the general case with N .

B. The case of $\text{SO}(N)$ with arbitrary odd N

The previous analysis suggests that it is possible to decompose the full exchange operator $\tau_k^{[N]}$ into two parts. This is indeed the case. We find that the operator $\tau_k^{[N]}$ can be decomposed as

$$\tau_k^{[N]} = \sigma_k^{[N]} h_k^{[N]}, \quad (4.14)$$

where $\sigma_k^{[N]}$ and $h_k^{[N]}$ are defined by using the notation introduced before as

$$\sigma_k^{[N]} \equiv \left(\frac{1}{\sqrt{2}} \right)^{N-1} (1 + \Gamma_k^{(2)} + \Gamma_k^{(4)} + \dots + \Gamma_k^{(N-1)}), \quad (4.15)$$

$$h_k^{[N]} \equiv \frac{1}{\sqrt{2}} (1 + \Gamma_k^{(N)}). \quad (4.16)$$

Note that $\sigma_k^{[N]}$ and $h_k^{[N]}$ are $\text{SO}(N)$ invariant, and $\sigma_k^{[N]}$ and $h_\ell^{[N]}$ are commutative for any k and ℓ . One can readily verify the decomposition (4.14). By using Eqs. (4.15) and (4.16), one can check that the product $\sigma_k^{[N]} h_k^{[N]}$ is equal to the last equation in Eq. (4.6) if one uses Eqs. (4.2) and (4.3).

First of all, let us discuss the properties of $h_k^{[N]}$. The analysis for the case $N = 3$ suggests that if one treats multiple Majorana fermions in a vortex in a unit, then the action of $\tau_k^{[N]}$ will be essentially equivalent to the Ivanov operator. This motivates us to introduce the following ‘‘singlet Majorana operator’’ locally defined on the k th vortex:

$$\bar{\gamma}_k \equiv \frac{1}{N!} e^{i\frac{\pi}{4}(N-1)} \epsilon^{a_1 a_2 \dots a_N} \gamma_k^{a_1} \gamma_k^{a_2} \dots \gamma_k^{a_N}, \quad (4.17)$$

which is manifestly invariant under the $\text{SO}(N)$ transformation. The phase factor is included so that the operator becomes self-conjugate $(\bar{\gamma}_k)^\dagger = \bar{\gamma}_k$, and satisfies the Clifford algebra $\{\bar{\gamma}_k, \bar{\gamma}_\ell\} = 2\delta_{k\ell}$. Notice that these properties of $\bar{\gamma}_k$ are the same as those of a single Majorana operator. For $N = 3$, one finds $\bar{\gamma}_k = i\gamma_k^1 \gamma_k^2 \gamma_k^3$, and the operator $h_k^{[3]}$ can be compactly expressed as $h_k^{[3]} = \frac{1}{\sqrt{2}} (1 + \bar{\gamma}_{k+1} \bar{\gamma}_k)$, which has the same structure as the Ivanov operator (2.4). For arbitrary odd N , we find that $h_k^{[N]}$ can also be expressed as

$$h_k^{[N]} = \exp\left(\frac{\pi}{4} \bar{\gamma}_{k+1} \bar{\gamma}_k\right) = \frac{1}{\sqrt{2}} (1 + \bar{\gamma}_{k+1} \bar{\gamma}_k) \quad (4.18)$$

by noting the relation $\Gamma_k^{(N)} = \bar{\gamma}_{k+1} \bar{\gamma}_k$. Then, $h_k^{[N]}$ works on $\bar{\gamma}_\ell$ as

$$h_k^{[N]} \bar{\gamma}_k (h_k^{[N]})^{-1} = \bar{\gamma}_{k+1}, \quad (4.19)$$

$$h_k^{[N]} \bar{\gamma}_{k+1} (h_k^{[N]})^{-1} = -\bar{\gamma}_k, \quad (4.20)$$

$$h_k^{[N]} \bar{\gamma}_\ell (h_k^{[N]})^{-1} = \bar{\gamma}_\ell \quad (\ell \neq k, k+1), \quad (4.21)$$

for an arbitrary odd $N \geq 3$. Interestingly, this is the same as the transformations (2.5)–(2.7) induced by the Ivanov operator with $N = 1$. Therefore, the operator $h_k^{[N]}$ for the singlet Majorana operator $\bar{\gamma}_k$ is equivalent to the Ivanov operator τ_k for the single Majorana operator γ_k .

Next, we discuss the properties of the operators $\sigma_k^{[N]}$. Similarly to the $N = 3$ case, $\sigma_k^{[N]}$ are generators of the Coxeter group. Indeed, it can be confirmed that $\sigma_k^{[N]}$ satisfy (see Appendix for details)

$$(\sigma_k^{[N]})^2 = 1, \quad (4.22)$$

$$(\sigma_k^{[N]} \sigma_\ell^{[N]})^3 = 1 \quad \text{for } |k - \ell| = 1, \quad (4.23)$$

$$(\sigma_k^{[N]} \sigma_\ell^{[N]})^2 = 1 \quad \text{for } |k - \ell| > 1, \quad (4.24)$$

and these relations induce the same Coxeter group for $n = 2m$ vortices. We thus have found that, for an arbitrary odd N , the operators $\sigma_k^{[N]}$'s again obey the Coxeter relations of A_{2m-1} .

In fact, one can “derive” the decomposition (4.14) by first defining $h_k^{[N]}$ in terms of the singlet Majorana operators $\bar{\gamma}_k$ as in Eq. (4.18), and then assuming the factorized form (4.14). The operator $\sigma_k^{[N]} = \tau_k^{[N]}(h_k^{[N]})^{-1}$ thus obtained indeed coincides with Eq. (4.15), and the commutativity of $\sigma_k^{[N]}$ and $h_\ell^{[N]}$ for any k and ℓ confirms the assumption of factorization. We thus have found that, for an arbitrary odd N , the exchange operator $\tau_k^{[N]}$ is expressed as a product of a generator of the Coxeter group A_{2m-1} (for the vortex number $n = 2m$) and the Ivanov operator for a single Majorana fermion.

C. Decomposition of the Majorana operators

Let us recall that the exchange of the Majorana operators γ_ℓ^a 's is originally defined as the operation T_k in Eq. (3.2). It is not apparently clear how the decomposed structure of the operator $\tau_k^{[N]}$ indeed works in the exchange operation T_k . To understand this, it is instructive to notice that the Majorana operator γ_k^a can be rewritten as

$$\gamma_k^a = \tilde{\gamma}_k^a \bar{\gamma}_k, \quad (4.25)$$

where $\tilde{\gamma}_k^a$ is a composite operator in the vector representation of $SO(N)$ defined locally on the k th vortex as

$$\tilde{\gamma}_k^a \equiv \frac{1}{(N-1)!} e^{i\frac{\pi}{4}(N-1)} \epsilon^{aa_1 \dots a_{N-1}} \gamma_k^{a_1} \dots \gamma_k^{a_{N-1}}, \quad (4.26)$$

and $\bar{\gamma}_k$ is the singlet Majorana operator defined in Eq. (4.17). The two operators $\tilde{\gamma}_k^a$ and $\bar{\gamma}_\ell$ commute with each other for any pair of k and ℓ . This expression allows us to extract, from the Majorana operator γ_k^a , the part of a singlet Majorana fermion $\bar{\gamma}_k$ whose properties are well understood. Notice that $\bar{\gamma}_k$ ($\tilde{\gamma}_k^a$) is composed of an odd number N (an even number $N-1$) of Majorana fermion operators.

Since $\tilde{\gamma}_k^a$ and $\bar{\gamma}_k$ are expressed in terms of the original Majorana operator γ_k^a , one can immediately find how they are transformed in the exchange T_k . Namely, we find the transformation of $\tilde{\gamma}_k^a$ and $\bar{\gamma}_k$ by T_k as

$$T_k : \begin{cases} \tilde{\gamma}_k^a & \rightarrow \tilde{\gamma}_{k+1}^a \\ \tilde{\gamma}_{k+1}^a & \rightarrow \tilde{\gamma}_k^a \end{cases} \quad \text{for all } a, \quad (4.27)$$

without a minus sign, and

$$T_k : \begin{cases} \bar{\gamma}_k & \rightarrow \bar{\gamma}_{k+1} \\ \bar{\gamma}_{k+1} & \rightarrow -\bar{\gamma}_k \end{cases} \quad \text{for all } a, \quad (4.28)$$

with a minus sign, while the rest $\tilde{\gamma}_\ell^a$ and $\bar{\gamma}_\ell$ ($\ell \neq k$ and $k+1$) are unchanged.³² It is easily checked that the simultaneous transformation of $\tilde{\gamma}_k^a$ and $\bar{\gamma}_k$ reproduces the transformation of γ_k^a in Eq. (3.2). Therefore, we observe from Eq. (4.27) that $\tilde{\gamma}_\ell^a$'s are transformed by a symmetric group S_{2m} , or the Coxeter group of A_{2m-1} (for the vortex number $n = 2m$), and from Eq. (4.28) that $\bar{\gamma}_\ell$'s are transformed as in the same way as the Ivanov operators for single Majorana fermions.³³

The exchange properties of $\tilde{\gamma}_k^a$ and $\bar{\gamma}_k$ in Eqs. (4.27) and (4.28) can be discussed at the operator level. Because $\sigma_k^{[N]}$ and $\bar{\gamma}_l$ ($h_k^{[N]}$ and $\tilde{\gamma}_l^a$) are commutative for any pair of k

and ℓ ,

$$[\sigma_k^{[N]}, \bar{\gamma}_\ell] = [h_k^{[N]}, \tilde{\gamma}_\ell^a] = 0, \quad (4.29)$$

the transformation $\tau_k^{[N]} \gamma_\ell^a (\tau_k^{[N]})^{-1}$ is decomposed as

$$\tau_k^{[N]} \gamma_\ell^a (\tau_k^{[N]})^{-1} = \{\sigma_k^{[N]} \tilde{\gamma}_\ell^a (\sigma_k^{[N]})^{-1}\} \{h_k^{[N]} \bar{\gamma}_\ell (h_k^{[N]})^{-1}\}. \quad (4.30)$$

Therefore, $\tilde{\gamma}_l^a$ and $\bar{\gamma}_l$ are transformed by $\sigma_k^{[N]}$ and $h_k^{[N]}$, respectively. From Eqs. (4.15) and (4.26), $\tilde{\gamma}_k^a$ is transformed as

$$\sigma_k^{[N]} \tilde{\gamma}_k^a (\sigma_k^{[N]})^{-1} = \tilde{\gamma}_{k+1}^a, \quad (4.31)$$

$$\sigma_k^{[N]} \tilde{\gamma}_{k+1}^a (\sigma_k^{[N]})^{-1} = \tilde{\gamma}_k^a, \quad (4.32)$$

$$\sigma_k^{[N]} \tilde{\gamma}_\ell^a (\sigma_k^{[N]})^{-1} = \tilde{\gamma}_\ell^a \quad (\ell \neq k, k+1), \quad (4.33)$$

without a minus sign. Thus, the operator $\sigma_k^{[N]}$ acting on $\tilde{\gamma}_l^a$ reproduces the transformation (4.27). We note that $\sigma_k^{[N]}$ can be expressed in terms of $\tilde{\gamma}_k^a$ only. On the other hand, $\bar{\gamma}_k$ is transformed by the operator $h_k^{[N]}$ like a single Majorana fermion as demonstrated in Eqs. (4.19)–(4.21), and hence $h_k^{[N]}$ reproduces the transformation (4.28).

To summarize this section, in correspondence to the product of $\tau_k^{[N]} = \sigma_k^{[N]} h_k^{[N]}$, the Majorana operator γ_k^a is also expressed by the product of the two parts, $\tilde{\gamma}_k^a$ obeying the Coxeter group given by $\sigma_k^{[N]}$ and $\bar{\gamma}_k$ obeying Ivanov's exchange given by $h_k^{[N]}$.

V. OPERATOR DECOMPOSITION AND MATRIX REPRESENTATION

So far, we have been discussing the factorized structure of the exchange operation of two vortices at the operator level. Everything was written in terms of the Majorana operators, and the decomposition into the Coxeter and Ivanov parts was naturally understood by using the Majorana operators. In contrast, in order to obtain the matrix representation, the usual procedure is to define Dirac fermion operators and use them in constructing the Hilbert space. Since the Dirac fermion operators are constructed from two Majorana fermions located separately at different vortices, it is not trivial if the factorized structure at the operator level is preserved in the matrix representation. For example, the Dirac fermion operator defined in Eq. (3.8) can not be decomposed similarly as the Majorana fermion operator as shown in Eq. (4.25). In this section, we are going to show that the decomposition holds even in the matrix representation in a suitable basis, and discuss the relationship between the the decompositions in the operator- and matrix-representation levels.

In the following, we discuss the case with $N = 3$ for simplicity, but present the procedures to obtain the matrix representation so that they can be easily extended to the case with any odd N .

A. Construction of the Hilbert space

Let us consider an even number $n = 2m$ of vortices. Then, we can construct m Dirac fermion operators Ψ_K^a ($a = 1, 2, 3$; $K = 1, \dots, m$), given in Eq. (3.8), in the

vector representation of SO(3). The Fock vacuum $|0\rangle$ is defined by the Dirac fermion operators as

$$\Psi_K^a |0\rangle = 0 \quad \text{for all } K \text{ and } a. \quad (5.1)$$

One can construct the basis of the Hilbert space by acting the Dirac fermion operators $\Psi_K^{a\dagger}$ on the Fock vacuum $|0\rangle$. The explicit forms of the basis were given in Ref. 12 for $N = 3$ and $n = 2, 4$. Now let us construct the Hilbert space in a way different from Ref. 12. To this end, we define the number operator for the triplet Dirac fermions of the K th pair of vortices by

$$\mathcal{N}_K^a \equiv \Psi_K^{a\dagger} \Psi_K^a. \quad (5.2)$$

A generic state for the K th pair of vortices can be expressed in terms of the eigenvalues of this number operator as

$$|\mathcal{N}_K^1, \mathcal{N}_K^2, \mathcal{N}_K^3\rangle_K, \quad (5.3)$$

where $\mathcal{N}_K^a = 0$ or 1 is the occupation number of the fermion created by $\Psi_K^{a\dagger}$ (here we use the same character for the operator and its eigenvalues). Then, the basis of the whole Hilbert space is composed of the tensor product of the states for each K :

$$\left\{ \bigotimes_{K=1}^m |\mathcal{N}_K^1, \mathcal{N}_K^2, \mathcal{N}_K^3\rangle_K \right\}. \quad (5.4)$$

B. Singlet Dirac operators

When $N = 3$, the singlet Majorana operators $\bar{\gamma}_k$ defined in Eq. (4.17) are given by

$$\bar{\gamma}_k = \frac{1}{3!} i \epsilon^{abc} \gamma_k^a \gamma_k^b \gamma_k^c. \quad (5.5)$$

By using these operators, we define singlet Dirac operators

$$\bar{\Psi}_K \equiv \frac{1}{2} (\bar{\gamma}_{2K-1} + i \bar{\gamma}_{2K}), \quad \bar{\Psi}_K^\dagger \equiv \frac{1}{2} (\bar{\gamma}_{2K-1} - i \bar{\gamma}_{2K}). \quad (5.6)$$

These operators play a particular role when they act on the states. Let us examine the action of the singlet Dirac operators $\bar{\Psi}_K$'s and $\bar{\Psi}_K^\dagger$'s on the Hilbert space (5.4), which is constructed by acting the triplet Dirac operators $\Psi_K^{a\dagger}$'s and Ψ_K^a 's on the Fock vacuum $|0\rangle$. To this end, we express the singlet Dirac operators $\bar{\Psi}_K$ and $\bar{\Psi}_K^\dagger$ in terms of the triplet Dirac operators Ψ_K^a and $\Psi_K^{a\dagger}$. For that purpose, we note that the singlet Majorana operators $\bar{\gamma}_{2K-1}$ and $\bar{\gamma}_{2K}$ defined in Eq. (4.17) can be written in two ways:

$$\begin{aligned} \bar{\gamma}_{2K-1} &= \bar{\Psi}_K + \bar{\Psi}_K^\dagger \\ &= \frac{1}{3!} i \epsilon^{abc} (\Psi_K^a + \Psi_K^{a\dagger}) (\Psi_K^b + \Psi_K^{b\dagger}) (\Psi_K^c + \Psi_K^{c\dagger}), \\ \bar{\gamma}_{2K} &= (\bar{\Psi}_K - \bar{\Psi}_K^\dagger) / i \\ &= \frac{1}{3!} i \epsilon^{abc} \left(\frac{1}{i} \right)^3 (\Psi_K^a - \Psi_K^{a\dagger}) (\Psi_K^b - \Psi_K^{b\dagger}) (\Psi_K^c - \Psi_K^{c\dagger}), \end{aligned} \quad (5.7)$$

respectively. From these two relations, we can express the singlet Dirac operators $\bar{\Psi}_K$ and $\bar{\Psi}_K^\dagger$ in terms of the triplet

Dirac operators Ψ_K^a and $\Psi_K^{a\dagger}$ as

$$\begin{aligned} \bar{\Psi}_K &= \frac{1}{3!} i \epsilon^{abc} (\Psi_K^a \Psi_K^b \Psi_K^{c\dagger} + \Psi_K^a \Psi_K^{b\dagger} \Psi_K^c + \Psi_K^{a\dagger} \Psi_K^b \Psi_K^c \\ &\quad + \Psi_K^{a\dagger} \Psi_K^{b\dagger} \Psi_K^{c\dagger}), \\ \bar{\Psi}_K^\dagger &= -\frac{1}{3!} i \epsilon^{abc} (\Psi_K^{a\dagger} \Psi_K^{b\dagger} \Psi_K^c + \Psi_K^{a\dagger} \Psi_K^b \Psi_K^{c\dagger} + \Psi_K^a \Psi_K^{b\dagger} \Psi_K^{c\dagger} \\ &\quad + \Psi_K^a \Psi_K^b \Psi_K^c). \end{aligned} \quad (5.8)$$

For the later purpose, we define the fermion number operator for the K th pair of vortices in terms of the singlet Dirac operators $\bar{\Psi}_K^\dagger$ and $\bar{\Psi}_K$:

$$\bar{\mathcal{N}}_K \equiv \bar{\Psi}_K^\dagger \bar{\Psi}_K. \quad (5.9)$$

The meaning of this number operator will be clarified below.

Now, we can discuss how the singlet Dirac operators $\bar{\Psi}_K$ and $\bar{\Psi}_K^\dagger$ act on the Fock states (5.3) defined by the triplet Dirac operators Ψ_K^a and $\Psi_K^{a\dagger}$. The action of the singlet Dirac operators $\bar{\Psi}_K$ and $\bar{\Psi}_K^\dagger$ on $|\mathcal{N}_K^1, \mathcal{N}_K^2, \mathcal{N}_K^3\rangle_K$ can be read off from Eqs. (5.8) as

$$\begin{aligned} \bar{\Psi}_K |0, 0, 0\rangle_K &= |1, 1, 1\rangle_K, \\ \bar{\Psi}_K |1, 0, 0\rangle_K &= \bar{\Psi}_K |0, 1, 0\rangle_K = \bar{\Psi}_K |0, 0, 1\rangle_K = 0, \\ \bar{\Psi}_K |0, 1, 1\rangle_K &= |1, 0, 0\rangle_K, \quad \bar{\Psi}_K |1, 0, 1\rangle_K = |0, 1, 0\rangle_K, \\ \bar{\Psi}_K |1, 1, 0\rangle_K &= |0, 0, 1\rangle_K, \quad \bar{\Psi}_K |1, 1, 1\rangle_K = 0 \end{aligned} \quad (5.10)$$

for $\bar{\Psi}_K$, and

$$\begin{aligned} \bar{\Psi}_K^\dagger |0, 0, 0\rangle_K &= 0, \quad \bar{\Psi}_K^\dagger |0, 0, 1\rangle_K = |1, 1, 0\rangle_K, \\ \bar{\Psi}_K^\dagger |0, 1, 0\rangle_K &= |1, 0, 1\rangle_K, \quad \bar{\Psi}_K^\dagger |1, 0, 0\rangle_K = |0, 1, 1\rangle_K, \\ \bar{\Psi}_K^\dagger |0, 1, 1\rangle_K &= \bar{\Psi}_K^\dagger |1, 0, 1\rangle_K = \bar{\Psi}_K^\dagger |1, 1, 0\rangle_K = 0, \\ \bar{\Psi}_K^\dagger |1, 1, 1\rangle_K &= |0, 0, 0\rangle_K \end{aligned} \quad (5.11)$$

for $\bar{\Psi}_K^\dagger$.

Let us define the total fermion number operator for the K th pair of vortices as the sum of the number operator \mathcal{N}_K^a in Eq. (5.2) of the triplet Dirac fermions:

$$\mathcal{F}_K \equiv \sum_{a=1}^N \mathcal{N}_K^a. \quad (5.12)$$

Then, we can deduce that the states annihilated by $\bar{\Psi}_K$ are those with odd eigenvalues of \mathcal{F}_K [see the second and fourth lines of Eq. (5.10)], while those with even eigenvalues of \mathcal{F}_K are annihilated by $\bar{\Psi}_K^\dagger$ [see the first and third lines of Eq. (5.11)]:

$$\bar{\Psi}_K^\dagger |\text{even}\rangle_K = 0, \quad \bar{\Psi}_K |\text{odd}\rangle_K = 0, \quad (5.13)$$

where $|\text{even}\rangle_K$ and $|\text{odd}\rangle_K$ are eigenstates of the parity operator

$$\mathcal{P}_K \equiv (-1)^{\mathcal{F}_K}, \quad (5.14)$$

namely,

$$\mathcal{P}_K |\text{even}\rangle_K = +1 |\text{even}\rangle_K, \quad \mathcal{P}_K |\text{odd}\rangle_K = -1 |\text{odd}\rangle_K. \quad (5.15)$$

It should also be noted that when $\bar{\Psi}_K^\dagger$ and $\bar{\Psi}_K$ do not annihilate the state, they create a state with opposite parity:

$$\bar{\Psi}_K^\dagger|\text{odd}\rangle_K = |\text{even}\rangle_K, \quad \bar{\Psi}_K|\text{even}\rangle_K = |\text{odd}\rangle_K. \quad (5.16)$$

These facts imply that the parity operator \mathcal{P}_K anticommutes with $\bar{\Psi}_K$ and $\bar{\Psi}_K^\dagger$:

$$\{\mathcal{P}_K, \bar{\Psi}_K\} = 0, \quad \{\mathcal{P}_K, \bar{\Psi}_K^\dagger\} = 0. \quad (5.17)$$

We claim that the fermion number operator $\bar{\mathcal{N}}_K$ is related with the parity operator \mathcal{P}_K as

$$\mathcal{P}_K = 1 - \bar{\mathcal{N}}_K. \quad (5.18)$$

Namely, the fermion number operator $\bar{\mathcal{N}}_K$ expresses the parity of the total fermion number for each index K . This relation results from Eq. (5.13), which is obvious from the structure of the operators shown in Eqs. (5.8).

By repeating the same argument, the above relation can be generalized to an arbitrary odd N as

$$\mathcal{P}_K = 1 - \bar{\mathcal{N}}_K \quad \text{for } N = 4\ell + 3, \quad (5.19)$$

$$\mathcal{P}_K = \bar{\mathcal{N}}_K \quad \text{for } N = 4\ell + 1. \quad (5.20)$$

C. Tensor-product structure of the Hilbert space

From Eqs. (5.10) and (5.11), we can say that the action of the singlet Dirac operators $\bar{\Psi}_K$ or $\bar{\Psi}_K^\dagger$ is a kind of ‘‘NOT’’ operation, which ‘‘flips’’ the fermion number for each component of the $\text{SO}(3)$ vector. Namely, under the action of $\bar{\Psi}_K$, one finds $|\mathcal{N}_K^1, \mathcal{N}_K^2, \mathcal{N}_K^3\rangle_K \rightarrow |1 - \mathcal{N}_K^1, 1 - \mathcal{N}_K^2, 1 - \mathcal{N}_K^3\rangle_K$. We can divide the states into pairs, in each of which the two states are related by the NOT operation. When $N = 3$, there are four pairs:

$$\begin{aligned} |0,0,0\rangle_K &\leftrightarrow |1,1,1\rangle_K, & |0,0,1\rangle_K &\leftrightarrow |1,1,0\rangle_K, \\ |0,1,1\rangle_K &\leftrightarrow |1,0,0\rangle_K, & |1,0,1\rangle_K &\leftrightarrow |0,1,0\rangle_K. \end{aligned} \quad (5.21)$$

It is important to note that the singlet Dirac operators $\bar{\Psi}_K$ and $\bar{\Psi}_K^\dagger$ induce the transition only within these pairs when they act on states (e.g., a transition between $|0,0,0\rangle_K$ and $|1,1,1\rangle_K$), but they never induce an interpair transition. This fact is essential in the following discussion.

Now, we are ready to discuss how the decomposition of the exchange operator $\tau_k^{[N]}$ results in the tensor-product structure in a matrix representation. From the analysis above, we can take the basis of the Hilbert space which is labeled by the parity \mathcal{P}_K of the number of fermions with index K , and an additional index m_K which labels the choice of a pair in Eq. (5.21). We denote the states by

$$|\mathcal{P}_K, m_K\rangle_K \equiv |\mathcal{P}_K\rangle_K \otimes |m_K\rangle_K. \quad (5.22)$$

Let us consider the matrix elements ${}_K\langle \mathcal{P}_K, m_K | \tau_k^{[N]} | \mathcal{P}'_K, m'_K \rangle_K$ of the exchange operator $\tau_k^{[N]}$ in this basis. Now we show that these matrix elements can be written as the tensor product of the Ivanov matrix and the Coxeter matrix:

$$\begin{aligned} &{}_K\langle \mathcal{P}_K, m_K | \tau_k^{[N]} | \mathcal{P}'_K, m'_K \rangle_K \\ &= {}_K\langle m_K | \sigma_k^{[N]} | m'_K \rangle_K \cdot {}_K\langle \mathcal{P}_K | h_k^{[N]} | \mathcal{P}'_K \rangle_K. \end{aligned} \quad (5.23)$$

Namely, the Coxeter operator $\sigma_k^{[N]}$ acts only on $|m_K\rangle_K$, while the Ivanov operator $h_k^{[N]}$ acts only on $|\mathcal{P}_K\rangle_K$. To prove this, we show the following two statements for any k and K :

1. $h_k^{[N]}$ acts as an identity on $|m_K\rangle_K$.
2. $\sigma_k^{[N]}$ acts as an identity on $|\mathcal{P}_K\rangle_K$.

First, we can see that the statement (i) is true in the following way. The Ivanov operator $h_k^{[N]}$ is written by the singlet Majorana operators $\bar{\gamma}_k$ [see Eq. (4.18)], and hence by the singlet Dirac operators $\bar{\Psi}_L$ and $\bar{\Psi}_L^\dagger$ [see Eq. (5.6)]. When the singlet Dirac operators $\bar{\Psi}_L$ or $\bar{\Psi}_L^\dagger$ act on states, the only state that can be created is the one in which the fermion numbers are flipped. So, the action of the singlet Dirac operators $\bar{\Psi}_L$ or $\bar{\Psi}_L^\dagger$ never induce an interpair transition. Therefore, the Ivanov operator $h_k^{[N]}$ does not change the index m_K , which labels the pair in Eq. (5.21).

Next, the statement (ii) is equivalent to the statement that the Coxeter operator $\sigma_k^{[N]}$ and the total fermion number operator $\bar{\mathcal{N}}_K$ commute. The total fermion number operator $\bar{\mathcal{N}}_K$ is written by the singlet Majorana operators $\bar{\gamma}_{2K-1}$ and $\bar{\gamma}_{2K}$, which commute with the Coxeter operator $\sigma_k^{[N]}$ as in Eq. (4.29). We thus find that the Coxeter operator $\sigma_k^{[N]}$ and the total fermion number operator $\bar{\mathcal{N}}_K$ also commute.

From these discussions, we have shown that the factorization of the exchange operator $\tau_k^{[N]}$ of vortices into the Ivanov operator $h_k^{[N]}$ and the Coxeter operator $\sigma_k^{[N]}$ results in the tensor-product structure in the matrix representation.

VI. SUMMARY AND DISCUSSION

We have considered non-Abelian statistics of vortices, each of which has N Majorana zero-energy states inside its core on which an $\text{SO}(N)$ symmetry acts. We have investigated how the degenerate states induced by zero modes are transformed under an exchange of neighboring vortices. We have shown that, for an arbitrary odd N , the exchange operator $\tau_k^{[N]}$ defined in Eq. (3.4), generating the exchange of two neighboring vortices, can be factorized into two parts $\tau_k^{[N]} = \sigma_k^{[N]} h_k^{[N]}$ as seen in Eq. (4.14). The part which is given by $h_k^{[N]}$ defined in Eq. (4.16) is essentially equivalent to the exchange operator introduced by Ivanov. If it is expressed in terms of the composite singlet Majorana operator $\bar{\gamma}_k$ defined in Eq. (4.17), then it has the same form as the exchange operator τ_k in the case of the single Majorana fermion. The other operator $\sigma_k^{[N]}$ defined in Eq. (4.15) is a new part. We have proven in the Appendix that they constitute the Coxeter group of the type A_{2m-1} (the symmetric group S_{2m}) for $n = 2m$ vortices. We have also shown in Sec. V that the factorization of the exchange operators results in the tensor-product structure in its matrix representation in a suitable basis.

The $\text{SO}(N)$ symmetry considered in this paper is the largest symmetry group in the presence of N Majorana fermions. Whether the symmetry is $\text{SO}(N)$ or its subgroups depends on the details of the systems. For instance, a higher (pseudo)spin S representation of $\text{SO}(3)$ contains $2S + 1$ Majorana fermions, but the symmetry acting on them does not have to be $\text{SO}(2S + 1)$. Also, the representation does not have to be irreducible; for instance, four Majorana fermions may be decomposed into one

singlet and one triplet of SO(3), but the symmetry group does not have to be SO(4). It will be interesting to extend our results to general representations including reducible representations. An extension to general groups also remains as an interesting future problem to be explored.

For vortices with an even number N of Majorana fermions with the SO(N) symmetry, we have not found any meaningful factorization of the exchange operator $\tau_k^{[N]}$ so far. It remains as a future problem to identify the non-Abelian statistics for even N Majorana fermions. When the symmetry group inside the vortex core is restricted to the unitary subgroup $U(N/2) \subset SO(N)$, N Majorana fermions can be rearranged into $N/2$ complex Dirac fermions in each vortex. In this case, the situation is rather different because Dirac fermions are locally defined, and we do not need to define Dirac fermions nonlocally by using two spatially separated vortices. Nevertheless, we found a rather different kind of non-Abelian statistics in the case of $N = 2$ with the U(1) symmetry²⁵ and $N = 4$ with the U(2) symmetry.²⁶ Identifying the statistics for the general even N of $N/2$ Dirac fermions also remains as a future problem.

Finally, it will be important to look for actual condensed matter systems realizing multiple Majorana fermions in the vortex core, and to study an impact of our results on applications to topological quantum computations.

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APPENDIX: PROOF OF COXETER RELATIONS FOR ARBITRARY ODD N

In this Appendix, we give a proof that the operators $\sigma_k^{[N]}$ defined in Eq. (4.15) satisfy the Coxeter relations in Eqs. (4.23) and (4.24).

1. $(\sigma_k^{[N]}\sigma_\ell^{[N]})^3 = 1$ for $|k - \ell| = 1$

In this section, we show Eq. (4.23). Let us note that the cube of the product of $\tau_k^{[N]}$ and $\tau_{k+1}^{[N]}$ is written as

$$(\tau_k^{[N]}\tau_{k+1}^{[N]})^3 = (\sigma_k^{[N]}h_k^{[N]}\sigma_{k+1}^{[N]}h_{k+1}^{[N]})^3. \quad (\text{A1})$$

Since $\sigma_k^{[N]}$ commutes with $h_k^{[N]}$ and $h_{k+1}^{[N]}$, the relation above is written as

$$(\tau_k^{[N]}\tau_{k+1}^{[N]})^3 = (\sigma_k^{[N]}\sigma_{k+1}^{[N]})^3 (h_k^{[N]}h_{k+1}^{[N]})^3. \quad (\text{A2})$$

The left-hand side of Eq. (A2) is equal to -1 , which can be shown as follows:

$$\begin{aligned} (\tau_k^{[N]}\tau_{k+1}^{[N]})^3 &= \prod_{a=1}^N \left\{ \frac{1}{\sqrt{2}}(1 + \Gamma_k^a) \frac{1}{\sqrt{2}}(1 + \Gamma_{k+1}^a) \right\}^3 \\ &= \prod_{a=1}^N \left\{ \frac{1}{2}(1 + \Gamma_k^a + \Gamma_{k+1}^a + \Gamma_k^a\Gamma_{k+1}^a) \right\}^3 \\ &= \prod_{a=1}^N \frac{1}{2}(-1 + \Gamma_k^a + \Gamma_{k+1}^a + \Gamma_k^a\Gamma_{k+1}^a) \\ &\quad \times \frac{1}{2}(1 + \Gamma_k^a + \Gamma_{k+1}^a + \Gamma_k^a\Gamma_{k+1}^a) \\ &= (-1)^N \\ &= -1, \end{aligned} \quad (\text{A3})$$

where in the third line we have used the relation

$$\begin{aligned} &\left\{ \frac{1}{2}(1 + \Gamma_k^a + \Gamma_{k+1}^a + \Gamma_k^a\Gamma_{k+1}^a) \right\}^2 \\ &= \frac{1}{2}(-1 + \Gamma_k^a + \Gamma_{k+1}^a + \Gamma_k^a\Gamma_{k+1}^a), \end{aligned} \quad (\text{A4})$$

which follows from the anticommuting property of Γ_k^a and Γ_{k+1}^a . In the fourth line, we have again used the anticommuting property.

On the other hand, we can also show that $(h_k^{[N]}h_{k+1}^{[N]})^3$ is equal to -1 , as

$$\begin{aligned} (h_k^{[N]}h_{k+1}^{[N]})^3 &= \left\{ \frac{1}{2}(1 + \Gamma_k^{(N)} + \Gamma_{k+1}^{(N)} + \Gamma_k^{(N)}\Gamma_{k+1}^{(N)}) \right\}^3 \\ &= \frac{1}{2}(-1 + \Gamma_k^{(N)} + \Gamma_{k+1}^{(N)} + \Gamma_k^{(N)}\Gamma_{k+1}^{(N)}) \\ &\quad \times \frac{1}{2}(1 + \Gamma_k^{(N)} + \Gamma_{k+1}^{(N)} + \Gamma_k^{(N)}\Gamma_{k+1}^{(N)}) \\ &= -1, \end{aligned} \quad (\text{A5})$$

where we have used

$$\begin{aligned} &\left\{ \frac{1}{2}(1 + \Gamma_k^{(N)} + \Gamma_{k+1}^{(N)} + \Gamma_k^{(N)}\Gamma_{k+1}^{(N)}) \right\}^2 \\ &= \frac{1}{2}(-1 + \Gamma_k^{(N)} + \Gamma_{k+1}^{(N)} + \Gamma_k^{(N)}\Gamma_{k+1}^{(N)}). \end{aligned} \quad (\text{A6})$$

From Eqs. (A2), (A3), and (A5), we can conclude $(\sigma_k^{[N]}\sigma_\ell^{[N]})^3 = 1$ for $|k - \ell| = 1$.

2. $(\sigma_k^{[N]}\sigma_\ell^{[N]})^2 = 1$ for $|k - \ell| > 1$

To prove Eq. (4.24), we first note that by squaring both sides of Eq. (4.14) and using the relation

$$\left(\frac{1}{\sqrt{2}}(1 + \Gamma_k^a) \right)^2 = \frac{1}{2}(1 + 2\Gamma_k^a - 1) = \Gamma_k^a, \quad (\text{A7})$$

one finds

$$\Gamma_k^{(N)} = (\sigma_k^{[N]})^2 \Gamma_k^{(N)}. \quad (\text{A8})$$

Then, by multiplying $(\Gamma_k^{(N)})^{-1}$ on both sides from right, we obtain

$$(\sigma_k^{[N]})^2 = 1. \quad (\text{A9})$$

It follows that $(\sigma_k^{[N]}\sigma_\ell^{[N]})^2 = 1$ for $|k - \ell| > 1$ since $\sigma_k^{[N]}$ and $\sigma_\ell^{[N]}$ with $|k - \ell| > 1$ commute. We thus have shown that the operators $\sigma_k^{[N]}$ satisfy the Coxeter relation of the type A_{2m-1} .

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²⁸The structure of the Hilbert space in the presence of an odd number of vortices is recently discussed in Ref. 24.

²⁹We use $k, \ell = 1, \dots, n$ to label the vortices and the trapped Majorana fermions, while $K, L = 1, \dots, n/2 = m$ to label the Dirac fermions which are constructed from two Majorana fermions.

³⁰One may allow for mixture of indices a under the exchange of two vortices, but here we discuss the simplest case where such mixing does not occur.

³¹A Coxeter group S is defined as a group with distinct generators $s_i \in S$ ($i = 1, 2, 3, \dots$) satisfying the following two conditions (Ref. 13): (a) $s_i^2 = 1$ and (b) $(s_i s_j)^{m_{i,j}} = 1$ with a positive integer $m_{i,j} \geq 2$ for $i \neq j$. It should be noted that condition (a) gives $m_{i,i} = 1$. Elements $m_{i,j}$ can be summarized as the Coxeter matrix $(M)_{ij} = m_{i,j}$. In our case of the $n = 2m$ vortices, the Coxeter matrix is given by a $(2m - 1) \times (2m - 1)$ matrix, the elements of which are 1 (diagonal elements, $m_{i,i} = 1$), 3 (adjacent elements, $m_{i,i+1} = m_{i+1,i} = 3$), and 2 (all the others).

³²In general, a composite operator made by an even (odd) number of the Majorana operators is transformed as in Eq. (4.27) [Eq. (4.28)].

³³If one considers the case when N is an even number, one may define a composite operator by $\bar{\gamma}'_k \equiv \frac{1}{N!} e^{i\frac{\pi}{4}N} \gamma_k^1 \gamma_k^2 \dots \gamma_k^N$. The operator is self-conjugate $(\bar{\gamma}'_k)^\dagger = \bar{\gamma}'_k$, but does not satisfy the Clifford algebra. Furthermore, since N is even, the operation T_k gives the transformation $\bar{\gamma}'_k \rightarrow \bar{\gamma}'_{k+1}$, $\bar{\gamma}'_{k+1} \rightarrow \bar{\gamma}'_k$, and the rest $\bar{\gamma}'_\ell$ ($\ell \neq k$ and $k + 1$) unchanged. Hence, the composite operator $\bar{\gamma}'_k$ for even N does not transform like a singlet Majorana fermion operator.