Conductance of one-dimensional quantum wires with anomalous electron wave-function localization

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We study the statistics of the conductance g through one-dimensional disordered systems where electron wave functions decay spatially as $|\psi| \sim \exp(-\lambda r^{\alpha})$ for $0 < \alpha < 1$, λ being a constant. In contrast to the conventional Anderson localization where $|\psi| \sim \exp(-\lambda r)$ and the conductance statistics is determined by a single parameter, the mean free path, here we show that when the wave function is anomalously localized ($\alpha < 1$), the full statistics of the conductance is determined by the average $\langle \ln g \rangle$ and the power α. Our theoretical predictions are verified numerically by using a random hopping tight-binding model at zero energy, where due to the presence of chiral symmetry in the lattice there exists an anomalous localization; this case corresponds to the particular value $\alpha = 1/2$. To test our theory for other values of α, we introduce a statistical model for random hopping in the tight-binding Hamiltonian.

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I. INTRODUCTION

The phenomena of electron wave-function localization—Anderson localization—in a disordered media has captured the attention of physicists for several decades.[1–3] Nowadays signatures of localization have been found in different physical systems. For instance, experiments with light, acoustic waves, microwaves, and cold atoms have reported evidence of localization.[4,5]

In the standard Anderson localization problem, electron wave functions are localized exponentially in space:

$$|\psi| \sim \exp(-\lambda r), \tag{1}$$

where λ can be identified as the inverse of the localization length. For practical purposes, it is more convenient to define the localization length through measurable transport quantities; for a system of length L, the localization length is defined by the exponential decay of the dimensionless conductance g, or transmission. Since $g \propto |\psi(L)/\psi(0)|^2$ we have that $g \propto \exp(-2\lambda L)$. Thus, the inverse localization length λ is usually estimated by the relation

$$\langle -\ln g \rangle = 2\lambda L, \tag{2}$$

i.e., the average $\langle \ln g \rangle$ is a linear function of L in the standard electron localization problem. Within a noninteracting electron model, a scaling approach of localization has successfully described the statistical properties of electronic transport.[6–9] Within this approach and assuming weak disorder (mean free path much larger than the Fermi wavelength), it has been found that the complete distribution of the dimensionless conductance is determined by a single parameter: the inverse localization length,[10] given by Eq. (2). In general, one might say that there is a good understanding of the statistical properties of the transport in the Anderson localization problem in one-dimensional (1D) and quasi-one-dimensional disordered systems.

On the other hand, anomalous localization of electron wave functions has been found in 1D disordered systems,[11–14] against the general idea that in 1D systems all the electronic eigenstates are always exponentially localized. This problem has been much less studied than the above standard localization phenomena. For instance, a disordered system described by a random hopping tight-binding model was studied in Ref. 11, where it was found that the typical conductance ($\exp\langle \ln g \rangle$) behaves as

$$g_{\text{typ}} \propto \exp(-\lambda L^{1/2}). \tag{3}$$

This unconventional localization of electrons (also named delocalization[14]) can be explained by the presence of a symmetry in the lattice, the so-called chiral symmetry,[13,14] which makes the energy spectrum symmetric around zero energy.[11] The effects of the chiral symmetry in a disordered system were studied also within a scaling approach to localization.[15,16] It was found that there is no exponential localization of the conductance and the logarithm of g is not self-averaging, while the ensemble average $\langle \ln g \rangle$ is not proportional to L, as in the standard Anderson localization, but to $L^{1/2}$, i.e., $\langle \ln g \rangle \propto L^{1/2}$. A similar delocalization has been found in disordered superconducting wires,[16–20] where the Bogoliubov–de Gennes Hamiltonian has additional symmetries.[21] Delocalization at zero energy has been also studied using tight-binding models of spinless fermions with particle-hole symmetric disorder[22] and in 1D systems in the context of phase transitions in random XY spin chains,[23] which is mapped onto the so-called random mass Dirac model; within this model, it was also found[24,25] that $\langle \ln g \rangle \propto L^{1/2}$. In addition, statistical properties of the conductance in two-dimensional (2D) systems under the presence of chiral symmetry has been studied in Ref. 26.

In the present paper we show that the complete distribution of the conductance for anomalous transport (nonstandard exponential localization) can be determined by the value of the average $\langle \ln g \rangle$ and the power α of its dependence on length L, i.e., $\langle \ln g \rangle \propto L^{\alpha}$. Thus, within a model of noninteracting electrons, the microscopic details of the systems (Hamiltonian) do not enter into the description of the statistical properties of the transport, in this sense, the description is *universal*. Our theoretical model is based on a previous study of the conductance statistics of 1D disordered quantum wires where the random configuration of potential scatterers along the wire follows a distribution with a long tail (Levy-type distribution).[27] However, in that paper, the analysis of the transport was restricted to disordered wires where information on the Lévy-type distribution was explicitly introduced into the disorder configuration of the scatterers. Here, we do not need a Lévy-type disorder configuration but a mechanism to produce anomalous localization of the electron wave function, within a single-electron model, e.g., the chiral symmetry. Thus, as we show in this work, the results in Ref. 27 can be applied in general to disordered systems where electron anomalous localization is present. This larger scope of such statistical analysis was overlooked in Ref. 27.

The remainder of this paper is as follows: After presenting a brief review of the results for wires with a Lévy-type disorder, we introduce the random hopping tight-binding model where at zero energy anomalous localization is present. The numerical results of this model will be compared with our theoretical predictions; in particular, we are interested in the conductance distribution. The numerical results from the random hopping tight-binding model at zero energy corresponds to a special case of our theory ($\alpha = 1/2$). To go further and verify our results in a more general way, we introduce a statistical model for the random hopping which allows to study different degrees of localization characterized by the value of α. We finally summarize our results and give some conclusions in the last part of the paper.

II. THEORETICAL MODEL

As we have mentioned, our theoretical model of this work is based on a study of coherent transport in the presence of Lévy-type disorder.[27] We briefly mention that Lévy-type random processes are described by a density probability $q_{\alpha,c}(x)$ with a long tail: for large x, $q_{\alpha,c}(x) \sim c/x^{1+\alpha}$ with $0 < \alpha < 2$ and c being a constant. These kinds of distributions are also known by mathematicians as α-stable distributions.[28–31] Notice that the first and second moments diverge for $0 < \alpha < 1$. Motivated by the realization of experimentally controlled Lévy processes in the so-called Lévy glasses,[32] in Ref. 27 a model was developed to describe the statistical properties of the conductance through a 1D quantum wire where electrons suffer multiple scattering due to scatterers placed along the wire in a random way according to a Lévy-type distribution (see Refs. 33–37 for other examples where Lévy processes have been studied in connection to transport problems). It was found in Ref. 27 that the full statistics of the conductance is determined by the average $\langle \ln g \rangle$ and the exponent α of the power-law tail in the macroscopic limit ($L \gg c^{1/\alpha}$). In particular, it was shown that the complete distribution of conductances $P_\xi(g)$, with $\xi = \langle \ln g \rangle$, is given by

$$P_\xi(g) = \int_0^\infty p_{s(\alpha,\xi,z)}(g) q_{\alpha,1}(z) dz, \tag{4}$$

for $\alpha < 1$, where $q_{\alpha,c}$ is the probability density function of the Lévy-type distribution supported in the positive semiaxis, $s(\alpha,\xi,z) = \xi/(2z^\alpha I_\alpha), I_\alpha = 1/2 \int_0^\infty z^{-\alpha} q_{\alpha,1} dz$, and

$$p_s(g) = \frac{s^{-\frac{3}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{s}{4}}}{g^2} \int_{y_0}^\infty dy \frac{y e^{-\frac{y^2}{4s}}}{\sqrt{\cosh y + 1 - 2/g}}, \tag{5}$$

where $y_0 = \mathrm{arcosh}(2/g - 1)$. Also, it was shown that the average of the logarithm of the conductance depends on L as

$$\langle \ln g \rangle \propto L^\alpha, \tag{6}$$

for $0 < \alpha < 1$, while for values $1 \leqslant \alpha < 2$ the linear behavior ($\langle \ln g \rangle \propto L$) is recovered. From the same model one can also find that the conductance average behaves as

$$\langle g \rangle \propto L^{-\alpha}, \tag{7}$$

for $0 < \alpha < 1$, in contrast to the exponential dependence with L in the standard localized regime. The most interesting effects of anomalous localization are seen for values $0 < \alpha < 1$, so we concentrate in this region, although the case $1 \leqslant \alpha < 2$ can be analyzed within the same theoretical framework.

III. ANOMALOUS LOCALIZATION: $\alpha = 1/2$

Next we consider the tight-binding model with nearest-neighbor random hopping, at zero energy, described by the Hamiltonian

$$H = \sum_n t_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n), \tag{8}$$

where $c_n^\dagger$ and c_n are creation and annihilation operators for spinless fermions, and $t_n (> 0)$ are the random hopping elements sampled from a distribution of the form $P(t) = 1/wt$, $\exp(-w/2) \leqslant t \leqslant \exp(w/2)$, where w denotes the strength of the disorder. This is the so-called logarithmic off-diagonal disorder.[11] As we have mentioned, the model described by Eq. (8) has been found to present unconventional localized states at zero energy,[11–14,22,24,25] whereas for nonzero energy standard localized states are present. To illustrate this fact, we have calculated the conductance within the Landauer-Büttiker approach. In Fig. 1 we show the ensemble average $\langle \ln g \rangle$ as a function of the length of the system (in units of the lattice constant) at zero and nonzero energies. As we can observe $\langle \ln g \rangle \propto L^{1/2}$ at zero energy (main frame), while a linear dependence on L is obtained at finite energy (lower inset), restoring the standard Anderson localization. Additionally, in the upper inset of Fig. 1 we show the average of the conductance $\langle g \rangle$ at zero energy, which depends on the length as $L^{-1/2}$, as given by Eq. (7).

We now show that the complete distribution of conductance is described by Eq. (4). As we have claimed, in order to compare the theoretical and numerical results, we only need the information of the value $\langle \ln g \rangle$ and its power dependence on L, which are taken from the numerical simulation; thus, there are no free parameters in our theory. In Figs. 2 and 3 we show the distribution of the conductance obtained from the numerical simulations (histograms) for two different strengths of disorder and the corresponding theoretical distributions (solid lines) accordingly to Eq. (4). Note that we plot $P(\ln g)$ in the main frames, instead of $P(g)$, since for very insulating cases the details of the distributions are better seen in this

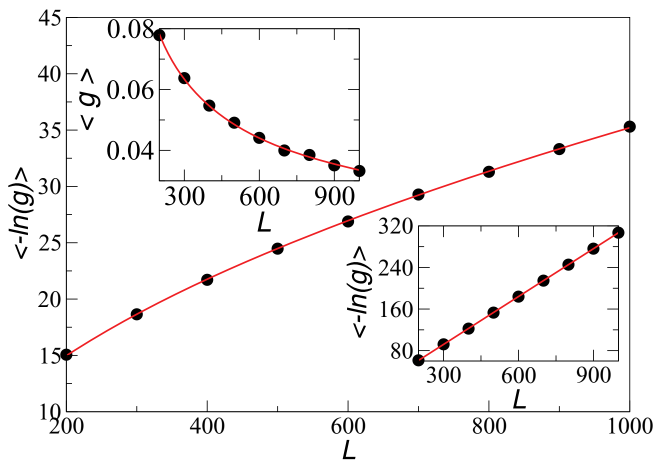

FIG. 1. (Color online) $\langle -\ln g\rangle$ as a function of the length L at energy $E=0$ for strength of disorder $w=2.5$. The solid line is obtained by fitting the data (dots) according to Eq. (6) with $\alpha=1/2$. Upper inset: $\langle g\rangle$ as a function of the length L (for the same parameters in the main frame). The solid line is fitted to the numerical data assuming that $\langle g\rangle \propto L^{-1/2}$. A good agreement is seen. Lower inset: $\langle -\ln g\rangle$ for a linear chain with off-diagonal logarithmic disorder as a function L at energy $E=0.1$ and strength of disorder $w=2.5$ (50 000 realizations). As expected, a linear behavior is observed, indicating Anderson localization.

way. For the smaller case of strength disorder ($w=0.35$) in Fig. 2 we have included $P(g)$ in an inset. Here we can observe two peaks at $g=0$ and $g=1$, which is due to the existence of strong sample-to-sample conductance fluctuations, i.e., in our ensemble a considerable amount of samples behaves as insultors ($g \ll 1$), whereas another important amount of them behaves as ballistic samples ($g \approx 1$). This behavior is very

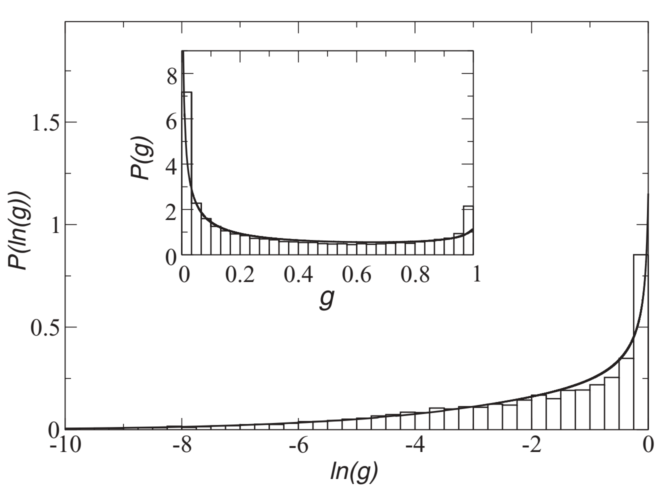

FIG. 2. The distribution of $\ln g$ for a system of length $L=400$ with off-diagonal logarithmic disorder, at energy $E=0$ and strength of disorder $w=0.35$ (50 000 realizations). From the numerical data $\langle -\ln g\rangle = 2.1$. Using this information the theoretical distribution (solid line) is calculated with $\alpha=1/2$, Eqs. (4) and (5). Inset: $P(g)$ for the same case as in the main frame. The coexistence of insulating and ballistic regimes are manifested by the presence of two peaks at $g=0$ and 1. As we can see, the theory (solid line) gives correctly the trend of the numerical results (histograms).

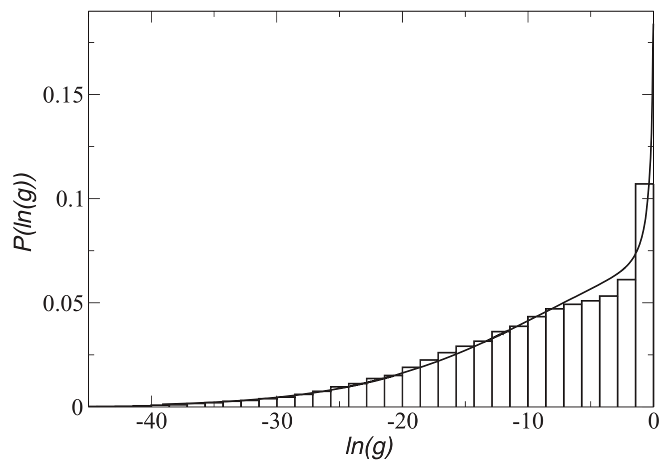

FIG. 3. The distribution of $\ln g$ for strength of disorder $w=1.2$. $\langle -\ln g\rangle = 9.7$ corresponds to a more insulating case than the previous one (Fig. 2), while the power-law dependence on L remains 1/2 (Fig. 1). A good agreement is seen between the numerical histogram and the corresponding theoretical distribution (solid line).

robust in the sense that if we increase the length of the system or the disorder degree, the peak at $g=1$ survives. This is not seen in the conventional 1D electron localization problem. In Fig. 3 we increase the strength of disorder to $w=1.2$. Thus, for both strengths of disorder, Figs. 2 and 3 show that our theory gives correctly the trend of the numerical distribution. We might see a small difference between numerics and theory in the inset of Fig. 2 at $g \approx 1$, but we would like to remark that there is no free parameter in our theory. Therefore our model with $\alpha = 1/2$ describes correctly the statistics of the conductance when anomalous localization of the wave function is of the form $|\psi| \sim \exp(-\lambda L^{1/2})$. However, this is a special case for our model. We would like to explore different exponential power decays α of the wave function.

IV. ANOMALOUS LOCALIZATION: ARBITARY α

In order to investigate different anomalous-localization degrees of the wave function, we introduce a statistical model for the nearest-neighbor random hopping model, Eq. (8). In fact, what we need is a model that will induce large fluctuations of the conductance. A way to introduce such large fluctuations is to consider the hopping t_n as a random variable that follows a distribution with a long tail, i.e., a Lévy-type distribution, and keeping fixed the total sum of the hopping elements: $T=\sum_n t_n$. By varying the value of T we can change the degree of the localization of the disordered samples. We have verified numerically that T acts similarly to the length L in the Lévy-type configurational disorder used in Ref. 27. However, the tight-binding model is more appropriate for numerical simulations. The study is carried out at nonzero energies in order to get rid of the effects of chiral symmetry.

With the above statistical model for the random hopping tight-binding Hamiltonian, we calculate the statistics of the conductance. The data are collected over an ensemble of 50 000 realizations of disorder. In Figs. 4 and 5 we show first the results for the average $\langle \ln g\rangle$ and $\langle g\rangle$ (insets) as a function of T where the random hopping elements are generated from

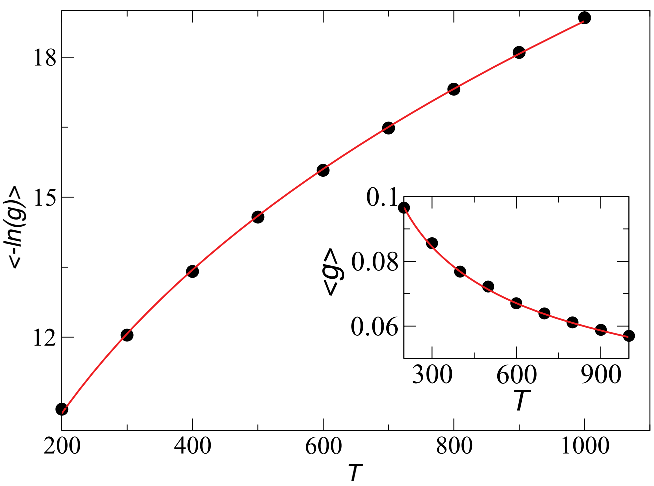

FIG. 4. (Color online) $\langle -\ln g\rangle$ and $\langle g\rangle$ (inset) as a function of the variable T for $\alpha = 1/3$ in the statistical model for the tight-binding Hamiltonian (see text). 50 000 realizations are considered and energy $E = 0.1$. The solid lines are obtained by fitting the power dependence $T^{1/3}$ and $T^{-1/3}$ for $\langle -\ln g\rangle$ and $\langle g\rangle$, respectively, as predicted by the theoretical model.

two different Lévy-type distributions with tail decay exponents $\alpha = 1/3$ and $3/4$. We can see that indeed $\langle \ln g\rangle \propto T^{\alpha}$ and $\langle g\rangle \propto T^{-\alpha}$, for both values of α. Keeping in mind that T plays a similar role as L in our configurational disorder model,[27] we expect that the wave function is anomalously localized as $|\psi| \sim \exp(-\lambda L^{1/3})$ and $|\psi| \sim \exp(-\lambda L^{3/4})$, for $\alpha = 1/3$ and $3/4$, respectively.

We now show that the distribution of the conductance is described by Eq. (4). For $\alpha = 3/4$ and two different values of T, in Figs. 6 and 7 we compare the numerical simulations (histograms) and the corresponding theoretical results (solid line). The case in Fig. 6 is less insulating than the one in Fig. 7, so we plot in an inset the distribution $P(g)$. For the more insulating case (Fig. 7), we can observe a nonconventional shape of the distribution $P(\ln g)$. By nonconventional shape

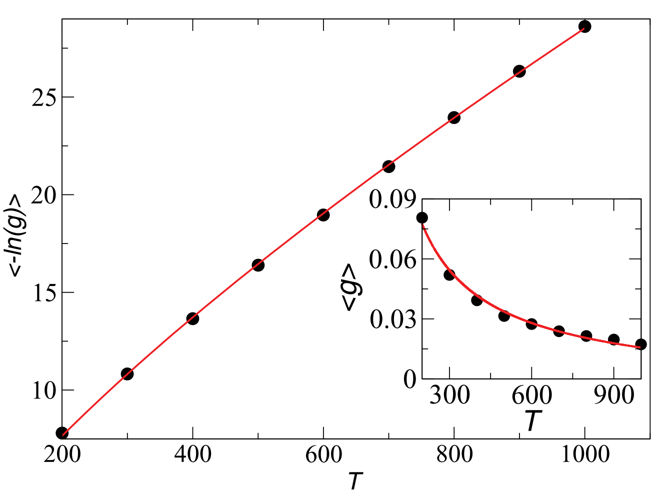

FIG. 5. (Color online) Numerical data (dots) of $\langle -\ln g\rangle$ and $\langle g\rangle$ (inset) from the tight-binding model at energy $E = 0.1$ and $\alpha = 3/4$. The solid lines are fitted assuming that $\langle -\ln g\rangle \propto T^{3/4}$ and $\langle g\rangle \propto T^{-3/4}$, in agreement with the model, Eqs. (6) and (7).

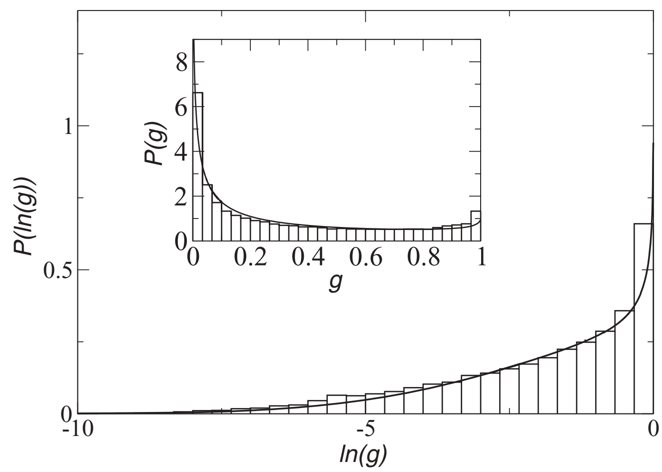

FIG. 6. The numerical distribution $P(\ln g)$ (histograms) for $\alpha = 3/4$ with $E = 0.1$, $T = 35$, and $\langle -\ln g\rangle = 2.0$. Inset: $P(g)$ for the same parameters of the main frame. The solid line is obtained accordingly to Eq. (4). A good agreement between numerical and theoretical (solid line) results is seen.

we mean the non-Gaussian shape of the distribution; we recall that for the standard Anderson localization a log-normal distribution is expected in the insulating regime. Thus, from both Figs. 6 and 7 we can see that the trend of the numerical distributions is well described by our theory. Finally in Fig. 8 we show the distribution $P(\ln g)$ for $\alpha = 1/3$. Here we also note the nonconventional shape of the distribution, which is a consequence of the anomalously large conductance fluctuations.

V. CONCLUSIONS

To conclude, in this work we have shown that the complete statistics of the conductance of a 1D disordered system, when electron wave functions are anomalously localized ($\psi \sim \exp(-\lambda r^{\alpha})$, $0 < \alpha < 1$), is determined by the exponent α and the average $\langle \ln g\rangle$. In contrast, in the standard Anderson localization, the knowledge of $\langle \ln g\rangle$ is enough to describe

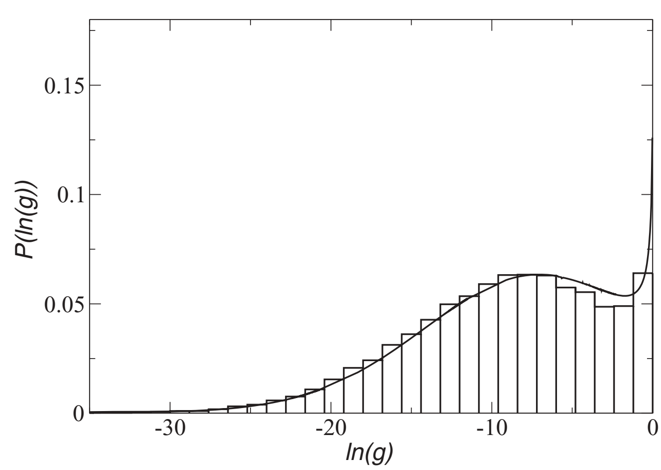

FIG. 7. The distribution of $\ln g$ for $\alpha = 3/4$, $T = 250$ and $E = 0.1$. For this case $\langle -\ln g\rangle = 9.3$. We can see that the theoretical result (solid line) describes correctly the numerical distribution.

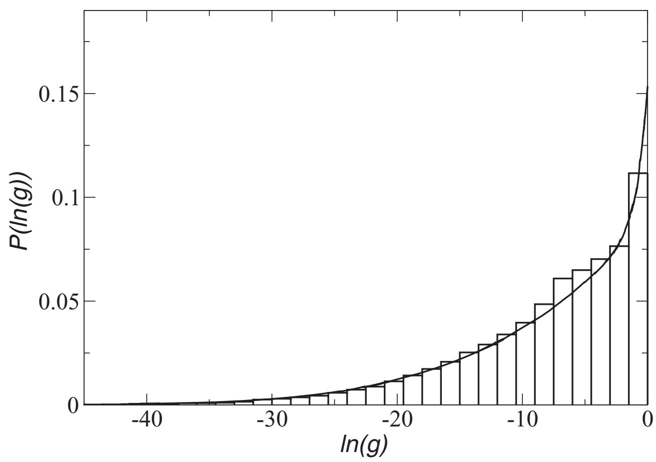

FIG. 8. The distribution of $\ln g$ for $\alpha = 1/3$, $T = 100$, and energy $E = 0.1$. $\langle -\ln g \rangle = 9.4$ for this case. Comparison with the corresponding theoretical distributions is shown. A good agreement between theory and numerics is seen.

the statistical properties of the conductance. We have verified our results for different values of α. For the particular case of $\alpha = 1/2$, we have used a random hopping tight-binding Hamiltonian at zero energy to verify our predictions since it is well known that nonexponential localization in this model is present due to the existence of a chiral symmetry on the lattice. In order to study other degrees of anomalous localization (different values of α) we have introduced a statistical model for the hopping in a tight-binding Hamiltonian that promotes the presence of large fluctuations of the conductance. We remark that our theoretical model does not make any reference to a specific Hamiltonian system and there is no free parameter; the information needed in our theoretical model (α and $\langle \ln g \rangle$) is extracted from the numerical simulation. On the other hand, as we have restricted our study to 1D systems (one channel), we think an extension to multichannel systems is of interest since other regimes of transport, e.g., the diffusive regime, can be analyzed. Finally, the conductance statistics in the conventional Anderson localization problem has been extensively studied, and we hope this work helps in the understanding of a much less studied topic in quantum transport: the statistical properties of the conductance when electron wave functions are anomalously localized.

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