PHYSICAL REVIEW B **85**, 241307(R) (2012)

Typology for quantum Hall liquids

S. A. Parameswaran,[1,2,*] S. A. Kivelson,[3,†] E. H. Rezayi,[4,‡] S. H. Simon,[5,§] S. L. Sondhi,[1,∥] and B. Z. Spivak[6,¶]

[1]*Department of Physics, Princeton University, Princeton, New Jersey 08544, USA*
[2]*Department of Physics, University of California, Berkeley, California 94720, USA*
[3]*Department of Physics, Stanford University, Stanford, California 94305, USA*
[4]*Department of Physics, California State University, Los Angeles, California 90032, USA*
[5]*Rudolf Peierls Centre for Theoretical Physics, 1 Keble Road, Oxford University, OX1 3NP, UK*
[6]*Department of Physics, University of Washington, Seattle, Washington 98195, USA*

(Received 10 August 2011; published 20 June 2012)

There is a close analogy between the response of a quantum Hall liquid (QHL) to a small change in the electron density and the response of a superconductor to an externally applied magnetic flux—an analogy which is made concrete in the Chern-Simons Landau-Ginzburg (CSLG) formulation of the problem. As the types of superconductors are distinguished by this response, so too for QHLs: A typology can be introduced which is, however, richer than that in superconductors owing to the lack of any time-reversal symmetry relating positive and negative fluxes. At the boundary between type I and type II behavior, the CSLG action has a "Bogomol'nyi point," where the quasiholes (vortices) are noninteracting—at the microscopic level, this corresponds to the behavior of systems governed by a set of model Hamiltonians which have been constructed to render exact a large class of QHL wave functions. All types of QHLs are capable of giving rise to quantized Hall plateaux.

DOI: 10.1103/PhysRevB.85.241307 PACS number(s): 73.43.−f, 64.75.Jk, 74.25.Uv

A transparent way in which to understand many properties of different quantum Hall phases is via the field theory of a two-dimensional charged superfluid coupled to a fictitious Chern-Simons (CS) gauge field.[1–3] A consequence of CS electrodynamics is that charges are bound to a fixed number of flux quanta. This equivalence of flux and charge implies that the condensed state—which exhibits the Meissner effect, perfect conductivity, and quantized vortex excitations—corresponds to an incompressible phase with a quantized Hall conductance, whose quasiparticles carry fractional electric charge and statistics. While the original Chern-Simons Landau-Ginzburg (CSLG) theory provides a convenient description of the Laughlin states and the Haldane-Halperin hierarchy[4,5] as condensates of composite bosons,[1] a parallel treatment in terms of composite fermions[6] extends the CS approach to describe fractional quantum Hall (FQH) phases seen in the vicinity of even-denominator filling factors as condensates of fermion pairs.[7,8] Additional results, such as a global phase diagram in which the plateau transitions are related to an underlying superconductor to insulator transition,[9] can also be derived within the CSLG formalism. There is thus a useful mapping between superconductivity and the FQHE.

Superconductors famously come in two varieties, which differ in their response to external magnetic fields: Type I superconductors phase separate into superconducting and normal regions, with flux concentrated in the latter, while type II superconductors form an Abrikosov lattice of vortices, each carrying a single flux quantum. The analogy between superconductors and FQH phases suggests that there is a similar distinction between type I and type II QH liquids, manifested in dramatically different patterns of charge organization upon doping. While in the clean limit, type II QH liquids exhibit Wigner crystallization of fractionally charged excitations, their type I cousins would exhibit phase separation. This quite general dichotomy was pointed out only recently, when it was argued that type I behavior occurs in paired QH states when the pairing scale is weak.[10] While the focus of that work was the Pfaffian phase in the vicinity of filling factor $\nu = 5/2$, the results generalize implicitly to all paired states.

In this Rapid Communication, we expand significantly on this work. First, we identify a class of "Bogomol'nyi points" of the CSLG theory which occur at the seperatrix between type I and type II behavior where the quasiholes are noninteracting even while the charged excitation spectrum is gapped. (Such field theories are also referred to as "self-dual" points, for reasons we will discuss below.) In the microscopic, lowest Landau level (LLL) formulation they correspond to special Hamiltonians introduced with the purpose of rendering particular model wave functions exact ground states. While some of what we say in both settings is not new, their connection has not been discussed before. Second, we observe that such self-dual points can be weakly perturbed to yield various types of QHLs. These include the traditional type I and II liquids, frustrated type I liquids, which exhibit short-distance phase separation frustrated by long-range repulsion,[10] and others which we will discuss below. Especially striking are type I-II QHLs which exhibit type I behavior for one sign of doping and type II for the other—this type might generalize to time-reversal (**T**) breaking superconductors as well. Note that all types of QHLs exhibit quantized transport plateaux as a function of magnetic field or density, at least upon including weak disorder.

Chern-Simons Landau-Ginzburg theory. Near $\nu = 1/k$, the description of QHLs as superconducting states of flux-charge composites is formalized in terms of a (bosonic) composite field ϕ that binds k flux quanta to an electron and interacts with a CS gauge field $(a_0, \mathbf{a})$ which encodes the electron statistics, as captured by the (gauge-fixed) Euclidean Lagrangian density

$$\mathcal{L} = \bar{\phi} D_\tau \phi + \frac{|\mathbf{D}\phi|^2}{2m} + \lambda(|\phi|^2 - \rho)^2 + i a_0 \frac{\nabla \times \mathbf{a}}{k\Phi_0}, \quad (1)$$

where $D_0 = \partial_\tau - ia_0$ and $D_\mu = \partial_\mu - i(a+A)_\mu$ are the covariant derivatives, the external field $B = \nabla \times \mathbf{A}$, the filling is $\nu \equiv \frac{\rho\Phi_0}{B} = \frac{1}{k}$, with $\Phi_0 = hc/e$ the quantum of flux, and a_0 is a nondynamical field which enforces the flux attachment constraint

$$b \equiv \nabla \times \mathbf{a} = -k\Phi_0|\phi|^2. \tag{2}$$

The static extrema of (1) satisfy (2) and the CSLG equations of motion

$$-\frac{\mathbf{D}^2\phi}{2m} + 2\lambda(|\phi(\mathbf{r})|^2 - \rho)\phi = 0, \tag{3a}$$

$$\mathbf{e} \equiv -\nabla a_0 = k\Phi_0\hat{\mathbf{z}} \times \mathbf{j}, \tag{3b}$$

where $\mathbf{j} = \frac{1}{m}\text{Im}[\bar{\phi}\mathbf{D}\phi]$ is the current density. The integral of $\mathcal{L}$ over space gives the energy of this state. Precisely at $\nu = 1/k$—corresponding to the pristine QHL—these are solved by the uniform condensate configuration $\phi = \sqrt{\rho}e^{i\theta}$, $\mathbf{a} = -\mathbf{A}$, and $a_0 = 0$, which yields a state with zero energy.

The Bogomol'nyi point. In search of vortex excitations we employ the Bogomol'nyi trick, $|\mathbf{D}\phi|^2 = |D_\pm\phi|^2 \mp (b + B)|\phi|^2 \pm m\nabla \times \mathbf{j}$ (where $D_\pm \equiv D_x \pm iD_y$) to express $\mathcal{L}$ in (1) for any time-independent field configuration subject to the constraint (2) as

$$\mathcal{L} = \frac{|D_\pm\phi|^2}{2m} + \left(\lambda \pm \frac{k\Phi_0}{2m}\right)(|\phi|^2 - \rho)^2 \mp \frac{(B - B^*)|\phi|^2 + B^*(\rho - |\phi|^2)}{2m}, \tag{4}$$

where $B^* \equiv k\Phi_0\rho$. Since the total charge is the integral over space of $|\phi|^2$, and is a conserved quantity, the term on the second line makes a configuration-independent contribution to the energy. Thus, the optimal vortex solutions are those that minimize the first two terms. At the Bogomol'nyi point,[11] $\lambda = \lambda_B \equiv k\Phi_0/2m$, the quartic term cancels for the lower sign in Eq. (4), so this reduces to finding solutions to the first-order equation

$$D_-\phi \equiv (D_x - iD_y)\phi = 0, \tag{5}$$

along with the constraint (2).[12] Equation (5) represents a form of self-duality as it can be rewritten as $D_a\phi = i\epsilon_{ab}D_b\phi$.

In Eq. (1), ρ is formally related to the zero of energy which we are free to choose so that ρ is equal to the average charge density. In this case, such a solution has an energy density $\mathcal{E} = \frac{(B-B^*)\rho}{2m}$. From Eq. (5) it is straightforward to show that $\mathbf{j} = \frac{1}{2m}\hat{\mathbf{z}} \times \nabla|\phi|^2$. Thus, the final extremal condition, Eq. (3b), is solved by $a_0 = \lambda_B|\phi|^2 + \text{const}$ where the constant is related to the chemical potential.

The energy of the self-dual solutions of a specified vorticity is *independent* of the spatial distribution of that vorticity. As our signs correspond to vortices being quasiholes, it follows that such self-dual quasiholes do not interact at $\lambda = \lambda_B$. The quasielectron (antivortex) solutions are *not* self-dual. These interactions have not been studied in detail although it is clear that they have a range of order the magnetic length, $\ell_B = (2\pi\Phi_0/B^*)^{1/2}$ and, from Ref. 13, that they are repulsive at very short distances and in their tails. However, as their order parameter (density) profile is radially nonmonotone, one cannot at present rule out an intermediate regime of attraction. We will assume that the quasielectrons at the self-dual point repel at all distances, although almost nothing that we say will depend on this assumption as we will clarify below. Finally, the gap to qh-qe pairs is positive[13] and the quadratic mode frequencies are all positive, whence the self-dual point is stable.

Perturbing the Bogomol'nyi points. It follows from the above that at the Bogomol'nyi points, QHLs are type II for quasielectron doping and agnostic for quasihole doping. We now study the effect of small perturbations.

(1) If we change decrease/increase λ by a small amount $\delta\lambda$, the quasiholes now experience an attractive/repulsive interaction resulting in a type I/type II QHL for hole doping. The quasielectron interaction will not change sign for sufficiently small $\delta\lambda$ of either sign. The result is a symmetric type II QHL for $\delta\lambda > 0$ and an asymmetric type I-II QHL for $\delta\lambda < 0$ (i.e., type I for quasihole and type II for quasielectron doping[14]).

(2) While the restriction to a local scalar self-interaction is natural in a superconductor, a more general nonlocal density-density interaction $\delta\mathcal{L} = \frac{1}{2}\int d\mathbf{r}'(|\phi(\mathbf{r})|^2 - \rho)v(\mathbf{r} - \mathbf{r}')(|\phi(\mathbf{r}')|^2 - \rho)$ is natural in the context of the Hall effect. Perturbing the Bogomol'nyi point with a long-range repulsive interaction (e.g., Coulomb) clearly results in a symmetric type II QHL.

(3) Let us perturb the self-dual model with a term of this form in which $v(\mathbf{r})$ is (a) attractive, (b) weak enough to not close the gap to making a quasielectron-quasihole pair, and (c) has a range $\gg \ell_B$. Now the quasiholes *and* quasielectrons attract at long distances which is sufficient for both to phase separate at finite densities and thus to exhibit (macroscopic) type I behavior for both signs of doping. However, for quasielectron doping, this macroscopic type I behavior hides the competition between the short-ranged repulsion at the self-dual point and the longer ranged attraction. If the attraction is sufficiently weak, then within the quasiparticle-rich region, the quasiparticles still form a Wigner crystal, a behavior analogous to what has been called type 1.5 in the superconducting context in Ref. 15.

(4) More generally, perturbing with additional, nonmonotone interactions can generate various forms of charge order upon doping.

Analogy with superconductors. As noted previously, in superconductors the Bogomol'nyi point marks the boundary between type I and II behavior. The key difference from the QH case is that *both* $D_\pm$ can be used to obtain vortex *and* antivortex solutions which are, naturally, related to each other simply by time-reversal conjugation. Thus near the Bogomol'nyi point and indeed more generally, $\mathbf{T}$-invariant superconductors exhibit a symmetric response to flux doping. This raises the interesting question of whether $\mathbf{T}$-breaking superconductors can exhibit asymmetric flux doping—for instance, type I-II behavior. Conversely, we note that weakly coupled paired QH states give rise to a LG theory of essentially the superconducting form[10] where the symmetry in doping can be traced to the particle-hole symmetry about the Fermi surface of the parent composite Fermi liquid. In this limit the paired states exhibit two different length scales—the pairing (coherence) length and the screening length (penetration depth)—and thus exhibit symmetric frustrated type I behavior at weak coupling where the coherence length greatly exceeds the penetration depth.

Microscopic models. We now turn to a large class of microscopic models for FQH states which realize the key properties of the Bogomol'nyi point of the CSLG theory and thus can be perturbed to yield type I QH fluids in exactly the same fashion. These are not new models—they have been constructed historically to render various desirable wave functions exact ground states, starting with the work of Refs. 4 and 16. However, the connection of these models to Bogomol'nyi points in the CSLG theory has not been made before to our knowledge.

An illustrative example of how this works is the $\nu = 1/3$ state, in many ways the prototypical FQH state. At this filling the ideal Laughlin state is the (essentially) unique[17] ground state of the model "hard core" or "pseudopotential" Hamiltonian[4,16] $H_{1/3} = \sum_{i<j} \nabla^2 \delta^{(2)}(\mathbf{r}_i - \mathbf{r}_j)$. All available evidence is consistent with the proposition that at exactly $\nu = 1/3$ the ground state is separated from all excited states by a gap that remains nonzero in the thermodynamic limit. Further, *all* states with a given number of quasiholes are degenerate, or in other words, the quasiholes do not interact. Quasielectrons on the other hand do interact, although their interaction has not been analytically computed. We have numerically evaluated the energy spectrum of a system of 12 electrons on a sphere, at a flux density corresponding to two quasielectrons; the results are shown in Fig. 1. Looking at the lowest energy states as a function of total angular momentum—which is inversely proportional to the distance between quasielectrons—we see that they exhibit a short-ranged interaction consistent with a hard core, an intermediate attraction, and an asymptotic repulsion. We note that Beran and Morf[18] have also studied the quasiparticle interactions in the 1/3 state for pseudopotentials tuned near the Coulomb point, which also exhibits nonmonotonic features. These features of the model Hamiltonian clearly parallel those of the Bogomol'nyi point of the CSLG theory; it follows that the model Hamiltonian is on the border between type I and II for hole doping and is type II/type 1.5 for electron doping. Therefore, we may follow the strategy adopted previously: By perturbing about $H_{1/3}$ with a weak, longer ranged interaction, we can make the quasiparticles either attract or repel without

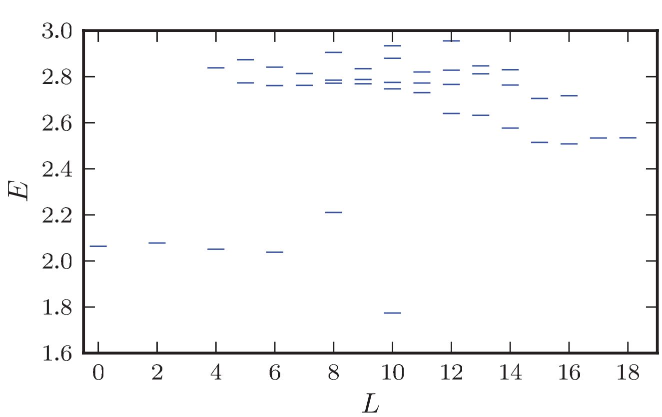

FIG. 1. (Color online) Eigenvalues of $H_{1/3}$ for $N = 12$ electrons on a sphere and flux corresponding to $\nu = 1/3 +$ two quasielectrons. Two-quasielectron states have total angular momentum L ($0 \leqslant L \leqslant 12$), upon which their separation depends inversely. The "hard-core" $L = 12$ energy merges with the continuum.

closing the gap and destabilizing the ground state. We have thus replicated our perturbative construction of various types of QH liquids at $\nu = 1/3$ within a LLL treatment.

The 1/3 Laughlin state is just one example of a much larger (indeed, infinite) class of microscopic wave functions inspired by conformal field theory which are gapped and exact ground states of short-range Hamiltonians. It is believed that these criteria are satisfied by all states belonging to the so-called Read-Rezayi (RR) sequence[19] and their particle-hole conjugates. The RR states have filling $\nu_{k,m} = k/(km + 2)$, where k is a nonzero positive integer, and m is odd for QH states of fermions; their wave functions obey a generalized Pauli principle, and they can be obtained as the densest zero-energy states of $k + 1$ body model Hamiltonians. The $k = 1$ case corresponds to the Laughlin states, while $k = 2$, $m = 1$ corresponds to the Moore-Read (Pfaffian) state.[20]

All these model Hamiltonians have noninteracting quasiholes and weakly (dominantly repulsively) interacting quasielectrons and thus exhibit the characteristics of a Bogomol'nyi point. It follows that by perturbing them we can find various members of our doping typology.

Plateau formation. Plateau formation in a QHL refers to the invariance of the $T \to 0$ conductivity tensor as the density is varied ("doped") over a nonzero range about the commensurate density, $\rho^* = B/k\Phi_0$. For this to occur, the doped charge must be pinned, so it does not contribute to the dc transport. For macroscopically type II fluids this localization can arise from disorder as commonly assumed in the theory of the QHE but potentially also from interactions alone.[21] For type I fluids with no disorder, the transport properties of the macroscopically phase separated state can depend on details of geometry and the nature of the boundary conditions. However in two dimensions arbitrarily weak disorder prevents macroscopic phase separation[22] and leads to plateau formation as discussed on phenomenological grounds in Ref. 23. We note that as the charge of the minimum deconfined charged excitation is the same for all fluids derived from the same parent state, all of them will exhibit a combination of activated and variable range hopping transport at low temperatures.

Experimental realizations. Thus far, we have been primarily concerned with a point of principle—establishing a doping typology of QH fluids. To this end, we have mostly considered model interactions which differ substantially from those in typical experimental systems. We now comment briefly on the prospects for experimental realizations of the new members of this typology: (1) Apart from potential cold-atom realizations, experimental systems involve repulsive Coulomb ($1/r$) or (in the presence of a nearby conducting plane) dipolar ($1/r^3$) interactions which limits the likely types to type II, or frustrated type I (which are thus macroscopically type II) QHLs. (2) Paired QHLs currently appear to be the most promising candidates for states which exhibit quasiparticle clumping, as they generically exhibit (frustrated) type I behavior at weak pairing.[10,24]

The prototypical paired state, the Pfaffian, can be tuned to weak coupling either by changing quantum well thickness[25] or using graphene samples with a screening plane.[26] Another example is a bilayer system, with each layer at $\nu = 1/2$. For large layer separation $d \gg \ell_B$, the ground state is simply two decoupled composite Fermi liquids and hence compressible, but pairing of composite fermions between layers becomes increasingly favored as d is decreased; for $d \to 0$ the ground state is an interlayer paired QH liquid.[27] At intermediate $d \gtrsim \ell_B$, the pairing gap will be small reflecting weak coupling so the resulting paired state must be type I. This is an example of a paired state that does not have a model microscopic Hamiltonian but nevertheless shows type I behavior in the appropriate limit.

Concluding remarks. In this Rapid Communication, we have established a typology for doped QH liquids by perturbing about special points where the quasiholes are noninteracting/weakly interacting and yet the QH state is protected by a gap. Conversely, we have identified these special points as poised on the boundary of type I/II behavior. For the CSLG theory this is tied to the mathematics of self-duality. It is interesting to ask whether the self-dual equations have meaning for model Hamiltonians such as $H_{1/3}$. It may be useful to note that at the self-dual point, $\mathbf{j}$ is the purely diamagnetic LLL current which *does* correctly yield the current in quasihole states, and that the band mass m drops out of the remaining equations consistent with purely LLL physics.[28]

Acknowledgments. We are grateful to D. Abanin, B. Estienne, Z. Papic, and R. Thomale for useful discussions. This work was supported in part by NSF Grants No. DMR-1006608 and No. PHY-1005429 (S.A.P., S.L.S.), No. DMR-0758356 (S.A.K.), and No. DMR-0704151 (B.Z.S.), and DOE Grant No. DE-SC0002140 (E.H.R.).

*sidp@berkeley.edu
†kivelson@stanford.edu
‡erezayi@calstatela.edu
§s.simon1@physics.ox.ac.uk
‖sondhi@princeton.edu
¶spivak@u.washington.edu

[1] S. C. Zhang, T. H. Hansson, and S. A. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).

[2] N. Read, Phys. Rev. Lett. **62**, 86 (1989).

[3] S. M. Girvin and A. H. MacDonald, Phys. Rev. Lett. **58**, 1252 (1987).

[4] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).

[5] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).

[6] A. Lopez and E. Fradkin, Phys. Rev. B **44**, 5246 (1991).

[7] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).

[8] M. Greiter, X.-G. Wen, and F. Wilczek, Nucl. Phys. B **374**, 567 (1992).

[9] S. A. Kivelson, D.-H. Lee, and S.-C. Zhang, Phys. Rev. B **46**, 2223 (1992).

[10] S. A. Parameswaran, S. A. Kivelson, S. L. Sondhi, and B. Z. Spivak, Phys. Rev. Lett. **106**, 236801 (2011).

[11] The alternative choice of $\lambda = -k\Phi_0/2m$ is unstable.

[12] G. Dunne, *Self-Dual Chern-Simons Theories*, Vol. 36 of Lecture Notes in Physics (Springer, Berlin, 1995).

[13] R. Tafelmayer, Nucl. Phys. B **396**, 386 (1993).

[14] If our assumption that the quasielectrons uniformly repel at $\lambda = \lambda_B$ turns out to be incorrect, the only effect is that type I-II behavior would be seen for $\delta\lambda > 0$.

[15] E. Babaev and M. Speight, Phys. Rev. B **72**, 180502 (2005).

[16] S. A. Trugman and S. Kivelson, Phys. Rev. B **31**, 5280 (1985).

[17] The caveat allows for topological degeneracies on closed manifolds, in this case 3^g for genus g.

[18] P. Béran and R. Morf, Phys. Rev. B **43**, 12654 (1991).

[19] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).

[20] The p-h conjugated RR states at $\bar{\nu}_{k,m} = 1 - \nu_{k,m}$ reverse the roles of quasiholes and quasielectrons and are ground states of conjugated Hamiltonians.

[21] Doped charges can be localized without disorder by forming quasiparticle/bubble crystals.

[22] Y. Imry and M. Wortis, Phys. Rev. B **19**, 3580 (1979).

[23] S. Kivelson and S. A. Trugman, Phys. Rev. B **33**, 3629 (1986).

[24] S. A. Parameswaran *et al.* (unpublished).

[25] Z. Papić, N. Regnault, and S. Das Sarma, Phys. Rev. B **80**, 201303 (2009).

[26] Z. Papić, R. Thomale, and D. A. Abanin, Phys. Rev. Lett. **107**, 176602 (2011).

[27] G. Möller, S. H. Simon, and E. H. Rezayi, Phys. Rev. B **79**, 125106 (2009).

[28] S. L. Sondhi, Ph.D. thesis, UCLA, 1992.