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Fractional Chern insulators and the W_∞ algebra

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A set of recent results indicates that fractionally filled bands of Chern insulators in two dimensions support fractional quantum Hall states analogous to those found in fractionally filled Landau levels. We provide an understanding of these results by examining the algebra of Chern band projected density operators. We find that this algebra closes at long wavelengths and for constant Berry curvature, whereupon it is isomorphic to the W_∞ algebra of lowest Landau level projected densities first identified by Girvin, MacDonald, and Platzman [Phys. Rev. B **33**, 2481 (1986)]. For Hamiltonians projected to the Chern band this provides a route to replicating lowest Landau level physics on the lattice.

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Introduction. Much of the theory of the quantum Hall effect is tied to the celebrated Landau levels (LLs)—the nondispersing energy states of free electrons in a uniform magnetic field. A fruitful line of inquiry has focused on separating quantum Hall physics from this weak lattice limit. In the case of the integer quantum Hall effect (IQHE), Thouless and co-workers[1] showed that the Hall conductance is quantized even in the presence of a strong periodic potential, where the energy bands are no longer flat, as long as the chemical potential lies in a gap between the bands. They did so by relating the conductance to a mathematical invariant, the first Chern number, associated with the filled bands. Subsequently, Haldane[2] showed, through an explicit construction of a tight-binding model of a "Chern band insulator" (CBI), that a quantized Hall conductance could be obtained even in the absence of a net magnetic field as long as the filled bands still came with nontrivial topology, i.e., nonzero Chern numbers. This thus established an equivalence between filled CBs and filled LLs. A natural question that then arises is whether such an equivalence also holds for partially filled CBs and partially filled LLs—specifically, whether the fractional quantum Hall effect (FQHE) arises in a partially filled CB in a lattice system.

A flurry of recent work has shed much light on this question. Several authors[3–6] have constructed models with nearly flat (nondispersing) Chern bands which allow interactions to dominate at partial fillings while leaving the gap to neighboring bands open. References 5,7, and 8 and, most convincingly, Regnault and Bernevig[9] have reported evidence for FQH states at $\nu = 1/3, 1/5$, and (for bosons) $1/2$ in finite-size studies of short-ranged interactions projected to these bands. A partial understanding of these numerical results is already available: Applying a generalized Pauli principle familiar from the lowest Landau level (LLL) FQHE rationalizes some aspects of the results,[9] while Qi[10] has constructed a fairly general recipe for translating familiar model wave functions and Hamiltonians from the LLL to CBs on cylinders by elegantly mapping Landau gauge eigenfunctions to particular Wannier functions.

In this Rapid Communication, we offer a complementary perspective on the CB-LLL equivalence. As in the earlier work we start by distinguishing the kinematical and dynamical aspects of obtaining QH physics in a CB. The latter involve an appropriate choice of band structure and interaction Hamiltonian but the former requires an appropriate identification of the algebra of operators in the two systems. Specifically, in both the CB and the LLL, the projected Hamiltonians that include interactions and scalar disorder involve only the projected particle density operators and thus the nontriviality of the physics enters precisely through the nontriviality of the algebraic relations between the different Fourier components of the densities. For the LLL the density operators obey the so-called W_∞ algebra first constructed by Girvin, Macdonald, and Platzman.[11] We show that at long wavelengths and in a natural limit (constant Berry curvature) densities projected to the CB obey the *same* algebra. This implies that *mutatis mutandis* LLL physics can be transported to the CB. We also note that this implies that the Cartesian components of the particle positions projected to the CB fail to commute, much as guiding center coordinates in the LLL. In what follows we review the W_∞ algebra for the LLL, show how it arises in a CB in $d = 2$, argue that it allows LLL physics (impurity and FQHE) to be translated to the CB, and discuss generalization of the density algebra to other topological insulators, which appears to us to be a promising route to understanding their fractional physics.

Projection to the lowest Landau level. We begin by recalling some details of the projection to the LLL[12] which captures the essential features of the FQHE in the high field limit.[13] Consider an electron in an external magnetic field $\mathbf{B} = -B\hat{\mathbf{z}}$, described by the Hamiltonian $H = \frac{1}{2m}(\mathbf{p} + \frac{e}{c}\mathbf{A})^2$. The motion of the electron can be separated into the fast cyclotron motion and the slower drift of the guiding center of its orbit. Working in symmetric gauge and with $\hbar = 1$, this can be implemented by decomposing its position $\mathbf{r}$ into two parts, $\mathbf{r} = (\frac{1}{2}\mathbf{r} + \hat{\mathbf{z}} \times \mathbf{p}) + (\frac{1}{2}\mathbf{r} - \hat{\mathbf{z}} \times \mathbf{p}) \equiv \boldsymbol{\eta} + \mathbf{R}$, which obey the commutation relations $[\eta_x, \eta_y] = i\ell_B^2, [R_x, R_y] = -i\ell_B^2, [\eta_i, R_j] = 0$, where $\ell_B = (c/eB)^{1/2}$ is the magnetic length. In terms of these variables, the Hamiltonian depends purely on the cyclotron coordinates $\boldsymbol{\eta}$, $H = \frac{\eta^2}{2m\ell_B^4}$, which (owing to the fact that the components of $\boldsymbol{\eta}$ are conjugate) has the familiar oscillator spectrum of a series of Landau levels, with $E_n = (n + \frac{1}{2})\omega_c$, where $\omega_c = eB/mc$ is the cyclotron frequency. Since H commutes with the guiding center coordinates $\mathbf{R}$, each energy level is extensively degenerate with a degeneracy given by N_Φ, the number of flux quanta threading the sample.

When the number of electrons N is smaller than N_Φ, we have a partially filled $n=0$ LL massively degenerate in the kinetic energy; we must therefore include the effect of interactions. A reasonable starting point is to project the interactions into the degenerate subspace; the effect of the projection is to replace $\boldsymbol{\eta}$-dependent expressions by their LLL expectation values, e.g., $\langle \boldsymbol{\eta}^2\rangle_{\rm LLL} = \ell_B^2$. The density operator $\rho_{\mathbf{q}} = e^{i\mathbf{q}\cdot\hat{\mathbf{r}}}$ is projected as $\mathcal{P}\rho_{\mathbf{q}}\mathcal{P} = e^{-q^2\ell_B^2/4}e^{i\mathbf{q}\cdot\mathbf{R}} \equiv e^{-q^2\ell_B^2/4}\overline{\rho}_{\mathbf{q}}$, where we have factored out a q-dependent constant in the definition of the projected density.[14] Since $[R_x,R_y] = -i\ell_B^2$, we can show that the $\overline{\rho}_{\mathbf{q}}$ satisfy the W_∞/Girvin-MacDonald-Platzman algebra[11]

$$\left[\overline{\rho}_{\mathbf{q}_1},\overline{\rho}_{\mathbf{q}_2}\right] = 2i\sin\left(\frac{\mathbf{q}_1\wedge\mathbf{q}_2\ell_B^2}{2}\right)\overline{\rho}_{\mathbf{q}_1+\mathbf{q}_2}, \tag{1}$$

where $\mathbf{q}_1\wedge\mathbf{q}_2 \equiv \hat{\mathbf{z}}\cdot(\mathbf{q}_1\times\mathbf{q}_2)$. This algebra arises in several contexts and has the interpretation of a quantum deformation of the algebra of area-preserving diffeomorphisms on the plane as well as that of magnetic translations in a uniform field as discussed, e.g., in Ref. 15.

Along with the above commutation relations, the Hamiltonian

$$H_{\rm LLL} = \frac{1}{2}\sum_{\mathbf{q}} V(\mathbf{q})e^{-q^2\ell_B^2/2}\overline{\rho}_{\mathbf{q}}\overline{\rho}_{-\mathbf{q}} + \sum_{\mathbf{q}} V_{\rm imp}(\mathbf{q})e^{-q^2\ell_B^2/4}\overline{\rho}_{-\mathbf{q}}\,, \tag{2}$$

which is the projection of a density-density interaction and an impurity potential into the LLL, completes the low-energy description of the fractional quantum Hall effect. [In passing from (1) to (2) we reinterpret $\overline{\rho}_{\mathbf{q}}$ as the density of a many electron system. As the latter is additive over the individual particle densities, it also obeys the algebra (1).] While still formidable, the LLL problem has been tackled by a variety of different approaches, most notably by guessing trial wave functions for FQH states in the clean limit. The essential point for this Rapid Communication is that if we can construct a similar low-energy description for a Chern insulator, then some aspects[16] of the familiar fractional quantum Hall technology can be applied—with suitable modifications to account for the different nature of the single-particle states—to the problem of interacting Chern insulators.

Projection to the Chern band. We begin by fixing notation and recalling some basic facts. An insulator with $\mathcal{N}$ bands is described by the single-particle Hamiltonian $H = \sum_{\mathbf{k},a,b} c^\dagger_{\mathbf{k},a}h_{ab}(\mathbf{k})c_{\mathbf{k},b}$, where $a,b = 1,2,\ldots,\mathcal{N}$ are sublattice/spin indices, and $\mathbf{k}$ is the crystal momentum restricted to the first Brillouin zone (BZ). (Here and below, we will explicitly indicate when repeated indices are summed over.) The solution of the $\mathcal{N}\times\mathcal{N}$ eigenvalue problem $\sum_b h_{ab}(\mathbf{k})u^\alpha_b(\mathbf{k}) = \epsilon_\alpha(\mathbf{k})u^\alpha_a(\mathbf{k})$ defines the energy bands and we will take the eigenvectors to be normalized, $\sum_a |u^\alpha_a(\mathbf{k})|^2 = 1$. The corresponding eigenstates are given by

$$|\mathbf{k},\alpha\rangle = \gamma^\dagger_{\mathbf{k},\alpha}|0\rangle \equiv \sum_a u^\alpha_a(\mathbf{k})c^\dagger_{\mathbf{k},a}|0\rangle. \tag{3}$$

Specializing to two dimensions, the Chern number of a given band α is computed as

$$C_\alpha = \frac{1}{2\pi}\int_{\rm BZ} d^2k\,\mathcal{B}_\alpha(\mathbf{k}). \tag{4}$$

Here, $\mathcal{B}_\alpha(\mathbf{k})$ is the Chern flux density (Berry curvature), defined as the curl of the Berry connection (Berry gauge potential), $\mathcal{B}_\alpha(\mathbf{k}) = \boldsymbol{\nabla}_{\mathbf{k}}\times\mathcal{A}_\alpha(\mathbf{k})$. In terms of the eigenstates, we have

$$\mathcal{A}_\alpha(\mathbf{k}) = i\sum_{b=1}^{\mathcal{N}} u^{\alpha*}_b(\mathbf{k})\boldsymbol{\nabla}_{\mathbf{k}}u^\alpha_b(\mathbf{k}). \tag{5}$$

A filled band with Chern number C_α yields a Hall conductance $\sigma_H = C_\alpha e^2/h$ regardless of whether it arises in a system with a net magnetic field[1] ("Hofstadter band") or zero net magnetic field[2] ("Haldane band"). We shall refer to both as Chern bands.

As announced, we are interested in the restriction of the kinematics to a single CB which can be implemented by use of the projection operator $\mathcal{P}_\alpha = |\mathbf{k},\alpha\rangle\langle\mathbf{k},\alpha|$. It follows that the density operator $\rho_{\mathbf{q}} = e^{i\mathbf{q}\cdot\hat{\mathbf{r}}}$ when projected onto the CB takes the form

$$\overline{\rho}_{\mathbf{q}} \equiv \mathcal{P}_\alpha\rho(\mathbf{q})\mathcal{P}_\alpha = \sum_{\mathbf{k}}\left[\sum_b u^{\alpha*}_b\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right)u^\alpha_b\left(\mathbf{k}-\frac{\mathbf{q}}{2}\right)\right]\gamma^\dagger_{\mathbf{k}+\frac{\mathbf{q}}{2},\alpha}\gamma_{\mathbf{k}-\frac{\mathbf{q}}{2},\alpha} \tag{6}$$

At long wavelengths $qa \ll 1$, we may expand $\sum_b u^{\alpha*}_b(\mathbf{k}+\frac{\mathbf{q}}{2})u^\alpha_b(\mathbf{k}-\frac{\mathbf{q}}{2}) \approx 1 - i\mathbf{q}\cdot\sum_b u^{\alpha*}_b(\mathbf{k})\frac{\nabla_{\mathbf{k}}}{i}u^\alpha_b(\mathbf{k}) \approx e^{i\int_{\mathbf{k}-\mathbf{q}/2}^{\mathbf{k}+\mathbf{q}/2}d\mathbf{k}'\cdot\mathcal{A}_\alpha(\mathbf{k}')}$, so that

$$\overline{\rho}_{\mathbf{q}}|\mathbf{k},\alpha\rangle \approx e^{i\int_{\mathbf{k}}^{\mathbf{k}+\mathbf{q}}d\mathbf{k}'\cdot\mathcal{A}_\alpha(\mathbf{k}')}|\mathbf{k}+\mathbf{q},\alpha\rangle. \tag{7}$$

In other words, for small q, $\overline{\rho}(\mathbf{q})$ implements parallel transport described by the Berry connection $\mathcal{A}_\alpha(\mathbf{k})$. Either from this observation or via a gradient expansion, we may show that at long wavelengths, the commutator of projected density operators at different wave vectors is

$$\left[\overline{\rho}_{\mathbf{q}_1},\overline{\rho}_{\mathbf{q}_2}\right] \approx i\,\mathbf{q}_1\wedge\mathbf{q}_2\sum_{\mathbf{k}}\left[\mathcal{B}_\alpha(\mathbf{k})\sum_b u^{\alpha*}_b(\mathbf{k}_+)u^\alpha_b(\mathbf{k}_-)\gamma^\dagger_{\mathbf{k}_+,\alpha}\gamma_{\mathbf{k}_-,\alpha}\right] \tag{8}$$

where we define $\mathbf{k}_\pm = \mathbf{k}\pm\frac{\mathbf{q}_1+\mathbf{q}_2}{2}$. Finally, let us assume that the local Berry curvature $\mathcal{B}_\alpha(\mathbf{k})$ can be replaced by its average

$$\overline{\mathcal{B}_\alpha} = \frac{\int_{\rm BZ}d\mathbf{k}\,\mathcal{B}_\alpha(\mathbf{k})}{\int_{\rm BZ}d\mathbf{k}} = \frac{2\pi C_\alpha}{A_{\rm BZ}} \tag{9}$$

over the BZ; here $A_{\rm BZ} = c_0^2/a^2$ is the area of the BZ, with a the lattice spacing and c_0 a numerical constant depending on the unit cell symmetry. This yields

$$\left[\overline{\rho}_{\mathbf{q}_1},\overline{\rho}_{\mathbf{q}_2}\right] \approx i\,\mathbf{q}_1\wedge\mathbf{q}_2\overline{\mathcal{B}_\alpha}\,\overline{\rho}_{\mathbf{q}_1+\mathbf{q}_2}, \tag{10}$$

which is identical to the long-wavelength limit of the density algebra (1) for the LLL, with $\overline{\mathcal{B}_\alpha}^{1/2} = \frac{\sqrt{2\pi C_\alpha}}{c_0}a$ playing the role of the magnetic length ℓ_B.

Equation (10) is our central result and various remarks are in order concerning it:

(1) We can define a coarse grained, projected, position operator as $\mathbf{r}_{\rm cg} \equiv x_{\rm cg}\hat{\mathbf{x}} + y_{\rm cg}\hat{\mathbf{y}} = \lim_{\mathbf{q}\to 0} \frac{\nabla_q}{i}\overline{\rho}_{\mathbf{q}}$. It follows from Eq. (10) that

$$[x_{\rm cg}, y_{\rm cg}] = -i\overline{\mathcal{B}_\alpha}. \tag{11}$$

which identifies the $\mathbf{r}_{\rm cg}$ with the guiding center position operator in the LLL.

(2) For a system of N unit cells there are N points in the BZ. If $\mathcal{B}_\alpha(\mathbf{k})$ is truly constant we can define a set of N parallel translation operators $T_{\mathbf{q}}$ for which (7) holds exactly:

$$T_{\mathbf{q}}|\mathbf{k},\alpha\rangle = e^{i\int_{\mathbf{k}}^{\mathbf{k}+\mathbf{q}} d\mathbf{k}'\cdot\mathcal{A}_\alpha(\mathbf{k}')}|\mathbf{k}+\mathbf{q},\alpha\rangle. \tag{12}$$

The algebra of the $T_{\mathbf{q}}$ is thus exactly of the W_∞ form (1) without a long-wavelength restriction. We note that the $T_{\mathbf{q}}$ are trivially isomorphic to magnetic translation operators for a system with N sites and flux $1/N$ per unit cell and it is straightforward to check that the states in the band form an N dimensional irreducible representation of their algebra.[17] From this perspective the idealization of a constant curvature CB hosts a W_∞ algebra whose long-wavelength generators coincide with the physical density operators.

(3) Alternatively, if the deviation of the Berry curvature from its average value is bounded, $|\mathcal{B}_\alpha(\mathbf{k}) - \overline{\mathcal{B}_\alpha}| < |\overline{\mathcal{B}_\alpha}| - \epsilon$, we may define a "smoothed" density operator which may be regarded as the projection of an operator $\rho^s(\mathbf{r})$ local in position space; for $qa \ll 1$ this gives a modified form of (7):

$$\overline{\rho}^s_{\mathbf{q}}|\mathbf{k},\alpha\rangle = \frac{\overline{\mathcal{B}_\alpha}}{\mathcal{B}_\alpha(\mathbf{k})} e^{i\int_{\mathbf{k}}^{\mathbf{k}+\mathbf{q}} d\mathbf{k}'\cdot\mathcal{A}_\alpha(\mathbf{k}')}|\mathbf{k}+\mathbf{q},\alpha\rangle. \tag{13}$$

At long wavelengths, the algebra of smoothed densities closes, and in this limit (10) is an *exact* equality when $\overline{\rho}_{\mathbf{q}}$ is replaced by $\overline{\rho}^s_{\mathbf{q}}$.

(4) Both the "Hofstadter" and "Haldane" problems give rise to (10), which unifies earlier lattice FQHE studies[18,19] with the ones considered here.

(5) This last observation can be used to get nearly constant curvature bands by approaching the Landau level limit on the lattice, i.e., by picking flux $1/q$ per plaquette and working the lowest subband at large q. While it is impossible to find a constant-curvature CB in models of Chern insulators with $\mathcal{N} = 2$ bands, it is possible to construct models with $\mathcal{N} > 2$ which host a CB with nearly constant Berry curvature.[20,21]

(6) Finally, readers familiar with the LLL problem will note that there the W_∞ algebra in a system with N states is generated by N^2 density operators while in the CB there are only N densities (or $T_{\mathbf{q}}$ if one wishes to work with a closed algebra). This distinction arises as the LLL is formally defined on a continuous space but is without fundamental dynamical significance as the relevant momenta $q\ell_B < 1$ are $O(N)$ in the LLL as well. For example, it was shown in Ref. 22 that keeping only this set of momenta keeps the entire physics of the quantum Hall localization transition in the LLL. However, this counting discrepancy does have the consequence that the algebra of the densities themselves *must* close in the LLL at *all* $\mathbf{q}$ which is not the case in the CB.

Interactions and disorder. We have argued above that it is possible to construct lattice models with nearly constant curvature for which the algebra (10) is realized to an excellent approximation. Evidently if we can ignore the variations in curvature and replace the $\overline{\rho}_{\mathbf{q}}$ by $T_{\mathbf{q}}$ in the Hamiltonian (2), we will have mapped accurately between the LLL and CB problems at the same fractional filling. Our remaining task is to argue that we can often ignore the residual variations anyway. For a problem with sufficient disorder, most notably the problem of localization within the CB, we expect that impurity scattering will effectively lead to an averaging of the curvature over the band. This is consistent with what is known about the QH transition in Chern insulators,[23] although a direct test in the projected CB would be desirable. A simple estimate for the requisite strength of disorder can be made by comparing the inverse mean free path $l^{-1}_{\rm MF}$ to the characteristic momentum-space scale $k_\sigma \sim |\overline{\nabla\mathcal{B}}|/\overline{\mathcal{B}}$ for variations of the Berry curvature; for disorder sufficiently strong that $l^{-1}_{\rm MF} \sim k_\sigma$, the random potential will scatter between points in the BZ that are k_σ apart, and thereby average their Berry curvature. For the formation of FQH states we appeal to the stability of topologically ordered states to arbitrary perturbations. Intuitively, we expect to be able to trade the extra pieces in the density commutators for terms in the Hamiltonian. For sufficiently constant curvature bands and sufficiently strong quantum Hall states, these extra terms should not destabilize the FQHE; numerical studies are in striking accord with this observation.[21,24]

Concluding remarks. This Rapid Communication has been concerned with establishing a correspondence between a CB and the LLL; the existence of QH physics in both problems is traced to a common source in the nontrivial, W_∞, algebra of long-wavelength densities. Intuitively, the nontriviality of the algebra is needed for the densities to do something other than simply condense and break translational symmetry when an interaction is introduced into a flattened band. This suggests that a program of identifying density algebras for other topologically nontrivial bands (sets of bands) in various dimensions could be fruitful in generating new physics upon inclusion of interactions when such bands are partially filled. Here we make a few comments in that direction.

Consider projecting onto a set of bands. For an interesting dynamical problem to arise it will be necessary for these bands to be nearly degenerate (possibly exactly on account of symmetry) and nearly flat, but we are again interested in the kinematics of projection. The projected density operator $\bar{\rho}_q$ now has a nontrivial structure in the band index, so that the analog of (7) is

$$\overline{\rho}_{\mathbf{q}}|\mathbf{k},\alpha\rangle \approx \sum_\beta \left[e^{i\int_{\mathbf{k}}^{\mathbf{k}+\mathbf{q}} d\mathbf{k}'\cdot\mathcal{A}(\mathbf{k}')}\right]_{\beta\alpha}|\beta,\mathbf{k}+\mathbf{q}\rangle, \tag{14}$$

where $\mathcal{A}_{\alpha\beta}(\mathbf{k}) \equiv \sum_b u^{\alpha *}_b(\mathbf{k})\frac{\nabla_{\mathbf{k}}}{i}u^\beta_b(\mathbf{k})$ is the non-Abelian vector potential, and the term in square brackets is understood as a matrix exponential. At long wavelengths,

$$\left[\overline{\rho}_{\mathbf{q}_1}, \overline{\rho}_{\mathbf{q}_2}\right] \approx i\,\mathbf{q}_1\wedge\mathbf{q}_2 \sum_{\mathbf{k},\alpha,\sigma,\beta} \mathcal{F}_{\alpha\sigma}(\mathbf{k})\left[\sum_b u^{\sigma *}_b(\mathbf{k}_+)u^\beta_b(\mathbf{k}_-)\gamma^\dagger_{\mathbf{k}_+,\sigma}\gamma_{\mathbf{k}_-,\beta}\right], \tag{15}$$

where $\mathcal{F} = d\mathcal{A} - i[\mathcal{A},\mathcal{A}]$ is the non-Abelian Berry curvature. Let us specialize to cases where $\mathcal{F}$ is nearly constant up to a gauge transformation. It can then be replaced by its BZ average $\overline{\mathcal{F}}$, and the long-wavelength algebra simplifies to an intertwined generalization of W_∞:

$$\left[\overline{\rho}_{\mathbf{q}_1},\overline{\rho}_{\mathbf{q}_2}\right] \approx i\,\mathbf{q}_1 \wedge \mathbf{q}_2 \overline{\mathcal{F}} \cdot \overline{\rho}_{\mathbf{q}_1+\mathbf{q}_2}, \tag{16}$$

where we have assumed a matrix product in the band indices.

In two dimensions, $\mathcal{A}(k)$ and $\mathcal{F}$ can always be globally diagonalized by gauge transformation. Hence in the long-wavelength limit, the density operators can be labeled by a new "band" index, such that densities with different indices always commute while those with the same index exhibit the standard W_∞ commutators. If the interactions are chosen to respect this structure, one obtains a mapping to a multicomponent quantum Hall system with different components potentially experiencing different strength magnetic fields. For example, in the case of topological insulators with time-reversal symmetry in two dimensions, a basis of time-reversal conjugate bands may always be found[25] where the two bands have odd Chern numbers, which are the same in magnitude and opposite in sign and this leads to the fractional quantum spin Hall effect.[26] We note though that generic interactions will not decompose naturally in such a basis. For $d > 2$, it is not in general possible to globally diagonalize the Berry connection and now the problem is intrinsically non-Abelian and worthy of further study. We conjecture that in $d = 4$ this approach will lead to fractional states for bands with a nonzero second Chern number, possibly related to the ones studied in Ref. 27.

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