Interacting topological phases and modular invariance

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We discuss $(2+1)$-dimensional topological superconductors with N_f left- and right-moving Majorana edge modes and a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. In the absence of interactions, these phases are distinguished by an integral topological invariant N_f. With interactions, the edge state in the case of $N_f = 8$ is unstable against interactions, and a $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant mass gap can be generated dynamically. We show that this phenomenon is closely related to the modular invariance of type II superstring theory. More generally, we show that the global gravitational anomaly of the nonchiral Majorana edge states is the physical manifestation of the bulk topological superconductors classified by $\mathbb{Z}_8$.

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I. INTRODUCTION

Topological insulators and superconductors are a gapped phase of matter with a stable gapless mode at their boundary. A classic example is the integer quantum Hall effect (IQHE), which exists for two spatial dimensions in the presence of a strong time-reversal symmetry breaking magnetic field.[1] A flurry of recent excitement came with the discovery of topological insulators in two and three dimensions in systems in the presence of strong spin-orbit coupling.[2–11] Unlike the IQHE, the topological character of these topological insulators (i.e., the stable gapless edge or surface modes) is protected by time-reversal symmetry (TRS). With a wider set of discrete symmetries in addition to TRS, such as particle-hole symmetries of various kinds realized in insulating and superconducting systems, one can ask if there is a topological distinction among gapped phases in the presence of such symmetries. The answer to this question is summarized in the systematic classification of topological insulators and superconductors.[11–14]

While these noninteracting topological phases are stable against arbitrary deformation of the Hamiltonian at the quadratic level, they could be fragile against fermion interactions. In the case of three-dimensional topological insulators, the topological invariant can be physically defined in terms of the topological magneto-electric effect with a quantized coefficient,[11] which can be evaluated for a generally interacting system in terms of the many-body Green's function.[15] For this reason, we can expect topological insulators to be stable against a general class of interactions. However, Refs. 16–18 also provided counter examples in the case of topological superconductors. It was demonstrated that in $(1+1)$-dimensional lattice Majorana fermion models, with a suitable choice of interactions, one can find an adiabatic path that connects what appears to be a topological phase at the quadratic level and a topologically trivial phase.

In this paper, we discuss a $(2+1)$-dimensional topological superconductor with N_f left- and right-moving Majorana edge modes, and a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry between them (see Sec. II). The similar/same models were studied independently in Refs. 19–22. In the absence of interactions, these phases are distinguished by an integral topological invariant, since they support an integral number of nonchiral edge modes ($=N_f$). With interactions, the edge state of the phase with $N_f = 8$ is unstable to interactions. Therefore the interacting phases of this model are classified by the $\mathbb{Z}_8$ topological class (see Sec. II). We argue that this phenomenon is closely related to the modular invariance of type II superstring (see Sec. IV). More generally, we show that the global gravitational anomaly or the modular noninvariance of the nonchiral Majorana edge states is the physical manifestation of the $(2+1)$ bulk topological superconductor (see Sec. III).

II. $\mathbb{Z}_2 \times \mathbb{Z}_2$ SYMMETRIC TOPOLOGICAL PHASES

A. Description of the model

The topological phases of our interest are in $(2+1)$ dimensions, and have $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with two conserved $\mathbb{Z}_2$ quantum numbers. A convenient way to describe these quantum numbers is to first consider systems with two conserved U(1) charges, and then later break the U(1) $\times$ U(1) symmetry down to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The two charges can be thought of as the total fermion number and the total S_z (the z component of spin-1/2 operator) quantum number, denoted by $N_\uparrow + N_\downarrow$, and $N_\uparrow - N_\downarrow$, respectively. We break the particle number conservation by introducing superconducting pair potential, so the system belongs to the Bogoliubov-de Genne (BdG) symmetry class (class D) of Altland Zirnbauer. Here, we deal with the pairing potential at the mean-field level, and regard it simply as a background. In effect, we are considering quadratic Hamiltonians of real fermions (BdG quasiparticles) instead of complex fermions. The pair potential breaks the electromagnetic U(1) symmetry, and the total fermion number $N_\uparrow + N_\downarrow$ is now conserved only modulo 2, i.e., the total fermion number parity $(-1)^{N_\uparrow + N_\downarrow}$ is conserved.

When the total S_z is conserved, the BdG Hamiltonians can be block diagonalized in the basis where S_z is diagonal (each block in the BdG Hamiltonians is a member of symmetry class A). We now relax the conservation of total S_z, and demand only the parity $(-1)^{N_\uparrow}$ [or $(-1)^{N_\downarrow}$] to be conserved; combined with the total fermion number parity conservation, the systems of our interest conserve two $\mathbb{Z}_2$ quantum numbers, $(-1)^{N_\uparrow}$ and $(-1)^{N_\downarrow}$. Even without strict conservation of S_z, at the quadratic level, the BdG Hamiltonians still remain block-diagonal since the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry does not allow any spin flip, i.e., any bilinear connecting spin-up and spin-down sectors. (So far, relaxing the S_z conservation down to the conservation of the two $\mathbb{Z}_2$ quantum numbers does not change the story much at the quadratic level, but it will make a big difference when we talk about interactions.)

These sub-blocks in the BdG Hamiltonians belong to the symmetry class A (the same symmetry class as IQHE) and their topological character is specified by the Chern number, $\mathrm{Ch}_\uparrow$ and $\mathrm{Ch}_\downarrow$, respectively; the topological classes of the system is characterized by a $\mathbb{Z}\times\mathbb{Z}$ topological number.

When $\mathrm{Ch}_\uparrow+\mathrm{Ch}_\downarrow\neq 0$, time-reversal symmetry (TRS) is necessarily broken, and a time-reversal symmetry breaking topological superconductor (in symmetry class D) is realized. This phase has nonzero thermal Hall conductance κ_{xy}, and when there is an edge, it supports an integer number ($=\mathrm{Ch}_\uparrow+\mathrm{Ch}_\downarrow$) of *chiral* Majorana fermions. This phase is robust against interactions as well as disorder.

The phase of our interest in this paper corresponds to the case with the vanishing total Chern number, $\mathrm{Ch}_\uparrow+\mathrm{Ch}_\downarrow=0$ (this is guaranteed when there is time-reversal symmetry), but with the nonzero spin Chern number, $\mathrm{Ch}_s:=(\mathrm{Ch}_\uparrow-\mathrm{Ch}_\downarrow)/2\neq 0$. A lattice model that realizes this situation can easily be constructed, by combining two copies of lattice chiral p-wave superconductors with opposite chiralities. (See, for example, Ref. 21.) Similarly to the case of $\mathrm{Ch}_\uparrow+\mathrm{Ch}_\downarrow\neq 0$, the phase with $\mathrm{Ch}_s\neq 0$ supports edge modes but unlike the case of $\mathrm{Ch}_\uparrow+\mathrm{Ch}_\downarrow\neq 0$, edge modes are *nonchiral*. Below, we will have a closer look at the edge modes.

Let us begin with the case of $\mathrm{Ch}_s=1$. The edge of the system consists of a single copy of Majorana fermion with both left- and right-moving chiralities, described by the following Euclidean Lagrangian:

$$\mathcal{L}=\frac{1}{4\pi}[\psi_L(\partial_\tau+iv\partial_x)\psi_L+\psi_R(\partial_\tau-iv\partial_x)\psi_R], \quad (1)$$

where τ is the imaginary time and x is the spatial coordinate parameterizing the edge; ψ_L (ψ_R) is the left- (right-) moving $(1+1)$ Majorana fermion, and v is the Fermi velocity. Here, one could think of the left-mover to carry "spin-up" and the right-mover to carry "spin-down" quantum numbers, respectively (or vice versa, depending on the sign of Ch_s). As emphasized before, however, we do not require the S_z quantum number to be conserved [$N_\uparrow$ (or $N_\downarrow$) is conserved only up to modulo 2]. This means, in particular, we do not have well-defined spin Hall conductance σ^s_{xy}. More generically, when $\mathrm{Ch}_s=N_f$, the edge is described by N_f flavor of Majorana fermions with both left- and right-moving chiralities:

$$\mathcal{L}=\frac{1}{4\pi}\sum_{a=1}^{N_f}\left[\psi^a_L(\partial_\tau+iv\partial_x)\psi^a_L+\psi^a_R(\partial_\tau-iv\partial_x)\psi^a_R\right]. \quad (2)$$

Since they are nonchiral, the gapless nature of the edge modes are not stable in the absence of any symmetry; one can find a suitable mass term that opens a gap. Since the bulk of the system respects $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry, this is inherited by the edge theory; we define two fermion parities in the edge theory,

$$G_L=(-)^{N_L}\quad\text{and}\quad G_R=(-)^{N_R}, \quad (3)$$

where $N_L(=N_\uparrow)$ [$N_R(=N_\downarrow)$] is the total left-moving (right-moving) fermion numbers at the edge. With the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry, at the quadratic level, all mass terms $\psi^a_L\psi^b_R$ are prohibited as they are odd under the left- or right-$\mathbb{Z}_2$ parity (G_L or G_R)—bulk topological phase is characterized by an integer, which is simply the number of branches of the (nonchiral) modes, N_f.

B. Effects of interactions

Beyond the quadratic level, we can write down interactions $\psi^a_L\psi^b_L\psi^c_R\psi^d_R$ that preserve $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry. The presence of such interactions can potentially destabilize the edge.[23] However, one would expect that the resulting gapped phase would spontaneously break $\mathbb{Z}_2\times\mathbb{Z}_2$; at the mean-field level, such interactions generate the expectation value $\langle\psi^a_L\psi^b_R\rangle\neq 0$ for some pair of flavor indices (a,b), and if so $\mathbb{Z}_2\times\mathbb{Z}_2$ conservation is violated.

When $N_f=8$ (more precisely, when $N_f\equiv 0 \bmod 8$), there is another type of interaction channel available that can potentially destabilize the edge—interactions in terms of "spin" or "disorder" operators. Let us first recall the case of $N_f=1$, the Ising conformal field theory (CFT). In the quantum Ising model, we have two relevant operators; the transverse field, and the Zeeman field. The former, in the language of the two-dimensional classical Ising model, corresponds to the deviation from the critical temperature $(T-T_c)$ and is given by the fermion mass term $\psi_L\psi_R$. The latter, the Zeeman field, while it is a natural and local perturbation in terms of the Ising spin variable, is a nonlocal term when the model is viewed as a fermion model. This is so because of the Jordan-Wigner string. In fact, the operator product expansion between the Majorana fermion $\psi_{L,R}$ and the spin operator $\sigma_{L,R}$ has a branch cut, signaling they are not a local object in terms of fermions. In fact, the spin operator is a twist operator for the fermion field $\psi_{L,R}$; when σ is inserted, say, at the origin, when ψ makes a round trip around the origin, it picks up a minus sign.

When $N_f=8$, from spin and disorder operators, we can form $2^8=256$ possible products of $\sigma^a(z,\bar z)$ and $\mu^a(z,\bar z)$ ($a=1,\ldots,N_f$). These have conformal weight $(1/16,1/16)\times 8=(1/2,1/2)$, which is the conformal weight of free fermions. These fermions, which are different from the original fermions $\psi^a_{L,R}$, can be used to form a perturbation to the edge theory, which are local with respect to $\psi^a_{L,R}$. This is rooted in the triality symmetry of SO(8).[24] Assuming, for simplicity, that all eight Majorana fermions $\psi^a_{L,R}$ have the same Fermi velocity, the kinetic term of the edge theory enjoys SO(8) symmetry. The Majorana fermions $\psi^a_{L/R}$ belong to the vector representation of SO(8), $\mathbf{8}_v$. For SO(8), by "accident," spinor (ξ) and conjugate spinor (η) are also eight dimensional (denoted by $\mathbf{8}_s$ and $\mathbf{8}_c$, respectively), the triality symmetry permutes these three representations. The $2^8=256$ possible products of $\sigma^a(z,\bar z)$ and $\mu^a(z,\bar z)$ are precisely the (linear combination of) 64×4 primary fields $\xi^a_R\xi^b_L$, $\xi^a_R\eta^b_L$, $\eta^a_R\xi^b_L$, $\eta^a_R\eta^b_L$.[25] These SO(8) spinors can be described in terms of Abelian bosonization as follows: we pair up the vector fermions, and bosonize as

$$\begin{aligned}\psi^{2j-1}_L\pm i\psi^{2j}_L&\simeq\exp\left(\pm i\varphi^j_L\right),\\ \psi^{2j-1}_R\pm i\psi^{2j}_R&\simeq\exp\left(\pm i\varphi^j_R\right),\end{aligned} \quad (4)$$

($j = 1,\dots,4$). The $16 = 8 + 8$ fields

$$\exp \frac{i}{2}\left(\pm\varphi_L^1 \pm \varphi_L^2 \pm \varphi_L^3 \pm \varphi_L^4\right) \tag{5}$$

are the spinor ξ_L^a and η_L^a, with $\mathbb{Z}_2$ parity determined by the parity of the number of minus signs in the exponential. ξ_L^a is even under $\mathbb{Z}_2$ parity, while η_L^a and ψ_L^a are odd under $\mathbb{Z}_2$ parity.

Since ξ_L^a and ξ_R^a are even under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ parity, it is now possible to construct quadratic terms $\xi_L^a \xi_R^b$ that could gap the edge states without violating the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. We use the interaction term constructed in Ref. 16, which is given by the Euclidean Lagrangian

$$\mathcal{L}_{\text{int}} = -A\left(\sum_{a=1}^{7} \xi_L^a \xi_R^a\right)^2 - B\left(\sum_{a=1}^{7} \xi_L^a \xi_R^a\right)\xi_L^8 \xi_R^8, \tag{6}$$

where A and B are some constant. This interaction can, in fact, also be expressed in terms of the vector fermions $\psi_{L/R}^a$ because of triality, and hence be a local interaction. The SO(8) symmetry is broken down to SO(7) which leaves the spinor $\xi_{L/R}^8$ invariant.

This interaction, when $B < 0$ and $B < 2A$, gives rise to a unique ground state as we can see as follows: when $B \ll A$, because of the dominant SO(7) Gross-Neveu interaction term $-A(\sum_{a=1}^7 \xi_L^a \xi_R^a)^2$, the bilinear $\sum_{a=1}^7 \xi_L^a \xi_R^a$ develops an expectation value $\langle \sum_{a=1}^7 \xi_L^a \xi_R^a \rangle = iM$. The interaction can then behave effectively as a mass term for $\xi_{L/R}^8$, $\mathcal{L}_{\text{int}} \simeq -iBM\xi_L^8\xi_R^8$. Thus, when $B \ll A$, the model behaves essentially as a single copy of the Ising model. Depending on the sign of the induced mass $-BM$, it can be either in the low-temperature (symmetry broken) or the higher-temperature (paramagnetic) phase . To determine which phase is realized, we first note that when $B = 2A$, the interaction term is the SO(8) Gross-Neveu interaction. This then leads to a gapped phase with two-fold degenerate ground states because of chiral symmetry breaking. We would then conclude that when $B \ll A$ and $B > 0$ (and in fact, for the entire region of $B > 0$ and $B > -2A$), the model is in the low-temperature phase of the effective Ising model, with two-fold degenerate ground states. Next, we note that the sign of B can be flipped in the interaction (6) by $\xi_R^a \to -\xi_R^a$, the Kramers-Wannier duality transformation. Thus, we conclude, when $B \ll A$ and $B < 0$ (and in fact, for the entire region of $B < 0$ and $B < 2A$), the effective Ising model is in the high-temperature phase (paramagnetic phase) with unique ground state. It can be checked that the ground state does not violate the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.

The discussion above can be formulated in a language more familiar in the context of correlated electron systems. When $N_f = 8$, the eight Majorana fermions can be mapped onto four complex fermions of a lwo-leg ladder (see, for example, Refs. 26–33, and references therein) or the spin-3/2 Hubbard model,[34] with a suitable choice of basis states. Interactions of a two-leg ladder can be described by the on-site Hubbard interaction U, the rung interaction V and the rung exchange J. When $J = 4(U + V)$, the model is SO(5) symmetric at half-filling.[26] Furthermore, when $V = 0$ or $J = 4U$, the model is also SO(7) symmetric, which in a suitable basis can also be expressed as Eq. (6). This interaction can either lead to a unique rung singlet ground state or a two-fold degenerate staggered flux ground states.[30] The quantum phase transition between these states can be described by the transverse field Ising model,[30] or equivalently, by a single Majorana spinor, which is nothing but our spinor ξ^8. In this sense, the high-temperature or the paramagnetic phase of the ξ^8 spinor corresponds to the rung singlet state of a two-leg ladder, with a gap generated by interactions.[35]

Alternatively, one can postulate an interaction that is $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric and involves both spinors and conjugate spinors,

$$\mathcal{L}'_{\text{int}} = -A\left(\sum_{a=1}^{7} \xi_L^a \eta_R^a\right)^2 - B\left(\sum_{a=1}^{7} \xi_L^a \eta_R^a\right)\xi_L^8 \eta_R^8. \tag{7}$$

Following the same reasoning, this interaction gives rise to, when $B < 0$ and $B < 2A$, a unique ground state.

From these discussion, we conclude that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric topological phases, while it can support an integer number of nonchiral edge modes when noninteracting, interactions make them unstable when $N_f = 8$. Therefore interacting models falls into $\mathbb{Z}_8$ topological classes. In the following sections, we will look more into the reasons behind this stability/instability.

III. GLOBAL GRAVITATIONAL ANOMALY

A. Large gauge transformations in electromagnetism

Our analysis on the stability/instability of the topological phases so far relies on an explicit construction of an interaction term in terms of the twist (spin and disorder) operators. For the QHE and for the quantum spin Hall effect (QSHE), however, their stability (and also instability in the case of the QSHE) against interactions can be understood from a wider (more "topological") point of view;[1,36,37] it is the Laughlin's thought experiment (and its suitable extension to the QSHE), which we will review briefly below for our later discussion. For our situation, since the particle number and S_z quantum number are not conserved (conserved only mod 2), we cannot rely on the flux(es) of U(1) gauge field of charge or spin origin. We will, instead, try to make use of gravitational field.

Let us consider the QHE on a finite cylinder (which is topologically equivalent to an annulus). There are two edges, which we call "edge I" and "edge II." We thread a magnetic flux Φ into the "hole" of the cylinder. Starting from zero flux, let us gradually increase the flux. The Hamiltonian $H(\Phi)$ of the system, when Φ is not an integer multiple of the flux quantum Φ_0, is not gauge equivalent to the original Hamiltonian; the insertion of the flux is a physically effect, and not a gauge transformation. When flux is an integer multiple of flux quantum, however, the Hamiltonian goes back to itself, $H(\Phi) = H(\Phi + n\Phi_0)$ ($n \in \mathbb{Z}$). This is an example of *large* gauge transformations; the Hamiltonian with n extra flux quanta $n\Phi_0$ cannot be generated from the original flux Φ by a successive application of infinitesimal gauge transformation. Unlike an infinitesimal gauge transformation, to achieve such gauge transformation by an adiabatic process, one needs to generate physical flux during the process.

The same is true for the total partition function Z of the system as a function of flux Φ: it is invariant under a large gauge transformation $\Phi \to \Phi + n\Phi_0$,

$$Z(\Phi) = Z(\Phi + n\Phi_0). \tag{8}$$

However, in the QHE, a closer inspection tells us that in the adiabatic process where we increase the flux from Φ to $\Phi + \Phi_0$, say, an integer multiple of charge is pumped from edge I to edge II (or edge II to edge I). This means, if we focus on a single edge (edge I or edge II), instead of the combined system of the two edges, it looks as if the charge is not conserved.

Since the bulk is fully gapped, for adiabatic processes, it is meaningful to focus on excitations at the edges, neglecting gapped excitations in the bulk. The total partition function can then be written as

$$Z(\Phi) = \sum_{a,b} N_{ab} \chi_a^{\mathrm{I}}(\Phi) \chi_b^{\mathrm{II}}(\Phi), \tag{9}$$

where $\chi_a^{\mathrm{I,II}}(\Phi)$ is a (chiral) partition function for edge I, II, and N_{ab} is some coefficient. Each $\chi_a(\Phi)$ is not invariant under $\Phi \to \Phi + n\Phi_0$ ("spectral flow"), while the total partition function should be invariant. This gauge argument by Laughlin suggests the stability of the QHE against disorder and interactions. In the case of the QSHE, flux insertion argument can also be applied, and it was shown that a flux of $\Phi_0/2$ pumps fermion number parity and lead to spin-charge separation.[36]

To summarize, for a chiral edge theory of the QHE, charge is not conserved under an adiabatic process to achieve a large gauge transformation, $\Phi \to \Phi + n\Phi_0$, signaling pumping of electric charge and thus detecting the bulk topological insulator. For later purpose, this observation can be equivalently rephrased as follows: if we "force" a chiral edge theory to conserve N_{I} and N_{II} separately, where N_{I} (N_{II}) is the fermion number at edge I (edge II), then, the edge partition function $Z(\Phi)$ cannot be made invariant under $\Phi \to \Phi + \Phi_0$.

B. Large coordinate transformations in gravity

1. Perturbative and global gravitational anomalies

For systems where electrical charge is not conserved, we cannot rely on U(1) gauge (non-) invariance of the edge theory to diagnose the stability of the topological phase. A natural tool to address the stability/instability is, then, (non)invariance under diffeomorphism transformations (coordinate transformations). (See, for example, Refs. 38 and 39 and references therein).

Similar to the electromagnetic U(1) gauge field in nonsimply connected geometry, there are infinitesimal as well as large coordinate transformations when the spacetime manifold has nontrivial topology. That is, coordinate transformations that can be reached by successive infinitesimal transformations from the identity, and those that are not continuously connected to the identity, respectively.

The noninvariance of the system under infinitesimal coordinate transformations ("perturbative gravitational anomaly") means the violation of energy-momentum conservation, $\langle D^{\mu} T_{\mu\nu} \rangle \neq 0$, where $T_{\mu\nu}$ is the energy-momentum tensor and D^{μ} is the covariant derivative. When this happens at the boundary of some bulk system, the fact that energy-momentum cannot be made conserved within the boundary theory necessitates the presence of the bulk theory; energy-momentum at the boundary should be "leaking" into the bulk, and, in fact, this bulk is what we call a topological phase. (See, for example, Refs. 38 and 39, and also Ref. 40). For example, the chiral edge theory of a (fractional) quantum Hall fluid is anomalous under infinitesimal coordinate transformations.[41] This signals the topological property of the bulk with nonzero thermal Hall conductance κ_{xy}.[42–44]

Even when there is no perturbative gravitational anomaly, e.g., when the edge theory in question is nonchiral as in our example of the topological phases with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, the system may not be invariant under *large* diffeomorphism transformations ("global gravitational anomaly"[45]). Similarly to perturbative gravitational anomaly, we will argue below that the noninvariance of the edge theory under large coordinate transformations can also be used as a diagnose of the stability/instability of the topological phase.

2. Modular transformations on a torus

More specifically, we again assume the bulk is defined on a finite cylinder with two edges. The edges may support a chiral or nonchiral edge mode, which we assume is a chiral or nonchiral CFT. The CFT on one edge is defined on a torus $T^2 = S^1 \times S^1$ with the periodically identified spatial coordinate (parameterizing the edge), and the periodically identified (imaginary) time.

There are a set of large coordinate transformations on a two-dimensional torus, *modular transformations*, which form a group Γ.[46] The geometry of a flat torus is specified by two real parameters ("moduli"), or a single complex parameter $\tau = \omega_2/\omega_1$, the ratio of the two periods of the torus ($\mathrm{Im}\,\tau > 0$). Two different modular parameters τ and τ' can describe the same toroidal geometry if they are related by an integer linear transformation with unit determinant,

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad a,b,c,d \in \mathbb{Z}, \quad ad - bc = 1. \tag{10}$$

(Here, τ should not be confused with the imaginary time). Modular transformations belong to the infinite discrete group $\mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$. These transformation are generated by two generators, $T: \tau \to \tau + 1$ and $S: \tau \to 1/\tau$, satisfying the relations $S^2 = (ST)^3 = C$, where C is the charge conjugation matrix, satisfying $C^2 = 1$.

For a CFT on a torus, we can ask if it is invariant under modular transformations. Any CFT that is derived from the continuum limit of a two-dimensional lattice statistical-mechanical system (or equivalently a one-dimensional lattice quantum system) is expected to be anomaly free (modular invariant).[49] On the contrary, if a CFT in question is not modular invariant, it may not be realized, on its own, as a continuum limit of a local lattice system, and must be accompanied by some (topological) bulk theory.

Based on these observations, we are lead to claim that the global gravitational anomaly implies the presence of a topological bulk theory, in a way quite analogous to the previous illustration of the charge response; basically, we simply replace Φ by τ, and the large gauge transformation $\Phi \to \Phi + \Phi_0$ by modular transformations, $\tau \to \tau + 1$ and $\tau \to -1/\tau$. The partition function now depends on a complex parameter τ (the moduli parameter of the torus), $Z(\tau,\bar{\tau})$. The modular noninvariance of the partition function of a given edge signals the presence of a topological bulk theory. Note, however, that when the two edges (edge I and edge II) are combined, we should be able to achieve the modular invariance;[50–52] they can be gapped pairwise. Similarly to Eq. (9), we can write the total partition function in terms of a liner combination:

$$Z(\tau,\bar{\tau}) = \sum_{a,b} N_{ab}\chi^{\mathrm{I}}_a(\tau,\bar{\tau})\chi^{\mathrm{II}}_b(\tau,\bar{\tau}). \tag{11}$$

Each block $\chi^{\mathrm{I,II}}_a(\tau,\bar{\tau})$ can be nonmodular invariant, but the total partition function $Z(\tau,\bar{\tau})$ should be modular invariant.

If the system is defined at the microscopic level, in terms of fermions (electrons), the requirement that the total partition function $Z(\tau,\bar{\tau})$ to be modular invariant may be relaxed; In the presence of fermions, the partition function may not be invariant under T, but should still be invariant under T^2.[50–52] The modular transformations generated by S and T^2 form a subgroup [$=\Gamma(2)$] of the full modular group Γ.

3. Symmetry projection

When there is a set of symmetries, and when we talk about symmetry-protected topological phases, it makes sense to diagnose the system by an adiabatic process that does not violate the symmetries. For a unitary symmetry, a convenient way to enforce the symmetry in the adiabatic process is to *project* the total Hilbert space into a given subsector specified by a quantum number. We then ask if, for a given edge separately, each sector can be made modular invariant (i.e., free of global gravitational anomaly).

Inability to achieve this would mean a quantum number of some kind should be "pumped" from one edge to the other along an adiabatic process to generate a modular transformation. When both edges are included, the total systems without projection would be modular invariant. This would mean the symmetry (conservation of a quantum number) should be violated in the adiabatic process, and thus leads to pumping.

Let us have a further look at the projection procedure. When projected, certain states (states which are not singlet under a symmetry group in question) are removed from the original Hilbert space of the edge theory. From the state-operator correspondence in CFT, this means the corresponding operators are not allowed in the theory after projection. Such operators, $\mathcal{O}(z,\bar{z})$, say, in the original theory, can be added to the action S_0 describing the edge theory as a perturbation, $S_0 \to S_0 + \lambda \int d^2x\, \mathcal{O}(z,\bar{z})$, where λ is a coupling constant, and if $\mathcal{O}(z,\bar{z})$ is relevant in the renormalization group (RG) sense, it can destabilize the edge. As its corresponding state, the operator is not singlet under the symmetry group, and hence when added to the action, it explicitly breaks the symmetry. In the projected theory, such perturbations are prohibited.

C. Free complex fermion

To illustrate the spectral flow (noninvariance under large gauge transformations) and the modular noninvariance (global gravitational anomaly), and also for our later use, let us consider a single copy of left-moving *complex* fermion as an example. (We follow Refs. 46–48.) It is described by the Lagrangian

$$\mathcal{L}_L = \frac{1}{2\pi}\Psi^{\dagger}_L\,(\partial_\tau + v\partial_x)\,\Psi_L. \tag{12}$$

The path integral for a single copy of complex fermion can be considered with boundary conditions in space and time directions:

$$\begin{aligned}\Psi_L(\tau,x+\ell) &= (-1)e^{2\pi i\lambda}\Psi_L(\tau,x),\\ \Psi_L(\tau+T^{-1},x) &= (-1)e^{-2\pi i\mu}\Psi_L(\tau,x),\end{aligned} \tag{13}$$

where T^{-1} is the inverse temperature, and the system is defined on a spatial circle of circumference ℓ; μ and λ specify the boundary condition for the space and time directions, respectively. In particular, if the chiral Lagrangian (12) is interpreted as the edge theory of the IQHE, λ is related to the flux Φ in Sec. III A as $\Phi/\Phi_0 = \lambda$. The corresponding partition function is denoted as

$$Z^{\lambda}{}_{\mu}(\tau). \tag{14}$$

Here, τ is the modular parameter, and the upper script indicates the boundary condition in space direction whereas the lower script indicates the boundary condition in time direction. Later, when we consider real (Majorana) fermions, rather than complex fermions, we also use notation "0 (1/2)" = "A (P)" = antiperiodic (periodic) boundary condition.

The fermionic path integral (fermionic determinant) is evaluated as

$$Z^{\lambda}{}_{\mu}(\tau) = e^{2\pi i\lambda\mu}q^{-1/24}q^{\lambda^2/2} \times \prod_{n=1}^{\infty}(1+wq^{n-1/2})(1+w^{-1}q^{n-1/2}), \tag{15}$$

where $w = e^{2\pi i\mu}q^{\lambda}$. Here, the overall phase factor $e^{2\pi i\lambda\mu}$ is purely conventional; since we have an independent path integral for a given set of boundary conditions, there is no unique way to determine the relative (Boltzmann) weight between sectors with different boundary conditions. The factor $e^{2\pi i\lambda\mu}$ in Eq. (15) is a common choice, but this will not affect our discussion below.

1. Spectral flow

Let us derive Eq. (15) in the operator formalism, where the partition function with given boundary conditions is given by

$$Z^{\lambda}{}_{\mu}(\tau) = \mathrm{Tr}_{\lambda}[e^{-2\pi i\mu N_L}q^{H_L}], \quad q = e^{2\pi i\tau}, \tag{16}$$

where Tr_λ is the trace over the Hilbert space defined with the spatial boundary condition λ. Here,

$$N_L := \int_0^{\ell} dx\, \Psi^{\dagger}_L\Psi_L \tag{17}$$

is the total left-moving fermion number. Observe that, in the operator formalism, the periodic boundary condition in time is realized here by an insertion of operator $e^{-2\pi i\mu N_L}$.

The partition function can be evaluated explicitly by making use of the mode expansion

$$\Psi_L(x) = \sqrt{\frac{2\pi}{\ell}}\sum_{s\in\mathbb{Z}+1/2-\lambda} e^{-ix\frac{2\pi s}{\ell}}\Psi_s, \tag{18}$$

where Ψ_s and $\Psi^{\dagger}_s$ satisfy the commutation relation $\{\Psi_s,\Psi^{\dagger}_{s'}\} = \delta_{ss'}$. In terms of the mode expansion, we define the ground state $|0\rangle_\lambda$ for a given spatial boundary condition λ as a filled

Dirac sea,

$$\Psi_{n+1/2-\lambda}|0\rangle_\lambda = \Psi^\dagger_{-n-1/2+\lambda}|0\rangle_\lambda = 0 \quad \text{for} \quad n+1/2-\lambda > 0. \tag{19}$$

Let us further assign the fermion number to the ground state as

$$e^{i\phi N_L}|0\rangle_\lambda = e^{i\phi\lambda}|0\rangle_\lambda, \quad \phi \in \mathbb{Z}. \tag{20}$$

Similarly to discussion below Eq. (15), this assignment is purely conventional. With this assignment, we obtain the partition function (15).

The two boundary conditions λ_1 and λ_2 are in general physically distinct, and correspondingly, the two ground states, $|0\rangle_{\lambda_1}$ and $|0\rangle_{\lambda_2}$, belong to different Hilbert spaces. However, when $\lambda_1 - \lambda_2 =$ integer, these two systems are related by a large gauge transformation. Let us now consider an adiabatic process interpolating two boundary conditions, $\lambda = 0 \to \lambda = 1$, say. While these boundary conditions are large-gauge equivalent, the ground state might not evolve into itself (the ground state) under the adiabatic process ("spectral flow"): for example, let us start from $\lambda = 0$ and define the ground state as

$$\Psi_{n+1/2}|0\rangle_{\lambda=0} = \Psi^\dagger_{-n-1/2}|0\rangle_{\lambda=0} = 0 \quad \text{for} \quad n \geqslant 0. \tag{21}$$

As we change λ, we assume the ground state evolves continuously: it is always annihilated by $\Psi_{n+1/2-\lambda}$ with $n \geqslant 0$. We define the state obtained by this adiabatic process as $|0'\rangle_\lambda$. On the other hand, by definition, the ground state at $\lambda = 1$ is given by

$$\Psi_{n-1/2}|0\rangle_{\lambda=1} = \Psi^\dagger_{-n+1/2}|0\rangle_{\lambda=1} = 0 \quad \text{for} \quad n \geqslant 1, \tag{22}$$

i.e., it is annihilated by $\Psi_{n+1/2-\lambda}$ with $n = 1$. We conclude $|0'\rangle_{\lambda=1} = \Psi^\dagger_{-1/2}|0\rangle_{\lambda=1} \neq |0\rangle_{\lambda=1}$. This spectral flow is reflected in the noninvariance of the partition function under the adiabatic process.

2. Modular transformation

Let us now examine the transformation properties of the partition function under modular transformations. From Eq. (15),

$$\begin{aligned} Z^0{}_0(\tau+1) &= e^{-i\pi/12} Z^0{}_{1/2}(\tau), \\ Z^0{}_{1/2}(\tau+1) &= e^{-i\pi/12} Z^0{}_0(\tau), \\ Z^{1/2}{}_0(\tau+1) &= e^{i\pi/6} Z^{1/2}{}_0(\tau), \\ Z^{1/2}{}_{1/2}(\tau+1) &= e^{i\pi/6} Z^{1/2}{}_{1/2}(\tau), \end{aligned} \tag{23}$$

$$\begin{aligned} Z^0{}_0(-1/\tau) &= Z^0{}_0(\tau), \\ Z^0{}_{1/2}(-1/\tau) &= Z^{1/2}{}_0(\tau), \\ Z^{1/2}{}_0(-1/\tau) &= Z^0{}_{1/2}(\tau), \\ Z^{1/2}{}_{1/2}(-1/\tau) &= e^{-\pi i/2} Z^{1/2}{}_{1/2}(\tau). \end{aligned} \tag{24}$$

The partition function $Z^{1/2}{}_{1/2}(\tau)$ is actually zero identically, because of the zero mode of the Dirac operator with periodic boundary condition in both directions. Nevertheless, we have assigned formal transformation rules to $Z^{1/2}{}_{1/2}(\tau)$.

The transformation law for $\tau \to -1/\tau$ is what we expect classically (i.e., just exchanging space and time boundary conditions), but the transformation law for $\tau \to \tau + 1$ is somewhat unexpected in the sense that the partition function acquires a phase factor. The reason for this is that there is no diff-invariant way to define the phase of the path integral for purely left-moving fermions. For left- plus right-moving fermions with matching boundary conditions, the path integral can be defined by Pauli-Villars or other regulators. This is the same as the absolute square of the left-moving path integral, but leaves a potential phase ambiguity in that path integral separately. The phase represents a global gravitational anomaly, an inability to define the phase of the path integral such that it is invariant under large coordinate transformations. Of course, a single-left moving fermion has nonzero chiral central charge and so has an anomaly even under infinitesimal coordinate transformations, but the global anomaly remains even when a left- and right-moving fermions are combined (see below).

IV. EDGE THEORY OF $\mathbb{Z}_2 \times \mathbb{Z}_2$ SYMMETRIC TOPOLOGICAL PHASE

Let us now consider the edge theory of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric topological phase, Eq. (2). We focus on the case of $N_f = 2N$ and demonstrate that while when $N \neq 4$ (mod 4), there is a global gravitational anomaly, the case with $N = 4$ (mod 4) is anomaly free. In fact, this is deeply related to the modular invariance and the consistency of type II superstring theory.[48] [While our presentation below uses, in order to make use of our discussion in Sec. III C, the partition function $Z^\lambda{}_\mu(\tau)$ of a complex fermion, there is no fundamental reason to do so. The entire discussion can be constructed in terms of real (Majorana) fermions, without referring to complex fermions.]

Since there are various boundary conditions allowed for the fermionic edge theory, the partition function is given as a sum of sectors with different boundary conditions. Let us discuss this issue by using the operator formalism. By considering contributions from different spatial boundary conditions, we consider a sum

$$\sum_\alpha \mathrm{Tr}_\alpha\, [q^{H_\alpha}], \tag{25}$$

where the summation extends all possible spatial boundary conditions, and H_α is the Hamiltonian with a boundary condition specified by α. (Here in our problem, $\alpha =$ A,P). Since the modular transformation exchanges the spatial and time directions, Eq. (25) is not modular invariant; we have to consider contributions from different boundary conditions in the time direction as well. As we have seen, in the operator formalism, a different kind of boundary in time direction is achieved by an insertion of a unitary operator. Thus the partition function is given by

$$Z = \sum_{\alpha,\alpha'} \mathrm{Tr}_\alpha\, [U_{\alpha'} q^{H_\alpha}], \tag{26}$$

where U_α is some unitary operator. (In our case, U_α is the parity of the fermion number operators.) The partition function can also be written as

$$Z = \mathcal{N} \sum_\alpha \mathrm{Tr}_\alpha\, [P q^{H_\alpha}], \quad \text{where} \quad P := \frac{1}{\mathcal{N}} \sum_\alpha U_\alpha. \tag{27}$$

Under the assumption that the set of unitary operators $\{U_\alpha\}_{\alpha=1,\ldots,\mathcal{N}}$ form a group, one verifies that

$$U_\alpha P = P U_\alpha = P^2 = P. \tag{28}$$

Thus P is a projection operator.

As we have seen, for the fermionic edge theory, the unitary operators that we need to change boundary conditions are the fermion number parity operators,

$$U_\alpha = 1, \quad (-1)^{N_L}, \quad (-1)^{N_R}, \quad (-1)^{N_L+N_R}, \tag{29}$$

where $N_L = \sum_{i=1}^N N_L^i$ and $N_R = \sum_{i=1}^N N_R^i$ are the total left- and right-moving fermion numbers, respectively. The sum (the projection operator) is then

$$\begin{aligned} P &= \frac{1}{4}[1 + (-1)^{N_L} + (-1)^{N_R} + (-1)^{N_L+N_R}] \\ &= \frac{1+(-1)^{N_L}}{2}\,\frac{1+(-1)^{N_R}}{2} =: P_{\mathrm{GSO}}. \end{aligned} \tag{30}$$

This operator projects, for each of the left- and right-moving sectors, onto the space of a definite fermion number parity [the Gliozzi-Scherk-Olive (GSO) projection]. Observe that this projection acts on the left- and right-moving sectors separately.

For the left-moving sector with $\alpha = 0 = \mathrm{A}$ spatial boundary condition in Eq. (26),

$$\begin{aligned} Z_{\mathrm{A}}(\tau) &= \mathrm{Tr}_{\mathrm{A}}\,[P_{\mathrm{GSO}} q^{H_{\mathrm{A}}}] = \mathrm{Tr}_{\mathrm{A}}\big[P_{\mathrm{GSO}} q^{H_{\mathrm{A}}^1+\cdots+H_{\mathrm{A}}^N}\big] \\ &= \tfrac{1}{2}\mathrm{Tr}_{\mathrm{A}}\big[q^{H_{\mathrm{A}}^1+\cdots+H_{\mathrm{A}}^N}\big] + \tfrac{1}{2}\mathrm{Tr}_{\mathrm{A}}\big[e^{\pi i N_L} q^{H_{\mathrm{A}}^1+\cdots+H_{\mathrm{A}}^N}\big] \\ &= \tfrac{1}{2}\big[{Z^0}_0(\tau)^N \pm {Z^0}_{1/2}(\tau)^N\big]. \end{aligned} \tag{31}$$

The sign $\pm$ in the last line indicates a possible ambiguity in assigning the fermion number parity to the ground state $|0\rangle_{\mathrm{A}}$ in the $\alpha = \mathrm{A}$ sector; see discussion around Eq. (15). While we adopted a particular choice for the fermion number parity in Eq. (15), here we leave other possibilities open in order to illustrate such ambiguity does not affect our conclusion. Similarly, for $\alpha = 1/2 = \mathrm{P}$ spatial boundary condition,

$$\begin{aligned} Z_{\mathrm{P}}(\tau) &= \mathrm{Tr}_{\mathrm{P}}\,\big[P_{\mathrm{GSO}} q^{H_{\mathrm{P}}}\big] \\ &= \tfrac{1}{2}\big[{Z^{1/2}}_0(\tau)^N \pm {Z^{1/2}}_{1/2}(\tau)^N\big]. \end{aligned} \tag{32}$$

There is again a sign ambiguity $\pm$ here, regarding to the fermion number parity of the ground state in the $\alpha = \mathrm{P}$ sector.

The total partition function for the $N_f = 2N$ left-moving Majorana fermions $Z_L(\tau)$ is obtained by taking a linear combination of $Z_{\mathrm{A}}(\tau)$ and $Z_{\mathrm{P}}(\tau)$. The requirement that the total partition function is invariant under the S-modular transformation motivates us to consider the following relative weight between $Z_{\mathrm{A}}(\tau)$ and $Z_{\mathrm{P}}(\tau)$:

$$\begin{aligned} Z_L(\tau) = \tfrac{1}{2}\big[&{Z^0}_0(\tau)^N + s{Z^0}_{1/2}(\tau)^N \\ &+ s{Z^{1/2}}_0(\tau)^N + ss'{Z^{1/2}}_{1/2}(\tau)^N\big], \end{aligned} \tag{33}$$

where the signs $s,s' = \pm 1$ are related to the ambiguity of the fermion number parity of the ground states $|0\rangle_{\mathrm{A,P}}$, and to the relative weight between $Z_{\mathrm{A}}(\tau)$ and $Z_{\mathrm{P}}(\tau)$ when taking a linear combination.

Under T-modular transformation, the partition function is transformed as

$$\begin{aligned} Z_L(\tau) = s e^{i\frac{\pi N}{12}}\tfrac{1}{2}\big[&\big({Z^0}_0\big)^N + s\big({Z^0}_{1/2}\big)^N + e^{-i\frac{\pi N}{4}}\big({Z^{1/2}}_0\big)^N \\ &+ s' e^{-i\frac{\pi N}{4}}\big({Z^{1/2}}_{1/2}\big)^N\big](\tau+1), \end{aligned} \tag{34}$$

whereas under T^2,

$$\begin{aligned} Z_L(\tau) = e^{i\frac{\pi N}{6}}\tfrac{1}{2}\big[&({Z^0}_0)^N + s({Z^0}_{1/2})^N \\ &+ s e^{-i\frac{\pi N}{2}}\big({Z^{1/2}}_0\big)^N + ss' e^{-i\frac{\pi N}{2}}\big({Z^{1/2}}_{1/2}\big)^N\big](\tau+2). \end{aligned} \tag{35}$$

Thus, when $N = 4$, we thus achieve the modular covariance, $Z_L(\tau) \to Z_L(\tau) = e^{i2\pi/3} Z_L(\tau+2)$. Combined with the right-moving part of the partition function, $Z_R(\bar\tau)$, the total partition function $Z(\tau,\bar\tau) = Z_R(\bar\tau) Z_L(\tau) = |Z_L(\tau)|^2$ is then invariant under T^2,

$$Z(\tau,\bar\tau) = Z(\tau+2,\bar\tau+2). \tag{36}$$

Similarly, when $N = 4$, by choosing $s = -1$, we thus achieve the modular covariance, $Z_L(\tau) \to Z_L(\tau) = (-1)e^{i\pi/3} Z_L(\tau+1)$. Combined with the right-moving part of the partition function, $Z_R(\bar\tau)$, the total partition function is then modular invariant,[53]

$$Z(\tau,\bar\tau) = Z(\tau+1,\bar\tau+1). \tag{37}$$

In the Lagrangian (2), the fermions $\psi^a_{R,L}$ are in the vector representation of SO(8), $\mathbf{8}_v$. In the context of superstring theory, this is the Ramond-Neveu-Schwarz (RNS) model of the superstring in the light-cone gauge. The Lagrangian does not completely specify the spectrum, and we need to impose the boundary conditions; the fermions $\psi^a_{R,L}$ obey either antiperiodic (NS) or periodic (R) boundary condition. Furthermore, we have used the GSO projection (30), which leads to type IIB and type IIA theories. Because of triality, one can rewrite the $\psi^a_{R,L}$ theory in terms of spinors $\xi^a_{R,L}$ and $\eta^a_{R,L}$ as well. Technically, this means we first bosonize the RNS fermions $\psi^a_{R,L}$ and refermionize, to obtain ξ^a and η^a, spinor ($\mathbf{8}_s$), and conjugate spinors ($\mathbf{8}_c$)—this is the Green-Schwarz (GS) formalism of the superstring. The two spinors, ξ^a and η^a, are distinguished by chirality operator of SO(8); spinor ξ^a has positive chirality and conjugate spinor η^a has negative chirality. When rewritten in terms of these spinors, in type IIB theory, we have left-moving and right-moving spinors, and the Lagrangian is given by

$$\mathcal{L} = \frac{1}{4\pi}\sum_{a=1}^{N_f=8}\big[\xi^a_L(\partial_\tau + iv\partial_x)\xi^a_L + \xi^a_R(\partial_\tau - iv\partial_x)\xi^a_R\big]. \tag{38}$$

Similarly, in type IIA theory, we have left-moving spinor and right-moving conjugate spinors, and the Lagrangian is given by

$$\mathcal{L} = \frac{1}{4\pi}\sum_{a=1}^{N_f=8}\big[\xi^a_L(\partial_\tau + iv\partial_x)\xi^a_L + \eta^a_R(\partial_\tau - iv\partial_x)\eta^a_R\big]. \tag{39}$$

Unlike the vector fermions $\psi^a_{R,L}$, the spinors obey periodic boundary condition only:

$$\begin{aligned}\xi^a_L(x+\ell) &= \xi^a_L(x), \quad \xi^a_R(x+\ell)=\xi^a_R(x),\\ \eta^a_L(x+\ell) &= \eta^a_L(x), \quad \eta^a_R(x+\ell)=\eta^a_R(x),\end{aligned} \tag{40}$$

where the system is defined on a spatial circle of circumference ℓ. Because of this, there is no need for projection. One can compare the spectrum of the RNS theory with GSO projection, and the GS theories; they match precisely.

We conclude this section with a discussion on the "Ising projection." As emphasized before, we have two separate projections for the left- and right-moving sectors. This should be contrasted to the projection with respect to the total fermion parity $(-1)^{N_L+N_R}$, which is described by the "diagonal" projection operator

$$P_0 = \frac{1+(-1)^{N_L+N_R}}{2}. \tag{41}$$

For $2N$ flavor of Majorana fermions, the resulting total partition function

$$\tfrac{1}{2}\big[\big|Z^0{}_0(\tau)\big|^N + \big|Z^0{}_{1/2}(\tau)\big|^N + \big|Z^{1/2}{}_0(\tau)\big|^N \mp \big|Z^{1/2}{}_{1/2}(\tau)\big|^N\big] \tag{42}$$

is invariant for any N because the phases cancel in the absolute values. The Ising model can be viewed as an example of the above partition function. (Only minor difference is that we have been mainly using the complex fermions, instead of Majorana fermions.) The Ising partition function is given by

$$Z_{\text{Ising}} = \tfrac{1}{2}\big[\big|\chi^0{}_0\big|^2 + \big|\chi^{1/2}{}_0\big|^2 + \big|\chi^0{}_{1/2}\big|^2 \pm \big|\chi^{1/2}{}_{1/2}\big|^2\big]. \tag{43}$$

Here, $\chi^\lambda{}_\mu(\tau)$ is the partition function of a left-moving Majorana (not complex) fermion with boundary conditions specified by λ and μ. As illustrated above, this partition function can be obtained by considering the following projection: $Z_{\text{Ising}} = \text{Tr}_{\text{A}\oplus\text{P}}\,[P_0\, q^{H_L} \bar{q}^{H_R}]$.[54]

V. DISCUSSION

The modular invariance plays a major role in CFT[49,55,56] and also in string theory. Its importance in chiral topological phases such as the fractional QHE has also been emphasized.[51,52]

Partly motivated by recent discoveries of nonchiral topological phases,[57] such as the QSHE, we studied in this paper an implication of modular invariance in nonchiral topological phases protected by discrete symmetries. Quite generically, a nonchiral edge theory can be gapped by some perturbation by "coupling" the left- and right-moving sectors. This is implied from the fact that a nonchiral CFT, when its left- and right-moving parts are properly combined, can be made modular invariant. In the presence of a certain symmetry condition, however, there is a constraint on perturbations which are allowed to be added to the action. In an extreme case, the symmetry constraint completely removes perturbations, in which case the gapless nature of the edge theory can be protected. This suggests that if the way we glue the left- and right-moving sectors were to be consistent with the symmetry condition, we would not be able to achieve modular invariance. For the particular example, we investigated in this work, there is $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry which allows us to decompose the Hilbert space into different sectors with different quantum numbers. After this decomposition, we studied if each sector can be made modular invariant separately. Even though we have looked at a particular example of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric topological phase, we expect the proposal using the modular invariance as a diagnostic tool for more general topological phases without local (perturbative) anomalies.

We close with several comments. (i) For the bulk of the paper, we have discussed mainly modular invariance/noninvariance of nonchiral CFTs. A chiral CFT can also be modular invariant/noninvariant on its own as well. A well-known example is a collection of N copies of chiral complex fermions or $2N$ copies of chiral Majorana fermions. Let us consider the partition function given by the following combination:

$$\begin{aligned}&\tfrac{1}{2}\big\{\big[Z^0{}_0(\tau)\big]^N + \big[Z^0{}_{1/2}(\tau)\big]^N + \big[Z^{1/2}{}_0(\tau)\big]^N\big\}\\ &\quad = \tfrac{1}{2}\big\{e^{iN\pi/12}\big[Z^0{}_{1/2}\big]^N + e^{iN\pi/12}\big[Z^0{}_0\big]^N\\ &\qquad + e^{-iN\pi/6}\big[Z^{1/2}{}_0\big]^N\big\}(\tau+1).\end{aligned} \tag{44}$$

The chiral central charge is $c_L = N$. The partition function is clearly S-modular invariant. In order to achieve invariance under T transformation, we need, at least, $N = 8k$ copies of fermions, where k is a positive integer. If we consider $16k$ chiral Majorana fermions or $8k$ complex fermions, the partition function is modular *covariant*. In particular, when $k = 1$, the chiral central charge is $c_L = 8$. (When bosonized, this is the partition function of the compactifed bosons on the root lattice E_8). If we cube this partition function, we achieve the true modular invariance with $c_L = 24$. The chiral topological phase with $2N$ copies of chiral Majorana fermions at its edge was discussed in the context of the honeycomb lattice Kitaev model.[58] A similar kind of mod 16 periodicity was observed in the bulk topological properties (non-Abelian statistics of quasiparticles in the bulk depends on the bulk Chern number mod 16).

(ii) We have used symmetry projection as a diagnostic tool to study the stability of noninteracting, symmetry-protected, topological phases. Instead, it is also possible to think of a topological phase with gauge interactions in the bulk. In this case, projections are performed dynamically in the bulk and in the edge theories. One of such models in the bulk would look like the two copies of the honeycomb lattice Kitaev model[58] with opposite chiralities.

(iii) While robust in the presence of a certain set of symmetries, nonchiral edges are in general susceptible to symmetry breaking perturbations. In particular, one can study the response of the edge theory to a local perturbation, such as a single impurity, or to a topological defect at the edge, which would reflect topological properties of the bulk. (See, for example, Refs. 59 and 60 for the edge state of the QSHE.) For the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric topological phase, such local impurity problems in the edge state, in the long-wave length limit, may correspond to D branes.

(iv) Finally, there are topological phases that are not accompanied by a gapless edge state. Whether or not these topological phases can be understood in terms of quantum anomalies of some kind is an open question.

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[54]As before, there is a sign ambiguity $\pm$ here. However, we can actually change this sign by Kremers-Wannier duality—so the sign convention is fixed once we fix convention for the spin σ and disorder μ operators.[47]

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